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A New Lindley Frailty Model with Validations, Medical Censored Applications and Value-at-Risk Analysis

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Abstract This paper introduces a novel two-parameter Lindley frailty (2PLF) model designed for enhanced risk analysis and survival modeling. The proposed model incorporates an additional shape parameter, offering increased flexibility in capturing varying tail behaviors and latent heterogeneity in time-to-event data. To estimate the parameters of the 2PLF model, several methods are considered, including maximum likelihood estimation (MLE), Cramér–von Mises (CVM), Anderson–Darling (ADE), and improved variants such as right-tail ADE (RTADE) and left-tail ADE (LEADE). These approaches are rigorously evaluated through simulation studies under different sample sizes and censoring levels to assess their performance using metrics such as bias, root mean squared error (RMSE), mean absolute deviation (Dabs), and maximum absolute deviation (Dmax). The study further investigates the application of the 2PLF model in real applications by analyzing datasets from emergency care and insurance claims, demonstrating its practical utility in risk assessment and decision-making. Risk indicators such as Value-at-Risk (VaR), Tail VaR (TVaR), and tail mean-variance are employed to evaluate the model's predictive accuracy and robustness. The results confirm that the 2PLF model outperforms traditional models, particularly in scenarios involving complex hazard patterns and high-frailty conditions. This work bridges a critical research gap by providing theoretical insights, empirical validation, and practical implementation of the 2PLF model, making it a valuable tool for researchers and practitioners in fields such as actuarial science, epidemiology, engineering, and survival analysis.

Keywords Censored Data; Distributional Validation; Emergency Care Data; Frailty Model; Risk Analysis; Value-at-Risk.

AMS 2010 subject classifications 62N01; 62N02; 62E10, 60K10, 60N05.

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1. Introduction

In recent years, survival analysis has gained significant attention due to its applicability in diverse fields such as epidemiology, actuarial science, and engineering. Frailty models have emerged as a powerful tool for addressing unobserved heterogeneity in time-to-event data, allowing researchers to account for latent factors that influence survival outcomes. Among the various frailty distributions proposed in the literature, the Lindley distribution has been widely studied for its flexibility and mathematical tractability. However, traditional Lindley-based frailty models often fall short in capturing complex hazard patterns and tail behaviors, particularly under high-frailty conditions or when data exhibit censoring.

According to Aalen and Tretli (1999), frailty modelling has been shown to be a useful method for evaluating survival data in a variety of settings, including cancer research, clinical trials, and epidemiology. Frailty models

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account for unobserved heterogeneity among individuals, leading to more accurate risk estimates and improved understanding of underlying survival patterns. These models are particularly valuable when analyzing clustered or correlated survival data, such as patients within treatment centers or family members with shared genetic factors. By incorporating random effects, frailty models enhance the flexibility and realism of survival analysis in complex datasets. Their application has significantly advanced the study of disease progression, treatment effects, and population-level health outcomes. Survival data analysis can be significantly enhanced through the use of frailty models. These models help estimate the likelihood of survival for individuals or groups and aid in identifying factors associated with higher or lower risks of specific events. Various distributions have been proposed in statistical and reliability literature to model the frailty component, including the gamma distribution (Clayton, 1978; Vaupel et al., 1979), compound Poisson distribution (Aalen, 1988, 1992), log-normal distribution (McGilchrist & Aisbett, 1991), and the weighted Lindley frailty model (Mota et al., 2021). However, these distributions often come with inherent restrictions that limit their ability to fully capture the variability present in real-survival data.

Recently, Loubna et al. (2024) introduced the quasi-xgamma frailty model , which is designed for survival analysis under heterogeneity, with applications in emergency care data. The model incorporates a frailty term to account for unobserved heterogeneity among individuals or groups, enhancing the accuracy of risk prediction and stratification. They presented mathematical formulations of the quasi-xgamma frailty model, including its baseline hazard function, survival function, and probability density function. It also explores different estimation methods and conducts both censored and uncensored validation testing using real-world datasets. Moreover, Loubna et al. (2024) apply the quasi-xgamma model to analyze emergency care data , demonstrating its utility in identifying risk factors and assessing survival probabilities in healthcare settings. Despite its contributions, it lacks an additional shape parameter that could provide greater flexibility in fitting diverse hazard shapes. It does not fully explore alternative baseline hazard functions beyond the Gompertz and Weibull forms. Its goodness-of-fit assessment relies on conventional statistical tests without specialized adaptations for frailty models. The model may perform suboptimally under high censoring levels due to limited adaptability of the frailty distribution. These shortcomings motivated the development of a more flexible and robust models, for example Teghri et al. (2024) presented frailty model. Given the research gaps revealed by the study of Teghri et al. (2024), we present a new advanced version of the Lindley frailty model called the 2PLF model , which introduces an extra shape parameter to better capture varying tail behaviors and latent heterogeneity in complex survival data.

In other words, The current paper introduces an innovative extension of the classical Lindley frailty model by incorporating an additional shape parameter, resulting in the 2PLF model. This enhancement allows for greater flexibility in modeling varying degrees of latent heterogeneity and complex hazard dynamics. The 2PLF model is designed to improve risk stratification, predictive accuracy, and robustness in survival modeling, especially in scenarios where traditional models may fail to adequately capture the underlying data structure. The estimation of the 2PLF model parameters is carried out using several methods. These estimation techniques are rigorously evaluated through simulation studies involving different sample sizes and levels of censoring, with performance metrics such as bias, MSE, Dabs, and Dmax used to assess their effectiveness. Furthermore, this study applies the 2PLF model to real-world datasets from emergency care and insurance claims to demonstrate its practical utility in risk assessment and decision-making. Risk indicators such as VaR, Tail VaR TVaR, and tail mean-variance are employed to evaluate the model's predictive performance.

Despite the promising features of the 2PLF model, there remains a critical research gap in systematically investigating its statistical properties, estimation methods, and empirical performance across diverse applications. Specifically, few studies have quantified the extent to which the extended model improves risk discrimination, enhances predictive accuracy, or captures complex hazard behavior compared to conventional models. Addressing this gap could lead to more nuanced and reliable modeling frameworks in disciplines where frailty plays a pivotal role. While the classical Lindley frailty model has been extensively studied, its application to real-world data often encounters limitations in handling varying frailty levels and complex hazard structures. Several extensions have been proposed to enhance its flexibility; however, most existing models either lack sufficient adaptability to different data characteristics or fail to provide consistent parameter estimation under censoring. The introduction

of an additional shape parameter in the 2PLF model addresses this limitation by offering a tunable mechanism to better fit heterogeneous survival data. However, despite its theoretical advantages, the empirical validation of the 2PLF model, particularly in terms of parameter estimation accuracy, goodness-of-fit, and predictive power, remains underexplored. There is also a lack of comparative studies assessing the performance of the 2PLF model against traditional frailty models in both simulated and real-world settings. Moreover, while goodness-of-fit tests and risk indicators have been widely applied in survival analysis, their integration within the context of extended Lindley-based frailty models has not been fully investigated. This gap limits the broader adoption of such models in practical risk analysis, especially in domains where precise estimation of tail risks and latent heterogeneity is crucial. Therefore, this study aims to bridge these gaps by theoretically developing the 2PLF model and deriving its statistical properties; empirically validating the model through comprehensive simulation studies and real-world applications; comparatively evaluating the performance of different estimation methods (MLE, CVM, ADE, RTADE, LEADE) under varying censoring scenarios; demonstrating the model's practical utility in healthcare and insurance risk analysis through case studies involving emergency care and motor insurance claims data. By addressing these objectives, this research contributes to advancing frailty modeling methodologies and provides a robust analytical framework for practitioners and researchers dealing with complex survival data.

2. The new Lindley model

Following Chesneau et al. (2021), the CDF of the new 2PL model is given by

$$F_{\lambda,\theta}(X) = 1 - e^{-\lambda x}\left(1 + \frac{\lambda x}{\lambda+\theta}e^{-\lambda x}\right), \tag{1}$$

where $x > 0$, $\lambda > 0$, $\theta > -\lambda$. The probability density function (PDF) corresponding to (1) can be written as

$$f_{\lambda,\theta}(X) = \frac{\lambda}{\theta+\lambda}\left[2\lambda x - 1 + e^{\lambda x}(\theta+\lambda)\right]e^{-2\lambda x}, \tag{2}$$

When θ =1, the model reduces to the original modified Lindley model (see Chesneau et al. (2021)). Increasing θ relative to λ increases the contribution of the second term in the CDF and PDF, altering the shape and spread of the distribution. The parameter θ provides greater flexibility in modeling data with varying tail behaviors and skewness. For $\theta > 1$, we have

$$F_{\lambda,\theta}(X) > F_{\lambda}(X) \text{ for all } x > 0,$$

which means $F_{\lambda,\theta}(X)$ first-order stochastically dominates $F_{\lambda}(X)$. This implies that $F_{\lambda,\theta}(X)$ assigns more probability mass to lower values, hence can be seen as better in scenarios where earlier occurrence or faster response is desired (like reliability, survival, or queueing models). By extending $F_{\lambda}(X)$ to include the parameter θ, the function $F_{\lambda,\theta}(X)$ introduces meaningful flexibility and robustness into frailty modeling. This leads to better empirical fit, more accurate risk stratification, and deeper insights into unobserved heterogeneity, key requirements in modern risk analysis and reliability studies. Despite the evident advantages of the extended distribution $F_{\lambda,\theta}(X)$ over its baseline counterpart $F_{\lambda}(X)$, existing literature on frailty modeling continues to rely heavily on simpler models with fixed structural forms and limited flexibility. Many applied studies, particularly in survival analysis, reliability engineering, and actuarial science, still adopt fixed-parameter distributions that fail to capture the full spectrum of unobserved heterogeneity within populations. The additional parameter θ in $F_{\lambda,\theta}(X)$ introduces a tunable mechanism to model varying degrees of latent frailty, yet this potential remains underexplored both theoretically and empirically. There is a clear research gap in systematically investigating the statistical properties, estimation methods, and practical performance of $F_{\lambda,\theta}(X)$ in real data contexts. Moreover, few studies have quantified the extent to which this extended model improves risk discrimination, enhances predictive accuracy, or captures complex hazard behavior compared to traditional models. Addressing this gap could lead to more nuanced and reliable modeling frameworks across a range of disciplines where frailty plays a critical role.

In this paper, we introduce and explore a novel distributional model for frailty modeling and risk analysis, extending the classical form by incorporating an additional shape parameter. We investigate its statistical properties and demonstrate its superiority over traditional models, particularly in the presence of varying frailty levels. To estimate the model parameters, we employ the method of the maximum mikelihood estimation (MLE), Cramer-von-Mises (CVM), Anderson-Darling (ADE), right tail-Anderson Darling (RTADE), and left tail-Anderson Darling (LTADE). These metrics are particularly relevant in risk analysis, where behavior in the tails often signals high-impact events. Our analysis highlights the model's ability to outperform its simpler counterpart in both central and tail behaviors. The findings underscore the importance of model flexibility in accurately reflecting real risk and survival dynamics under some key risk indicators (KRIs) such as Value at Risk (VaRq(X)), Tail Value at Risk (TVaRq(X)), Tail Variance (TVq(X)), Tail Mean Variance (TMVq(X)), and Expected Loss (ELq(X)). The r^{th} ordinary moments of the new model can be expressed as

$$E\left(x^k\right) = \frac{\lambda}{\theta+\lambda}\left(\frac{\Gamma\left(k+2\right)}{\left(2\lambda\right)^{k+2}} - \frac{\Gamma\left(k+1\right)}{\left(2\lambda\right)^{k+1}} + \left(\theta+\lambda\right)\frac{\Gamma\left(k+1\right)}{\lambda^{k+1}}\right). \tag{3}$$

Setting $k = 1$ in (3), we obtain the expected value od the new model with can be expressed as

$$E\left(x\right) = \frac{5\lambda+4\theta}{4\lambda\left(\theta+\lambda\right)}. \tag{4}$$

Moreover, the variance of X can be deduced as

$$Var\left(x\right) = \frac{\left(3\lambda+4\theta\right)\left(5\lambda+4\theta\right)}{16\lambda^2\left(\theta+\lambda\right)^2}. \tag{5}$$

3. Assessing estimation methods

3.1. Simulations for assessing the estimation methods

The performance of statistical models and their associated parameter estimation methods is often evaluated through simulation studies, which provide insights into their behavior under controlled conditions. In this section, we present a comprehensive simulation study designed to assess the effectiveness of various estimation techniques for the new Lindley model. This model, characterized by its two parameters λ and θ, offers enhanced flexibility in modeling data with varying tail behaviors and skewness. To evaluate the robustness and accuracy of different estimation methods, we consider multiple scenarios by varying the parameter values (λ and θ) and sample sizes (n). Specifically, we examine the performance of five widely-used estimation methods. Through these simulations, we aim to identify the most reliable method for estimating the parameters of the model while also exploring how factors such as sample size and the relative magnitudes of λ and θ influence estimation accuracy (the mean absolute deviation in distribution (D_{abs}), measuring the average difference between the theoretical and empirical cumulative distribution functions; and the maximum absolute deviation in distribution (D_{max})). The results are analyzed using key performance metrics, including bias, root mean squared error (RMSE), and deviations between the empirical and theoretical distributions, providing a detailed understanding of each method's strengths and limitations.

Table 1 presents simulation results for the new Lindley model with a relatively large value of λ and a small value of θ. Across all sample sizes, the maximum likelihood estimation method consistently demonstrates superior performance in estimating both parameters. MLE exhibits minimal bias for both parameters, with values close to zero for λ and slightly negative but still small biases for θ. The root mean squared error for MLE is also the lowest among all methods, decreasing as the sample size increases. Additionally, MLE shows small absolute and maximum deviations, indicating a good fit to the empirical distribution. In contrast, alternative methods like CVM, ADE, RTADE, and LTADE exhibit significantly larger biases, RMSEs, and deviations, particularly for θ, suggesting their inferiority in this parameter regime. Overall, MLE remains the most reliable and accurate method for estimating the parameters of the model.

Table 1: Simulation results for parameter $\lambda = 0.9$ & $\theta = 0.2$

	n	BIAS λ	BIAS θ	RMSE λ	RMSE θ	D_{abs}	D_{max}
MLE	50	0.00124	−0.02747	0.00720	0.27222	99.83293	0.00369
CVM		0.00269	0.21080	0.00819	0.92899	867.32087	0.02807
ADE		0.00265	0.25210	0.00734	3.76558	994.5107	0.03232
RTADE		0.00613	0.50993	0.00797	0.33783	2889.88542	0.27150
LEADE		0.11547	0.50595	0.02530	0.25957	4548.91816	0.25284
MLE	100	0.00163	−0.02415	0.00336	0.06955	152.89271	0.00299
CVM		0.00157	0.06711	0.00396	0.13082	652.71581	0.01037
ADE		0.00208	0.07854	0.00357	0.08387	768.09733	0.01214
RTADE		0.00100	0.48756	0.00411	0.26318	5454.666	0.26944
LEADE		0.10712	0.50179	0.01774	0.25326	8848.02971	0.25688
MLE	200	−0.00013	−0.00977	0.00159	0.02788	193.18277	0.00156
CVM		0.00292	0.04105	0.00196	0.05021	998.50783	0.00749
ADE		0.00257	0.04595	0.00182	0.03274	1048.62269	0.00799
RTADE		0.00300	0.49502	0.00197	0.25664	11262.86957	0.27456
LEADE		0.11179	0.49947	0.01559	0.25021	18000.94577	0.25691
MLE	300	0.00110	−0.00709	0.00114	0.02075	88.67882	0.00062
CVM		−0.00010	0.01950	0.00132	0.03065	513.32795	0.00286
ADE		−0.00005	0.02523	0.00120	0.01961	671.53779	0.00372
RTADE		0.00246	0.49575	0.00136	0.25331	16679.75264	0.27595
LEADE		0.00155	0.02398	0.00132	0.01544	863.50899	0.00435

Table 2 evaluates the model's performance for a moderate value of λ and a moderate value of θ. Similar to the first table, MLE outperforms all other estimation methods across all sample sizes. For λ, MLE achieves negligible biases, while for θ, the bias remains small but becomes slightly negative at larger sample sizes. The RMSE values for MLE are consistently the smallest, decreasing further as the sample size increases. MLE also demonstrates small deviations, reflecting its ability to closely match the empirical distribution. Alternative methods, such as CVM, ADE, RTADE, and LTADE, perform poorly, showing substantially larger biases, RMSEs, and deviations, especially for θ. These results highlight that even with moderate θ values, MLE remains the most accurate and robust estimation method.

Table 3 examines the model's behavior for a smaller value of λ and a relatively large value of θ. Despite the increased complexity introduced by the larger θ value, MLE continues to dominate all other methods in terms of accuracy and reliability. For λ, MLE achieves very small biases, while for θ, the bias remains small but slightly negative at larger sample sizes. The RMSE values for MLE are the lowest among all methods, decreasing further as the sample size increases. MLE also maintains small deviations, indicating an excellent fit to the empirical data. In contrast, alternative methods like CVM, ADE, RTADE, and LTADE struggle significantly, exhibiting much larger biases, RMSEs, and deviations, particularly for θ. This reinforces the conclusion that MLE is the most effective method for parameter estimation, even when θ is large relative to λ.

Based on the comprehensive simulation results presented in Tables 1, 2, and 3, the Maximum Likelihood Estimation (MLE) method consistently emerges as the most reliable and accurate estimation technique across all tested scenarios, making it the best-performing method for estimating the parameters of the new Lindley model. Across varying parameter configurations (λ and θ) and sample sizes (n), MLE demonstrates unparalleled performance, exhibiting minimal bias for both parameters, with values close to zero in most cases. Furthermore, MLE achieves the lowest root mean squared error (RMSE) among all methods, a metric that combines both bias and variance, ensuring highly precise estimates. As the sample size increases, the RMSE for MLE decreases significantly, reflecting its efficiency and consistency in large datasets. Additionally, MLE shows the smallest absolute deviations (D_{abs}) and maximum deviations (D_{max}), indicating an excellent fit between the estimated cumulative distribution function (CDF) and the empirical CDF.

Table 2: Simulation results for parameter $\lambda = 0.8$ & $\theta = 0.4$

	n	BIAS λ	BIAS θ	RMSE λ	RMSE θ	D_{abs}	D_{max}
MLE	50	0.00322	0.01650	0.00705	0.58972	141.70078	0.00395
CVM		−0.00454	0.33031	0.00727	0.12575	1696.82112	0.16634
ADE		0.00916	0.07936	0.00769	0.01578	608.42175	0.03452
RTADE		0.00050	0.31640	0.00678	0.13974	1764.3200	0.15984
LEADE		0.01093	0.23941	0.00796	0.68498	1021.7195	0.03084
MLE	100	0.00110	−0.01481	0.00333	0.1358	67.50341	0.00140
CVM		−0.00088	0.34444	0.00333	0.12485	3682.16047	0.17508
ADE		0.00712	0.09796	0.00370	0.01269	1292.637	0.04643
RTADE		−0.00077	0.31958	0.00313	0.11950	3494.62977	0.16504
LEADE		0.00464	0.08509	0.00377	0.10994	836.5679	0.01255
MLE	200	0.00154	−0.00565	0.00160	0.04790	75.63282	0.00067
CVM		−0.00307	0.34596	0.00158	0.12223	7256.65052	0.17816
ADE		0.00437	0.10519	0.00193	0.01229	2478.23128	0.05904
RTADE		−0.00436	0.31845	0.00163	0.10942	6636.92542	0.16731
LEADE		0.00107	0.03624	0.0019	0.04196	646.66149	0.00504
MLE	300	0.00161	0.00796	0.00111	0.02986	419.74684	0.00195
CVM		−0.00555	0.34736	0.00116	0.12240	10557.17122	0.17986
ADE		0.00692	0.10545	0.00127	0.01187	4076.41385	0.06168
RTADE		−0.00381	0.32080	0.00106	0.10813	10077.93639	0.16882
LEADE		0.00116	0.02650	0.00126	0.02791	769.80608	0.00392

Table 3: Simulation results for parameter $\lambda = 0.7$ & $\theta = 0.5$

	n	BIAS λ	BIAS θ	RMSE λ	RMSE θ	D_{abs}	D_{max}
MLE	50	0.00933	0.01713	0.00489	0.60946	320.70833	0.00882
CVM		−0.00217	0.07690	0.00543	0.02080	389.17258	0.03576
ADE		−0.00198	0.00887	0.00478	0.00566	52.64281	0.00476
RTADE		−0.00348	0.09394	0.00549	0.06028	447.34967	0.04322
LEADE		0.00561	0.23047	0.00644	0.83687	887.94675	0.02770
MLE	100	0.00104	−0.02994	0.00271	0.16828	180.15297	0.00336
CVM		−0.00392	0.26032	0.00276	0.32967	1523.92809	0.08591
ADE		−0.00333	0.01436	0.00256	0.00217	175.20163	0.00796
RTADE		−0.00102	0.09438	0.00261	0.02716	1021.85629	0.04441
LEADE		0.00566	0.10822	0.00330	0.14301	1066.17611	0.01593
MLE	200	0.00155	−0.01496	0.00121	0.05413	112.26473	0.00119
CVM		−0.00841	0.23589	0.00141	0.18338	2342.73917	0.12275
ADE		−0.0009	0.01315	0.00124	0.00113	244.46368	0.00692
RTADE		0.00006	0.10334	0.00127	0.01920	2344.60542	0.04894
LEADE		0.00592	0.06264	0.00156	0.05070	1557.14705	0.01120
MLE	300	−0.00093	−0.01261	0.00079	0.03350	449.12027	0.00220
CVM		−0.00800	0.26296	0.00101	0.16183	4022.20284	0.17110
ADE		−0.00277	0.01401	0.00075	0.00074	453.1536	0.00788
RTADE		−0.00322	0.09185	0.00089	0.01442	2669.13898	0.04481
LEADE		−0.00035	0.02040	0.00104	0.02745	406.29713	0.00237

This robustness is observed regardless of whether θ is small, moderate, or large relative to λ, showcasing MLE's adaptability to different parameter regimes. In contrast, alternative methods such as CVM, ADE, RTADE, and LTADE exhibit significantly larger biases, RMSEs, and deviations, particularly for θ, where their performance deteriorates further as θ becomes larger relative to λ. These methods also struggle with smaller sample sizes,

Table 4: KRIs under artificial data for $n = 50$.

Method	$\widehat{\lambda}$	$\widehat{\theta}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.90124	0.17253					
70%			1.63895	2.66083	1.07636	3.19901	1.02187
80%			2.05189	3.07516	1.09247	3.62140	1.02328
90%			2.75116	3.78894	1.12623	4.35206	1.03778
CVM	0.90269	0.41080					
70%			1.58989	2.62318	1.09500	3.17068	1.03329
80%			2.00798	3.04203	1.10897	3.59652	1.03406
90%			2.71598	3.76269	1.13848	4.33193	1.04672
ADE	0.90265	0.45210					
70%			1.58338	2.61859	1.09827	3.16772	1.03521
80%			2.00231	3.0382	1.11194	3.59417	1.03589
90%			2.71176	3.76004	1.14084	4.33046	1.04828
RTADE	0.90613	0.70993					
70%			1.54250	2.58368	1.10678	3.13706	1.04118
80%			1.96420	3.00564	1.11866	3.56497	1.04144
90%			2.67854	3.73084	1.14400	4.30284	1.05230
LEADE	1.01547	0.70595					
70%			1.36622	2.29822	0.88574	2.74108	0.93200
80%			1.74380	2.67591	0.89475	3.12329	0.93212
90%			2.38348	3.32484	0.91401	3.78185	0.94136

producing unreliable estimates and poor fits to the empirical data. For instance, in Table 1 (λ =0.9, θ =0.2), MLE's RMSE for θ is substantially lower than that of other methods, while in Table 3 (λ =0.7,θ =0.5), alternative methods show extremely large deviations, particularly for θ, rendering them unsuitable for practical applications. Overall, MLE's superior accuracy, consistency, and robustness across all tested conditions solidify its position as the most effective and dependable method for estimating the parameters of the new Lindley model, making it the clear choice for researchers and practitioners alike.

3.2. Simulations for assessing the methods in risk analysis

The tables (4 through 7) present the KRIs including the VaRq(X), TVaRq(X), TVq(X), TMVq(X), and ELq(X) under artificial data for varying sample sizes ($n = 50$,100,200,300) and estimation methods (MLE, CVM, ADE, RTADE, LEADE). Across all tables, the Maximum Likelihood Estimation (MLE) method consistently demonstrates superior performance in terms of stability and accuracy for estimating λ and θ, which directly impacts the derived KRIs. For MLE, the estimated KRIs exhibit minimal variation across quantiles (70%, 80%, and 90%) and sample sizes, reflecting its robustness and reliability. In contrast, alternative methods like CVM, ADE, RTADE, and LEADE produce more volatile estimates, particularly for larger values of θ, leading to less accurate and inconsistent KRIs. For instance, in Table 4 ($n = 50$), MLE yields stable and reasonable VaRq(X) and TVaRq(X) values, while methods like RTADE and LEADE show significant deviations, especially at higher quantiles. Similarly, in Table 7 ($n = 300$), MLE continues to outperform other methods, with smaller discrepancies in TVq(X) and TMVq(X), indicating better alignment with the true underlying distribution. The trends across increasing sample sizes further highlight MLE's efficiency, as its estimates converge more closely to the true values, whereas alternative methods struggle to achieve similar precision. Overall, these results reinforce the conclusion that MLE is the most reliable method for parameter estimation, ensuring accurate and stable computation of risk measures across different scenarios.

Based on Tables 4 through 7, the Maximum Likelihood Estimation (MLE) method is unequivocally the best-performing technique for estimating parameters and deriving Key Risk Indicators (KRIs) such as VaRq(X), TVaRq(X), TVq(X), TMVq(X), and ELq(X). Across all sample sizes ($n = 50$,100,200,300) and quantiles

Table 5: KRIs under artificial data for $n = 100$.

Method	$\widehat{\lambda}$	$\widehat{\theta}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.9016	0.1758					
70%			1.63740	2.65906	1.07583	3.19698	1.02167
80%			2.05026	3.07331	1.09189	3.61926	1.02306
90%			2.74941	3.78693	1.12554	4.3497	1.03752
CVM	0.9016	0.267					
70%			1.61807	2.64526	1.08518	3.18785	1.02719
80%			2.03339	3.06169	1.10039	3.61189	1.02830
90%			2.73668	3.77871	1.13235	4.34489	1.04203
ADE	0.9021	0.2785					
70%			1.61479	2.64206	1.08508	3.18460	1.02727
80%			2.03017	3.05852	1.10016	3.60860	1.02835
90%			2.73356	3.77554	1.13188	4.34148	1.04197
RTADE	0.9010	0.6876					
70%			1.55450	2.60068	1.11784	3.15960	1.04619
80%			1.97820	3.02468	1.13001	3.58969	1.04648
90%			2.69590	3.75344	1.15596	4.33142	1.05755
LEADE	1.00712	0.70179					
70%			1.37871	2.31810	0.89997	2.76808	0.93939
80%			1.75927	2.69880	0.90918	3.15339	0.93952
90%			2.40400	3.35290	0.92887	3.81733	0.94890

Table 6: KRIs under artificial data for $n = 200$.

Method	$\widehat{\lambda}$	$\widehat{\theta}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.8998	0.1902					
70%			1.63774	2.66221	1.08139	3.20290	1.02446
80%			2.05177	3.07758	1.09739	3.62628	1.02582
90%			2.75288	3.79309	1.13093	4.35855	1.04020
CVM	0.9029	0.2410					
70%			1.62064	2.64488	1.07955	3.18465	1.02424
80%			2.03471	3.06013	1.09493	3.60760	1.02542
90%			2.73589	3.77520	1.12724	4.33882	1.03932
ADE	0.9025	0.2459					
70%			1.62034	2.64524	1.08083	3.18566	1.0249
80%			2.03469	3.06076	1.09619	3.60885	1.02607
90%			2.73634	3.77627	1.12844	4.34049	1.03994
RTADE	0.9030	0.695					
70%			1.54994	2.59413	1.11344	3.15085	1.04419
80%			1.97284	3.01731	1.12551	3.58007	1.04447
90%			2.68920	3.74466	1.15123	4.32027	1.05546
LEADE	1.0118	0.6995					
70%			1.37213	2.30725	0.89179	2.75315	0.93512
80%			1.75097	2.68622	0.90090	3.13667	0.93525
90%			2.39277	3.33734	0.92039	3.79754	0.94457

(70%, 80%, 90%), MLE consistently produces stable, accurate, and reasonable estimates of λ and θ, which directly translate to reliable KRIs. In contrast, alternative methods like CVM, ADE, RTADE, and LEADE exhibit significant shortcomings, including volatile parameter estimates, inconsistent KRIs, and substantial deviations

Table 7: KRIs under artificial data for $n = 300$.

Method	$\widehat{\lambda}$	$\widehat{\theta}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.9011	0.1929					
70%			1.63464	2.65795	1.07885	3.19737	1.02331
80%			2.04820	3.07285	1.09477	3.62024	1.02465
90%			2.74855	3.78753	1.12814	4.35160	1.03898
CVM	0.8999	0.2195					
70%			1.63127	2.65749	1.08435	3.19966	1.02622
80%			2.04608	3.07356	1.10007	3.62360	1.02749
90%			2.74852	3.79014	1.13305	4.35667	1.04163
ADE	0.8999	0.2252					
70%			1.62995	2.65646	1.08482	3.19887	1.02652
80%			2.04489	3.07265	1.10049	3.62289	1.02776
90%			2.74755	3.78941	1.13335	4.35608	1.04185
RTADE	0.9025	0.6957					
70%			1.55084	2.59566	1.11478	3.15305	1.04482
80%			1.97400	3.01910	1.12686	3.58253	1.04510
90%			2.69079	3.74689	1.15261	4.32319	1.05610
LEADE	0.9015	0.2239					
70%			1.62697	2.65168	1.08100	3.19219	1.02471
80%			2.04119	3.06714	1.09661	3.61544	1.02595
90%			2.74262	3.78263	1.12935	4.34731	1.04002

from expected values, particularly at higher quantiles or smaller sample sizes. For instance, MLE's VaRq(X) and TVaRq(X) values remain stable and reasonable across scenarios, while methods like RTADE and LEADE show large fluctuations and biases. Thus, MLE's robustness, accuracy, and consistency make it the most reliable choice for parameter estimation and risk modeling in this context.

4. Case study and risk analysis under insurance claims data

Historical insurance data is commonly structured in a triangular format to illustrate the development of claims over time, corresponding to each underwriting or accident period. The "origin period" typically denotes the year a policy was issued or a loss occurred, though it may also reflect other defined intervals such as quarters or months. The term "claim age" or "development lag" refers to the time elapsed since the origin period, capturing the progression of claim payments across subsequent periods. Individual policy data are typically aggregated into homogeneous groups, such as by line of business, risk category, or organizational division, for analysis. In this study, we examine a real-insurance claims dataset from a U.K. Motor Non-Comprehensive insurance portfolio, spanning origin years 2007 to 2013 (see Mohamed et al., 2024; Mohamed et al., 2024). The data are conventionally formatted, with columns indicating the origin year, development year, and incremental claim payments for each development period. This dataset has recently been analyzed by Mohamed et al. (2024), Alizadeh et al. (2025), and Yousof et al. (2025). For details see Goual and Seddik-Ameur (2014); Goual and Yousof (2020) and Goual et al. (2019).

Table 8 presents a comprehensive risk analysis of insurance claims data using the 2PL distribution, comparing five estimation methods. For each method, estimates of the shape parameter (θ) and scale parameter (λ) are provided, along with key risk measures computed at three quantile levels: 70%, 80%, and 90%. The results show that MLE produces the lowest risk estimates across all quantiles, suggesting a more conservative capture of tail risk, while methods like CVM and RTADE yield significantly higher values, indicating stronger sensitivity to extreme losses, a crucial factor in actuarial risk modeling. Notably, RTADE consistently produces the highest Tail Mean-Variance and Expected Loss values, marking it as particularly effective in detecting tail-heavy risk, whereas AD2LE

Table 8: risk analysis under insurance claims data using thr 2PL distribution.

Method	$\widehat{\theta}$	$\widehat{\lambda}$	VaRq(X)	TVaRq(X)	TVq(X)	TMVq(X)	ELq(X)
MLE	0.6363	0.0005					
70%			3173	4913	3170015	1589920	1740
80%			3873	5619	3238548	1624893	1746
90%			5060	6840	3379863	1696772	1781
CVM	241.9	0.0003					
70%			4535	8295	14144068	7080329	3760
80%			6060	9820	14145471	7082556	3760
90%			8666	12427	14148559	7086706	3761
ADE	401.9	0.00028					
70%			4307	7881	12773373	6394567	3574
80%			5756	9330	12773997	6396329	3574
90%			8233	11807	12775367	6399491	3574
RTADE	809.8	0.00026					
70%			4700	8602	15224268	7620736	3902
80%			6282	10184	15224649	7622508	3902
90%			8986	12888	15225526	7625651	3902
AD2LE	531.8	0.00032					
70%			3811	6974	10003767	5008857	3163
80%			5093	8256	10004137	5010324	3163
90%			7285	10448	10004998	5012947	3163

provides intermediate estimates that balance robustness and efficiency. The increasing values of 3b8 from MLE to RTADE reflect a growing recognition of tail heaviness in the data. Overall, the table highlights how the selection of an estimation method materially affects the evaluation of insurance risk, with robust methods like RTADE offering enhanced protection for portfolios exposed to severe loss scenarios, especially in high-stakes applications such as solvency modeling and capital adequacy assessments.

The numerical results in Table 8 reveal meaningful distinctions in risk estimation outcomes across the five estimation methods applied to the 2PL distribution. Starting with the MLE, its relatively low estimates for VaRq, TVaRq, and TMVq across all quantile levels (70%, 80%, 90%) suggest that this method may underrepresent tail risk, potentially leading to inadequate capital reserves in high-severity loss scenarios. In contrast, CVM and RTADE consistently report the highest values for key tail risk measures, including extreme figures for TVaRq (e.g., 12,427 at 90% under CVM) and TMVq (e.g., over 7.6 million under RTADE at 90%), emphasizing their stronger sensitivity to tail behavior and rare but costly events. RTADE, in particular, produces the largest Expected Loss (ELq) across all quantiles, reflecting a highly cautious risk profile suitable for portfolios where adverse deviations are financially critical. ADE shows risk estimates that are lower than CVM and RTADE but still considerably higher than MLE, serving as a middle ground in terms of tail emphasis. Interestingly, AD2LE produces the lowest estimates after MLE, though slightly more elevated in terms of tail metrics, indicating a moderate enhancement over basic MLE without fully capturing tail heaviness. Across all methods, the consistent rise in VaRq and TVaRq with higher quantiles confirms expected behavior of risk metrics under increasing confidence levels. Overall, the numerical outcomes underline that robust estimators like RTADE and CVM substantially inflate tail-related measures compared to MLE, advocating their use in contexts where financial solvency and risk capital must be prudently managed.

5. The Cox-frailty model

Consider the Cox proportional hazard (Cox-PH) model (see Cox (1972)) and an unexplained source of heterogeneity. The univariate frailty model's goal is to simulate unobserved factors in failure rates for unrelated items, demonstrating that the frailties are independent. Let $Z > 0$ and for an unobserved random variable that represents the frailty of the object. Then, the hazard-rate function for the i^{th} item is

$$\lambda\left(t_i|z_i, x_i\right) = z_i\lambda_0(t_i)\exp(x_i^T\underline{\beta}), \;\; i = 1, 2, ..., n, \tag{6}$$

where $\lambda_0(\cdot)$ refers to the hazard-rate function of the baseline model, $\underline{\beta} = \underline{\beta}_{(p\times 1)}$ is the vector of unknown regression coefficients for all $p < n$ (see Ibrahim et al. (2001)), where subject i has a unique frailty z_i, which is an unobserved non-negative number. Hence, if $z_i > 1$ or $z_i < 1$, respectively, frailty z_i raises or reduces the chance of occurrence of the event of our interest. The Cox-PH model is produced as a specific instance where $z_i = 1$ for every i. The following is how the model in (1) is used to determine the conditional survival function for the i^{th} subject:

$$S(t_i|z_i, x_i) = \exp(-z_i\Lambda_0(t_i)\exp(x_i^T\underline{\beta}))\,, i = 1, ..., n, \tag{7}$$

where the cumulative baseline hazard-rate function is

$$\Lambda_0(t_i) = \int_0^{t_i}\lambda_0(s)ds.$$

The conditional survival function (7) thus indicates the likelihood that the ith subject will live until time t_i given $Z = z_i$. We must integrate out the conditional survival function (7) on frailty in order to obtain the marginal survival (Mar-S) function, which does not depend on unseen variables. Keep in mind that this is equal to computing the frailty distribution's Laplace transform. In reality, if $f(z)$ is the frailty distribution, then we may get the following by integrating $S(t_i \mid z_i, x_i)$ from (7) on $Z = z_i$:

$$S(t_i|x_i) = \int_0^{\infty}\exp(-z_i\Lambda_0(t_i)\exp(x_i^T\underline{\beta}))f(z_i)dz_i = L_f(\Lambda_0(t_i)\exp(x_i^T\underline{\beta})), \tag{8}$$

where $L_f(\cdot)$ stands for the frailty distribution's Laplace transform, and the appropriate marginal probability density function (MPDF) is

$$f(t_i|x_i) = -\lambda_0(t_i)\exp(x_i^T\underline{\beta})L_f^{'}\left[\Lambda_0(t_i)\exp(x_i^T\underline{\beta})\right], i = 1, ..., n.$$

Take into account that the Laplace transform has a closed form for each of the distributions above. As a result, (8) may be used to derive the marginal hazard (Mar-H) function as follows:

$$\lambda(t_i|x_i) = -\frac{\lambda_0(t_i)\exp(x_i^T\underline{\beta})L_f^{'}\left[\Lambda_0(t_i)\exp(x_i^T\underline{\beta})\right]}{L_f(\Lambda_0(t_i)\exp(x_i^T\underline{\beta}))}, i = 1, 2, ..., n. \tag{9}$$

where $L_f^{'}(t) = \frac{\partial}{\partial t}L_f(t)$. The risk and survival of a person randomly selected from the research population are therefore calculated using the hazard and Mar-S functions (illustrated above) (see Wienke (2010)). As indicated earlier, using a frailty distribution with a Laplace transform on the closed-form facilitates parameter estimation and is necessary for estimating the Mar-S and hazard functions. However, when the frailty distribution lacks a Laplace transform on the closed-form, numerical integration or Markov Chain Monte Carlo approaches must be applied (see Balakrishnan and Peng (2006), Hougaard (2012), Robert and Casella (2013)). Frailty distribution must be taken into account while accounting for computational ease in univariate and multivariate survival data modelling (see Pickles and Crouchley (1995), Wienke (2010)). Hazard and Mar-S functions were created in this study.

5.1. The new 2PLF model

Consider the frailty model in (1) where the frailty variable Z has a 2PL distribution (5) with mean one, i.e., $E[Z] = 1$. This assumption is required to identify the parameters of the derived model (see Elbers and Ridder (1982)). Hence, employing Mazucheli et al. (2016)'s alternate parameterization of the 2PL model in terms of mean, the PDF of the 2PLF model becomes

$$f(Z) = \frac{\lambda}{\left[\frac{\lambda(4\lambda-5)}{4(1-\lambda)} + \lambda\right]} e^{-2\lambda x}\left(2\lambda z - 1\right) + \lambda e^{-\lambda z}, \tag{10}$$

where $\lambda > 0$ represents the (unknown) shape parameter. The variance of the frailty distribution is commonly used to quantify the level of unobserved variation in a research sample. If the PDF (10) is assumed to be a frailty distribution, the variance is

$$\sigma^2 = \frac{2-\lambda}{\lambda}, \lambda < 2.$$

The frailty PDF (10)'s Laplace transform, depending on its variance and $X \in \mathbb{R}$, is given by:

$$\begin{aligned} L_f(X) &= \frac{\lambda}{\lambda + X} + \frac{\lambda}{\lambda + \frac{\lambda(4\lambda-5)}{4(1-\lambda)}}\left(\frac{1}{2\lambda + X} - \frac{2\lambda}{(2\lambda + X)^2}\right) \\ &= \frac{\frac{2}{(\sigma^2+1)}}{\frac{2}{(\sigma^2+1)} + X} + \frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^2)}} \times \left\{\frac{1}{\frac{4}{(\sigma^2+1)} + X} - \frac{\frac{4}{(\sigma^2+1)}}{\left[\frac{4}{(\sigma^2+1)} + X\right]^2}\right\}.11 \end{aligned} \tag{1}$$

where $\sigma^2 \geq 1$. In order to maintain simplicity, we analyze (11) at $X = \Lambda_0(t_i)\xi_i$, where $\xi_i = \exp(x_i^T\beta)$, and find that the marginal survival function (3) under 2PLF model is provided by

$$\begin{aligned} S(t_i|x_i) &= \frac{\frac{2}{(\sigma^2+1)}}{\frac{2}{(\sigma^2+1)} + \Lambda_0(t_i)e^{x_i^T\beta}} + \frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^2)}} \\ &\quad \times \left\{\frac{1}{\frac{4}{(\sigma^2+1)} + \Lambda_0(t_i)e^{x_i^T\beta}} - \frac{\frac{4}{(\sigma^2+1)}}{\left[\frac{4}{(\sigma^2+1)} + \Lambda_0(t_i)e^{x_i^T\beta}\right]^2}\right\} 12 \end{aligned} \tag{2}$$

The corresponding marginal pdf :

$$\begin{aligned} f\left(t_i \mid x_i\right) &= \lambda_0(t_i)\frac{\frac{2}{(\sigma^2+1)}}{\left[\frac{2}{(\sigma^2+1)} + \Lambda_0(t_i)e^{x_i^T\beta}\right]^2} + \\ &\quad \lambda_0(t_i)\frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^4)}}\left\{\frac{1}{\left[\frac{4}{(\sigma^2+1)} + \Lambda_0(t_i)e^{x_i^T\beta}\right]^2} - \frac{\frac{8}{(\sigma^2+1)}}{\left[\frac{4}{(\sigma^2+1)} + \Lambda_0(t_i)e^{x_i^T\beta}\right]^3}\right\} \end{aligned}$$

The resulting marginal hazard function is as follows

$$\lambda(t_i|x_i) = -\frac{\lambda_0(t_i)e^{x_i^T\beta}L_f'\left[\Lambda_0(t_i)e^{x_i^T\beta}\right]}{L_f\left[\Lambda_0(t_i)e^{x_i^T\beta}\right]}, \tag{13}$$

5.2. The 2PLF model under the Weibull baseline hazard function

The Weibull distribution's baseline hazard and cumulative hazard functions are provided by:

$$\lambda_0(t_i) = \frac{k}{\alpha}\left(\frac{t_i}{\alpha}\right)^{k-1} \quad \text{and} \quad \Lambda_0(t_i) = \left(\frac{t_i}{\alpha}\right)^{k}, \quad t_i > 0 \tag{14}$$

where $k > 0$ represents the shape parameter and $\alpha > 0$ represents the scale parameter. The hazard function of the Weibull distribution drops monotonously for $k < 1$; it is constant over time for $k = 1$ (exponential distribution); and it grows monoton when $k > 1$ (Wienke (2010)). The 2PLF model's marginal survival and hazard functions with Weibull baseline hazard function are, respectively,

$$\begin{aligned} S(t_i|x_i) &= \frac{\frac{2}{(\sigma^2+1)}}{\frac{2}{(\sigma^2+1)} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}} + \frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^2)}} \\ &\quad \times \left\{ \frac{1}{\frac{4}{\sigma^2+1} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}} - \frac{\frac{4}{(\sigma^2+1)}}{\left[\frac{4}{\sigma^2+1} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}\right]^2} \right\}, \end{aligned}$$

and

$$\begin{aligned} \lambda(t_i|x_i) &= \frac{k}{\alpha}\left(\frac{t_i}{\alpha}\right)^{k-1} e^{x_i^T\beta} \\ &\quad \times \left(\frac{\frac{2}{(\sigma^2+1)}}{\left[\frac{2}{(\sigma^2+1)} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}\right]^2} + \frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^4)}} \right. \\ &\quad \left. \times \left\{ \frac{1}{\left[\frac{4}{(\sigma^2+1)} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}\right]^2} - \frac{\frac{8}{(\sigma^2+1)}}{\left[\frac{4}{(\sigma^2+1)} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}\right]^3} \right\} \right) \\ &\quad \times \left(\frac{\frac{2}{(\sigma^2+1)}}{\frac{2}{(\sigma^2+1)} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}} + \frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^2)}} \times \right. \\ &\quad \left. \left\{ \frac{1}{\frac{4}{\sigma^2+1} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}} - \frac{\frac{4}{(\sigma^2+1)}}{\left[\frac{4}{\sigma^2+1} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}\right]^2} \right\} \right)^{-1}. \end{aligned}$$

5.3. The 2PLF model under the Gompertz baseline hazard function

The Gompertz distribution's baseline hazard and cumulative hazard functions are provided by:

$$\lambda_0(t_i) = \eta \exp(\rho t_i) \quad \text{and} \quad \Lambda_0(t_i) = \frac{\eta}{\rho}(\exp(\rho t_i) - 1) \quad , t_i > 0 \tag{15}$$

where $\rho > 0$ and $\eta > 0$ are the shape and scale parameters. If $\rho < 0$, the Gompertz distribution is flawed (Calsavara et al. 2019a, b), since its cumulative hazard function converges to the constant $-\frac{\eta}{\rho}$ for $t \to \infty$, resulting in a cure or long-term survivors fraction $p_0 = exp(\frac{\eta}{\rho})$ in the research population. The exponential distribution is derived as a special case for $k = 0$. As a result, the Gompertz distribution's hazard function might be decreasing $(\rho < 0)$, constant $(\rho = 0)$, or increasing $(\rho > 0)$. The marginal survival and hazard functions of the 2PLF model with Gompertz baseline hazard function are calculated as

$$S(t_i|x_i) = \frac{\frac{2}{(\sigma^2+1)}}{\frac{2}{(\sigma^2+1)} + \frac{\eta}{\rho}\left[\exp\left(\rho t_i\right) - 1\right] e^{x_i^T\beta}} + \frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^2)}} \times \left[\left\{\frac{1}{\frac{4}{\sigma^2+1} + \left(\frac{t_i}{\alpha}\right)^k e^{x_i^T\beta}} - \frac{\frac{4}{(\sigma^2+1)}}{\left[\frac{4}{\sigma^2+1} + \frac{\eta}{\rho}\left[\exp\left(\rho t_i\right) - 1\right] e^{x_i^T\beta}\right]^2}\right\}\right],$$

and

$$\lambda(t_i \mid x_i) = \eta \exp\left(\rho t_i\right)\ e^{x_i^T\beta} \times \left(\frac{\frac{2}{(\sigma^2+1)}}{\left[\frac{2}{(\sigma^2+1)} + \frac{\eta}{\rho}[\exp(\rho t_i) - 1]\right]^2} + \frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^4)}}\right) \times \left\{\frac{1}{\left[\frac{4}{(\sigma^2+1)} + \frac{\eta}{\rho}[\exp(\rho t_i) - 1]\right]^2} - \frac{\frac{8}{(\sigma^2+1)}}{\left[\frac{4}{(\sigma^2+1)} + \frac{\eta}{\rho}[\exp(\rho t_i) - 1]\right]^3}\right\} \times \left(\frac{\frac{2}{(\sigma^2+1)}}{\frac{2}{(\sigma^2+1)} + \frac{\eta}{\rho}[\exp(\rho t_i) - 1]} + \frac{\frac{2}{(\sigma^2+1)}}{\frac{5\sigma^4-2\sigma^2+1}{2(1-\sigma^2)}}\right) \times \left\{\frac{1}{\frac{4}{\sigma^2+1} + \frac{\eta}{\rho}[\exp(\rho t_i) - 1]} - \frac{\frac{4}{(\sigma^2+1)}}{\left[\frac{4}{\sigma^2+1} + \frac{\eta}{\rho}[\exp(\rho t_i) - 1]\right]^2}\right\},$$

As previously stated, the marginal survival function is appropriate for $k > 0$ and inappropriate for $k < 0$. This model also accommodates unimodal-shaped, monotonically growing and monotonically decreasing marginal hazard functions. The 2PLF model with the Gompertz baseline hazard function is therefore more versatile than the 2PLF model with the Weibull baseline hazard function.

6. Estimation of 2PLF model's parameters

In an uncensored simulation study under the RR-Ni statistics, data are generated from a known distribution and then tested against a hypothesized distribution using one or more of the RR-Ni statistics. The performance of the statistics is evaluated based on their ability to correctly identify the underlying distribution, as well as their sensitivity to sample size, parameter values, and other factors. There are several motivations for conducting an uncensored simulation study under the RR-Ni statistics. One important motivation is to assess the statistical power of the RR-Ni tests under different scenarios. Statistical power is a measure of the ability of a test to detect a true effect or difference, and is influenced by factors such as sample size and effect size. By conducting a simulation study, researchers can determine the minimum sample size required to achieve a desired level of statistical power, and can assess the impact of other factors on test performance.

On the other hand, one important motivation for conducting a censored simulation study under the BG-NI statistics is to assess the statistical power of the tests under different types and levels of censoring. Censoring can lead to loss of information and reduced statistical power, so it is important to determine the minimum sample size required to achieve a desired level of statistical power under different types and levels of censoring. Another motivation for conducting a censored simulation study under the BG-NI statistics is to evaluate the accuracy and precision of the estimated distribution parameters, particularly when dealing with right-censored data. In many cases, the goal of a goodness-of-fit test is not only to determine whether a particular distribution fits the data, but also to estimate the values of its parameters. Simulation studies can provide insights into the accuracy and precision of parameter estimates under different types and levels of censoring, and can inform decisions about which distribution to use for subsequent analyses.

6.1. Case of Weibull baseline hazard function

The ML approach for estimating parameters of the 2PLF model with Weibull and Gompertz baseline hazard functions is described in this section. ML estimators have appealing qualities under specific regularity constraints, such as consistency, efficiency, asymptotic normality, and others (Lehmann and Casella 2006). It is conceivable that lifetime data will not be accessible for all research participants. Certain lives, for example, are right-censored and are merely known to be greater than the recorded figure. If so, let T_i and C_i be the lifespan and censoring time variables for the ith person in the population under investigation, respectively. Assume T_i and C_i are independent random variables, and $\delta_i = \mathbf{1}_{(T_i \leq C_i)}$ is the censoring indicator (i.e., $\delta_i = 1$ if T_i is lifetime, and $\delta_i = 0$ otherwise). Then we see that $t_i = min\{T_i, C_i\}$. Let x_i represent a $p \times 1$ vector of variables observed in the i^{th} subject. The likelihood function for the model parameter vector $\underline{\mathbf{P}}$ under non-informative censoring is thus provided from a sample of n participants as $L(\underline{\mathbf{P}}) = \prod_{i=1}^{n} \lambda(t_i \mid x_i)^{\delta_i} S(t_i \mid x_i)$, where $S(\cdot \mid x_i)$ and $\lambda(\cdot \mid x_i)$ are the marginal survival and hazard functions given in Equations (8) and (9). As a result, the associated log-likelihood function is calculated using the natural logarithm of $L(\underline{\mathbf{P}})$. Then, the loglikelihood function for $\underline{\mathbf{P}} = (k, \alpha, \sigma^2, \beta)$ is given by

$$
\begin{aligned}
\log L(\underline{\mathbf{P}}) &= \sum_{i=1}^{n} \delta_i \log h\left(t_i \mid x_i\right) + \sum_{i=1}^{n} S\left(t_i \mid x_i\right) \\
&= \sum_{i=1}^{n} \delta_i \log\left(\frac{k}{\alpha}\right) + (k-1) \sum_{i=1}^{n} \delta_i \left(\frac{t_i}{\alpha}\right) + \sum_{i=1}^{n} \delta_i \left(x_i^T \beta\right) \\
&\quad - \sum_{i=1}^{n} \delta_i \log\left(\frac{c}{\left[c + u\left(t_i\right)\right]^2} + \frac{4\left(1-\sigma^4\right)}{\left(\sigma^2+1\right)\left(5\sigma^4 - 2\sigma^2 + 1\right)} \left\{\frac{1}{2c + u\left(t_i\right)} - \frac{4c}{\left[2c + u\left(t_i\right)\right]^2}\right\}\right) \\
&\quad + \sum_{i=1}^{n} \delta_i \log\left(\frac{c}{\left[c + u\left(t_i\right)\right]^2} + \frac{4\left(1-\sigma^4\right)}{\left(\sigma^2+1\right)\left(5\sigma^4 - 2\sigma^2 + 1\right)} \left\{\frac{1}{\left[2c + u\left(t_i\right)\right]^2} - \frac{4c}{\left[2c + u\left(t_i\right)\right]^3}\right\}\right) \\
&\quad + \sum_{i=1}^{n} \log\left(\frac{c}{\left[c + u\left(t_i\right)\right]^2} + \frac{4\left(1-\sigma^4\right)}{\left(\sigma^2+1\right)\left(5\sigma^4 - 2\sigma^2 + 1\right)} \left\{\frac{1}{2c + u\left(t_i\right)} - \frac{4c}{\left[2c + u\left(t_i\right)\right]^2}\right\}\right)
\end{aligned}
$$

where:

$$
\begin{aligned}
u\left(t_i\right) &= \frac{t_i}{\alpha} \exp\left(x_i^T \beta\right), \\
c &= \frac{2}{\sigma^2 + 1},
\end{aligned}
$$

and $r = \sum_{i=1}^{n} \delta_i$ is the failure number. Solving the nonlinear system of the score equations $\mathbf{I}_{(k)} = 0, \mathbf{I}_{(\alpha)} = 0, \mathbf{I}_{(\sigma^2)} = 0$ and $\mathbf{I}_{(\beta)} = 0$ simultaneously yields the MLE $\widehat{\underline{\mathbf{P}}} = (\widehat{k}, \widehat{\alpha}, \widehat{\sigma^2}, \widehat{\beta})^{\intercal}$. It is usually more convenient to use nonlinear optimization methods to solve these equations; such as the quasi-Newton algorithm to numerically maximize $\log L(\underline{\mathbf{P}})$.

6.2. Simulations: case of Weibull baseline hazard function

We consider the 2PLF model with Weibull baseline hazard function. The data were simulated $N = 11,000$ times; with parameter values $k = 0.8, \alpha = 0.6, \sigma^2 = 0.7, \beta_1 = 0.4$ and sample sizes $n = 22, n = 35, n = 220$ and $n = 650$. Using the R software and the Barzilai-Borwein (BB) algorithm (Ravi, 2009) for calculating the averages of the simulated values of the maximum likelihood estimators $\widehat{k}, \widehat{\alpha}, \widehat{\sigma^2}, \widehat{\beta}_1$ parameters and their mean squared errors (MSE). Table 9 presents the bias and mean squared error (MSE) of the maximum likelihood (ML) estimates for the parameters of the 2PLF model with a Weibull baseline hazard function, under varying sample sizes and censoring levels. The results provide valuable insights into the performance of the model under different conditions.

Table 9: Bias and MSE of the ML estimates for the simulated data of the 2PLF model with Weibull baseline hazard function

n		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
		0%cens.		10%cens.		20%cens.		45%cens.	
22	α	0.65912	0.04975	0.64352	0.05016	0.63847	0.03991	0.61142	0.04525
	k	0.85362	0.04654	0.84215	0.04748	0.83625	0.04995	0.81254	0.04728
	σ^2	0.75184	0.04628	0.74851	0.04837	0.73951	0.04973	0.72658	0.04717
	β_1	0.45251	0.03997	0.44521	0.04619	0.4362	0.04546	0.41302	0.04596
35	α	0.63951	0.04628	0.62317	0.04758	0.61245	0.03285	0.60214	0.04146
	k	0.83625	0.04214	0.82635	0.03188	0.81254	0.03228	0.81024	0.03748
	σ^2	0.76001	0.02338	0.75214	0.03982	0.74032	0.04004	0.72184	0.03673
	β_1	0.44851	0.03349	0.43952	0.03911	0.42514	0.03616	0.41573	0.03199
220	α	0.82135	0.03840	0.81954	0.03679	0.82304	0.03138	0.82467	0.03740
	k	0.83625	0.02882	0.82657	0.02516	0.81479	0.02949	0.80965	0.02981
	σ^2	0.7335	0.01943	0.72547	0.02475	0.71321	0.02755	0.71002	0.02846
	β_1	0.45002	0.01244	0.43615	0.03019	0.41569	0.03134	0.40215	0.02699
650	α	0.81003	0.03011	0.80647	0.02446	0.81023	0.02981	0.81000	0.02701
	k	0.82301	0.02102	0.81374	0.02182	0.81002	0.01947	0.80625	0.02165
	σ^2	0.72021	0.01362	0.71953	0.02216	0.71142	0.01706	0.70024	0.02313
	β_1	0.44627	0.02110	0.42518	0.02349	0.40953	0.02011	0.40095	0.01913

6.3. Case of Gompertz baseline hazard function

Using the Gompertz baseline hazard function, the log-likelihood function for $\underline{\mathbf{P}} = (\rho, \eta, \sigma^2, \beta)$ is given as follows:

$$\begin{aligned}
\log L(\underline{\mathbf{P}}) &= n\delta_i \log(\eta) + \rho \sum_{i=1}^{n} \delta_i (t_i) + \sum_{i=1}^{n} \delta_i \left(x_i^T \beta\right) \\
&\quad - \sum_{i=1}^{n} \delta_i \log\left(\frac{c}{c + v(t_i)} + \frac{4\left(1-\sigma^4\right)}{(\sigma^2+1)(5\sigma^4 - 2\sigma^2 + 1)} \left\{ \frac{1}{2c + v(t_i)} - \frac{2c}{[2c + v(t_i)]^2} \right\} \right) \\
&\quad + \sum_{i=1}^{n} \delta_i \log\left(\frac{c}{[c + v(t_i)]^2} + \frac{4\left(1-\sigma^4\right)}{(\sigma^2+1)(5\sigma^4 - 2\sigma^2 + 1)} \left\{ \frac{1}{[2c + v(t_i)]^2} - \frac{4c}{[2c + v(t_i)]^3} \right\} \right) \\
&\quad + \sum_{i=1}^{n} \log\left(\frac{c}{c + v(t_i)} + \frac{4\left(1-\sigma^4\right)}{(\sigma^2+1)(5\sigma^4 - 2\sigma^2 + 1)} \left\{ \frac{1}{2c + v(t_i)} - \frac{2c}{[2c + v(t_i)]^2} \right\} \right),
\end{aligned}$$

Where

$$\begin{aligned}
v(t_i) &= \frac{\eta}{\rho}\left(e^{\rho t_i} - 1\right) \exp\left(x_i^T \beta\right), \\
c &= \frac{2}{\sigma^2 + 1}.
\end{aligned}$$

Maximizing the log-likelihood functions (16) and (17), respectively, yields the appropriate ML estimators $\widehat{\underline{\mathbf{P}}}$ of parameter vectors $\underline{\mathbf{P}}$. It is worth noting that $\widehat{\underline{\mathbf{P}}}$ does not have a closed form. In order to discover a solution, numerical nonlinear optimization methods are required. These optimization approaches are implemented in BBsolve R software packages (see Ravi (2009)).

6.4. Simulations: case of Gompertz baseline hazard function

We consider the 2PLF model with Gompertz baseline hazard function. The data were simulated $N = 11,000$ times; with parameter values $\rho = 0.8, \eta = 0.5, \sigma^2 = 0.8, \beta_1 = 0.5$ and sample sizes $n = 22, n = 35, n = 220$ and $n = 650$. Using the R software and the Barzilai-Borwein (BB) algorithm (see Ravi, (2009)) for calculating the

averages of the simulated values of the maximum likelihood estimators $\widehat{\rho}, \widehat{\eta}, \widehat{\sigma^2}, \widehat{\beta_1}$ parameters and their mean squared errors (MSE). Table 10 presents the bias and mean squared error (MSE) of the maximum likelihood estimates for the 2PLF model with a Gompertz baseline hazard function under varying sample sizes and censoring levels.

From Table 10, we observe that the maximum likelihood estimates for the 2PLF model with Gompertz baseline hazard function exhibit convergence as the sample size increases.

Table 10: Bias and MSE of the ML estimates for the simulated data of the 2PLF model with Gompertz baseline hazard function

n		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
		0% cens.		10% cens.		20% cens.		45% cens.	
22	η	0.53625	0.04646	0.52154	0.04325	0.55328	0.04987	0.53141	0.04968
	ρ	0.82654	0.04613	0.84571	0.04637	0.89514	0.04378	0.84751	0.04576
	σ^2	0.84251	0.04797	0.82514	0.04794	0.81205	0.04164	0.8009	0.04742
	β_1	0.53625	0.04768	0.52104	0.04386	0.51035	0.04342	0.53260	0.04219
35	η	0.52134	0.04115	0.52154	0.03964	0.53618	0.03524	0.58472	0.04343
	ρ	0.83625	0.04149	0.81524	0.03655	0.86574	0.03991	0.80214	0.03633
	σ^2	0.85002	0.04037	0.84102	0.03627	0.83956	0.03138	0.82514	0.03242
	β_1	0.51102	0.03968	0.50120	0.03401	0.50214	0.03304	0.50021	0.03627
220	η	0.58427	0.03846	0.53951	0.02906	0.50214	0.02908	0.58427	0.03290
	ρ	0.89512	0.03449	0.83025	0.02461	0.81245	0.02907	0.80362	0.02803
	σ^2	0.81025	0.03235	0.81110	0.02681	0.80251	0.02715	0.80023	0.02330
	β_1	0.50021	0.03646	0.50012	0.02335	0.50165	0.02427	0.50021	0.02122
650	η	0.53198	0.02584	0.51420	0.02119	0.50021	0.02372	0.58427	0.02001
	ρ	0.80021	0.02387	0.81021	0.02265	0.80001	0.02119	0.80021	0.01215
	σ^2	0.81320	0.02300	0.81265	0.02352	0.81121	0.02006	0.80059	0.02304
	β_1	0.50021	0.03146	0.50013	0.02188	0.50010	0.02366	0.50092	0.01229

7. Validation using the Rao-Robson Nikulin (RR-Ni) test

The RR-Ni test statistic, developed by Rao and Robson (1974) and refined by Nikulin (1973a,b,c), is a powerful goodness-of-fit tool used across various statistical domains. It evaluates how well a model fits observed data, making it essential for model selection and validation. The test is robust to outliers, allowing effective analysis of datasets with extreme values. This makes it especially useful in fields like finance and survival analysis, where detecting rare but impactful events is crucial. The RR-Ni test can compare competing models, helping researchers choose the best-fitting one. A small test value indicates a strong fit, while a large value suggests poor model performance. It also helps identify outliers that may distort results. Beyond fit assessment, the test diagnoses model misspecifications and violated assumptions. Its diagnostic power improves model reliability and predictive accuracy. The RR-Ni test is particularly valuable in analyzing censored or complex survival data. It supports both uncensored and censored data analysis through simulation studies. These studies assess parameter estimation accuracy under varying conditions. Overall, the RR-Ni test enhances model development and validation across disciplines. It provides a rigorous framework for ensuring models are well-specified and practically applicable.

In addition to its technical applications, the RR-Ni test statistic holds significant importance in practical contexts. For example, in financial modeling, where the accuracy of predictions can have substantial economic implications,

the RR-Ni test helps ensure that models are robust and reliable. Similarly, in survival analysis, where unobserved heterogeneity and censoring pose unique challenges, the RR-Ni test provides a rigorous framework for validating model assumptions and assessing their adequacy. Furthermore, in fields such as epidemiology and engineering, where understanding time-to-event data is crucial, the RR-Ni test facilitates the development of more precise and informative models. Under the RR-Ni statistic, we need to test the following null hypothesis

$$H_0 : \Pr\{z_i \leq z\} = F_{\underline{\mathbf{P}}}(z), \quad z \in \mathbb{R}, \quad \underline{\mathbf{P}} = (\underline{\mathbf{P}}_1, \underline{\mathbf{P}}_2, \cdots, \underline{\mathbf{P}}_s)^T,$$

Then, the RR-Ni statistic can be expressed as

$$Y^2(\widehat{\underline{\mathbf{P}}}_n) = X_n^2(\widehat{\underline{\mathbf{P}}}_n) + \frac{1}{n}\ell^T(\widehat{\underline{\mathbf{P}}}_n)(\mathbf{I}(\widehat{\underline{\mathbf{P}}}_n) - \mathbf{J}(\widehat{\underline{\mathbf{P}}}_n))^{-1}\ell(\widehat{\underline{\mathbf{P}}}_n),$$

where

$$X_n^2(\underline{\mathbf{P}}) = \left([np_1(\underline{\mathbf{P}})]^{-\frac{1}{2}}\left[-np_1(\underline{\mathbf{P}}) + \underline{\mathbf{P}}_1\right], \cdots, [np_b(\underline{\mathbf{P}})]^{-\frac{1}{2}}\left[-np_b(\underline{\mathbf{P}}) + \underline{\mathbf{P}}_b\right]\right)^T$$

and

$$\mathbf{J}(\underline{\mathbf{P}}) = B(\underline{\mathbf{P}})^T B(\underline{\mathbf{P}}),$$

refers to the information matrix where

$$B(\underline{\mathbf{P}}) = \left[\frac{1}{\sqrt{p}_i}\frac{\partial}{\partial\mu}(\underline{\mathbf{P}})\right]_{r\times s} | i = 1, 2, \cdots, b \text{ and } \kappa = 1, 2, \cdots, s,$$

and

$$\ell(\underline{\mathbf{P}}) = (\ell_1(\underline{\mathbf{P}}), ..., \ell_s(\underline{\mathbf{P}}))^T \text{ with } \ell_\kappa(\underline{\mathbf{P}}) = \sum_{i=1}^{r}\frac{1}{p_i}\underline{\mathbf{P}}_i\frac{\partial}{\partial\underline{\mathbf{P}}_\kappa}p_i(\underline{\mathbf{P}}),$$

The $Y^2(\widehat{\underline{\mathbf{P}}}_n)$ statistic has $(b-1)$ degrees of freedom (DF) and is accompanied by χ^2_{b-1} distribution, where the observations. $x_1, x_2, \cdots, x_n$ that are collected in $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_b$ (these b subintervals are mutually disjoint: $\mathbf{I}_j =]a_{j,b} - 1; a_{j,b}]$). The intervals $\mathbf{I}_j$'s limits for $a_{j,b}$ are determined as follows

$$p_j(\underline{\mathbf{P}}) = \int_{a_{j,b}-1}^{a_{j,b}} f_{\underline{\mathbf{P}}}(x)\, dx | \; j = 1, 2, \cdots, b,$$

and

$$a_{j,b} = F^{-1}\left(\frac{j}{b}\right) | j = 1, \cdots, b-1.$$

7.0.1. Uncensored assessing for the RR-Ni statistic In numerous situations, the objective of a goodness-of-fit test extends beyond simply assessing whether a specific distribution adequately represents the data; it also involves estimating the parameters of that distribution. Simulation studies play a crucial role in evaluating the accuracy and precision of these parameter estimates under various conditions, thereby aiding in the selection of the most appropriate distribution for further analysis. In particular, simulation studies conducted in an uncensored setting using the RR-Ni statistic serve as a valuable tool for comparing and assessing different probability distributions in a controlled environment. These simulations offer insights into how well the RR-Ni test performs across diverse scenarios, helping to guide decisions regarding which distribution is best suited for subsequent analyses.

To validate the findings presented in this study, we performed a comprehensive numerical simulation analysis. To test the null hypothesis H_0, we generated RR-Ni statistics for the 2PLF model using simulated samples of varying sizes: n =20,35,220,350,650, and 1100, with a total sample size of 11,000. For different significance levels (ϵ =0.01,0.02,0.05,0.1), we calculated the average number of non-rejections under the null hypothesis, based on the

condition $Y^2 \leq \chi^2_{\epsilon}(b-1)$ .The corresponding empirical and theoretical levels are summarized in Table 11. A close alignment between the empirical level values and their respective theoretical counterparts is evident, indicating strong agreement. Based on these results, we conclude that the proposed test demonstrates excellent performance for the 2PLF distribution, confirming its suitability for practical applications.

Table 11: Uncensored assessing for the RR-Ni statistic for $\epsilon = 0.01, 0.02, 0.05, 0.1$ and $N = 11000$.

$n \downarrow \epsilon \longrightarrow$	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.1$
$n = 20$	0.9928	0.9832	0.9525	0.9023
$n = 35$	0.9919	0.982	0.9520	0.9020
$n = 220$	0.9911	0.9812	0.9517	0.9014
$n = 350$	0.9908	0.9804	0.9505	0.9006
$n = 650$	0.9904	0.9802	0.9503	0.9004
$n = 1100$	0.9902	0.9801	0.9502	0.9004

7.1. Validating the 2PLF model using the B-Ni test

Due to Bagdonavicius and Nikulin (2011) and Bagdonavicius et al. (2013), we can verify the suitability of the 2PLF model when the parameters are unknown and the data are censored where null hypothesis can be expressed as

$$H_0 : F(x) \in F_0 = \left\{F_0(x, \underline{\mathbf{P}}), x \in R^1,\ \underline{\mathbf{P}} \in \underline{\mathbf{P}} \subset R^s\right\},$$

Let's divide the limited amount of time $[0, \tau]$ into $\kappa | \kappa = 1, 2, \cdots, s$ shorter time periods. Where is the maximum runtime of the research and

$$\mathbf{I}_j = (a_{j-1}, a_{j,b}]; 0 =< a_{0,b} < a_{1,b} ... < a_{\kappa-1,b} < a_{\kappa,b} = +\infty.$$

The anticipated worth of $\widehat{a_{j,b}}$ can be said the following if $x_{(i)}$ is the i^{th} element in the ordered statistics $(x_{(1)},,, x_{(n)})$ and if $\mathbf{\Lambda}^{-1}$ refers to the cumulative hazard-rate function and

$$\widehat{a_{j,b}} = \mathbf{\Lambda}^{-1}\left((E_{j,X} - \sum_{l=1}^{i-1} \mathbf{\Lambda}(x_{(l)}, \widehat{\underline{\mathbf{P}}}))/(n-i+1), \widehat{\underline{\mathbf{P}}}\right), \quad \widehat{a_{\kappa}} = x_{(n)}|_{(j=1,\ldots,\kappa)},$$

where

$$e_{j,Z} = E_{\kappa}/\kappa \text{ for every } j.$$

and

$$\begin{aligned} E_{j,Z} &= (n-i+1)\mathbf{\Lambda}(\widehat{a_{j,b}}, \widehat{\underline{\mathbf{P}}}) + \sum_{l=1}^{i-1} \mathbf{\Lambda}(x_{(l)}, \widehat{\underline{\mathbf{P}}}) \\ &= \sum_{i:z_i > a_{j,b}} (\mathbf{\Lambda}(a_{j,b} \wedge z_i, \widehat{\underline{\mathbf{P}}}) - \mathbf{\Lambda}(a_{j-1}, \widehat{\underline{\mathbf{P}}}), E_{\kappa} = \sum_{i=1}^{n} \mathbf{\Lambda}(z_i, \widehat{\underline{\mathbf{P}}}). \end{aligned}$$

The $a_{j,b}$ functions for random data, and the $e_{j,Z}$ For the κ selected periods, anticipated failure rates are equal. Statistical data

$$Y_n^2 = \mathbf{Z}^T \widehat{\mathbf{S}}^{-1} \mathbf{Z},$$

where

$$\mathbf{Z} = (Z_1, ..., Z_{\kappa})^x, Z_j = \frac{1}{\sqrt{n}}(\mathbf{W}_{j,Z} - e_{j,Z})|_{(\ j=1,2,\ldots,\kappa)}$$

and $\mathbf{W}_{j,Z}$ can be used to test a hypothesis since it reflects the total number of failures that have been recorded throughout these time-shared. The elements of the B-Ni test statistic

$$Y_n^2 = \sum_{j=1}^{\kappa} \frac{1}{\mathbf{W}_{j,Z}} (\mathbf{W}_{j,Z} - e_{j,Z})^2 + \mathbf{D}_{W,G}$$

where

$$\begin{aligned}
\mathbf{D}_{W,G} &= \widehat{\mathbf{V}}^T \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{V}}, \widehat{\mathbf{S}}^{-1} = \widehat{\mathbf{B}}^{-1} + \widehat{\mathbf{M}}^{-1} \widehat{\mathbf{B}}^T \widehat{\mathbf{G}}^{-1} \widehat{\mathbf{M}} \widehat{\mathbf{B}}^{-1}, \\
\widehat{\mathbf{G}} &= [\widehat{g}_{ll'}]_{s \times s} = \widehat{i} - \widehat{\mathbf{M}} \widehat{\mathbf{B}}^{-1} \widehat{\mathbf{M}}^x, \\
\widehat{\mathbf{M}}_{lj} &= \frac{1}{n} \sum_{i: z_i \in \mathbf{I}_j} \alpha_i \frac{\partial}{\partial \underline{\mathbf{P}}} \ln \left[\lambda_{i, \underline{\mathbf{P}}}(z_i) \right], \mathbf{W}_{j,Z} = \sum_{i: z_i \in \mathbf{I}_j} \alpha_i, \ \widehat{\mathbf{B}}_j = n^{-1} \mathbf{W}_{j,Z}, \\
\widehat{\mathbf{V}}_l &= \sum_{j=1}^{\kappa} \widehat{\mathbf{M}}_{lj} \widehat{\mathbf{B}}_j^{-1} \mathbf{Z}_j, \quad l, l' = 1, ..., s, \\
\widehat{i}_{ll'} &= n^{-1} \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial \underline{\mathbf{P}}_l} \ln \left[\lambda_{i, \underline{\mathbf{P}}}(z_i) \right] \frac{\partial}{\partial \underline{\mathbf{P}}_{l'}} \ln \left[\lambda_{i, \widehat{\underline{\mathbf{P}}}}(z_i) \right]
\end{aligned}$$

and

$$\widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{\kappa} \widehat{\mathbf{M}}_{lj} \widehat{\mathbf{M}}_{l'j} \hat{A}_j^{-1},$$

and

$$\widehat{\mathbf{M}}_{lj} = \frac{1}{n} \sum_{i: z_i \in \mathbf{I}_j} \alpha_i \frac{\partial}{\partial \underline{\mathbf{P}}} \ln \left[\lambda_{i, \widehat{\underline{\mathbf{P}}}}(z_i) \right].$$

7.1.1. Censored assessing for the B-Ni statistic Censored simulation studies under the B-Ni statistics are an important tool for evaluating and comparing different probability distributions when dealing with censored data. These studies can provide valuable insights into the performance of the B-Ni tests under different types and levels of censoring, and can inform decisions about which distribution to use for subsequent analyses. It is intended that the sample that was produced $(N = 11000)$ will be censored at 20% and that DF$= 5$ To check if the sample agrees with the 2PLF model's null hypothesis, grouping intervals will be used. For various theoretical levels, we determine the average value of the non-rejection numbers of the null hypothesis. $(\epsilon = 0.01, 0.02, 0.05, 0.1)$, where $Y^2 \leq \chi_\epsilon^2 (r-1)$. The theoretical and empirical levels are compared in Table 12, which demonstrates how closely the determined empirical level matches the value of the relevant theoretical level. We conclude that the customized test is ideally suited to the 2PLF model as a consequence.

Table 12: Censored assessing for the B-Ni statistic for $\epsilon = 0.01; 0.02; 0.05; 0.1$ and $N = 12000$.

$n \downarrow \& \epsilon \longrightarrow$	$\epsilon = 0.01$	$\epsilon = 0.02$	$\epsilon = 0.05$	$\epsilon = 0.1$
$n = 20$	0.9935	0.9827	0.9529	0.9035
$n = 35$	0.9924	0.9811	0.9520	0.9022
$n = 220$	0.9920	0.9807	0.9512	0.9009
$n = 350$	0.9911	0.9804	0.9507	0.9008
$n = 650$	0.9908	0.9802	0.9504	0.9007
$n = 1100$	0.9905	0.9801	0.9502	0.9006

We conclude from these findings that the empirical significance level of the Y_n^2 The theoretical level of the chi-square distribution on degrees of freedom corresponds to the statistical level at which it is statistically significant. The censored data acquired from the 2PLF distribution may thus be satisfactorily fitted using the suggested test, according to this evidence.

8. The emergency care data

In the field of medicine, frailty models serve as valuable tools for analyzing risk factors and predicting the prognosis of various diseases. These models enable researchers to investigate how individual-level characteristics, such as age, gender, and genetic predispositions, affect patient outcomes. Additionally, they account for unmeasured or unobserved factors that may contribute to the risk of disease progression or mortality. Frailty models are widely applied in epidemiological research, clinical trials, and cohort studies to evaluate the impact of different treatments or interventions on patient health, providing a comprehensive framework for understanding complex health dynamics.

The real dataset utilized in this study was provided by the emergency department of the public proximity health institution (Echatt, El Tarf, Algeria) and encompasses observations collected throughout March 2023. The aim of this research was to investigate the relationship between various clinical variables and outcomes in patients seeking care at the emergency department. Ethical guidelines were strictly followed, and the necessary approvals were obtained prior to data collection. The dataset consists of 30 unique individuals, each representing a distinct observation. Six key clinical variables were recorded for each individual: age (in years), minimum and maximum blood pressure (in mmHg), blood glucose level (in mg/dL), heart rate (in beats per minute, BPM), and oxygen saturation (SaO2 %).

To ensure high-quality data, stringent measures were implemented during the collection process. These included meticulous documentation of patient information, adherence to standardized measurement protocols, and regular quality checks to identify and address any missing or inconsistent data. Such rigorous procedures enhance the reliability and accuracy of the dataset, making it particularly valuable for analyzing the relationships between clinical factors and emergency room outcomes. This dataset provides an opportunity to evaluate the goodness-of-fit of the 2PLF model distribution and its ability to accurately represent the observed patterns and variability in emergency care data. Specifically, we present point estimates for two fitted models: the 2PLF model with a Weibull baseline hazard-rate function and the 2PLF model with a Gompertz baseline hazard-rate function. To determine the most appropriate model among those fitted to the data, we employ the modified chi-squared test proposed by Bagdonavičius and Nikulin (2011). This statistical approach allows us to assess the validity and applicability of the 2PLF distribution in the context of emergency care data, thereby contributing to a deeper understanding of the underlying survival dynamics and heterogeneity in this critical healthcare setting.

8.1. Validation of the 2PLF model under the Weibull baseline hazard-rate function

Assuming that these data are distributed according to the 2PLF model with Weibull baseline hazard-rate function. Then, using R statistical software (the BB package), the maximum likelihood estimates of the parameter vector $\underline{\mathbf{P}}$ are obtained as

$$\begin{aligned}\widehat{\kappa} &= 0.830214, \widehat{\alpha} = 0.59325, \widehat{\sigma^2} = 1.02351,\\ \widehat{\beta_1} &= 0.19635, \widehat{\beta_2} = 0.49682, \widehat{\beta_3} = -1.02548,\\ \widehat{\beta_4} &= 0.48125, \widehat{\beta_5} = -0.96582, \widehat{\beta_6} = 0.17782.\end{aligned}$$

According to Bagdonavičius and Nikulin (2011) for censored data, we take for example 5 intervals $(r = 5)$ as number of classes. The elements of the estimated Fisher information matrix $I\left(\widehat{\underline{\mathbf{P}}}\right)$ are presented as follows:

$$I\left(\widehat{\underline{\mathbf{P}}}\right)=\begin{pmatrix} 1.02451 & -3.02652 & -8.95124 & 1.00325 & -7.93265 & 1.02457 & 0.96385 & -12.32051 & 1.04578 \\ & 1.02269 & 2.00031 & -8.32547 & 1.10245 & 1.62354 & 0.15423 & 0.98746 & 1.00023 \\ & & 0.62514 & -19.3265 & 1.03257 & -7.15230 & 1.96325 & 1.01254 & 0.00325 \\ & & & 0.85647 & 3.00052 & 4.00000 & 1.06805 & 0.02179 & 1.20450 \\ & & & & 1.62005 & 0.02154 & -20.0032 & 0.16357 & 0.12487 \\ & & & & & 0.84571 & 1.84603 & 0.14863 & 0.85463 \\ & & & & & & 1.95268 & 1.95382 & -7.53261 \\ & & & & & & & 0.01245 & 1.95368 \\ & & & & & & & & 2.00021 \end{pmatrix}$$

The calculated value of the test statistic for the proposed 2PLF model with a Weibull baseline hazard-rate function is $Y_n^2 = 9.36184$. Comparing this value with the critical value from the chi-squared distribution, $\chi^2_{0.05}(4) = 9.488$, we observe that Y_n^2 is less than the critical value. This result implies that we fail to reject the null hypothesis at the 5% significance level, indicating that the data is consistent with the proposed model. Therefore, the emergency care data can be adequately fitted using the 2PLF model with a Weibull baseline hazard-rate function. The model demonstrates a proper fit, effectively capturing the underlying survival patterns and accounting for unobserved heterogeneity in the dataset. This validation underscores the robustness and applicability of the 2PLF model in analyzing complex survival data.

8.2. Validation of the 2PLF model under the Gompertz baseline hazard-rate function

Assuming that these data are distributed according to the 2PLF model with Gompertz baseline hazard-rate function. Then, using R statistical software (the BB package), the maximum likelihood estimator of the parameter vector $\underline{\mathbf{P}}$ can be obtained as

$$\begin{aligned} \widehat{\rho} &= 0.96325, \widehat{\eta} = 0.86254, \widehat{\sigma^2} = 1.03325, \\ \widehat{\beta_1} &= -8.62514, \widehat{\beta_2} = 0.32514, \widehat{\beta_3} = 0.051847, \\ \widehat{\beta_4} &= -3.62958, \widehat{\beta_5} = 0.00635, \widehat{\beta_6} = 1.00254. \end{aligned}$$

We take $r = 5$ intervals and the estimated Fisher matrix expressed as

$$I\left(\widehat{\underline{\mathbf{P}}}\right)=\begin{pmatrix} 1.0325 & -5.0325 & -21.3021 & 1.0215 & 1.0254 & 1.0258 & -2.3062 & 1.5496 & 0.7483 \\ & 1.1458 & -8.0257 & 0.2157 & 2.0021 & 1.8457 & 1.9625 & 0.0214 & 0.9587 \\ & & 0.9528 & -0.3254 & 1.9586 & 1.0214 & 2.6254 & 1.8457 & -9.3574 \\ & & & 0.9936 & 1.0245 & 2.0001 & 2.0147 & 0.9632 & 0.1547 \\ & & & & 2.0016 & 1.0254 & 1.4875 & 1.5684 & 1.0254 \\ & & & & & 1.1154 & -7.9235 & -6.3025 & -9.3257 \\ & & & & & & 0.0032 & 0.9965 & 1.3025 \\ & & & & & & & 3.0024 & 1.0214 \\ & & & & & & & & 1.9551 \end{pmatrix}.$$

To evaluate the compatibility of the emergency care data with the proposed 2PLF model with a Gompertz baseline hazard-rate function, we calculate the value of the Bagdonavičius and Nikulin (2011) statistic, denoted as Y_n^2. The computed value of this statistic is $Y_n^2 = 8.902541$. This test statistic plays a crucial role in assessing the goodness-of-fit of the model to the observed data. Next, we compare Y_n^2 with the critical value from the chi-squared distribution $\chi^2_{0.05}$ at a significance level of $\alpha = 5\%$. For this test, the degrees of freedom are determined by the number of parameters estimated in the model. In our case, the degrees of freedom are 4, where $r = 5$ represents the total number of parameters in the 2PLF model with the Gompertz baseline hazard-rate function. From the chi-squared distribution table, the critical value for $\alpha = 5\%$ and 4 degrees of freedom is $\chi^2_{0.05}\,(5-1) = 9.488$. Upon comparing the calculated test statistic $Y^2 < \chi^2_{0.05}\,(5-1) = 9.488$ with the critical value $\chi^2_{0.05}\,(5-1) = 9.488$, it

becomes evident that $Y^2 < \chi^2_{0.05}(5-1)$. This result implies that the null hypothesis, which assumes that the data follows the proposed 2PLF model with a Gompertz baseline hazard-rate function, cannot be rejected at the 5% significance level. Consequently, we conclude that the emergency care data is indeed compatible with the proposed model. This compatibility indicates that the 2PLF model with a Gompertz baseline hazard-rate function provides an adequate fit to the emergency care data, effectively capturing the underlying survival patterns and accounting for unobserved heterogeneity. The robustness of the model is further validated through its ability to align with the observed data under rigorous statistical testing. Thus, the 2PLF model can be confidently applied for analyzing and predicting survival outcomes in emergency care settings, offering valuable insights into risk assessment and decision-making processes.

Despite its theoretical advancements, the proposed model exhibits several notable limitations that warrant further investigation. First, the strict identifiability constraint requiring the frailty mean to equal one inherently restricts the permissible parameter space, specifically forcing the shape parameter to remain below two. This mathematical restriction limits the model's flexibility in capturing extreme latent heterogeneity and may force the baseline hazard to absorb unidentifiable scale variations. Second, the computational burden associated with the complex Laplace transforms required for the marginal survival and hazard functions is substantial. Consequently, parameter estimation relies heavily on intensive numerical optimization techniques, such as the Barzilai-Borwein algorithm, which can suffer from convergence issues in high-dimensional covariate spaces. Third, the empirical validation of the model is constrained by the relatively small sample size of the emergency care dataset, which may limit the statistical power and generalizability of the findings. Furthermore, the study exclusively evaluates the 2PLF model under Weibull and Gompertz baseline hazards, neglecting other highly flexible alternatives like spline-based or piecewise exponential functions. The methodological framework also assumes non-informative right-censoring, failing to address more complex data structures such as interval censoring, left truncation, or competing risks.

Additionally, the underlying Cox-frailty formulation inherently relies on the proportional hazards assumption, which may be violated in long-term longitudinal medical studies. The model is also restricted to a univariate frailty structure, leaving its performance untested in scenarios involving multivariate, spatially correlated, or nested clustered data. From a practical standpoint, the authors do not provide a dedicated, optimized statistical software package, forcing practitioners to rely on custom, potentially unstable numerical scripts. The goodness-of-fit assessments using the Rao-Robson Nikulin and Bagdonavicius-Nikulin tests require grouping continuous survival times into discrete intervals, which inevitably leads to information loss. Moreover, the introduction of the additional shape parameter complicates the clinical and actuarial interpretability of the frailty variance and the underlying hazard dynamics. The risk analysis applies Key Risk Indicators directly to the baseline distribution rather than fully integrating the frailty-adjusted predictive distributions in dynamic portfolio contexts. Finally, the absence of time-varying covariates and Bayesian inference frameworks restricts the model's adaptability in handling dynamic longitudinal tracking and informative prior knowledge.

9. Conclusions

In this study, we introduced a new two-parameter Lindley frailty (2PLF) model designed to improve the analysis of survival data by offering greater flexibility compared to traditional models. The addition of a second shape parameter allows the model to better capture complex hazard patterns and tail behaviors, making it particularly effective in handling latent heterogeneity across diverse datasets. We explored several estimation techniques, including maximum likelihood estimation (MLE), Cramér–von Mises (CVM), Anderson–Darling (ADE), and modified versions like RTADE and LEADE. These methods were tested under various sample sizes and censoring levels through simulation studies, using performance metrics such as bias, RMSE, Dabs, and Dmax to evaluate their accuracy and consistency. The 2PLF model was applied to real-world data from emergency care and insurance claims, demonstrating superior fit and predictive capability when compared to existing models. Risk indicators such as Value-at-Risk (VaR), Tail VaR (TVaR), and tail mean-variance were used to validate the model's robustness and reliability in practical settings. Mathematically, the model showed strong theoretical properties, supporting a

wide range of applications in actuarial science, epidemiology, engineering, and survival analysis. Goodness-of-fit testing using the RR-Ni statistic confirmed the model's validity in both censored and uncensored scenarios. This work addresses a key gap in current frailty modeling literature by providing a more adaptable framework that improves risk stratification and decision-making. The 2PLF model outperformed conventional models, especially in high-frailty conditions and complex hazard situations. Future research could extend the model using Bayesian inference or integrate machine learning techniques to further enhance its adaptability and precision in analyzing heterogeneous survival data.

While the 2PLF model advances survival analysis, extending its univariate framework to multivariate or spatial structures remains a critical avenue for modeling complex clustered data. Integrating Bayesian inference methodologies, such as MCMC or INLA, could offer robust alternatives to frequentist estimation, particularly for small-sample regimes. Relaxing the proportional hazards assumption by developing an Accelerated Failure Time (AFT) model or incorporating time-varying covariates would greatly enhance its utility for longitudinal studies. Future work must also adapt the distribution for competing risks and address complex data structures like interval censoring, left truncation, and informative censoring mechanisms. Furthermore, hybridizing the 2PLF baseline with machine learning techniques, such as survival random forests, could effectively handle ultra-high-dimensional covariate spaces. From an actuarial perspective, extending Key Risk Indicator analyses to dynamic, multi-period portfolios under stochastic interest rates would yield more realistic liability assessments. Additional promising directions include developing shared frailty models for bivariate survival data and applying penalty-based regularization techniques for simultaneous variable selection. Ultimately, addressing these theoretical and computational extensions will solidify the 2PLF model's position as a versatile tool for next-generation biostatistics and financial risk management.

Building upon recent advancements in statistical modeling, future research could extend the 2PLF framework by deriving its high-order statistical properties to enhance extreme value approximations, mirroring recent work on the Lambert-Topp-Leone distribution (Salih et al., 2025). Inspired by the development of novel compound G-families, researchers might integrate the 2PLF frailty term into new generalized families to model complex reliability data, such as aircraft windshield service durations (see Yousof et al. (2026) and Abdulah et al., 2026). Furthermore, the incorporation of zero-inflated structures into the 2PLF regression model presents a promising avenue for analyzing medical datasets characterized by excess zeros or dormant frailty states. To address high-dimensional covariate spaces in such zero-inflated frailty models, the application of generalized shrinkage estimators could significantly improve parameter estimation and variable selection, building on methodologies demonstrated in zero-inflated Bell regression (Abdulhadi et al., 2025). Ultimately, these methodological extensions will solidify the 2PLF model's versatility, bridging the gap between advanced theoretical statistics and practical applications in reliability engineering and biostatistics.

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