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A Fuzzy ARDL Model for Estimating the Dynamic Relationship between Government Revenues and Fiscal Balance

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Abstract The relationship between government revenues and the fiscal balance represents a central pillar in the analysis of fiscal sustainability. However, its modeling faces a fundamental challenge in the form of structural uncertainty, which is not captured by point estimates in traditional models such as ARDL, as these models assume structural stability that is inconsistent with the nature of rentier economies. The current study aims to develop a fuzzy framework by constructing a Fuzzy Autoregressive Distributed Lag (FARDL) model. This is achieved through integrating the Autoregressive Distributed Lag (ARDL) approach with fuzzy logic theory, thereby enabling the incorporation of uncertainty into the inherent structure of the economic relationship rather than confining it to the stochastic error term. The fuzzy parameters are estimated using a Quadratic Programming (QP) algorithm, and the proposed model is empirically applied to quarterly data for the Iraqi economy over the period (2013-2025). The results provide evidence of a long-run equilibrium relationship between government revenues and the fiscal balance, alongside a persistent structural tendency toward fiscal deficits. Government revenues are shown to have a positive impact on the fiscal balance, however, this impact is uncertain due to revenue uncertainty and resource efficiency. The results of stability tests reveal the dynamic stability of the model, suggesting the existence of a self-equilibrating mechanism. Additionally, the difference between the ARDL and FARDL models shows a descriptive superiority of the fuzzy model in terms of forecasting performance (in-sample and out-sample) with statistically equivalent predictive efficiency. At the same time, the FARDL model offers a more flexible representation with the possibility of interval estimates and multiple scenarios, thus improving the analysis of fiscal sustainability under structural uncertainty.

Keywords ARDL, FARDL, Public Revenues, Fiscal Balance, Quadratic programming

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1. Introduction

Fiscal surplus/deficit is one of the most conspicuous macroeconomic indicators hence the financial stability of the financial state and the sustainability of fiscal policy since it represents the ability of the government to maintain a balance between government revenues and expenditure obligations over time [1]. Current fiscal losses have proven to have consequences on the levels of national debts, macroeconomic stability, inflation as well as the exchange rates [2]. Therefore, achieving balance between government incomes and spending is a core goal that all governments are universally aiming at achieve and the dual goals of strengthening the financial stability and increasing fiscal space to both respond to economic shocks. Evidence-based studies indicate that many sovereign states tend to face fiscal deficits which is defined as the gap between government revenues and government expenditures over time. However, the scale and orientation of the economic consequences that follow are largely dependent on the cause of the deficit. When the deficit can be ascribed to increased state spending on critical infrastructure that fosters both short-term and long-term economic benefits, the revenue flows that the deficit will generate can sufficiently offset the deficit in the long run, thus reducing its risks of being a long-term menace to the economy. On the other hand, deficits so caused by sudden declines in price of export commodities, shrinkages in tax base or partisan spending often results in severe macro-economic shocks that undermines national stability [3], [4].

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Although the fiscal surplus or deficit is theoretically defined as the difference between public revenues and government expenditures, public revenues represent the most volatile and flexible component in determining variations in the budget balance, especially in oil economies where the government's revenue base is largely dependent on oil revenues that are tied to the international oil prices; by contrast, government expenditures tend to be relatively more rigid and less adjustably in the short run, given existing government commitments, which makes the fiscal surplus or deficit largely susceptible to fluctuations in public revenues. In this respect, government revenues usually consist of oil revenues or natural resource rents (in rentier economies), followed by different tax revenues, plus other revenues such as government fees, administrative fees, and revenues from state-owned enterprises [5]. Most economic literature has focused on examining the relationship between government revenues and government expenditures within what is commonly referred to as the Revenue-Expenditure Nexus literature [6], [7]. Many studies have employed a variety of econometric models, including traditional linear models such as the Autoregressive Distributed Lag (ARDL) models, co-integration approaches, and causality tests. For instance, one study analyzing the relationship between government expenditure and public revenues in the Gulf Cooperation Council (GCC) countries during the period (1975-2000), using co-integration techniques and the Error Correction Model (ECM), aimed to identify the direction of causality between the two variables. The results revealed the existence of a long-run equilibrium relationship between revenues and expenditures, along with evidence of unidirectional causality in most cases running from revenues to government expenditure, indicating that expenditure levels were largely determined by fluctuations in oil revenues in the sampled countries [8]. The results of a study that sought to analyze the long-run relationship and dynamic interactions between government revenues and government expenditures in Nigeria during the period (1970-2008), using the Bounds Testing approach within the ARDL framework, revealed the existence of a long-run relationship when government expenditure was treated as the dependent variable. However, no long-run relationship was found when revenues were considered the dependent variable, thereby supporting the Tax-Spend Hypothesis [9]. Similarly, another study applied to the Iranian economy over the period (1978-2011), employing the Toda-Yamamoto Granger causality test, provided further empirical support for the same hypothesis [10]. In contrast, another study examined the relationship between government revenues and public expenditure in Tanzania during the period (2000-2017), using quarterly data and several econometric tests, including the Johansen co-integration test and the Granger causality test. The results indicated the absence of a co-integration relationship between the two variables, while supporting the Spend-Tax Hypothesis, as the direction of causality was found to run from government expenditure to government revenues, suggesting that the government determines expenditure levels prior to mobilizing public revenues [11]. Instead, another study evaluated the nonlinear relationship between state spending and local revenues in the six Central and Eastern Europe nations, namely Croatia, the Czech Republic, Hungary, Poland, Romania, and Slovenia, during the 1995-2016 period by using quarterly data and a Nonlinear Autoregressive Distributed Lag (NARDL) co-integration model. The analysis revealed that fiscal synchronization was present in Slovenia, the Spend-Tax hypothesis was confirmed in the Czech Republic, and the Tax-Spend hypothesis was proven in Croatia and Hungary but Romania and Poland displayed fiscal neutrality in the long-run. In addition, the asymmetric long-run effects were evident in some states with short-run asymmetries dominating all through the sample [4]. Similarly, the other study investigated the asymmetric relationship between government revenues and the government expenditure in Turkish economy throughout the period (2006Q1-2019Q3) using the quarterly data and an asymmetric causality test. The results were in support of the fiscal synchronization hypothesis in the case of total revenues and expenditures, and the Spend-Tax hypothesis was validated with tax-revenues alone. The results also indicated the causality in both directions and asymmetry effects in that, as revenues decrease, government expenditures also get cut [12].

Although there is methodological variety in the available literature on the topic of applied studies, most studies have focused their attention on the relationship between government revenues and public expenditure in the Revenue-Expenditure Nexus model where the two variables are viewed as the main units of analysis to test the causal directions as well as explaining the government fiscal decision-making processes. By comparison, the aggregate variable, the general budget balance (fiscal surplus/deficit) which is a product of the dynamic relationship between revenues and government spending has received relatively little attention in the econometric analysis more generally, and in dynamic econometric modeling in particular. This narrow attention could be in part explained by the fact that the current vision of the fiscal surplus or deficit was presented as a simple result of accounting generated by the discrepancy between the revenues and expenditures instead of a behavioral variable that could be directly modeled. However, this point of view may hinder a better comprehension of the actual dynamics of fiscal sustainability, as the balance of the budget in reality is the reaction of the fiscal system to the economic shocks and temporal interdependences of fiscal policy, which makes its direct modeling more suitable to the study of long-term fiscal sustainability. Furthermore, the established literature has largely used traditional econometric models, which are based on point-parameter estimation and, thus, implicitly, assume structural invariance of economic associations, as well as the possibility of estimating parameters in a statistically significant manner [13]. In that kind of a paradigm, the uncertainty is coded through the error term and confidence ranges that represent the probabilistic uncertainty of the stochastic aspect of the model [14]. However, macro-fiscal adjustments, particularly in rentier economies can be characterized by a high level of structural uncertainty which is characterized by the volatility of revenues, exogenous shocks, new institutional arrangements and shifting over time of fiscal policy orientations [15]. In turn this assumption of the constant level of the parameters that drive economic dynamics is limiting in explaining observed fiscal behavior [16]. To overcome these limitations, recent research has shifted focus to building more flexible models that attempt to capture uncertainty more broadly, either in a probabilistic or dynamic setting. This is done by making parameters non-constant, time-varying or switching between regimes, which will be elaborated on in the next subsection.

In terms of the stated research gaps, the paper is going to suggest the creation of a dynamic econometric model to investigate the nexus of government revenues and the overall budget balance. This framework is realized via extension of the Autoregressive Distributed Lag (ARDL) approach, through incorporation of the fuzzy logic theory, hence resulting to the Fuzzy Autoregressive Distributed Lag (FARDL) model. The associated model is helpful in exploring the dynamics of fiscal sustainability when the uncertainty in the structure exists.

The main scientific contributions of this study can be summarized in three principal aspects:

1. This work also provides the methodological contribution in the sense of developing the Fuzzy Autoregressive Distributed Lag (FARDL) model and thus allows incorporating structural uncertainty directly into the parameters of the dynamic system.
2. Another empirical contribution of the research, though, is that it uses the FARDL framework to examine the dynamic interdependence between the government revenues and the overall budget balance in the Iraqi economy; it is characterized by high degree of uncertainty inherent in the instability of the oil revenues and the subsequent external economic shocks.
3. Lastly, the paper provides a different analytical prism to fiscal scholarship by explicitly modeling the impact of government revenues on the overall balance of the budget (fiscal surplus/deficit) as an autonomous dynamic variable, thereby making it easier to understand the dynamics of fiscal sustainability as compared to traditional Revenue-Expenditure Nexus approaches.

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1.1. Structural Uncertainty and the Choice of Fuzzy ARDL

Economic uncertainty has been recently defined as a latent variable corresponding to the dynamics of the variance of economic and financial variables. It cannot be effectively represented as a directly observable variable. Its impacts are not only on variance, but also on the stability of the relationships among the variables, having non-constant and/or not well identified parameters [17]. In this research, uncertainty is used to refer to a latent variable that cannot be directly measured and arises from real economic phenomena such as variability in oil revenues, intermittent fiscal policy actions and regional and international geopolitical events. The effect of public revenues on the budget surplus/deficit is assumed to exist but the direction and the magnitude of this effect is uncertain. As such, this uncertainty is captured by fuzzy coefficients that express ranges of values rather than a single value, thus capturing the variability in the effect without specifying a particular magnitude or direction. This is in line with the long-run equilibrium hypothesis in the fiscal sustainability literature [18].

Building on this idea, many studies have tackled the modelling of this form of uncertainty in a number of ways, including the (Bayesian VAR) model. This approach considers the parameters as random variables that follow probability distributions by combining prior knowledge with data in a Bayesian framework, rather than as constants, thus capturing the uncertainty of the parameters. While this approach has proven to be effective in enhancing the

statistical stability of models, it still assumes that uncertainty can be defined in terms of a known or estimated probability distribution [19]. Similarly, other research has shifted towards modeling structural uncertainty using Time-Varying Parameter (TVP) models, which assume that the model parameters are not fixed but instead vary over time in response to the changing economic environment. This is done within a dynamic probabilistic framework that enables the tracking of the evolution of the parameters over time [20]. Building on these models, more sophisticated models have been developed that combine regime switching and dynamic variation in model parameters, such as Markov-Switching (MS) models [21], and hybrid models such as the Markov-Switching Autoregressive (MSAR) model. These models allow for economic relationships to switch between regimes (such as normal and crisis) with certain probabilities, allowing for abrupt changes [22]. Furthermore, models like (MSAR-TVP) combine dynamic changes within regimes with sudden shifts between regimes, in order to better capture the dynamics of the economy and to enhance forecasting accuracy [23]. Other research has also tackled the representation of structural uncertainty based on the (Bai-Perron) test for structural breaks in time series, by assuming that parameters are changing at some dates, where changes are assumed to be abrupt at some dates [24]. However, while significant, these approaches are still based on the idea that uncertainty can be represented with a degree of accuracy either in the form of probability or dynamics. This may not be suitable in the case of ambiguity that arises from the very inability to determine the true values of the parameters in the first place, rather than merely their uncertainty or change over time.
In a similar vein, some research has attempted to overcome this kind of uncertainty by using fuzzy regression techniques, which treat parameters as intervals or fuzzy numbers instead of single-point values, thus enabling the incorporation of uncertainty and vagueness in the data and economic relationships [25]. Although fuzzy regression models are important and efficient in performance [26], most of them have been used in static or non-dynamic settings, which means that they are unable to account for complex temporal relationships in economic time series. In this sense, the (FARDL) model stands out as it does not simply model the dynamics of the parameters or try to restrict them, but rather treats them as fuzzy intervals that capture a range of uncertainty. This offers a more accurate representation of structural uncertainty due to the nature of economic relationships, especially in an uncertain and unmeasurable environment.

1.2. Research problem

The relationship between government revenues and the general budget balance in rentier economies, particularly the Iraqi economy, faces a methodological challenge stemming from structural uncertainty linked to oil price volatility, shifts in fiscal policies, and economic shocks. Additionally, geopolitical conflicts and disturbances further disrupt resource flows and exacerbate financial fluctuations, leading to pronounced instability in public budget dynamics. In such an unstable environment, traditional econometric models such as the Autoregressive Distributed Lag (ARDL) model may be limited in their ability to capture this non-deterministic behavior due to their reliance on fixed parameters. In contrast, the Fuzzy ARDL (FARDL) model offers a more flexible framework that accommodates uncertainty through fuzzy parameters, thereby potentially improving the characterization of the relationship between government revenues and the general budget balance.
Based on this, the following questions may be identified to formulate the research problem:

1. Can the fuzzy parameters in the FARDL model capture the range of potential responses of the general budget balance to revenue shocks more flexibly than conventional models?
2. Does the FARDL model achieve superior predictive performance, in terms of forecast accuracy and prediction interval coverage, compared to the traditional ARDL model?

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1.3. Research Objective

The study aims to develop a Fuzzy ARDL model by integrating the ARDL methodology with fuzzy logic in order to represent uncertainty within the model parameters. It also seeks to analyze the dynamic relationship between government revenues and the public budget balance in Iraq over the period (2013-2025).

1.4. Research Importance

The importance of the present work is that it gives the importance of the general budget balance as one of the main indicators of fiscal sustainability, thus going beyond the traditional revenue expenditure model The study also suggests a more flexible econometric system that supports a structural uncertainty, particularly, in economies that strongly depend on oil incomes.

1.5. Research Hypothesis

The hypothesis is that the Fuzzy Autoregressive Distributed Lag (FARDL) model is better in relation to the traditional ARDL model in estimating the connection between fiscal balance and government revenues. This hypothesis is empirically tested by establishing whether the FARDL model gives lower values to both in-sample and out-of-sample forecast error measures and its capacity to give fuzzy estimates in intervals indicative of uncertainty in contrast to the traditional ARDL model forecasts given as point estimates.

1.6. Research Gap

Most studies have focused on the relationship between government revenues and government expenditures within the Revenue-Expenditure Nexus framework without directly modeling the fiscal balance. Moreover, the existing literature has not addressed the application of the FARDL model to represent structural uncertainty in fiscal sustainability analysis.

2. Methodology

The current research employs the Fuzzy Autoregressive Distributed Lag (FARDL) model to examine the dynamic correlation between the government revenues and fiscal balance under the condition of structural uncertainty. The method tries to integrate ARDL model with fuzzy logic to harness the uncertainty in the model structure. The Quadratic Programming (QP) method is used to estimate the fuzzy parameters.

2.1. Fuzzy logic theory

Fuzzy logic, first proposed by Lotfi A. Zadeh in 1965 [27], is a mathematical paradigm of encoding non-probabilistic uncertainty with variable degrees of membership to elements in the closed unit interval $[0, 1]$, avoiding the traditional dichotomous classification based on the absolute membership or non-membership. This paradigm makes it easy to describe situations that cannot be accurately defined in deterministic terms using numerical measures, since instead of a point estimate variables or model parameters are represented as fuzzy sets that contain a range or continuum of plausible values. Accordingly, the theoretical foundations of fuzzy regression models are based on this principle, where uncertainty is represented in the model structure itself, and not confined to the statistical error term.

2.2. Fuzzy Regression Analysis

The concept of fuzzy regression was first introduced in 1982 by (Tanaka et al) [28]. As an extension of traditional regression models designed to address situations in which the relationship between variables is characterized by ambiguity or structural instability [29]. Fuzzy regression is considered an appropriate alternative in cases where some assumptions or properties of the conventional model are not satisfied [14], or when insufficient data are available to construct a reliable econometric model. Moreover, it allows researchers to relax strict model assumptions such as normally distributed residuals, independence, and homoscedasticity [30].

Unlike probabilistic regression models, which attribute uncertainty to random disturbances represented in the error term, fuzzy regression allows explicit description of structural indeterminacy represented by the model parameters. In this paradigm, the coefficients are estimated as fuzzy numbers, which represent a set of possible values as opposed to a single point estimate [31]. This methodology provides an appropriate methodological tool to use in economic studies whereby the structures of time are faced with challenges of institutional reform, fiscal restructuring, and economic shocks. Fuzzy regression assumes that non-conformance between observed and estimated values are due to ambiguity and imprecision in specification of the structural model i.e. inability to estimate the system parameters with certainty, not due to measurement error or missing influential variables, as is usually assumed in classical regression [32]. The different compounds of fuzzy regression models have been advanced depending on the type of variables and parameters used [33], This study adopts the case of crisp variables, where all variables are expressed in precise numerical values, while the coefficients are represented in a fuzzy manner to capture the inherent uncertainty in the relationship between the variables.

3. Fuzzy Autoregressive Distributed Lag Model – FARDL

The steps for constructing the FARDL model are represented in the following stages:

3.1. Model Specification

The functional specification of the FARDL model could be expressed as a fuzzy linear expression integrated into a linear system associated with one dependent variable Y_t, and one explanatory variable X_t. The two variables are all crisp and their respective lagged values are also crisp and precisely determined. The model also includes a constant and a deterministic trend component. In case of a sample space with T observations, the model is as follows:

$$\widetilde{Y}_t = \widetilde{\alpha}_0 + \widetilde{\psi}_t + \sum_{j=1}^{p} \widetilde{\phi}_j Y_{t-j} + \sum_{i=0}^{q} \widetilde{\theta}_i X_{t-i}, \quad t = 1, 2, \ldots, T$$
$$Y, X \sim I(0) \text{ or } I(1), \quad j = 1, 2, \ldots, p, \quad i = 0, 1, \ldots, q \tag{1}$$

Where: Y_t: represents the financial surplus or deficit. X_t: represents the government revenue. (j) : represents the lagged values of the dependent variable and (i) represents the lagged values of the independent variable. $Z = [1 \quad t \quad Y_{t-j} \quad X_{t-i}]^t$:represents the system inputs and is a non-fuzzy (crisp) vector.
$\tilde{A} = \left[\tilde{\alpha}_0 \quad \tilde{\psi} \quad \tilde{\phi}_j \quad \tilde{\theta}_i\right]^t$:represents the system parameter vector, which is expressed in the form of symmetric triangular fuzzy numbers.

3.2. FARDL Model Selection and Lag Order Determination

The process of determining the appropriate lag order in the FARDL model is conceptually similar to that of the conventional model in terms of importance. However, the key difference lies in the mechanism used for selecting the optimal lag structure in fuzzy models, where statistical procedures based on probabilistic distributions cannot be applied. This is due to the absence of a criterion that allows matching the proposed models with the true distribution $g(Y)$ assumed to generate (Y_t), and the lack of a well-defined probability distribution describing the relationship precisely, as is the case with criteria such as the Akaike Information Criterion (AIC) .Accordingly, the lag order of the FARDL model is determined by relying on the autocorrelation function (ACF) to identify the lags of the dependent variable, while the lags of the independent variable are determined using the cross-correlation function (CCF). The CCF test is employed to examine the influence of past values of one time series on another by computing the linear correlation across different time shifts, based on the following expression [34]:

$$c(l) = \frac{\sum (X_t - \bar{X})(Y_{t+l} - \bar{Y})}{\sqrt{\sum (X_t - \bar{X})^2 \sum (Y_{t+l} - \bar{Y})^2}} \tag{2}$$

In this case, (l) refers to the time lag or change between the two time series (i.e. between the dependent variable and the independent variable). It is important to note that to use this test it is a necessary requirement that one

has an integrated series of order $I(0)$. When the test is applied to trending series, it is likely that the high values of correlation will spuriously indicate that there exists a strong relationship between the two series and that such correlation is due to autocorrelation in each individual series as opposed to the relationship between the two series. This means that the correlation patterns and behavior are dependent on the inherent characteristics and inner hierarchy of both series [35] hence to acquire a pure measure which shows the actual relationship between the two series, it is a necessary condition that both series are stationary [34].
If the two series exhibit a common trend, it becomes necessary to perform pre-whitening, which is a preliminary preparation procedure carried out prior to applying the test to ensure that the identified causal relationship is free from the effects of autocorrelation and deterministic trends present in each time series [36]. This procedure ensures the removal of both the general trend and the autocorrelation structure of the series under examination separately.

3.3. Estimation of FARDL Model

The analysis uses interval regression on Quadratic Programming framework proposed by (Tanaka & Lee) [37] as the aim of estimating parameters of the FARDL model and the corresponding spread of each parameter $\{A' = \alpha', c'\}$. This methodology uses mathematical optimization strategies to find parameter ranges which give optimal correspondence between observed and estimated values in a fuzzy environment. Quadratic Programming method stands out as a method that combines the central tendency property of the least squares estimation with the possibility construct of fuzzy regression, thus balancing the accuracy of estimation and the extent of fuzzy uncertainty. The Quadratic Programming problem can therefore be defined as symmetric, triangular fuzzy coefficient based on an objective function which will reduce the total dispersion of the model and this is as formulated below [38]:

The quadratic programming model is defined as follows:

$$\begin{array}{ll} \min J = (c_{\alpha_0}\,|X_0|)^2 + (c_{\psi}\,|t\,|)^2 + \sum\limits_{j=1}^{p}\left(c_{\Phi_j}\,|Y_{t-j}|\right)^2 + \sum\limits_{k=1}^{K}\sum\limits_{i=0}^{q_k}\left(c_{k\theta_{k\,i}}\,|X_{k,t-i}|\right)^2 + \xi\alpha'\alpha & \\ \to \min J = c'\,|Z|\,|Z|'c + \xi\alpha'\alpha & \\ Subjectto & \\ Y_t \geq \;\; \alpha' Z_t - (1-h)\,c'\,|Z_t| & , Z_t = [\,1, t, Y_{t-1}, ...Y_{t-p}, X_t, X_{t-1}, ...X_{t-q}]' \\ Y_t \leq \;\; \alpha' Z_t + (1-h)\,c'\,|Z_t| & , c' \geq 0 \end{array} \tag{3}$$

Here (ξ) represents a positive number with a very small value used to transform the objective function into a quadratic form [37]. The formulation of the Quadratic Programming (QP) problem in the FARDL model is constructed within a framework of linear constraints that ensure the actual observations are contained within the estimated fuzzy bounds, such that each observation lies within the interval defined by the lower and upper limits. In addition, a non-negativity condition is imposed on the spread coefficients $c \geq 0$, ensuring the logical consistency of the estimation and preventing unrealistic or unbounded solutions. From a methodological perspective, these constraints help regulate the behavior of the model and reduce the likelihood of Overfitting, as they prevent uncontrolled expansion in coefficient estimation or the generation of excessively wide fuzzy intervals.
The objective function in Equation (3) contributes to producing a spread vector that is more flexible and less sharp for the model parameters. It seeks to reduce the degrees of spread of the fuzzy parameters, which in turn lowers the overall fuzziness of the model. At the same time, it takes into consideration that the membership degree of each observed value (Y_t) must be greater than or equal to the value of the $(\alpha - cut)$, that is:

$$\begin{gathered} \mu_{Y^*}(Y_t) \geq h, \quad h = (\alpha - cut) \\ \text{for all } t = 1, 2, \ldots, T \end{gathered} \tag{4}$$

This criterion simply expresses the fact that the fuzzy output (response variable) of the model must cover all observed values with a specified degree of the $(\alpha - cut)$. This requirement is referred to as the threshold condition, and the inequality representing the threshold condition can be reformulated as follows:

$$(1-h)c'|Z| - |Y - Z'\alpha| \geq 0, \quad Z \neq 0 \tag{5}$$

It must be observed that the motivation behind the use of triangular membership functions is both due to their simplicity of computation, as well as the paucity of information available to the linguistic variable (model parameters). In cases where there is a need to describe a given linguistic term in a fuzzy logic context, the simplest representation is one which has a central (full) value with upper and lower spread limits. Besides, the use of triangular functions in the modeling of linguistic variables allows the achievement of entropy balance where entropy is taken as the measure of the level of ambiguity in the fuzzy set [39].
The membership function of the fuzzy parameters $(\tilde{A_J})$ is given, according to the methodology described above, in the following form [40]:

$$\mu_{A_i}(a_i) = \begin{cases} 1 - \frac{\alpha_i - a_i}{c_i} & , \alpha_i - c_i < a_i < \alpha_i + c_i \\ 0, & other\ wise \end{cases} \quad i = 1, 2, ..., k \quad , c_i > 0 \tag{6}$$

Where: a_i represents the fuzzy parameter, as illustrated in Figure 1 , c_i represents the spread of the fuzzy parameter, α_i It is the center of the fuzzy set and k represents the number of model parameters.

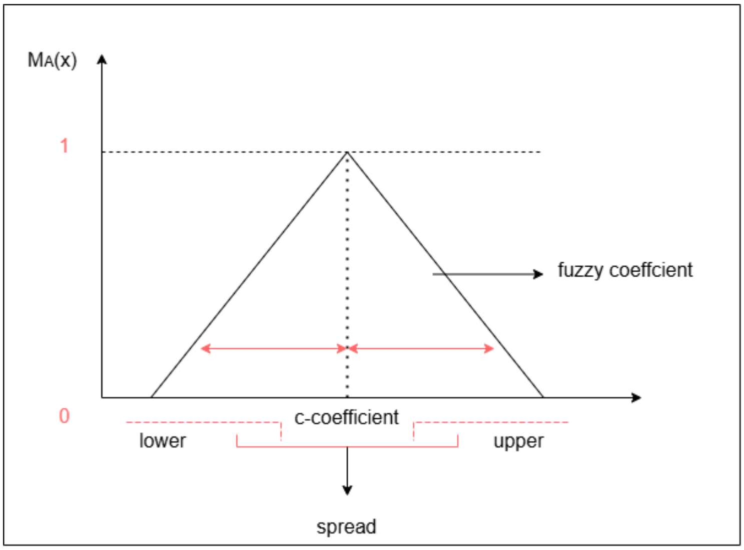


Figure 1. illustrates the triangular membership function of the fuzzy parameter

Accordingly, the fuzzy coefficients can be represented as ordered pairs in the form of (central parameter, spread bounds) as follows:

$$\begin{aligned} &A' = \{\alpha', c'\} \\ &where \quad \alpha' = (\alpha_0^c, \psi^c, \Phi_j^c, \theta_i^c)' \quad , j = 1, 2, ..., p \\ &c' = (c^{\alpha_0}, c^{\psi}, c^{\Phi_j}, c^{\theta_i})' \quad , i = 0, 1, ..., q \end{aligned} \tag{7}$$

Where: (j) : represents the lagged values of the dependent variable, (i) represents the lagged values of the independent variable, α_k^c denotes the central parameter and c^k represents the spread of the parameter.
Accordingly, the FARDL model can be expressed in the following form

$$\begin{aligned} &\widetilde{Y}_t = \widetilde{\alpha}_0 + \widetilde{\psi}t + \sum_{j=1}^{p} \widetilde{\Phi}_j Y_{t-j} + \sum_{i=0}^{q} \widetilde{\theta}_i X_{t-i} \\ &\Rightarrow \widetilde{Y}_t = \widetilde{\alpha}_0 + \widetilde{\psi}t + \widetilde{\Phi}_1 Y_{t-1} + \cdots + \widetilde{\Phi}_p Y_{t-p} + \widetilde{\theta}_0 X_t + \widetilde{\theta}_1 X_{t-1} + \cdots + \widetilde{\theta}_q X_{t-q} \quad = \widetilde{A}Z, \quad t = 1, 2, \ldots, T \end{aligned} \tag{8}$$

Since $(\tilde{A})$ in Equation (8) is a vector composed of fuzzy coefficients with centers α_k and spreads (c^k) the dependent variable (response variable) is also considered a fuzzy set. Based on Derivation (1) in [28], its membership function takes a symmetric triangular form and is given as follows:

$$\mu_{Y*}(Y) = \begin{cases} 1 - \frac{|Y - Z'\alpha|}{c'|Z|} & , c'\,|Z| \neq 0, \\ 0, & , c'\,|Z| = 0 \quad , Y \neq Z'\alpha \\ 1, & , c'\,|Z| = 0 \ \ , Y = Z'\alpha \end{cases} \quad where \quad |Z| = \begin{bmatrix} 1 & |t| & |Y_{t-j}| & |X_{t-i}| \end{bmatrix}' \tag{9}$$

The efficiency of the estimated model may be assessed in terms of the Root Mean Squared Error (RMSE) as a measure of the distance between the actual values and the estimated values from the model, in terms of central parameters rather than residuals (in a probabilistic sense). Similarly, a measure of fuzziness is used in the case of the fuzzy model. In order to improve the assessment of the forecast performance in a more complete way, Theil's U statistic is applied to compare the model's efficiency with that of a naïve model [25] [41]. Further, the Coverage Ratio is used to measure the capacity of the confidence intervals (probabilistic in the case of the ARDL model and fuzzy in the case of the FARDL model) to include the actual values [42]. Finally, the Diebold-Mariano Test is used to test the statistical significance of differences between forecasts from different models, with the emphasis being placed on the fact that this test compares differences in forecast accuracy between the forecasts themselves, not on the quality, efficiency or validity of the models; this test is only used to compare the relative accuracy of the forecasts [43].

4. Experimental results

4.1. Data and Variables

Data and Variables This analysis uses quarterly data of the Iraqi economy during the time (2013Q1-2025Q3) as a whole of 51 observations. The fiscal balance is a dependent variable represented as (Fiscal Balance) and the independent variable or the variable under study is the public revenues represented as (Public Revenues). The data are in the form of quarterly flows and not cumulative stocks and are stated in trillion Iraqi dinars. Data were taken on the official site of the Central Bank of Iraq www.cbi.iq. The dataset was divided into two subsets: 86% of the observations were set aside to create a training set whereas the remaining 14% was set aside as the testing set.

4.2. Unit Root Test

The stationarity of the time series was examined using the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test. The results presented in Table (1) indicate that the study variables are stationary at level and do not contain a unit root, confirming that the stationarity condition is satisfied and justifying proceeding to the estimation of the ARDL and FARDL models to analyze the dynamic relationship between the variables.

Table 1. presents the results of the KPSS test

Variables	**Level**		**Integration Degree**
	LM-Stat.	**Critical Values 5%**	
Intercept			
Public Revenues	0.382288	0.463	I(0)
Surplus/ Deficit	0.081684	0.463	
Trend & Intercept			
Public Revenues	0.142957	0.146	I(0)
Surplus/ Deficit	0.081196	0.146	

4.3. Estimation of the ARDL Model

ARDL(3, 4) specification along with its associated Error Correction Model (ECM) was estimated. Results reported in Table (2) reveal that the long-run effect of public revenues is positive and significant with the long-run relationship supported by the F-bounds test (F = 13.475). The negative and significant error-correction coefficient is an indication of a quick reversion to long-run equilibrium dynamics. As the goodness-of-fit statistics in Table (3) show that the model has satisfactory explanatory power, diagnostic tests in Table (4) show that the model has violated the residual normality, and we should assume that it has marginal heteroskedasticity.

Table 2. Estimation results of the ARDL model and Error Correction Model (ECM).

Form	**Variable**	**Coefficient**	**Std. Error**	**t-Statistic**	**Prob.**
Deterministic Components	Constant	-14.93199	5.743904	-2.599624	0.0143
	Trend	-0.309956	0.125056	-2.478537	0.019
Long Run	Constant	-7.33306415			
	Trend	-0.152215367			
	Public Revenues	0.512217	0.09134	5.607808	0.000
ECM	CointEq(-1)	-2.036299	0.401301	-5.074248	0.000
Short Run	D(Y(-1))	0.625319	0.301053	2.077102	0.0465
	D(Y(-2))	0.345192	0.198307	1.740696	0.092
	D(X)	0.690791	0.207857	3.323392	0.0024
	D(X(-1))	-0.331263	0.22901	-1.446505	0.1584
	D(X(-2))	0.004204	0.193335	0.021744	0.9828
	D(X(-3))	0.583901	0.187005	3.122383	0.004

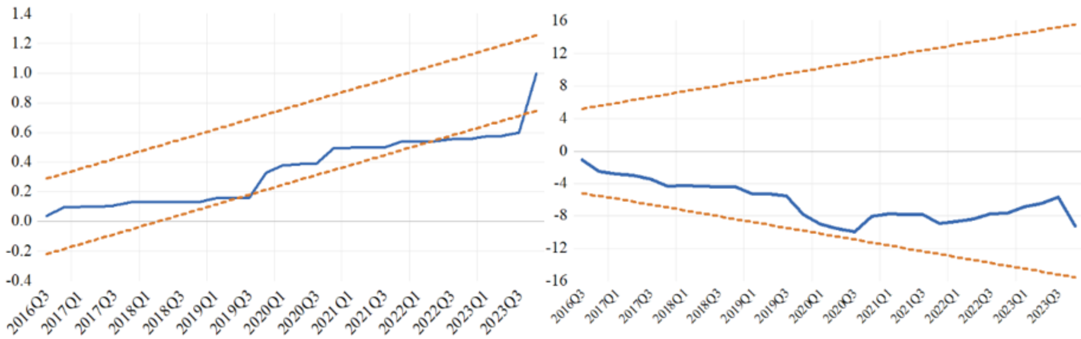
F-Bounds Test				
	Signif. 5%			
F-statistic	Pesaran et al.		Narayan.	
	LB-bound I(0)	UP-bound I(1)	LB-bound I(0)	UP-bound I(1)
13.475	5.56	7.3	7.135	7.98

Table 3. presents the summary of estimation statistics for the ARDL model.

Statistic	**Value**
R-squared	0.484311
Adjusted R-squared	0.329605
S.E. of regression	6.850328
Sum squared resid	1407.81
Log likelihood	-127.9758
F-statistic	3.130515
Prob(F-statistic)	0.008896
Mean dependent var	1.499086
S.D. dependent var	8.36654
Akaike info criterion	6.898788
Schwarz criterion	7.321008
Hannan–Quinn criterion	7.051449
Durbin–Watson stat	1.915772
RMSE	5.933

Table 4. presents the results of the diagnostic tests for the ARDL model.

Breusch–Godfrey Test			
F-statistic	1.0354	Prob(F)	0.4079
Obs*R-squared	5.4961	Prob(Chi-Square)	0.2401
Ramsey Test			
F-statistic	1.6313	Prob(F)	0.2117
Likelihood ratio	2.1891	Prob(Chi-Square)	0.139
Breusch–Pagan–Godfrey Test			
F-statistic	1.7565	Prob(F)	0.1193
Obs*R-squared	13.8042	Prob(Chi-Square)	0.1295
White Test			
F-statistic	2.2155	Prob(F)	0.0495
Obs*R-squared	15.9708	Prob(Chi-Square)	0.0675
Normality Test			
JB-statistic	15.9239	Prob(JB)	0.0003



a) CUSUMSQ b) CUSUM

Figure 2. illustrates the structural stability test of the ARDL model.

Figure 2 shows the results of the CUSUM and CUSUMSQ diagnostics, showing changes in the direction of the ARDL coefficients over time and their closeness to important values at certain time periods. This observation raises the possibility of a temporal instability of the parameters which can be attributed to economic shocks and structural dynamics of the Iraqi economy. It does not refer to the failure of the standard econometric specification, but suggests that the assumption that the parameters are time-invariant is perhaps too draconian to reflect the fiscal interdependences being examined. In this regard, these results provide a methodological justification of the switch to a Fuzzy ARDL estimation framework, which allows the model coefficients to be realized in the range of numbers that allow to capture the structural uncertainty in time, as opposed to depicting the model coefficients as fixed over time.

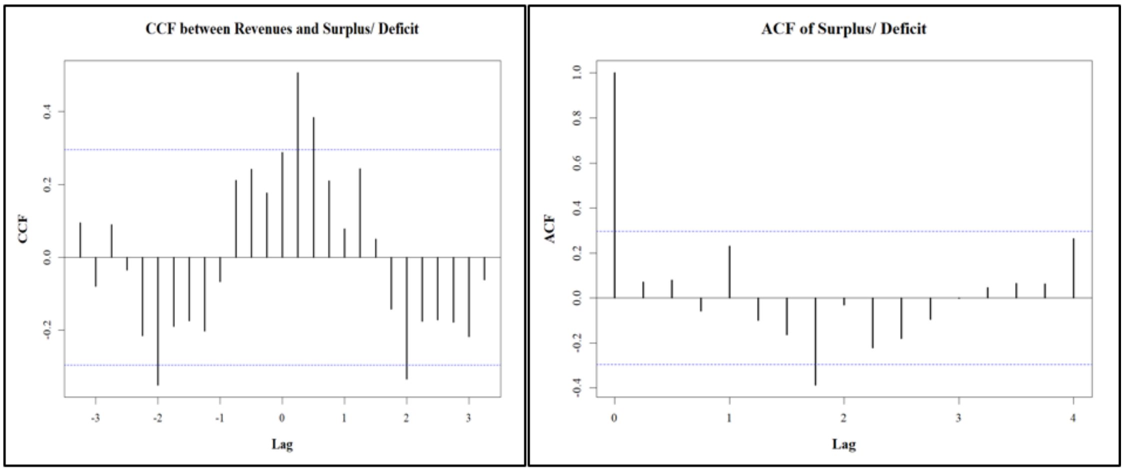
4.4. Estimation of the FARDL Model

After verifying the stationarity of the time series variables under study and specifying the general form of the model, and prior to determining the optimal lag orders and initiating the estimation process, the nature of the long-run co-integration relationship between the variables was examined using the LSTAR test. The results reported in Table (5) indicate the stationarity of the residuals of the co-integration relationship, suggesting the existence of a linear long-run equilibrium relationship and the absence of statistical evidence supporting a nonlinear transition relationship among the study variables. Accordingly, the appropriate methodological foundation for estimating the Fuzzy ARDL model is satisfied. $\alpha - cut$ parameter $(h = 0.1)$ has been chosen to indicate a low level of alpha-cut, consistent with the nature of the study. This option generalizes the structural responses without overstating this margin and thus demonstrates the realistic uncertainty the relationship between the two variables in a study, especially in cases of high oil price volatility and presence of geopolitical factors which in turn influence the revenues of the populace. Regarding the regularization parameter ξ, this was chosen as a very small non-negative positive quantity of $1 * e^{-8}$, with the purpose of curing multiple solution problems, and ensuring the numerical stability of the estimation, but not the decrease of the coefficients or change of their economic interpretation. Based on this, the numerical estimation procedures of the FARDL model were implemented using the R statistical programming environment.

Table 5. KPSS test results for the residuals of the LSTAR model.

Variables	**Level**		**Integration Degree**
	LM-Stat.	**Critical Values.**	
Fourier $(k = 1)$	0.109044	0.463	I(0)
Fourier $(k = 2)$	0.155843	0.463	I(0)
Fourier $(k = 5)$	0.195845	0.463	I(0)

Though the theoretical framework implies that such aspects of information criteria as AIC should not be used in the context of fuzzy models, and rather the autocorrelation (ACF) and cross-correlation (CCF) functions should be used to identify the order of the model, the outcome of these tests Figure 3 did not show a decisive lag structure. Thus, to achieve methodological rigor in estimation, a complementary and systematic process was assumed, which involves testing a small set of plausible lag combinations in a small range (up to a maximum of four lags) based on the initial hints given by the ACF and CCF without uncontrolled model proliferation. Moreover, the Root Mean Square Error (RMSE) was used as a strategically selected selection criterion (criteria by design) to assess the forecasting accuracy and goodness of fit since it offers a quantitative measure of the distance between the estimated and actual values, not a probabilistic measure of the distance based on the residual properties. The results of the estimation, based on consideration of (20) different specifications, showed that FARDL(4,4) model gives the best results in terms of minimizing the error in prediction, and thus, it is reasonable to choose it on the ground of measurable empirical evidence and not arbitrarily. Table (6) summarizes the estimation results of the FARDL(4,4) model.



a) CCF b) ACF

Figure 3. ACF and CCF Plots between Oil Prices and Budget Surplus/Deficit.

Table 6. Estimated coefficients of the FARDL model with long-run, short-run dynamics and model evaluation

Form	Variable	Coefficient	Spread	Upper	Lower
Full Dynamic	Intercept	-12.6136	0.0028	-12.6108	-12.6163
	Trend	-0.2843	0.1099	-0.1743	-0.3942
	Y(-1)	-0.3484	0.0305	-0.3180	-0.3789
	Y(-2)	-0.2325	0.0053	-0.2271	-0.2378
	Y(-3)	-0.2980	0.0126	-0.2854	-0.3105
	Y(-4)	0.1620	0.0019	0.1639	0.1601
	X	0.6930	0.1095	0.8024	0.5835
	X(-1)	-0.0123	0.1142	0.1018	-0.1265
	X(-2)	0.2836	0.0744	0.3580	0.2092
	X(-3)	0.5095	0.0749	0.5844	0.4346
	X(-4)	-0.6107	0.1069	-0.5038	-0.7176
Long Run	Constant	-7.3470		-7.5669	-7.1396
	Trend	-0.1656		-0.1046	-0.2231
	X	0.5027		0.8057	0.2168

Continued on next page

Table 6. Estimated coefficients of the FARDL model (continued)

Form	Variable	Coefficient	Spread	Upper	Lower
Error Correction	CointEq(-1)	-1.7168		-1.6666	-1.7671
Short Run	D(Y(-1))	0.3684		0.3486	0.3882
	D(Y(-2))	0.1360		0.1215	0.1504
	D(Y(-3))	-0.1620		-0.1639	-0.1601
	D(X)	0.6930		0.8024	0.5835
	D(X(-1))	-0.1824		-0.4385	0.0738
	D(X(-2))	0.1012		-0.0806	0.2830
	D(X(-3))	0.6107		0.5038	0.7176
Model Evaluation	RMSE		5.876		
	FD		28.413		

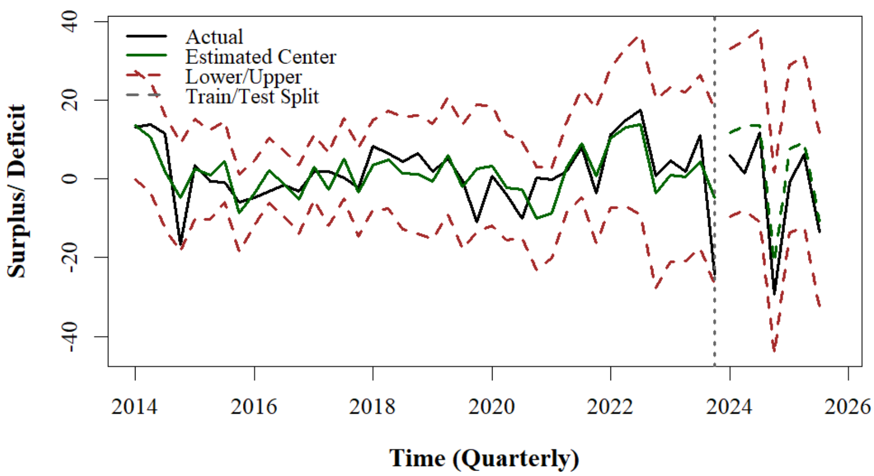


Figure 4. Actual vs. Estimated Fiscal Balance (In-sample and Out-sample)

4.5. Analysis of the Estimation Results of the FARDL(4,4) Model

The estimated results of the Fuzzy Autoregressive Distributed Lag (FARDL) model, as reported in Table (6), show that the dynamic adjustment of the budget surplus/deficit displays a certain degree of persistence over time, since the lagged coefficients of the dependent variable are mostly negative and have relatively small fuzzy ranges, especially at the first lags. This implies the existence of an endogenous mechanism which corrects for short-run fiscal imbalances, hereby providing evidence consistent with fiscal sustainability. The narrow spreads of these coefficients reflect a low degree of uncertainty, indicating relative stability in the self-adjustment dynamics of the surplus/deficit, Moreover, the results of the model's dynamic stability test support this finding, as the roots of the characteristic equation are all located outside the unit circle, which implies that the stability condition is satisfied, thereby confirming the stability of the system and its convergence back to equilibrium in the long run following shocks.

On the other hand, the public revenue coefficients show a mixed pattern, in which the contemporaneous effect of revenues seems positive and unambiguous, while the signs and sizes of the lagged effects differ, implying an asymmetric fiscal response. It is also noted that some lags, such as $X(-1)$, are associated with relatively large spreads compared to their central values, with their fuzzy ranges including zero, implying the short-run impact of revenues may be uncertain or statistically undetermined. This suggests that the effect of some lags of revenues cannot be statistically separated from zero in the fuzzy range, reflecting what can be called implicit insignificance, where the sign of the effect is not determined conclusively rather than just imprecise. This is not a shortcoming of the model, but rather an indication of a high level of uncertainty surrounding the actual fiscal response in the short run, which can be justified by the fact that the revenue coefficients are related to very volatile external factors, such as oil prices and geopolitical shocks, which influence public revenues in an erratic way. This, in turn, relates to the nature of structural uncertainty in rentier economies, where the fiscal response is more vulnerable to shocks, resulting in a broader set of values for the corresponding coefficients.

It is important in this context to distinguish between two types of uncertainty: the first is parameter uncertainty, arising from limited information or estimation precision, and the second is structural uncertainty, resulting from changes in the underlying economic relationship over time. Within the FARDL framework, the spreads reflect a combination of these two types; however, their economic interpretation is more closely associated with structural uncertainty, particularly in the case of revenue related variables, given their volatile nature and dependence on external factors. Moreover, the variation in the magnitude of spreads across coefficients reveals that uncertainty is not homogeneous throughout the model but is instead concentrated in specific components. While the lagged coefficients of the dependent variable exhibit relatively narrow ranges, the revenue coefficients especially at longer lags are characterized by wider intervals. This suggests that the impact of revenue shocks is less certain in the short run but becomes more pronounced and stable in the long run. This pattern clearly indicates temporal asymmetry in the relationship, whereby the effects of revenue shocks are highly ambiguous and indeterminate in the short run, yet become more stable and identifiable over the long run, This is consistent with the overall degree of fuzziness (FD) of the model, which amounts to (28.413), reflecting the level of structural uncertainty inherent in the estimated parameters.

This difference could be explained in the context of the specifics of the Iraqi economy, where oil price volatility and unpredictability in fiscal policies, as well as alterations related to political and economic circumstances, result in the effects of revenues being hard to foresee in the short-run, but more predictable in the long-run as fiscal decisions become more consistent.

The long-run results support this conclusion, as the revenue coefficient exhibits a positive value of (0.5027), within a fuzzy interval ranging from (0.2168) to (0.8057), indicating a relatively stable positive effect of revenues on the budget surplus/deficit in the long run, despite the persistence of some degree of uncertainty. This suggests that, although the impact of revenues is not clearly identifiable in the short run, it gradually materializes over time.

As for the intercept term, it exhibits a spread of (0.0028), which is extremely small relative to its central value of (-12.614), making it close to a crisp value. This does not indicate a weakness in representing uncertainty; rather, it reflects that this coefficient represents a relatively stable structural component of the economy, namely the general tendency toward a fiscal deficit, which is less sensitive to short-run fluctuations. Therefore, there is no statistical necessity to represent it with a high degree of fuzziness. From a methodological perspective, this suggests that the optimization problem within the quadratic programming framework did not require expanding the spread of this parameter, as the solution achieved the minimum possible level of fuzziness while satisfying the model constraints and limiting estimation imprecision. This is consistent with the literature, where the intercept can be treated as a crisp parameter or one with a very limited spread to avoid introducing unnecessary fuzziness into the model when not statistically justified [30], which supports the low spread obtained for this coefficient in the present estimation. In addition, the trend coefficient was estimated at (-0.284), exhibiting a negative sign in both its central and bound values, indicating a downward time trend in the behavior of the surplus/deficit. This reflects the influence of structural and cumulative long-run factors that extend beyond short-term cyclical fluctuations.

With regard to the error correction coefficient, its estimated value reached (-1.72), indicating the presence of a strong adjustment mechanism; however, it exceeds unity in absolute value, suggesting an overshooting adjustment pattern. This can be interpreted within the context of the Iraqi economy, where sharp fluctuations in revenues lead to non-gradual fiscal responses, characterized by rapid expansion in public spending during periods of high oil prices, followed by sharp corrective measures when prices decline. This pattern of unbalanced adjustment results in a temporary overshooting of the equilibrium level before reverting back, reflecting the cyclical and irregular nature of fiscal policy in rentier economies, particularly Iraq. Moreover, the negative sign of the error correction term indicates the existence of an adjustment mechanism toward long-run equilibrium, which is consistent with the economic relationship between government revenues and the budget surplus/deficit. In the fuzzy context, the persistence of this negative sign within its interval further reinforces this interpretation. However, the absence of an explicit significance test for the error correction term within the fuzzy framework represents one of the methodological limitations of this study.

These results can be viewed from the perspective of the fiscal reaction function proposed by (Henning Bohn) [44] as a test of fiscal sustainability, which is obtained when the government adjusts its primary balance in response to imbalances in a manner that stabilizes public debt. While the current study does not explicitly model public debt, the negative lagged coefficients and the existence of evidence supporting a stable long-run equilibrium relationship imply the presence of an implicit fiscal reaction function which helps to dampen imbalances and restore equilibrium, thus indirectly supporting the fiscal sustainability hypothesis.

In the short run, the results indicate that the contemporaneous change in revenues $D(X)$, exerts a strong and positive effect on the budget surplus/deficit, with a central value of (0.6930) and a fuzzy interval ranging from (0.5835) to (0.8024), suggesting a stable and well-defined impact. This reflects that immediate increases in revenues directly contribute to improving the fiscal balance before being offset by subsequent spending adjustments. In contrast, some lagged differences, such as $D(X(-1))$ and $D(X(-2))$, exhibit fuzzy intervals that include zero, implying that their effects cannot be statistically distinguished from zero within these ranges. This reflects what may be described as implicit insignificance and an indeterminate direction of impact. This can be attributed to the fact that fiscal responses to revenues during these periods are influenced by variations in public spending decisions and their timing, whereby revenues may sometimes be directed toward improving the fiscal balance, and at other times toward financing expenditure expansions, leading to overlapping positive and negative effects and resulting instability. This pattern indicates that the short-run impact of revenues is neither constant nor linear, but rather characterized by volatility and dependence on the context of fiscal decision-making, rather than solely on the magnitude of revenues. Meanwhile, the variable $D(X(-3))$ exhibits a positive effect, ranging from a lower bound of (0.504) to an upper bound of (0.718), indicating that part of the impact of revenues does not materialize immediately but rather emerges after several time periods. This pattern reflects the presence of a time lag in the transmission of revenue effects to the budget, which is consistent with the nature of fiscal policy in the Iraqi economy, where increases in revenues are not instantly translated into immediate improvements in the fiscal balance, but are instead associated with gradual or delayed spending decisions. The coefficients of the lagged dependent variable indicate a dynamic pattern characterized by short-run persistence followed by gradual adjustment. The positive signs in the initial lags reflect inertia in the behavior of the budget surplus/deficit, while the negative sign in the third lag indicates the beginning of the adjustment process toward equilibrium. This pattern implies that fiscal adjustment does not occur immediately, but rather takes place through a gradual mechanism with a time lag.

4.6. *Model Evaluation and Performance Comparison*

To compare the forecasting performance of the ARDL and FARDL models, a set of forecasting performance measures was employed, as reported in Table (7).

Table 7. Evaluation and Comparison of Forecast Performance for ARDL and FARDL Models.

Model	**RMSE**		**Theil's U**	**Coverage Ratio**	**DM Test**	
	In	**Out**			**DM Statistic**	**p-value**
ARDL	5.933	7.7885	0.2964	86%	1.1485	0.2945
FARDL	5.876	6.9145	0.2597	100%		

As shown Table (7), the results suggest that the predictive performance of the FARDL(4,4) model is better than the ARDL(3,4) model in quantitative terms. In particular, the FARDL model has lower RMSE in both in-sample (5.876) vs. (5.933) and out-of-sample (6.91445) vs. (7.78846) periods, which suggests that it is more accurate in capturing the actual values. Similarly, Theil's U is lower for the FARDL model (0.25968) than for the ARDL model (0.29644), indicating a higher efficiency of the forecasts compared to the naïve model in both cases, with a clear superiority of the fuzzy model. Furthermore, the FARDL model has a higher probability of including the actual values in the forecast intervals (100% vs. 86% for the ARDL model), which makes it a more suitable model for uncertainty representation. On the other hand, the Diebold-Mariano Test results show that there is no statistically significant difference between the forecasts of the two models (p-value = 0.29447), suggesting that the predictive efficiency of the two models is statistically equivalent, despite the descriptive advantages of the FARDL model. Hence, we can conclude that the FARDL model offers a more flexible approach to analysis and forecasting, not only in terms of predictive accuracy but also by generating three possible scenarios (best case, most likely case and worst case). This characteristic provides a wider view for decision-makers than the conventional point forecasts that only give a single value, thereby improving decision-making efficiency in an uncertain environment.

5. Conclusion

In using the classical ARDL model to estimate the relationship between the state fiscal balance and public revenues, the results showed that there is a long-run equilibrium relationship that has an acceptable explanatory power on the relationship between the economic dynamics of the two variables. Diagnostic tests showed that most model assumptions were met even though there were certain restricted deviations manifested through non-normality of residuals and marginal indications of heteroskedasticity. These results demonstrate the methodological constraints of the conventional econometric models that are of a point-estimation type. We have, in furthering the analysis, chosen a Fuzzy Autoregressive Distributed Lag (FARDL) specification, where the original crisp data set is used and the model parameters are represented in a fuzzy way. This kind of strategy implies that the fuzziness is extended to the inferred economic relationships as opposed to the actual observations. In its turn, the impact of both explanatory variables is described by an approximation interval, rather than a point estimate, thus, representing various possible economic states and allowing a more flexible analysis of the relationship under investigation, FARDL(4,4) model has the best results in reducing the error in prediction and hence it is justifiable to select it based on empirical evidence (measurable) and not by chance. The results of the two models' predictive performance, and in line with the RMSE criterion, suggest a descriptive advantage of the FARDL model in-sample and out-sample. The Theil's U statistic also indicates that it is more predictive efficient than the naïve model. Moreover, the fuzzy model is more capable of capturing the actual values within the forecast intervals, which increases its flexibility in modelling uncertainty. On the other hand, the Diebold-Mariano test results show that the predictive efficiency of the two models does not differ, despite the fact that the FARDL model performs better in describing the data. which can be explained by the small sample size and the low power of the test. Finally, the economic analysis shows that government revenues have a positive impact on the surplus/deficit balance, both in the short and long term. But the impact of this effect is subject to the efficiency of financial resource allocation and the use of funds to attain fiscal stability and promote economic growth. The intercept also appears to be relatively stable, while a structural imbalance and a deficit tendency are also evident. The findings further provide indications of a long-run equilibrium relationship, as the error correction term takes negative values at both the central estimate and its fuzzy bounds, reflecting the system's tendency to return to equilibrium after shocks. However, this result should be interpreted as indicative dynamic evidence, given that fuzzy parameters are not subject to conventional statistical significance tests, which constitutes a methodological limitation that should be taken into consideration. In terms of the dynamic behavior of the model, the stability condition is met, with the roots outside the unit circle, which indicates the stability of the model. As such, it can be argued that the FARDL model provides a more flexible and informative approach to analysis and forecasting, not only in terms of providing more accurate forecasts, but also in terms of producing three possible scenarios for the dependent variable (lower bound, central estimate, and upper bound). This characteristic offers a more comprehensive view to decision-makers than traditional point forecasts, thus improving the efficiency of decision-making in a dynamic economic environment.

Methodological Constraints and Study Limitations

The research is constrained by the fact that it only deals with crisp variables and demonstrates uncertainty as fuzzy parameters in the ARDL model, not being extended to other fuzzy elements like lag selection in a fuzzy setting. Further, the existing literature lacks a detailed framework of implementation of co-integration tests in a fuzzy context, as well as the fact that statistical tools of measuring the significance of fuzzy parameters, such as, the error correction term and the long-run and short-run coefficients are scarce. Moreover, some of the model parameters, including the cut parameter h, were chosen in a deliberate way, informed by methodological choices and data characteristics, and was not computed in terms of numerical estimation algorithms or a traditional probabilistic model. In this regard, the results of this research are a practical step towards the creation of more unified fuzzy models which is an encouraging course of research in the future.

Recommendations

Based on the results of the FARDL model, the study recommends adopting more flexible fiscal policies that take into account the nature of uncertainty surrounding economic relationships, particularly in oil-dependent economies. The central values of the coefficients represent the expected behavior of the relationship between revenues and the budget deficit and can be used as a reference basis in policy design, while the fuzzy spreads reflect the range of uncertainty associated with this relationship. Accordingly, policymakers are advised to utilize these results within a scenario analysis framework, where the lower bounds are used to design conservative fiscal policies that account for worst-case scenarios, while the upper bounds are used to evaluate more optimistic scenarios. Furthermore, relationships characterized by wide spreads should be treated with caution, as they reflect a higher degree of uncertainty, and therefore should not be relied upon decisively in policymaking.

REFERENCES

1. E. Trachanas and C. Katrakilidis, “Fiscal deficits under financial pressure and insolvency: Evidence for italy, greece and spain,” *Journal of Policy Modeling*, vol. 35, no. 5, pp. 730–749, 2013.
2. H. A. Al-Zeaud, “The causal relationship between government revenue and expenditure in jordan,” *International Journal of Management and Business Research*, vol. 18, no. 2, p. 117, 2015.
3. X.-L. Soo, J. S.-H. Kueh, R. Ab-Rahim, and J. T.-H. Yau, “Public debt, current account balance, and fiscal balance: The threshold effects in selected asean countries,” *Business and Economic Research*, vol. 13, no. 3, pp. 23–39, 2023.
4. M. Karakas, T. Turan *et al.*, “The government spending-revenue nexus in cee countries: Some evidence for asymmetric effects,” *Prague Economic Papers*, vol. 28, no. 6, pp. 633–647, 2019.
5. M. S. De Clerck and T. Wickens, *Government finance statistics manual 2014*. International Monetary Fund, 2019.
6. P. K. Narayan and S. Narayan, “Government revenue and government expenditure nexus: evidence from developing countries,” *Applied Economics*, vol. 38, no. 3, pp. 285–291, 2006.
7. A. Adedokun, “The effects of oil shocks on government expenditures and government revenues nexus in nigeria (with exogeneity restrictions),” *Future Business Journal*, vol. 4, no. 2, pp. 219–232, 2018.
8. U. Fasano and Q. Wang, “Testing the relationship between government spending and revenue: Evidence from gcc countries,” 2002.
9. O. Aregbeyen and T. M. Ibrahim, “Testing the revenue and expenditure nexus in nigeria: An application of the bound test approach,” *European Journal of Social Sciences*, vol. 27, no. 3, pp. 374–380, 2012.
10. M. Mehrara and A. A. Rezaei, “The relationship between government revenue and government expenditure in iran,” *International Journal of Academic Research in Business and Social Sciences*, vol. 4, no. 3, p. 171, 2014.
11. K. Kazungu *et al.*, “The nexus between government expenditure and revenue in tanzania,” *Asian Journal of Economic Modelling*, vol. 7, no. 4, pp. 158–170, 2019.
12. H. F. Kaya and Ş. N. Arslan, “The government revenue–expenditure nexus: Asymmetric causality test,” *Current Research in Social Sciences*, vol. 6, no. 2, pp. 170–178, 2020.
13. M. A. Hussein and M. Y. Hmood, “Nonstationary time series models using ardl and nardl estimation,” *Journal of Economics and Administrative Sciences*, vol. 30, no. 141, pp. 439–456, 2024.
14. S. Donoso, N. Marin, and M. A. Vila, “Quadratic programming models for fuzzy regression,” in *Proceedings of international conference on mathematical and statistical modeling in honor of Enrique Castillo*, 2006.
15. S. A. Rasheed, “The impact of oil price volatility on economic growth and stability in iraq through the public expenditure for the period (2003-2020),” *International Journal of Professional Business Review: Int. J. Prof. Bus. Rev.*, vol. 8, no. 6, p. 17, 2023.
16. R. Neck, D. Blueschke, and V. Blueschke-Nikolaeva, “Optimal fiscal policy in times of uncertainty: a stochastic control approach,” *Empirica*, vol. 52, no. 1, pp. 99–120, 2025.
17. A. Carriero, T. E. Clark, and M. Marcellino, “Measuring uncertainty and its impact on the economy,” *Review of Economics and Statistics*, vol. 100, no. 5, pp. 799–815, 2018.
18. I. Chugunov, A. Vatulov, and Y. Markuts, “General government revenue in the system of fiscal regulation,” *Investment Management & Financial Innovations*, vol. 17, no. 1, p. 134, 2020.
19. B. Zhang and B. H. Nguyen, “Real-time forecasting of the australian macroeconomy using flexible bayesian vars,” 2020.
20. Q. Ding, J. Huang, and H. Zhang, “The time-varying effects of financial and geopolitical uncertainties on commodity market dynamics: A tvp-svar-sv analysis,” *Resources Policy*, vol. 72, p. 102079, 2021.
21. M. Guidolin, “Markov switching models in empirical finance,” 2011.
22. P. J. Deschamps, “Comparing smooth transition and markov switching autoregressive models of us unemployment,” *Journal of Applied Econometrics*, vol. 23, no. 4, pp. 435–462, 2008.

23. S. Inayati, N. Iriawan, and Irhamah, “A markov switching autoregressive model with time-varying parameters,” *Forecasting*, vol. 6, no. 3, pp. 568–590, 2024.
24. E. Ngobeni and C. L. Muchopa, “Structural change in the south african agricultural sector: Bai-perron modelling,” *Scientific African*, vol. 21, p. e01732, 2023.
25. F. Gkountakou and B. Papadopoulos, “The use of fuzzy linear regression and anfis methods to predict the compressive strength of cement,” *Symmetry*, vol. 12, no. 8, p. 1295, 2020.
26. K. Ostad-Ali-Askari, “Evaporation and production efficiency modeling using classical least square regression and fuzzy linear recurrence,” *Available at SSRN 4761502*, 2021.
27. L. A. Zadeh, “Fuzzy sets,” *Information and control*, vol. 8, no. 3, pp. 338–353, 1965.
28. H. Asai, S. Tanaka, and K. Uegima, “Linear regression analysis with fuzzy model,” *IEEE Trans. Systems Man Cybern*, vol. 12, pp. 903–907, 1982.
29. P. Pandit, P. Dey, and K. Krishnamurthy, “Comparative assessment of multiple linear regression and fuzzy linear regression models,” *SN Computer Science*, vol. 2, no. 2, p. 76, 2021.
30. K. J. Kim, H. Moskowitz, and M. Koksalan, “Fuzzy versus statistical linear regression,” *European Journal of Operational Research*, vol. 92, no. 2, pp. 417–434, 1996.
31. F.-M. Tseng, G.-H. Tzeng, H.-C. Yu, and B. J. Yuan, “Fuzzy arima model for forecasting the foreign exchange market,” *Fuzzy sets and systems*, vol. 118, no. 1, pp. 9–19, 2001.
32. E. C. Özelkan and L. Duckstein, “Multi-objective fuzzy regression: a general framework,” *Computers & Operations Research*, vol. 27, no. 7-8, pp. 635–652, 2000.
33. A. F. Shapiro, “Fuzzy regression models,” *Article of Penn State University*, vol. 102, no. 2, pp. 373–383, 2005.
34. G. Tóth and B. Balogh, “How to identify cross correlations: a statistical test with time lag and its application on air-pollutant time series,” *Journal of Chemometrics*, vol. 26, no. 3-4, pp. 125–133, 2012.
35. M. J. Alonso, S. Wolf, R. B. Jørgensen, H. Madsen, and H. M. Mathisen, “A methodology for the selection of pollutants for ensuring good indoor air quality using the de-trended cross-correlation function,” *Building and Environment*, vol. 209, p. 108668, 2022.
36. S. Bisgaard and M. Kulahci, “Quality quandaries: Studying input-output relationships, part i,” *Quality Engineering*, vol. 18, no. 2, pp. 273–281, 2006.
37. H. Tanaka and H. Lee, “Interval regression analysis by quadratic programming approach,” *IEEE Transactions on Fuzzy Systems*, vol. 6, no. 4, pp. 473–481, 2002.
38. E. H. Aboudi *et al.*, “Estimating the fuzzy regression panel data model based on approximate bayesian computation with application,” *Iraqi Statisticians journal*, pp. 163–176, 2025.
39. W. Pedrycz, “Why triangular membership functions?” *Fuzzy sets and Systems*, vol. 64, no. 1, pp. 21–30, 1994.
40. D. T. Redden and W. H. Woodall, “Properties of certain fuzzy linear regression methods,” *Fuzzy sets and Systems*, vol. 64, no. 3, pp. 361–375, 1994.
41. K. GOHAIN, “Evaluation of theils u: A naïve forecast application,” *Quantum Journal of Engineering, Science and Technology*, vol. 2, no. 5, pp. 26–31, 2021.
42. P. F. Christoffersen, “Evaluating interval forecasts,” *International economic review*, pp. 841–862, 1998.
43. F. X. Diebold, “Comparing predictive accuracy, twenty years later: A personal perspective on the use and abuse of diebold–mariano tests,” *Journal of Business & Economic Statistics*, vol. 33, no. 1, pp. 1–1, 2015.
44. H. Bohn, “The behavior of us public debt and deficits,” *the Quarterly Journal of economics*, vol. 113, no. 3, pp. 949–963, 1998.