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Kernel Density Estimation for Random Effects Models driven by Weighted Fractional Brownian Motion

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Abstract This paper deals with the nonparametric estimation of the random effects from stochastic differential equations of the form $dX(t) = S(X(t), \psi)\, dt + \varepsilon dB_{a,b}(t)$, $X(0) = 0$, $\varepsilon \in (0, 1]$, where $B_{a,b}$ is the weighted fractional Brownian motion. The drift function $S(\cdot, \cdot)$ is known and possibly nonlinear. Based on continuous observations of n copies independent and identically distributed of X (known as subjects characterized by different random effects $\psi = \psi_i$), and by using a two-stage kernel approach we formulate selected estimators of the common density f of the random effects ψ_i. Both the L^2-risk bounds and the selection of the bandwidths are discussed. The proposed estimators are illustrated by numerical simulations of two concrete examples, where the random effects are either Gaussian or Beta distributed.

Keywords Weighted fractional Brownian motion, Stochastic differential equation , Random effect, Kernel estimator, Bandwidth selection

AMS 2010 subject classifications 60G22, 60G60

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1. Introduction

The noise in many biological, physical or economical random phenomena are often described by the simple Brownian motion (Bm) or by the fractional Brownian motion (fBm) (see e.g. [1], [2], and [3]). The two-sided fBm B_H is formally defined as a centered Gaussian process with covariance

$$\mathbb{E}(B_H(t)B_H(s)) = \frac{V_H}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad \text{for all } t, s \in \mathbb{R},$$

where V_H is some nonnegative constant. The fBm exhibits two properties (self-similarity and long range dependence) which are of great importance in many disciplines. However, its reliance one single parameter (known as the Hurst index) or its stationarity of increments presents a drawback. As a result various extensions of the fBm, such as bi-fractional Brownian motion, sub-fractional Brownian motion and weighted fractional Brownian motion (wfBm) have been proposed. The wfBm appeared in a limit of occupation time fluctuations of certain particle systems for some values of the parameters (e.g., [4, Theorem 2.2]). It has no stationary increments in general. In contrast, its increments are asymptotically stationary for long time intervals, but not for short ones (e.g., [5, Theorem 2.4]. This implies more

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complexity in the dependence structure, which allows to capture the intricate behaviors observed in financial markets. The problem of arbitrage opportunities arises in fractional Gaussian processes, including the fBm and the wfBm (see, [6]). However, by using the idea of mixing these processes with the Brownian motion (e.g., [7]) the problem can be solved. In recent works the mixed wfBm was adopted to describe European options [8, 9].

Stochastic differential equations with random effects (**SDERE**) are increasingly used in the biomedical fields and have proved to be adequate tools for the study of repeated measurements with variability between subjects. In SDERE setting, the problem of inference is of particular importance. Within the parametric framework, the maximum likelihood estimation is often considered to estimate the parameters of the random effects (known as population parameters). When the noise in SDERE is described by an fBm, one lacks the Markovian property and the construction of the likelihood function becomes very difficult. Even, in the Brownian case the transition densities are unkown in most cases, while the likelihood has to be marginalized over the distribution of the random effects. This often leads to multidimensional integrals with no closed form. To overcome these intractability issues, many approximations of the likelihood have been proposed, based on linearization or Laplace's approximation. As an alternative the authors in [10] have tackled the Bayesian inference with emphasis on MonteCarlo algorithms. Other methods have been proposed in the context of multilevel logistic regression [11]. For some specific distributions of random effects (Gaussian or gamma) the estimation of the population parameters lead to exact formula (e.g., [12, 13, 14]). Recently, the authors in [15] have used the method of moments to address the estimation of the population parameters in general context. Under some regularity assumptions on the drift and diffusion functions, the estimators were formulated and analyzed. The theoretical results are supported by numerical simulations and illustrated through an empirical study on Asian financial data.In the non-parametric framework, two-stages methods of estimations have been proposed. The idea is to proceed in two steps: **(a)** Estimate the random effects from the available (discrete or continuous) observations. **(b)** Plug the obtained estimates into the well known density estimators type, such as, the kernel estimators [16, 17], the deconvolving estimators [16, 18], or the projection estimators [19, 20]. For these types of estimators, the computations depend on a bandwidth to be carefully chosen from the data. For that purpose, several selection methods of the bandwidth have been proposed, such as the Goldenshluger and Lepski method [21], or the standard cross-validation type method. Note that in recent works [17, 20, 22] the noise in SDERE is described by an fBm, but [22] is only the reference where the small diffusion scenario is treated. The present work deals with the nonparametric estimation of random effects from the observations governed by stochastic differential equations of the form

$$dX(t) \quad = \quad S\left(X(t),\psi\right)dt + \varepsilon dB_{a,b}(t),\ X(0)=0,\ \varepsilon\in(0,1] \text{ and } t\geq 0.$$

where $S\left(\cdot,\cdot\right)$ is a known function (possibly nonlinear) and ψ is the random effect of interest. Given n i.i.d copies of X (known as subjects), and under some regularity assumptions (including dissipativity $\mathbf{H}_1$), we build selected kernel density estimators of the density f of ψ. The mean integrated squared risk bound of these estimators is rigorously established, with selection of the bandwidths being discussed. Our theoretical results are illustrated by numerical simulations showing the good performance of the proposed estimators.

The rest of the paper is organized as follows. In Section 2, we recall some basic properties of the wfBm. In Section 3 we construct a class of selected estimators of the random effects and analyze their $\mathbb{L}^2-$risk. To this end, kernel-based approach is applied twice with different kernels K_h and $G_{h'}$. By means of K_h kernels we estimate the random effects ψ_i, while $G_{h'}$ kernels were used to obtain the estimates of the common density of ψ_i. In order to improve the accuracy of the proposed estimators, a particular attention is devoted to the adaptation of the bandwidth parameter $h'>0$ in Subsection 3.3. A special case is discussed in Subsection 3.4, with stochastic Langevin equations being implemented for two different types of distributions of random effects ψ_i in Subsection 4. The performance of the proposed selected kernel density estimators are illustrated in figures 1, 2 and 3. Some concluding remarks and perspectives are presented in Section 5. For the clarity of exposure, we relegate all proofs to Appendix

section 5.

The following notations are systematically used in this paper: $x \wedge y := \min(x, y)$, $x \vee y := \max(x, y)$, $(x)_+ = \max(x, 0)$ and $\mathcal{B}(\cdot, \cdot)$ is the Beta function. We denote by $\mathcal{O}(.)$, $\mathbb{E}$, $\mathbb{V}ar$ and $\to$ the usual big-oh, the expectation, the variance under the probability $\mathbb{P}$ and the simple convergence, respectively. We denote by $Card(A)$ the cardinal of any set A and by $\mathbb{1}_A$ its indicator function. Also $\|f\|$ and $\|f\|_\infty$ will denote $\mathbb{L}^2$-norm and the supremum norm of f on its own space, respectively. The symbol $\lfloor x \rfloor$ denotes the greatest integer strictly less than the real number x, and M^T denote the transpose of the matrix M.

2. Weighted fractional Brownian motion

This section is devoted to basic properties and the stochastic calculus of the wfBm. The wfBm $B_{a,b} = (B_{a,b}(t), t \geq 0)$ with parameters $a > -1$, $|b| < 1$ and $|b| < a + 1$, was first considered in [23] as a centered Gaussian process with the covariance function given by:

$$R_{a,b}(s,t) := \int_0^{t\wedge s} u^a \left((t-u)^b + (s-u)^b\right) du, \quad \text{for all } s, t \geq 0. \tag{1}$$

Clearly, if $a = b = 0$, $B_{a,b}$ coincides with the standard Brownian motion. For $a = 0$, (1) reduces to

$$R_{a,b}(s,t) = \frac{1}{b+1}\left(t^{b+1} + s^{b+1} - |t-s|^{b+1}\right), \quad \text{for all } s, t \geq 0,$$

which corresponds to the covariance of a fBm with Hurst index $H = (b + 1)/2$. According to [23], [24] and [25] the wfBm has the following properties:

(i) $\mathbb{E}(B_{a,b}(t)) = 0$ and $\mathbb{V}ar\,(B_{a,b}(t)) = 2\int_0^t u^a(t-u)^b du$, for all $t \geq 0$.
(ii) The process $B_{a,b}$ is self-similar with the index $\alpha = (a + b + 1)/2$; that is for every $c > 0$, the processes $(B_{a,b}(ct)\ ,\ t \geq 0)$ and $(c^\alpha B_{a,b}(t)\ ,\ t \geq 0)$ have the same finite dimensional probability law.
(iii) $B_{a,b}$ is neither a semi-martingale nor a Markov process unless $a = 0$ and $b = 0$.
(iv) The trajectories of $B_{a,b}$ are γ-Hölderian with $\gamma \in (0, (b + 1)/2)$.
(v) The trajectories of $B_{a,b}$ are continuous with the only exception of the case $a < 0$, $b < 0$ and $a + b = -1$, where $t \mapsto B_{a,b}(t)$ is discontinuous at 0.
(vi) $B_{a,b}$ has no stationary increments in general, and for all $s, t \geq 0$

$$c_{a,b}(t \vee s)^a |t-s|^{b+1} \leq \mathbb{E}|B_{a,b}(t) - B_{a,b}(s)|^2 \leq C_{a,b}(t \vee s)^a |t-s|^{b+1},$$

for some nonnegative constants $c_{a,b}$ and $C_{a,b}$.
(vii) $B_{a,b}$ exhibits long range dependence for $b > 0$ and short range dependence for $b < 0$.

Let $a > -1$ and $0 < b < (1 \wedge (a + 1))$. It is possible to construct a variety of stochastic calculus with respect to the wfBm associated with the Malliavin calculus. More studies and references in the context can be found in [26] and references therein. Fix $T > 0$ and let $\mathcal{H}$ be the Hilbert space associated with the wfBm defined as the closure of the linear space generated by indicator functions $\mathcal{E} := \left\{\mathbb{1}_{[0,t]}\ ,\ t \in [0, T]\right\}$ with respect to inner product $\langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]} \rangle_{\mathcal{H}} = R_{a,b}(s,t)$. The mapping $\varphi \mapsto W_{a,b}(\varphi) := \int_0^T \varphi(s) dB_{a,b}(s)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by $B_{a,b}$, and can be extended to the Hilbert space $\mathcal{H}$. Let $|\mathcal{H}| \subset \mathcal{H}$ be the subspace of measurable functions φ on $[0, T]$ such that $\|\varphi\|^2_{|\mathcal{H}|} = \int_0^T \int_0^T |\varphi(t)||\varphi(s)|\phi_{a,b}(t,s)dtds < \infty$, where $\phi_{a,b}(s,t) = b(t \wedge s)^a |t-s|^{b-1}$. It can be proved that $\left(|\mathcal{H}|, \|\cdot\|_{|\mathcal{H}|}\right)$ is a Banach space (e.g. [27], [28]). Moreover, $\mathbb{L}^2([0, T]) \subset \mathbb{L}^{2/(1+a+b)}([0, T]) \subset |\mathcal{H}| \subset \mathcal{H}$

and for all $\varphi, \tilde{\varphi} \in |\mathcal{H}|$,

$$\mathbb{E}\left(\int_0^T \varphi(s) dB_{a,b}(s) \int_0^T \tilde{\varphi}(s) dB_{a,b}(s)\right) = \int_0^T \int_0^T \varphi(u)\tilde{\varphi}(v)\phi_{a,b}(u,v) du dv. \tag{2}$$

3. Main results

For $i = 1, ..., n$, a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^i)_{t\geq 0}, \mathbb{P}\right)$ is considered. The filtration $(\mathcal{F}_t^i)_t$ is understood as the $\mathbb{P}$-completion of the filtration generated by the process. On the same space we define two independent sequences; $(B_{a,b}^i)_{i=1,...,n}$ a sequence of n independent wfBm's with a common indices a and b, and $(\psi_i)_{i=1,...,n}$, a sequence of n independent and identically distributed $(i.i.d)$ real random effects such that ψ_i is $\mathcal{F}_0^i$ measurable for all $i = 1, ..., n$ with finite moments. Now, we consider n subjects $\left(X^i(t), \mathcal{F}_t^i, t \geq 0\right)_{i=1,..n}$ with dynamics ruled by the following SDE's, with small diffusion:

$$dX^i(t) \quad = \quad S\left(X^i(t), \psi_i\right) dt + \varepsilon dB_{a,b}^i(t), \; X^i(0) = 0, \; \varepsilon \in (0, 1] \text{ and } t \geq 0. \tag{3}$$

where $S(\cdot, \cdot)$ is a known function and the random effects ψ_i are $\mathcal{F}_0^i$-measurable with common density f belonging to a class of functions to be specified later. The importance of the imposed restriction $0 < b < (1 \wedge (a + 1))$ is two folds: First, it is practically convenient, as the process exhibits long range dependence (see, the property (vii) in Section 3). Second, it allows the process $B_{a,b}$ to be γ-Hölderian with $1/2 < \gamma < (b + 1)/2$. As a result, the integrals with respect to $B_{a,b}$ should be interpreted as Young integrals, whenever the integrand is γ'-Hölderian with $\gamma' \geq 1/2$. Hence, the approximation of the stochastic integrals can be performed by means of Riemann Steiltjes sums. Moreover, we know (e.g. [29, Theorem 3.1 and Theorem 3.2]) that for all functions g and h α-Hölderian with $\alpha > 1/2$ we have, $g(t)h(t) = g(0)h(0) + \int_0^t g(u)dh(u) + \int_0^t h(u)dg(u)$.

We consider the limiting solution to (3), as $\varepsilon \to 0$, given by

$$\frac{dx^i(t)}{dt} = S\left(x^i(t), \psi_i\right), \quad x^i(0) = 0, \; t \geq 0, \tag{4}$$

and adopt the two following conditions that guarantees the existence and uniqueness of solutions for (3).

- $\mathbf{H}_1$ **(Dissipative condition).** The function $S(z, u)$ is continuously differentiable with respect to z and there exists $M > 0$ such that $(z - z')\left(S(z, u) - S(z', u)\right) \leq -M|z - z'|^2$ for all $u, z, z' \in \mathbb{R}$.
- $\mathbf{H}_2$ **(Polynomial growth condition).** There exist a positive constant $L > 0$ and nonnegative integer p such that $|S(z, u)| \leq L\left(1 + |z|^p + |u|^p\right)$, for all $z, u \in \mathbb{R}$.

3.1. Estimation of the density of the random effects ψ_i

In this subsection we construct a random approximation of $S\left(x^i(t), \psi_i\right)$, and derive from them suitable random approximations $\widehat{\psi}_{h,\varepsilon}^{(k)}(X^i)$, $k = 1, 2$ of ψ_i in $\mathbb{L}^2(\Omega)$. The subjects X^i, $i = 1, \cdots, n$ given in (3) are $i.i.d$, and can be considered as n independent copies of a process X satisfying the same equation (3) (say $X = X^1$). In what follows we focus on properties of X, thus all indices in (3) and (4) referring to different subjects are omitted.

Lemma 1
Let the assumptions $\mathbf{H}_1$ and $\mathbf{H}_2$ hold. The solution of (3) exists and almost surely unique. Moreover, there are some deterministic constants $c_1, c_2, c_3 > 0$ independent of T such that

$$\sup_{0\leq t\leq T}\left(|X(t)| \vee |x(t)|\right) \leq c_1 + c_2|\psi|^p + c_3 \sup_{0\leq t\leq T}|B_{a,b}(t)|^p, \; \textit{for all } T > 0, \tag{5}$$

where $x(.)$ is the associated limiting of $X(.)$ as solution of (4).

Lemma 2
Suppose that $\mathbf{H}_1$ and $\mathbf{H}_2$ hold true. Then,

$$\sup_{0\leq t\leq T}\mathbb{E}|X(t)-x(t)|^2\leq C_1\varepsilon, \tag{6}$$

where C_1 is nonnegative constant.

Lemma 3
Let the assumptions of Lemma 2 be fulfilled, then for all $T>0$ there exists $C_2>0$ such that,

$$\mathbb{E}|x(t)-x(s)|^2\leq C_2|t-s|^2,\quad \text{for all } s,t\in[0,T]. \tag{7}$$

Now we are ready to approximate the random function $S(x(t),\psi)$ and hereafter the random effect ψ ($\psi=\psi^1$). In the sequel $S(z,u)$ we will be denoted by $S_z(u)$ or $S_u(z)$ depending on the arguments of interest (that is, one of z and u is fixed as a parameter). To proceed we make the following assumptions

$\mathbf{H}_3$: There exists $M_1>0$ such that $\mathbb{E}|S(z,\psi)-S(z',\psi)|^2\leq M_1\mathbb{E}|z-z'|^2$, for all random variables z,z' with finite second moments .
$\mathbf{H}_4$: Assume that for each $z\in\mathbb{R}$, the function $S_z(\cdot)=S(z,\cdot)$ admits an inverse $S_z^{-1}(\cdot)$.

$\mathbf{H}_5$: Consider $K(\cdot)$ a function that verifies the following hypotheses:
(i) K is bounded and vanishing outside the interval $[A,B]$ with $A<B$.
(ii) $K(u)\geq 0$, $\int_A^B K(u)du=1$ and $\int_A^B|u|K(u)du<\infty$.

Remark 1
The condition $\mathbf{H}_3$ follows immediately if $S(z,\psi)$ is Lipschitz in z, uniformly in ψ. Tracking the constant M_1 in the proofs given in Appendix, we find that $\mathbf{H}_3$ can be replaced by the next weaker condition.

$\mathbf{H'}_3$: There exist $\gamma>0$ and a bivariate function $\mathcal{M}_1$ having polynomial growth such that

$$|S(z,\psi)-S(z',\psi)|\leq|\mathcal{M}_1(z,z')|\cdot|z-z'|^{\gamma},\quad\forall z,z'\in\mathbb{R}.$$

For fixed $T>0$ set

$$\widehat{Q}_{h,\varepsilon}(X,\tau)=\frac{1}{h}\int_0^T K\left(\frac{s-\tau}{h}\right)dX(s),\quad 0<\tau<T. \tag{8}$$

Lemma 4
Let $\mathbf{H}_3$, $\mathbf{H}_5$ and the assumptions of Lemma 2 be fulfilled. Consider the sequence $\widehat{Q}_{h,\varepsilon}(X,\tau)$ given by (8), we have

$$\mathbb{E}|\widehat{Q}_{h,\varepsilon}(X,\tau)-S(x(\tau),\psi)|^2\to 0,\quad \text{for all } \tau\in(0,T), \tag{9}$$

as $\varepsilon,h\to 0$ with $h^{b-1}\varepsilon^2\to 0$. Moreover, the convergence holds uniformly in τ if $a\geq 0$.

Thanks to Lemma 4 and the assumption $\mathbf{H}_4$, we can approximate the random effect ψ by one of the following statistics

$$\widehat{\psi}_{h,\varepsilon}^{(1)}(X):=S_{X(\tau)}^{-1}(\widehat{Q}_{h,\varepsilon}(X,\tau)) \tag{10}$$

$$\widehat{\psi}_{h,\varepsilon}^{(2)}(X):=S_{x(\tau)}^{-1}(\widehat{Q}_{h,\varepsilon}(X,\tau)), \tag{11}$$

By Lemma 4 and continuous mapping one can show that $\widehat{\psi}_{h,\varepsilon}^{(k)}$, $k=1,2$ converge toward ψ in the space $\mathbb{L}^2(\Omega)$. These estimates of ψ depend on a fixed $\tau\in(0,T)$. Each choice of τ produces a new estimate of ψ.

Theorem 1
Let $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{H}_5$ and the assumptions of Lemma 2 be fulfilled. Consider the estimators $\widehat{\psi}_{h,\varepsilon}^{(k)}(X)$, $k=1,2$ given by (10) and (11) and assume further that there is $M_2>0$ such that

$$(u-v)\left(S\left(z,u\right)-S\left(z,v\right)\right)\leq -M_2|u-v|^2, \quad \text{for all } z,u,v\in\mathbb{R}, \tag{12}$$

then,

$$\mathbb{E}\left|\widehat{\psi}_{h,\varepsilon}^{(k)}(X)-\psi\right|^2\leq A_1(\varepsilon+h^2+h^{b-1}\varepsilon^2), \quad k=1,2, \tag{13}$$

where A_1 is nonnegative constant.

The following result gives the optimal rate of convergence of $\widehat{\psi}_{h,\varepsilon}^{(k)}$, $k=1,2$ toward ψ. It is obtained by minimizing the upper bound given in (13) as a function of h with $0<b<1\wedge(1+a)$. The exact bandwidth producing this optimal rate is $h_*=\left(\frac{1}{2}(1-b)\right)^{\frac{1}{3-b}}\varepsilon^{2/(3-b)}$.

Corollary 1
Consider the estimators $\widehat{\psi}_{h,\varepsilon}^{(k)}(X)$, $k=1,2$ given by (10) and (11), and set $h=\varepsilon^{2/(3-b)}$, then, $\left\|\widehat{\psi}_{h,\varepsilon}^{(k)}(X)-\psi\right\|_{L^2(\Omega)}=\mathcal{O}(\varepsilon^{1/2})$, $k=1,2$, as $\varepsilon,h\to 0$ with $h^{b-1}\varepsilon\to 0$.

For each subject $X^i(t)$, $i=1,\cdots,n$ with associated limiting process $x^i(t)$, we define the estimates of ψ_i by (10) and (11) with $X=X^i$ and $x=x^i$. It is well known that if the random effects ψ_i, $i=1,2,\cdots,n$ are observed, then their common density f can be estimated (see e.g.[30]) by

$$\widetilde{f}_{h'}(y) \quad = \quad \frac{1}{n}\sum_{i=1}^{n}G_{h'}\left(y-\psi_i\right), \quad h'>0,\ y\in\mathbb{R}, \tag{14}$$

where $G_{h'}(y)=\frac{1}{h'}G\left(\frac{y}{h'}\right)$ and G is some suitable kernel to be specified later. Unfortunately, in real cases the random effects ψ_j are not directly observed, and we plan to replace them in (14) by their estimates $\widehat{\psi}_{h,\varepsilon}^{(k)}(X^i)$, $k=1,2$. So, the resulting estimators are defined by

$$\widehat{f}_{h'}^{(k)}(y) \quad = \quad \frac{1}{n}\sum_{i=1}^{n}G_{h'}(y-\widehat{\psi}_{h,\varepsilon}^{(k)}(X^i)),\ h'>0,\ y\in\mathbb{R},\ k=1,2. \tag{15}$$

In what follows we focus on the asymptotic behavior of $\widehat{f}_{h'}^{(1)}$, because the same procedures can be applied to get asymptotic results for $\widehat{f}_{h'}^{(2)}$.

3.2. Mean integrated squared risk bound of $\widehat{f}_{h'}^{(1)}$

In order to prove our principal results we make the following assumptions.

$\mathbf{H}_6$: The kernel G defined on $\mathbb{R}$ is differentiable such that $\int_{\mathbb{R}}G(u)du=1$ and $\|G\|+\|G'\|<\infty$.
$\mathbf{H}_7$: The kernel G is of order $l\geq 1$, that is,

$$\int_{\mathbb{R}}u^jG(u)du=0, \quad \text{for all } 1\leq j\leq l-1 \quad \text{and} \quad \int_{\mathbb{R}}u^lG(u)du\neq 0.$$

Assume further that f belongs to the Nikol'ski class $\mathrm{N}(\delta,R)$ defined as the set of functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ whose derivatives $f^{(l)}$ of order $l=\lfloor\delta\rfloor$ exist and satisfy

$$\left(\int_{\mathbb{R}}\left(f^{(l)}(x+t)-f^{(l)}(x)\right)^2dx\right)^{1/2}\leq R|t|^{\delta-l}, \text{ for all } t\in\mathbb{R}.$$

Theorem 2
Consider the estimator $\widehat{f}_{h'}^{(1)}$ given by (15). Let $\mathbf{H}_6$ and the assumptions of Theorem 1 be fulfilled. Then,

$$\mathbb{E} \parallel \widehat{f}_{h'}^{(1)} - f \parallel^2 \quad \leq \quad A_2 \left(\| f_{h'} - f \| + \frac{\|G\|^2}{nh'} + \frac{\|G'\|^2}{h'^3} \cdot \varepsilon \right), \tag{16}$$

where A_2 is nonnegative constant, and $f_{h'}$ the function defined by $f_{h'} = G_{h'} \star f$.

So, as a direct consequence of this theorem, we obtain the following result

Corollary 2
Let $\mathbf{H}_6$, $\mathbf{H}_7$ and the assumptions of Theorem 1 be fulfilled. Consider the estimator $\widehat{f}_{h'}^{(1)}$ given in (15). Set $h = \varepsilon^{2/(3-b)}$, $h' = \varepsilon^{1/(3+2\delta)}$ and $n \geq \varepsilon^{-(1+2\delta)/(3+2\delta)}$. Assume further that $\int_{\mathbb{R}} |u|^{\delta} |G(u)| du < \infty$. Then,

$$\mathbb{E} \parallel \widehat{f}_{h'}^{(1)} - f \parallel^2 = \mathcal{O}(\varepsilon^{2\delta/(3+2\delta)}). \tag{17}$$

3.3. Adaptation of the bandwidth h'

For the selection of the bandwidth h', we follow [16]. Given a collection $\mathbb{H}$ of estimators $\widehat{f}_{h'}^{(1)}$ depending on the bandwidth $h' > 0$. The best choice of h' is the one that minimizes the upper bound (16). But this is not possible, because the term $\|f_{h'} - f\|^2$ is unknown and has to be estimated. To choose a suitable h' from the data in non-asymptotic way, we use the Goldenshluger and Lipski's criterion [21]. The idea is to estimate $\|f_{h'} - f\|^2$ by the $\mathbb{L}^2$-distance between two estimators defined by (15). But this induces an error which has to be corrected by the variance term. For that purpose we set

$$A(h') \quad = \quad \sup_{\tilde{h} \in \mathbb{H}} \left(\left\| \widehat{f}_{h',\tilde{h}}^{(1)} - \widehat{f}_{\tilde{h}}^{(1)} \right\|^2 - V(\tilde{h}) \right)_+, \tag{18}$$

where $\widehat{f}_{h',\tilde{h}}^{(1)}(x) = G_{\tilde{h}} \star \widehat{f}_{h'}^{(1)}(x) = \frac{1}{n} \sum_{j=1}^{n} G_{\tilde{h}} \star G_{h'} \left(x - \widehat{\psi}_{h,\varepsilon}^{(1)}(X^j) \right)$ and $G_{\tilde{h}}(.)$ is the kernel given by (**??**) and

$$V(h') \quad = \quad \kappa_1 \frac{\|G\|^2}{nh'} + \kappa_2 \frac{\|G'\|^2}{h'^3} \cdot \varepsilon, \tag{19}$$

where κ_1 and κ_2 are two numerical constants. The bandwidth h' is selected as

$$\widehat{h'} \quad = \quad \arg \min_{h' \in \mathbb{H}} \{ A(h') + V(h') \}, \tag{20}$$

with $\mathbb{H}$ is finite discrete set of bandwidths h' such that $h' > 0$, $\frac{1}{nh'} \leq 1$, $\frac{\varepsilon}{h'^3} \leq 1$ and $Card\,(\mathbb{H}) \leq n$. Assume further that

$$\sum_{h' \in \mathbb{H}} h'^{-1/2} e^{-c/\sqrt{h'}} \quad < \quad \infty, \quad \text{for all } c > 0.$$

Then the following Theorem is obtained.

Theorem 3
Let $\mathbf{H}_6$ and the assumptions of Theorem 1 be fulfilled. Consider the estimator $\widehat{f}_{h'}^{(1)}$ given by (15) with $h' \in \mathbb{H}$. Then there exist two penality constants $\kappa_1, \kappa_2 > 0$ such that

$$\mathbb{E} \left\| \widehat{f}_{\widehat{h'}}^{(1)} - f \right\|^2 \quad \leq \quad A_3 \inf_{h' \in \mathbb{H}} \left(\|f_{h'} - f\|^2 + V(h') \right) + \frac{A_4}{n}, \tag{21}$$

where A_3 and A_4 are nonnegative constants independent of n.

Proof
We can proceed in a similar fashion as in [16, Theorem 3.1] to get (21). Thus, we omit its proof. □

3.4. Special Case

Let $S(z,u) = \boldsymbol{S}(z+u)$. Then all the assumptions required for our theoretical framework are met if $\boldsymbol{S}(\cdot)$ is continuously differentiable with bounded derivative and satisfies $\boldsymbol{S'}(z) \leq -M$, for all $z \in \mathbb{R}$ (for some constant $M > 0$). First, the dissipative and the polynomial growth conditions hold. In fact, by using the mean value theorem, for all $u, z, z' \in \mathbb{R}$ one has

$$\begin{aligned}
&(z-z')\left(S(z,u)-S(z',u)\right) = (z-z')\left(\boldsymbol{S}(z+u)-\boldsymbol{S}(z'+u)\right)\\
&= (z-z')^2\boldsymbol{S'}(c_*), \quad \text{for some } c_* \text{ being convex combination of } z+u \text{ and } z'+u\\
&\leq -M|z-z'|^2
\end{aligned}$$

and

$$\begin{aligned}
|S(z,u)-S(z',u)| &\leq |\boldsymbol{S'}(c_*)|\cdot|z-z'|\\
&\leq \| \boldsymbol{S'} \|_\infty \cdot |z-z'|.
\end{aligned}$$

Because of symmetry, the assumptions $\mathbf{H}_3$ and (12) follow by the same procedures. If $\boldsymbol{S}$ is not bounded, but instead has a polynomial growth, then the alternative assumption $\mathbf{H'}_3$ holds. In general, without $\mathbf{H}_4$ the random effects may not be identifiable. But for this special case we recognize that $\psi \mapsto \boldsymbol{S}(z+\psi)$ is strictly decreasing, because of the condition $\boldsymbol{S'}(z) \leq -M$, for all $z \in \mathbb{R}$. Therefore, $\mathbf{H}_4$ holds true. The concrete example for this special case is $\boldsymbol{S}(z) = -d_1 z + d_2$, where $d_1 > 0$ and $d_2 \in \mathbb{R}$. This corresponds to the widely used class of stochastic Langevin equations. In the next section, we implement the case $d_1 = 2$, $d_2 = 3$, in order to show the performance of the proposed selected kernel estimators for the common density f of the random effects. For this example we have $S_z^{-1}(u) = (-d_1 z + d_2 - u)/d_1$.

For the second example to be implemented, we choose the nonlinear function $\boldsymbol{S}(z) = -z - z^3$. Its derivative is unbounded polynomial function. So for this example $\mathbf{H'}_3$ holds true instead of $\mathbf{H}_3$. After performing some computations with the use of Cardano's formula, we obtain

$$S_z^{-1}(u) = \sqrt[3]{-\frac{u}{2} - \sqrt{\frac{u^2}{4}+\frac{1}{27}}} + \sqrt[3]{-\frac{u}{2} + \sqrt{\frac{u^2}{4}+\frac{1}{27}}} - z.$$

4. Numerical simulations

This section is devoted to numerical simulations of the model (3), with either $S(X^i(t),\psi_i) = -2(X^i(t)+\psi_i)+3$ or $S(X^i(t),\psi_i) = -(X^i(t)+\psi_i)-(X^i(t)+\psi_i)^3$. Here the random effects ψ_i, $i = 1,2,\cdots,n$ are i.i.d and incorporated to describe the differences between subjects. Below, we summarize the procedures used to construct selected estimates $\widehat{f}_{\widehat{h'}}$ of the common density f of ψ_i.

1. Based on continuous observations $X^i = \left(X^i(t), \mathcal{F}_t^i, 0 \leq t \leq T\right)$, $i = 1,2,\cdots,n$ and by using the K_h kernel-based approach we construct an estimate $\widehat{Q}_{h,\varepsilon}(X^i,\tau)$, for fixed $\tau \in (0,T)$ of the partially observed drift function $S(X^i(t),\psi_i)$, as $\varepsilon, h \to 0$ with $h^{b-1}\varepsilon^2 \to 0$.
2. By inverting the function $u \longmapsto S(\tau,u)$, for each fixed $\tau \in (0,T)$ we derive an estimate $\widehat{\psi}_{h,\varepsilon}^{(1)}(X^i)$ of ψ_i (depending on τ). In our simulations we take τ to ranges in $\{t_1, t_2, \cdots, t_m\}$, where $(t_j)_{j=0}^m$ is the grid time used to generate the wfBm $B_{a,b}$. This leads to a sequence of estimates of ψ_i denoted as $\widehat{\psi}_{h,\varepsilon,j}^{(1)}(X^i)$, $j = 1,2,\cdots,m-1$.

3. For each j, the G'_h kernel-based approach is applied with the use of the approximations $\widehat{\psi}^{(1)}_{h,\varepsilon,j}(X^i)$, $i = 1, 2, \cdots, n$ of the random effects $\psi_1, \psi_2, \cdots, \psi_n$ to get the selected estimate $\widehat{f}^{(1)}_{\widehat{h}',j}$ of f (depending on $\tau = t_j$).
4. **Selection of the bandwidths h and h'. (a)** An exact formula for the optimal bandwidth $h_* > 0$ can be found by minimizing the upper bound (13). Note this upper bound is a continuous function of h with infinite limit ∞ as $h \downarrow 0$ or $h \uparrow \infty$. So, a global minimum exists. **(b)** Given $h = h_*$, we use the Goldenshluger-Lepski method to select the remaining bandwidth h'.

For the ease of computations, at first stage we choose the rectangular kernels $K_h(u) = K(u/h)/h$, where $K(u) = \frac{1}{2}1_{(|u|\leq 1)}$. So, for $X = X^i$ and $\tau = t_j$ (8), simplifies to

$$\begin{aligned}
\widehat{Q}_{h,\varepsilon}(X^i, t_j) &= \frac{1}{h}\int_0^T K\left(\frac{s - t_j}{h}\right) dX^i(s) \\
&= \frac{1}{2h}\int_{t_j - h}^{t_j + h} dX^i(s) = \frac{1}{2h}\left(X^i(t_j + h) - X^i(t_j - h)\right) \\
&\simeq \frac{1}{2h}\left(X^i(t_{j+k}) - X^i(t_{j-k})\right), \quad \text{if } h \simeq kT/m,\ k = 1, 2, \cdots
\end{aligned}$$

Our results are illustrated by figures 1, 2 and 3 for the first example and by figures 4 and 5 for the second example. The selected kernel density estimates $\widehat{f}^{(1)}_{\widehat{h}',j}$ are given in green, while the exact density and its estimate based on true values of the random effects are given in black and red, respectively. The proposed selected estimates of f were drawn from the observations $X^i(t_j)$, $i = 1, 2 \cdots, n$; $j = 0, 1, \cdots, m$ for two types of distributions of random effects ψ_i. More precisely, either $\psi_i \sim \mathcal{N}(\mu, \omega^2)$ or $\psi_i \sim \mathcal{B}(\alpha, \beta)$ with specific values of the parameters: μ, ω^2, α and β as given in figures 1, 2 and 3. In all cases, the number of observations per subject is $m = 30$, while the horizon time is $T = 5$ for the first example (resp. $T = 1$ for the second example). For the simulations of the wfBm, we proceed as follows.

1. Consider the sampling size m and the time horizon T. We are concerned with sampling the wfBm $B_{a,b}$ at equidistant points $t_j = jT/m$, $j = 1, 2, \cdots, m$ (Say a vector $\mathbf{B}_{a,b} = (B_{a,b}(t_1), \cdots, B_{a,b}(t_m))^\top$). The value of $B_{a,b}$ at $t_0 = 0$ is identically zero.
2. Set the value of the parameters a, b and compute the covariance matrix $\Phi = (\Phi_{k,j})$ defined by $\Phi_{k,j} := R_{a,b}(t_k, t_j) = \int_0^{t_k \wedge t_j} u^a[(t_k - u)^b + (t_j - u)^b]du$, $k, j = 1, 2, \cdots, m$.
3. Perform the Cholesky decomposition to find the matrix L such that $\Phi = LL^\top$.
4. Generate an m-dimensional Gaussian vector $z \sim \mathcal{N}(0, I)$ and define the random vector of interest as $\mathbf{B}_{a,b} = Lz$.

Overall, for all cases in the two examples, the curves of the selected estimates $\widehat{f}^{(1)}_{\widehat{h}',j}$ and those based on true values of random effects are close to the curve of the exact density f. Moreover, the accuracy of estimation improves as both the number of subjects n and ε^{-1} tend to infinity. Note that the selected kernel estimates $\widehat{f}^{(2)}_{\widehat{h}',j}$ of f derived according to (11) are not implement here, because they correspond unrealistic scenario described by ordinary differential equations with random effects (see, (4)). It is worth to mention that the curves (in dashed green) appearing in figures 1, 2, 3, 4 and 5 are due to different values of $\tau \in \{t_1, t_2, \cdots, t_{m-1}\}$, rather than replications of the same estimator.

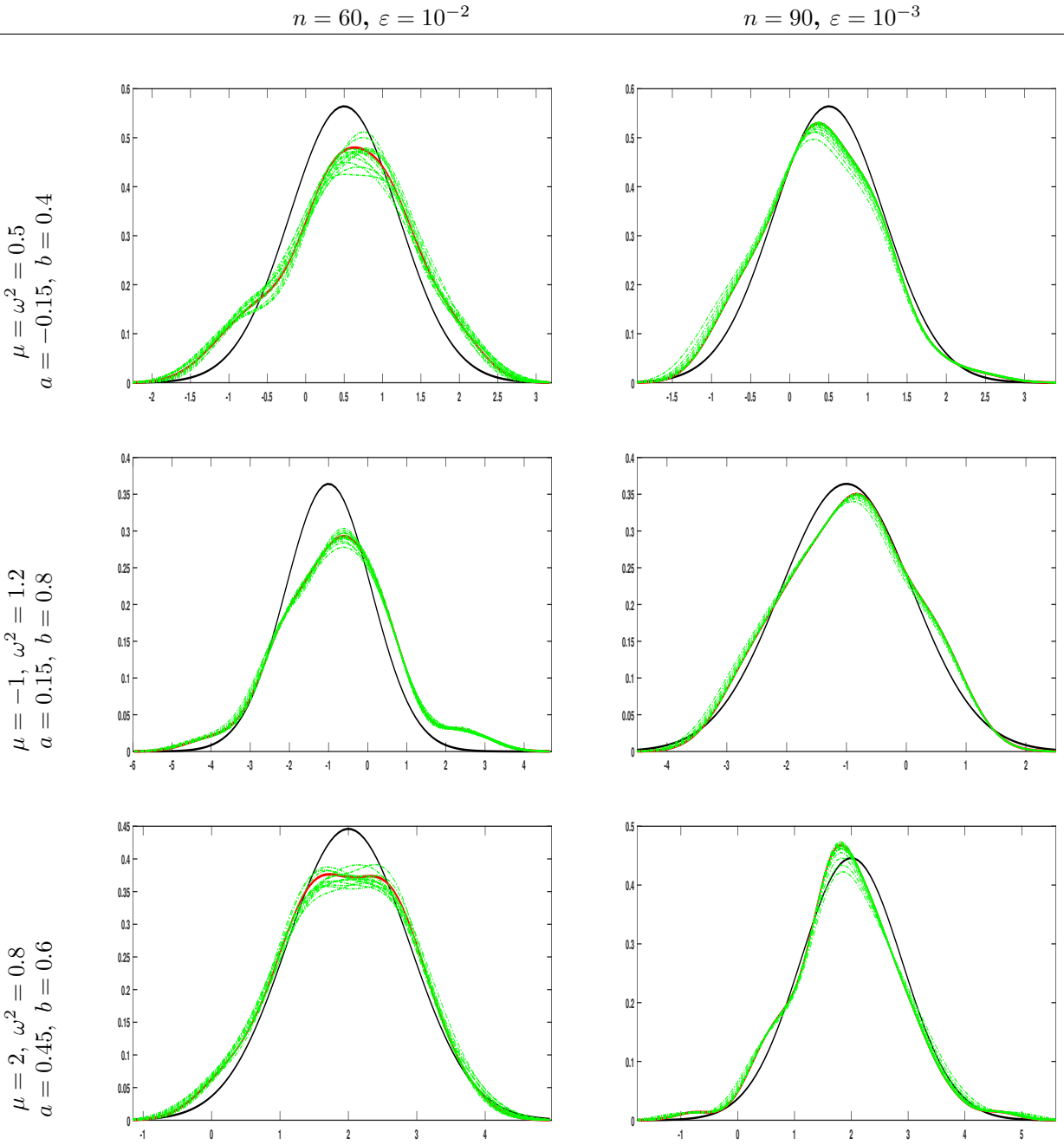


Figure 1. The selected kernel estimates $\widehat{f}^{(1)}_{\widehat{h}'}$ of the common density f of the random effects (in dashed green), the exact curve of f (in blak) and its estimate based on true values of random effects (in red) are displayed for different values of the population parameters (μ, ω^2) and the chosen parameters of the wfBm. Here the drift term of interest is $S(X(t), \psi_i) = -2(X(t) + \psi_i) + 3$ and the random effects are normally distributed: $\psi_i \sim \mathcal{N}(\mu, \omega^2)$. The number of subjects n and the value of ε are set $(n, \varepsilon) = (60, 10^{-2})$ in the left column and $(n, \varepsilon) = (90, 10^{-3})$ in the right column. (For interpretation of the references to color the reader is referred to the electronic version of this article).

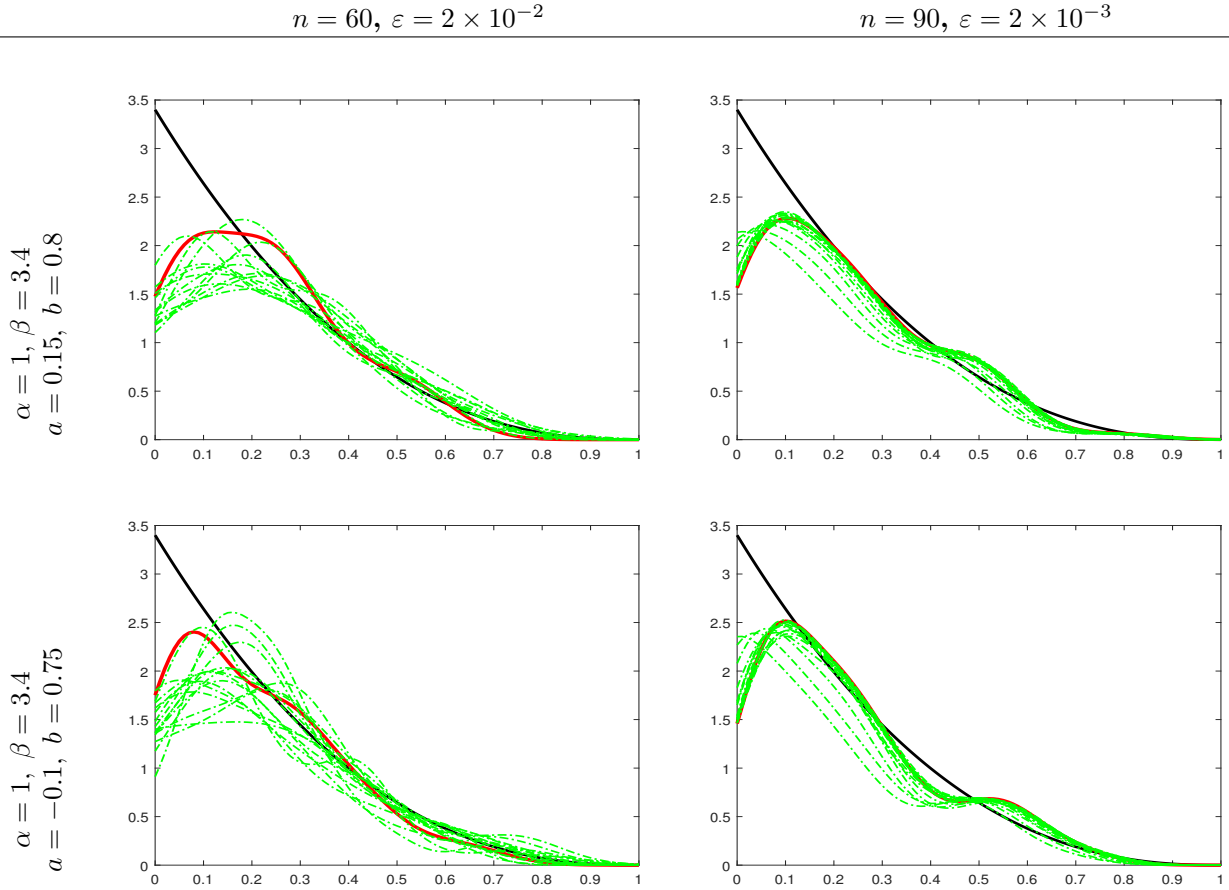


Figure 2. The selected kernel estimates $\widehat{f}^{(1)}_{\widehat{h}'}$ of the common density f of the random effects (in dashed green), the exact curve of f (in black) and its estimate based on true values of random effects (in red) are displayed for different values of the population parameters (α, β) and the chosen parameters of the wfBm. Here the drift term of interest is $S(X(t), \psi_i) = -2(X(t) + \psi_i) + 3$ and the random effects are beta distributed: $\psi_i \sim \mathcal{B}(\alpha, \beta)$. The number of subjects n and the value of ε are set $(n, \varepsilon) = (60, 2 \times 10^{-2})$ in the left column and $(n, \varepsilon) = (90, 2 \times 10^{-3})$ in the right column. (For interpretation of the references to color the reader is referred to the electronic version of this article).

5. Conclusion

In this paper, we studied a stochastic differential equation with random effects driven by wfBm. We proposed estimators for the random effects and constructed a kernel estimator for their common density. We also established convergence properties and risk bounds for the proposed estimators and discussed an adaptive bandwidth selection procedure. To our best knowledge, the problem of estimation for SDERE driven by the fBm has attracted few authors (see, [17, 22, 20, 14, 15]). Except [22] all these references neither consider the small diffusion scenario, nor assume the dissipative condition. Unlike [22], where the drift is linear in the random effects and bounded with respect to the observations, the present work addresses a general drift $S(X^i(t), \psi_i)$ (possibly nonlinear and not bounded in $X^i(t)$) such that $\psi_i \mapsto S(X^i(\tau), \psi_i)$ is invertible for some fixed $\tau \in (0, T)$. The dissipative condition allowed us to prove the convergence of the estimates $\widehat{\psi}^{(k)}_{h,\varepsilon}(X^i)$, $k = 1, 2$ toward ψ_i in $\mathbb{L}^2(\Omega)$-sense rather than the convergence

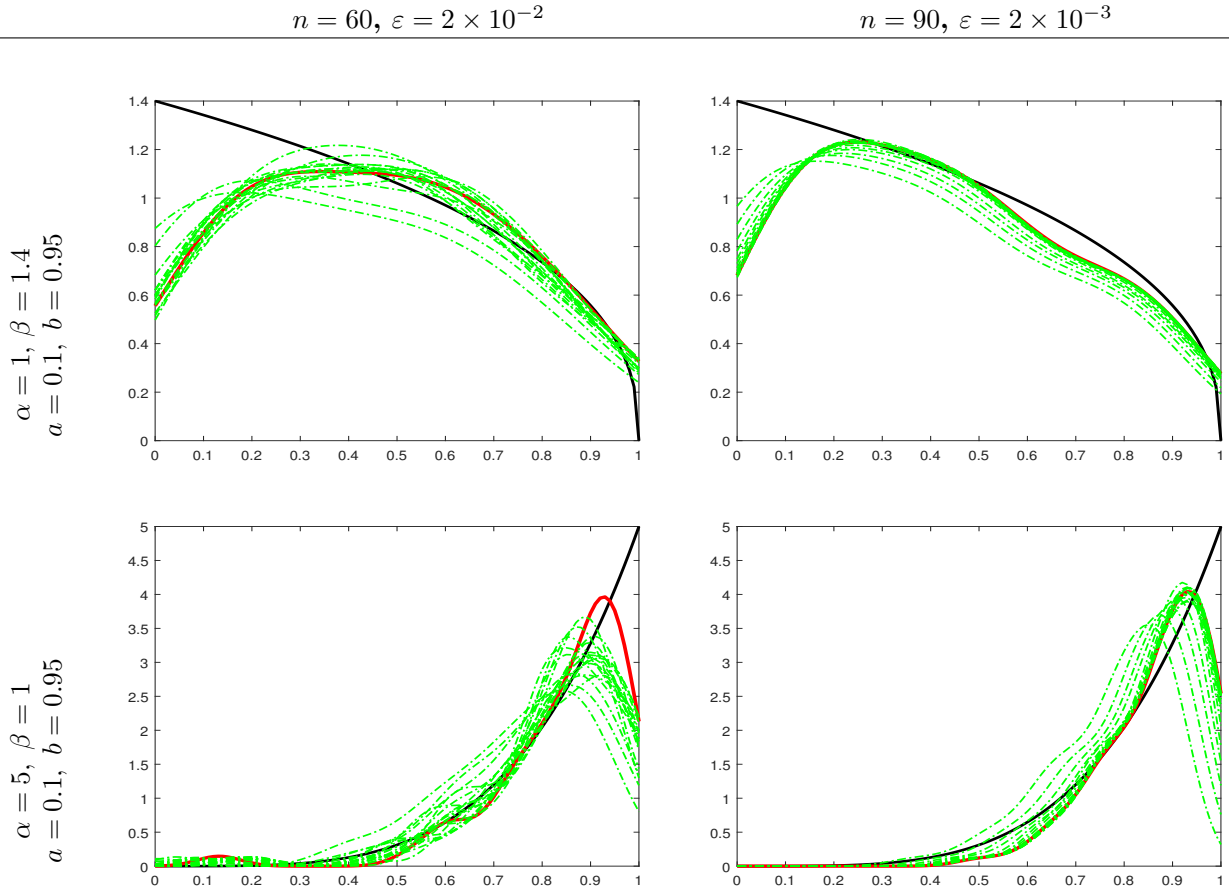


Figure 3. The selected kernel estimates $\widehat{f}^{(1)}_{\widehat{h}'}$ of the common density f of the random effects (in dashed green), the exact curve of f (in black) and its estimate based on true values of random effects (in red) are displayed for different values of the population parameters (α, β) and the chosen parameters of the wfBm. Here the drift term of interest is $S(X(t), \psi_i) = -2(X(t) + \psi_i) + 3$ and the random effects are beta distributed: $\psi_i \sim \mathcal{B}(\alpha, \beta)$. The number of subjects n and the value of ε are set $(n, \varepsilon) = (60, 2 \times 10^{-2})$ in the left column and $(n, \varepsilon) = (90, 2 \times 10^{-3})$ in the right column. (For interpretation of the references to color the reader is referred to the electronic version of this article).

in probability as established in [22]. Moreover, the associated rate of convergence (see, Corollary 1) of the approximations $\widehat{\psi}^{(k)}_{h,\varepsilon}(X^i)$ of the random effects is independent of the parameters of the wfBm $B_{a,b}$. It is obtained according to the exact optimal bandwidth $h_* > 0$ as previously shown. Note that in [22] the two-stages approach of estimation is applied under different assumptions, while the problem of selecting the bandwidths is not treated. So, the present work can be viewed as complementary results to [22].

As a perspective, it would be interesting to extend the proposed methodology to models with multidimensional random effects or to consider more general stochastic differential equations driven by other fractional or Gaussian processes. That is what we do in a future paper.

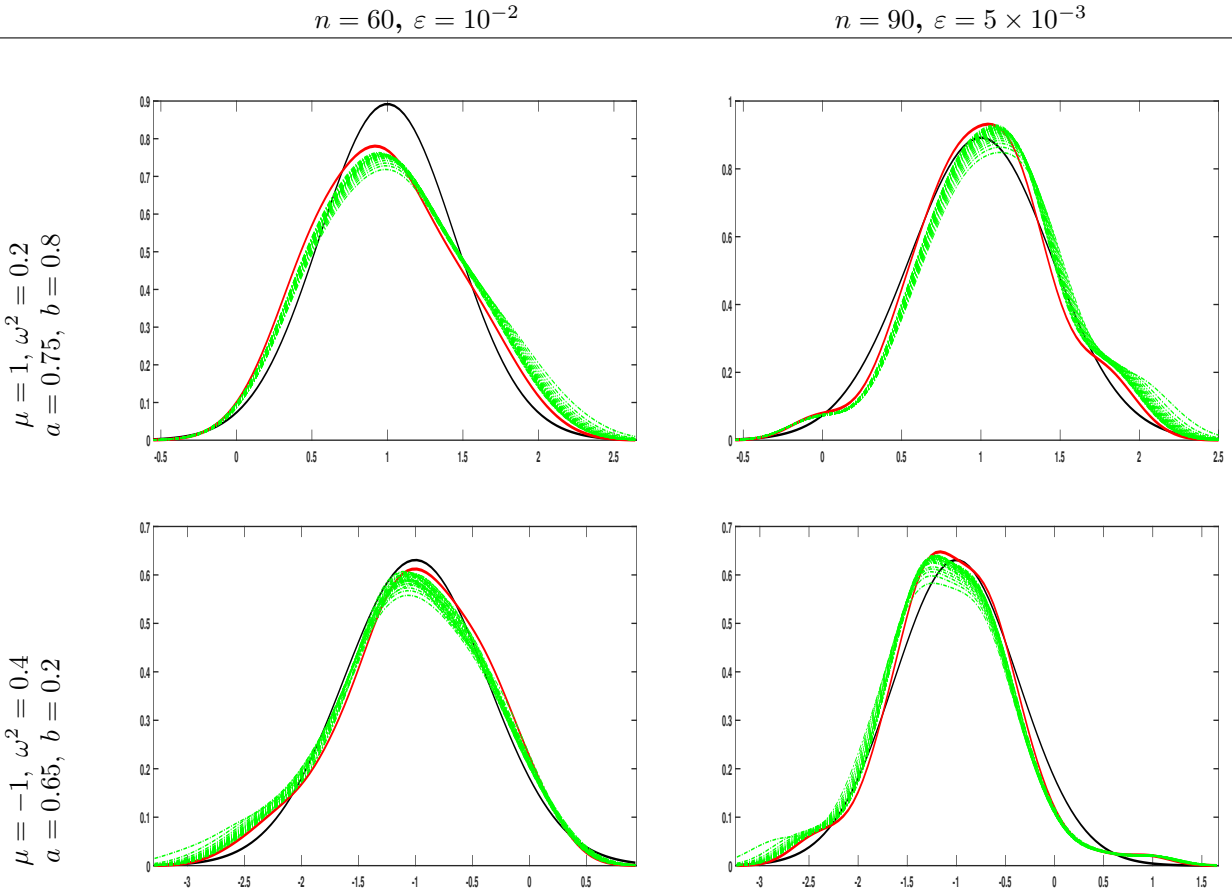


Figure 4. The selected kernel estimates $\widehat{f}^{(1)}_{\widehat{h}'}$ of the common density f of the random effects (in dashed green), the exact curve of f (in black) and its estimate based on true values of random effects (in red) are displayed for different values of the population parameters (α, β) and the chosen parameters of the wfBm. Here the drift term of interest is $S(X(t), \psi_i) = -(X(t) + \psi_i) - (X(t) + \psi_i)^3$ and the random effects are beta distributed: $\psi_i \sim \mathcal{N}(\mu, \omega^2)$. The number of subjects n and the value of ε are set $(n, \varepsilon) = (60, 10^{-2})$ in the left column and $(n, \varepsilon) = (90, 5 \times 10^{-3})$ in the right column. (For interpretation of the references to color the reader is referred to the electronic version of this article).

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Disclosure statement

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

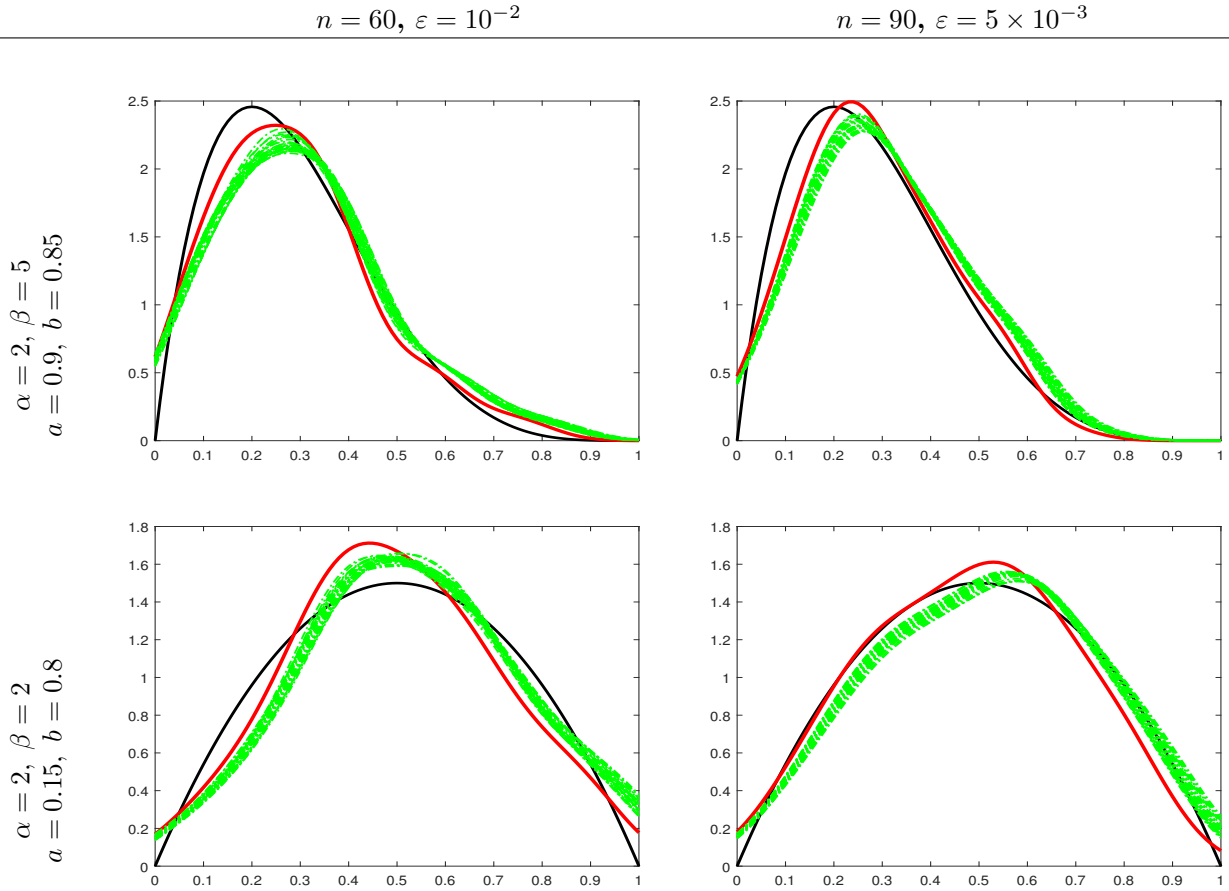


Figure 5. The selected kernel estimates $\widehat{f}^{(1)}_{\widehat{h}'}$ of the common density f of the random effects (in dashed green), the exact curve of f (in black) and its estimate based on true values of random effects (in red) are displayed for different values of the population parameters (α, β) and the chosen parameters of the wfBm. Here the drift term of interest is $S(X(t), \psi_i) = -(X(t) + \psi_i) - (X(t) + \psi_i)^3$ and the random effects are beta distributed: $\psi_i \sim \mathcal{B}(\alpha, \beta)$. The number of subjects n and the value of ε are set $(n, \varepsilon) = (60, 10^{-2})$ in the left column and $(n, \varepsilon) = (90, 5 \times 10^{-3})$ in the right column. (For interpretation of the references to color the reader is referred to the electronic version of this article).

Appendix

Throughout this section we simplify notations by omitting indices referring to different subjects, and consider $a > -1$, $0 < b < (1 \wedge (a + 1))$.

Proof of Lemma 1.
For the proof of this lemma we follow the same procedures as done in the proof of [31, Proposition 1]. First, formula (3) can be rewritten in integral form as

$$X(t) \quad = \quad \int_0^t S\left(X(r), \psi\right) dr + \varepsilon B_{a,b}(t),\ X(0) = 0,\ t \geq 0. \tag{22}$$

The process $X = (X(t), t \geq 0)$ defined by (22) exists if and only if the process $Z(t) := X(t) - \varepsilon B_{a,b}(t)$ satisfies the following equation $Z(t) = \int_0^t S\left(Z(r) + \varepsilon B_{a,b}(r), \psi\right) dr$, in pathwise sense; or equivalently,

$$\frac{dZ(t)}{dt} \quad = \quad g\left(t, Z(t)\right), \quad Z(0) = 0, \ t \geq 0, \tag{23}$$

where $g\left(t, z\right) \ = \ S\left(z + \varepsilon B_{a,b}(t), \psi\right), \ z \in \mathbb{R}$.
Fix ω and consider (23). Since g is continuous in t and continuously differentiable in z (this is inherited from the smoothness of $z \mapsto S(z,u)$), (23) admits a unique local solution in an interval $[0, \boldsymbol{\tau}(\omega)]$, where $\boldsymbol{\tau}(\omega)$ is some nonnegative constant. Thereby a local solution to (22) exists and is unique on the same interval. We shall prove that this local solution can be extended to a global solution on the interval $[0, \infty)$. Let us first establish (5). We focus on X because the same procedures can be applied to its associated limiting process which corresponds to $\varepsilon = 0$. Let $T > 0$ and suppose that Z is defined on $[0, T]$. Under $\mathbf{H}_1$ we have

$$\begin{aligned}
\frac{d}{dt}|Z(t)|^2 &= 2Z(t)\frac{dZ(t)}{dt} = 2Z(t)g\left(t, Z(t)\right) \\
&= 2Z(t)S\left(Z(t) + \varepsilon B_{a,b}(t), \psi\right) \\
&= 2\left(Z(t) + \varepsilon B_{a,b}(t) - \varepsilon B_{a,b}(t)\right)\left(S\left(Z(t) + \varepsilon B_{a,b}(t), \psi\right) - S\left(\varepsilon B_{a,b}(t), \psi\right)\right) \\
&\quad + 2Z(t)S\left(\varepsilon B_{a,b}(t), \psi\right) \\
&\leq -2M|Z(t)|^2 + 2Z(t)S\left(\varepsilon B_{a,b}(t), \psi\right).
\end{aligned}$$

Let $\eta \in (0, 2M)$ be fixed and set $M' = 2M - \eta$. The previous inequality leads, using the fact that $2uv \leq \eta u^2 + \frac{v^2}{\eta}$, for all $u, v \in \mathbb{R}$, to

$$\frac{d}{dt}|Z(t)|^2 \quad \leq \quad -M'|Z(t)|^2 + \frac{1}{\eta}\left(S\left(\varepsilon B_{a,b}(t), \psi\right)\right)^2,$$

which implies, using the assumption $\mathbf{H}_2$,

$$\begin{aligned}
|Z(t)|^2 &\leq \frac{1}{\eta}\int_0^t e^{-M'(t-r)}\left[S\left(\varepsilon B_{a,b}(r), \psi\right)\right]^2 dr, \\
&\leq \frac{3L^2}{\eta}\int_0^t e^{-M'(t-r)}\left(1 + \varepsilon^{2p}|B_{a,b}(r)|^{2p} + |\psi|^{2p}\right) dr.
\end{aligned}$$

The first inequality in the above equation is justified in [32, Lemma 1.12] and $Z(0) = 0$. It follows that

$$\begin{aligned}
\sup_{0\leq t\leq T}|X(t)|^2 &\leq 2\left(\varepsilon^2 \sup_{0\leq t\leq T}|B_{a,b}(t)|^2 + \sup_{0\leq t\leq T}|Z(t)|^2\right), \\
&\leq 2\left(\sup_{0\leq t\leq T}|B_{a,b}(t)|^2 + \frac{3L^2}{\eta M'}\left(1 + \sup_{0\leq t\leq T}|B_{a,b}(t)|^{2p} + |\psi|^{2p}\right)\right), \quad 0 < \varepsilon \leq 1.
\end{aligned}$$

Hence (5) is established. So, as consequence, the local solution to (3) can be extended see [33, Theorem 2.1.4]. □

Proof of Lemma 2.
Let $t \in [0,T]$ and set $U_t := X(t) - x(t)$, we have

$$\begin{aligned}
\frac{1}{2}U_t^2 &= \int_0^t U_r dU_r = \int_0^t (X(r) - x(r)) (dX(r) - dx(r)), \\
&= \int_0^t (X(r) - x(r)) (S(X(r), \psi) - S(x(r), \psi)) dr + \varepsilon \int_0^t U_r dB_{a,b}(r), \\
&\leq -M \int_0^t U_r^2 dr + \varepsilon \int_0^t U_r dB_{a,b}(r) \leq \varepsilon \int_0^t U_r dB_{a,b}(r). \qquad (24)
\end{aligned}$$

The first equality above is justified, because both $B_{a,b}(t)$ and U_t are γ-Hölderian with $\frac{1}{2} < \gamma < \frac{b+1}{2}$. On the other hand,

$$\begin{aligned}
\int_0^t U_r dB_{a,b}(r) &= U_t B_{a,b}(t) - \int_0^t B_{a,b}(r) dU_r \\
&= \varepsilon |B_{a,b}(t)|^2 + B_{a,b}(t) \int_0^t (S(X(r), \psi) - S(x(r), \psi)) dr \\
&\quad - \left(\int_0^t B_{a,b}(r) (S(X(r), \psi) - S(x(r), \psi)) dr + \varepsilon \int_0^t B_{a,b}(r) dB_{a,b}(r) \right) \\
&= \frac{\varepsilon}{2} |B_{a,b}(t)|^2 + B_{a,b}(t) \int_0^t (S(X(r), \psi) - S(x(r), \psi)) dr \\
&\quad - \int_0^t B_{a,b}(r) (S(X(r), \psi) - S(x(r), \psi)) dr \\
&= \frac{\varepsilon}{2} |B_{a,b}(t)|^2 + \int_0^t (B_{a,b}(t) - B_{a,b}(r)) (S(X(r), \psi) - S(x(r), \psi)) dr \\
&\leq \frac{1}{2} \left(\varepsilon |B_{a,b}(t)|^2 + \int_0^t (B_{a,b}(t) - B_{a,b}(r))^2 dr + \int_0^t (S(X(r), \psi) - S(x(r), \psi))^2 dr \right).
\end{aligned}$$

Taking expectation in the aforementioned inequalities and using the properties (i) and (vi) of wfBm (see section 2) yields

$$
\begin{aligned}
2\mathbb{E}\int_0^t U_r dB_{a,b}(r) &\leq 2\varepsilon\int_0^t r^a(t-r)^b dr + \int_0^t \mathbb{E}|B_{a,b}(t)-B_{a,b}(r)|^2 dr \\
&\quad + \int_0^t \mathbb{E}|S\left(X(r),\psi\right) - S\left(x(r),\psi\right)|^2 dr \\
&\leq 2\varepsilon\mathcal{B}\left(a+1,b+1\right)t^{a+b+1} + C_{a,b}t^a\int_0^t (t-r)^{b+1}dr \\
&\quad + \int_0^t \mathbb{E}|S\left(X(r),\psi\right) - S\left(x(r),\psi\right)|^2 dr \\
&\leq 2\varepsilon\mathcal{B}\left(a+1,b+1\right)t^{a+b+1} + C_{a,b}t^a\int_0^t (t-r)^{b+1}dr \\
&\quad + \int_0^t \mathbb{E}\left(|S(X(r),\psi)|^2 \vee |S(x(r),\psi)|^2\right) dr \\
&\leq 2\mathcal{B}\left(a+1,b+1\right)t^{a+b+1} + C_{a,b}t^a\int_0^t (t-r)^{b+1}dr \\
&\quad + 3L^2\int_0^t \mathbb{E}\left(1+\mathbb{E}|\psi|^{2p} + \mathbb{E}\left(|X(r)|\vee|x(r)|\right)^{2p}\right) dr \quad (\text{By } \mathbf{H}_2 \text{ and } \varepsilon \leq 1) \\
&\leq 2\mathcal{B}\left(a+1,b+1\right)T^{a+b+1} + \frac{C_{a,b}}{b+2}T^{a+b+2} \\
&\quad + 3L^2T\left(1+\mathbb{E}|\psi|^{2p} + \left(C_{1,p}+C_{2,p}\mathbb{E}|\psi|^{2p^2}\right)\right) < \infty, \qquad (25)
\end{aligned}
$$

where $C_{1,p}$, $C_{2,p}$ are nonnegative constants. The inequality (25) follows from (5). In fact, by using self-similarity of $B_{a,b}$ (see property (ii) of wfBm) and the fact that $|a+b|^{2p} \leq 2^{2p-1}\left(|a|^{2p}+|b|^{2p}\right)$ we obtain

$$
\begin{aligned}
\mathbb{E}\left(|X(r)|\vee|x(r)|\right)^{2p} &\leq \mathbb{E}\left(\sup_{r\leq T}(|X(r)|\vee|x(r)|)\right)^{2p}, \\
&\leq \mathbb{E}\left(c_1 + c_2|\psi|^p + c_3\sup_{0\leq t\leq T}|B_{a,b}(t)|^p\right)^{2p}, \quad (\text{By } (5)) \\
&\leq 2^{2p-1}\left(c_2^{2p}\mathbb{E}|\psi|^{2p^2} + \mathbb{E}\left(c_1 + c_3\sup_{0\leq t\leq T}|B_{a,b}(t)|^p\right)^{2p}\right) \\
&\leq C_{1,p} + C_{2,p}\mathbb{E}|\psi|^{2p^2}, \qquad (26)
\end{aligned}
$$

where

$$
C_{1,p} = 2^{4p-2}\left(c_1^{2p} + c_3^{2p}T^{2p^2\alpha}\mathbb{E}(\sup_{0\leq t\leq 1}|B_{a,b}(t)|^{2p^2})\right),
$$

$C_{2,p} = 2^{2p-1}c_2^{2p}$ and $\alpha = (a+b+1)/2$ (note that the finiteness of the expectation in the constant $C_{1,p}$ follows from [34, Theorem 1]. Combining (25) and (25) with (24) we get the desired result. □

Proof of Lemma 3.
Let $t, s \in [0,T]$ such that $t > s$. The Cauchy -Schwartz inequality gives

$$\begin{aligned}
\mathbb{E}|x(t)-x(s)|^2 &= \mathbb{E}\left|\int_s^t S\left(x(r),\psi\right)dr\right|^2, \\
&\leq (t-s)\int_s^t \mathbb{E}|S\left(x(r),\psi\right)|^2 dr \\
&\leq 3L^2(t-s)\int_s^t \left(1+\mathbb{E}|\psi|^{2p}+\mathbb{E}|x(r)|^{2p}\right)dr \\
&\leq 3L^2\left(1+\mathbb{E}|\psi|^{2p}+\left(\sup_{0\leq t\leq T}\mathbb{E}|x(t)|\right)^{2p}\right)|t-s|^2 \\
&\leq 3L^2\left(1+\mathbb{E}|\psi|^{2p}+C_{1,p}+C_{2,p}\mathbb{E}|\psi|^{2p^2}\right)|t-s|^2 \quad \text{(By (26))}
\end{aligned}$$

and the proof is complete. □

Proof of Lemma 4.
Consider $\widehat{Q}_{h,\varepsilon}(X,\tau)$ with $\tau \in (0,T)$ fixed. For sufficiently small h (say $h < h_0$), we have $[A,B] \subset (-\tau/h, (T-\tau)/h)$. Thus,

$$\int_{-\tau/h}^{(T-\tau)/h} K(u)S\left(X(\tau+hu),\psi\right)du = \int_A^B K(u)S\left(X(\tau+hu),\psi\right)du, \quad \text{for all } 0<h<h_0. \tag{27}$$

Note that $K(u)$ vanishes outside on the interval $[A,B]$. Hence,

$$\begin{aligned}
\widehat{Q}_{h,\varepsilon}(X,\tau)-S\left(x(\tau),\psi\right) &= \frac{1}{h}\int_0^T K\left(\frac{u-\tau}{h}\right)dX(u) - S\left(x(\tau),\psi\right) \\
&= \frac{1}{h}\int_0^T K\left(\frac{u-\tau}{h}\right)\left(S\left(X(u),\psi\right)du+\varepsilon dB_{a,b}(u)\right) - S\left(x(\tau),\psi\right) \\
&= \int_{-\tau/h}^{(T-\tau)/h} K(u)S\left(X(\tau+hu),\psi\right)du \\
&\quad -\int_A^B K(u)S\left(x(\tau),\psi\right)du + \frac{\varepsilon}{h}\int_0^T K\left(\frac{u-\tau}{h}\right)dB_{a,b}(u) \\
&= \int_A^B K(u)\left[S\left(X(\tau+hu),\psi\right)-S\left(x(\tau+hu),\psi\right)\right]du \\
&\quad +\int_A^B K(u)\left[S\left(x(\tau+hu),\psi\right)-S\left(x(\tau),\psi\right)\right]du + \frac{\varepsilon}{h}\int_0^T K\left(\frac{u-\tau}{h}\right)dB_{a,b}(u) \\
&:= I_1+I_2+I_3 \quad .
\end{aligned}$$

It follows that $\mathbb{E}\left|\widehat{Q}_{h,\varepsilon}(X,\tau) - S\left(x(\tau),\psi\right)\right|^2 \leq 3\left(\mathbb{E}I_1^2 + \mathbb{E}I_2^2 + \mathbb{E}I_3^2\right)$. We shall evaluate the two first expectations. Using Jensen inequality, we have

$$\begin{aligned}
\mathbb{E}I_1^2 &= \mathbb{E}\left(\int_A^B K(u)\left[S\left(X(\tau+hu),\psi\right) - S\left(x(\tau+hu),\psi\right)\right]du\right)^2 \\
&\leq \int_A^B K(u)\mathbb{E}\left[S\left(X(\tau+hu),\psi\right) - S\left(x(\tau+hu),\psi\right)\right]^2 du \\
&\leq M_1\int_A^B K(u)\mathbb{E}|X(\tau+hu) - x(\tau+hu)|^2 du \\
&\leq M_1 C_1\varepsilon \to 0, \text{ as } \varepsilon \to 0. \quad \text{(By Lemma 2)},
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}I_2^2 &= \mathbb{E}\left(\int_A^B K(u)\left[S\left(x(\tau+hu),\psi\right) - S\left(x(\tau),\psi\right)\right]du\right)^2 \\
&\leq \int_A^B K(u)\mathbb{E}\left[S\left(x(\tau+hu),\psi\right) - S\left(x(\tau),\psi\right)\right]^2 du \\
&\leq M_1\int_A^B K(u)\mathbb{E}|x(\tau+hu) - x(\tau)|^2 du \\
&\leq M_1 C_2 h^2\int_A^B u^2 K(u)du \to 0, \text{ as } h \to 0, \quad \text{(By Lemma 3)}
\end{aligned}$$

for the third, we have

$$\begin{aligned}
\mathbb{E}I_3^2 &= \frac{\varepsilon^2}{h^2}\mathbb{E}\left(\int_0^T K\left(\frac{u-\tau}{h}\right)dB_{a,b}(u)\right)^2 \\
&= \frac{\varepsilon^2}{h^2}\int_0^T\int_0^T K\left(\frac{u-\tau}{h}\right)K\left(\frac{v-\tau}{h}\right)\phi_{a,b}(u,v)dudv \quad \text{(By (2))} \\
&= \varepsilon^2\int_{-\tau/h}^{(T-\tau)/h}\int_{-\tau/h}^{(T-\tau)/h} K(u)K(v)\phi_{a,b}(\tau+hu,\tau+hv)dudv \\
&= \varepsilon^2\int_A^B\int_A^B K(u)K(v)\phi_{a,b}(\tau+hu,\tau+hv)dudv \quad (h < h_0) \\
&= h^{b-1}\varepsilon^2 \times b\int_A^B\int_A^B K(u)K(v)\left(\tau+h(u\wedge v)\right)^a|u-v|^{b-1}dudv \quad (h < \tau\wedge h_0) \\
&\leq b\tau^a h^{b-1}\varepsilon^2\int_A^B\int_A^B K(u)K(v)\left(1\vee(1+u\wedge v)^a\right)|u-v|^{b-1}dudv \to 0,
\end{aligned}$$

provided that $h^{b-1}\varepsilon^2 \to 0$, as both h and ε tend to zero. Formula (9) follows immediately from the above convergences. Clearly, the upper bounds of $\mathbb{E}I_1^2$ and $\mathbb{E}I_2^2$ are independent of $\tau \in (0,T)$. The last one $\mathbb{E}I_3^2$ becomes independent of τ when $a \geq 0$. Thus formula (9) holds uniformly in τ, if $a \geq 0$. □

Proof of Theorem 1

Consider the estimators $\widehat{\psi}_{h,\varepsilon}^{(i)}(X)$, $i = 1,2$ as given in (10) and (11). Using (12) with $u = \widehat{\psi}_{h,\varepsilon}^{(2)}(X)$ and $v = \psi$ gives

$$M_2\left|\widehat{\psi}_{h,\varepsilon}^{(2)}(X) - \psi\right|^2 \leq \left|\widehat{\psi}_{h,\varepsilon}^{(2)}(X) - \psi\right|\left|\widehat{Q}_{h,\varepsilon}(X,\tau) - S\left(x(\tau),\psi\right)\right|$$

which implies,

$$\begin{aligned}\mathbb{E}\left|\widehat{\psi}_{h,\varepsilon}^{(2)}(X)-\psi\right|^2 &\leq \mathbb{E}\left(\left|\widehat{\psi}_{h,\varepsilon}^{(2)}(X)-\psi\right|^2 \mathbb{1}_{\left\{|\widehat{\psi}_{h,\varepsilon}^{(2)}(X)-\psi|>0\right\}}\right)\\ &\leq M_2^{-2}\mathbb{E}\left|\widehat{Q}_{h,\varepsilon}(X,\tau)-S\left(x(\tau),\psi\right)\right|^2\\ &\leq D_1(\varepsilon+h^2+h^{b-1}\varepsilon^2)\end{aligned}$$

where

$$D_1 = M_2^{-2}\left(M_1(C_1\vee C_2)+b\tau^a\int_A^B\int_A^B K(u)K(v)\left(1\vee(1+u\wedge v)^a\right)|u-v|^{b-1}dudv\right).$$

The constants appearing in D_1 is due to the proof of Lemma 4. Similarly, we have

$$\begin{aligned}\mathbb{E}\left|\widehat{\psi}_{h,\varepsilon}^{(1)}(X)-\psi\right|^2 &\leq M_2^{-2}\mathbb{E}\left|\widehat{Q}_{h,\varepsilon}(X,\tau)-S\left(X(\tau),\psi\right)\right|^2\\ &\leq 2M_2^{-2}\left(\mathbb{E}\left|\widehat{Q}_{h,\varepsilon}(X,\tau)-S\left(x(\tau),\psi\right)\right|^2+\mathbb{E}|S\left(x(\tau),\psi\right)-S\left(X(\tau),\psi\right)|^2\right)\\ &\leq 2M_2^{-2}\left(\mathbb{E}\left|\widehat{Q}_{h,\varepsilon}(X,\tau)-S\left(x(\tau),\psi\right)\right|^2+M_1\mathbb{E}|x(\tau)-X(\tau)|^2\right)\\ &\leq 2M_2^{-2}\left[D_1\left(\varepsilon+h^2+h^{b-1}\varepsilon^2\right)+M_1C_1\varepsilon\right]\\ &\leq D_2(\varepsilon+h^2+h^{b-1}\varepsilon^2)\end{aligned}$$

with $D_2=2M_2^{-2}(D_1+M_1C_1)$. In the third inequality above we used $\mathbf{H}_3$. Finally, we conclude that (13) holds with $A_1=D_1\vee D_2$. □

Proof of Theorem 2

Consider the estimator $\widehat{f}_{h'}^{(1)}$ defined in (15). In what follows we will denote $\widehat{\psi}_{h,\varepsilon}^{(1)}:=\widehat{\psi}_{h,\varepsilon}^{(1)}(X^1)$. Since $\widehat{\psi}_j^{(1)}$, $j=1,\cdots,n$ are *i.i.d*, we have $\mathbb{E}\left(\widehat{f}_{h'}^{(1)}(y)\right)=\mathbb{E}\left[G_{h'}\left(y-\widehat{\psi}_{h,\varepsilon}^{(1)}\right)\right]$ and $\mathbb{E}\left(\widetilde{f}_{h'}(y)\right)=\mathbb{E}\left(G_{h'}\left(y-\psi_1\right)\right)=f_{h'}(y)$. Therefore,

$$\begin{aligned}\mathbb{E}\left\|\widehat{f}_{h'}^{(1)}-f\right\|^2 &= \mathbb{E}\left\|\widehat{f}_{h'}^{(1)}-\mathbb{E}\widehat{f}_{h'}^{(1)}\right\|^2+\left\|\mathbb{E}\widehat{f}_{h'}^{(1)}-f\right\|^2\\ &\leq \mathbb{E}\left\|\widehat{f}_{h'}^{(1)}-\mathbb{E}\widehat{f}_{h'}^{(1)}\right\|^2+2\left(\left\|\mathbb{E}\widehat{f}_{h'}^{(1)}-\mathbb{E}\widetilde{f}_{h'}\right\|^2+\left\|\mathbb{E}\widetilde{f}_{h'}-f\right\|^2\right)\\ &\leq 2\|f_{h'}-f\|^2+2\left\|\mathbb{E}\widehat{f}_{h'}^{(1)}-\mathbb{E}\widetilde{f}_{h'}\right\|^2+\mathbb{E}\left\|\widehat{f}_{h'}^{(1)}-\mathbb{E}\widehat{f}_{h'}^{(1)}\right\|^2. \qquad (28)\end{aligned}$$

Let us evaluate the last two terms on the right hand side of (28). First, by Taylor's formula with integral remainder we obtain

$$\begin{aligned}\mathbb{E}\widehat{f}_{h'}^{(1)}(y)-\mathbb{E}\widetilde{f}_{h'}(y) &= \mathbb{E}\left(G_{h'}\left(y-\widehat{\psi}_{h,\varepsilon}^{(1)}\right)\right)-\mathbb{E}\left(G_{h'}\left(y-\psi_1\right)\right)\\ &= \mathbb{E}\left(G_{h'}\left(y-\widehat{\psi}_{h,\varepsilon}^{(1)}\right)-G_{h'}\left(y-\psi_1\right)\right)\\ &= \frac{1}{h'^2}\mathbb{E}\left(\left(\widehat{\psi}_{h,\varepsilon}^{(1)}-\psi_1\right)\int_0^1 g(y,u)du\right)\end{aligned}$$

with $g(y,u) = G'\left(\frac{1}{h'}\left(y - \psi_1 + u(\psi_1 - \widehat{\psi}^{(1)}_{h,\varepsilon})\right)\right)$. Then,

$$\begin{aligned}
\left\|\mathbb{E}\widehat{f}^{(1)}_{h'} - \mathbb{E}\widetilde{f}_{h'}\right\|^2 &= \frac{1}{h'^4}\int_{\mathbb{R}}\left(\mathbb{E}\left(\left(\widehat{\psi}^{(1)}_{h,\varepsilon} - \psi_1\right)\int_0^1 g(y,u)du\right)\right)^2 dy \\
&\leq \frac{1}{h'^4}\int_{\mathbb{R}}\mathbb{E}\left(\left(\widehat{\psi}^{(1)}_{h,\varepsilon} - \psi_1\right)\int_0^1 g(y,u)du\right)^2 dy \\
&\leq \frac{1}{h'^4}\mathbb{E}\left(\left(\widehat{\psi}^{(1)}_{n} - \psi_1\right)^2\int_{\mathbb{R}}\left(\int_0^1 g(y,u)du\right)^2 dy\right) \\
&\leq \frac{1}{h'^4}\mathbb{E}\left(\left(\widehat{\psi}^{(1)}_{h,\varepsilon} - \psi_1\right)^2\left(\int_0^1\left(\int_{\mathbb{R}} g^2(y,u)dy\right)^{1/2}du\right)^2\right).
\end{aligned}$$

The last inequality can be justified by generalized Minkowski inequality (e.g. [30, Lemma A.1]). By change of variables $z = \frac{1}{h'}\left(y - \psi_1 + u(\psi_1 - \widehat{\psi}^{(1)}_n)\right)$, we get $\int_{\mathbb{R}} g^2(y,u)dy = \|G'\|^2 h'$, thus,

$$\begin{aligned}
\left\|\mathbb{E}\widehat{f}^{(1)}_{h'} - \mathbb{E}\widetilde{f}_{h'}\right\|^2 &\leq \frac{1}{h'^4}\mathbb{E}\left(\left(\widehat{\psi}^{(1)}_{h,\varepsilon} - \psi_1\right)^2\|G'\|^2 h'\right) \\
&\leq \frac{\|G'\|^2}{h'^3}\mathbb{E}\left(\widehat{\psi}^{(1)}_{h,\varepsilon} - \psi_1\right)^2 \\
&\leq \frac{\|G'\|^2}{h'^3}\cdot D_3\varepsilon \quad \text{(By Corollary 1)}
\end{aligned}$$

for some nonnegative constant D_3. There remains to evaluate the last term in (28). Set $\eta_{j,n}(y) = G_{h'}(y - \widehat{\psi}^{(1)}_{h,\varepsilon}(X^j)) - \mathbb{E}\left(G_{h'}(y - \widehat{\psi}^{(1)}_{h,\varepsilon}(X^j))\right)$, where $G_{h'}(y) = \frac{1}{h'}G\left(\frac{u}{h'}\right)$. $\eta_{j,n}(y)$, $j = 1, \cdots, n$ are $i.i.d$ centered random variables, the change of variables $\dfrac{y - \widehat{\psi}^{(1)}_{h,\varepsilon}(X^j)}{h'} = z$ in the second inequality below, we get

$$\begin{aligned}
\int_{\mathbb{R}}\mathbb{E}\left(\eta_{1,n}(y)\right)^2 dy &\leq \int_{\mathbb{R}}\mathbb{E}\left(G_{h'}(y - \widehat{\psi}^{(1)}_{h,\varepsilon}(X^j))\right)^2 dy \\
&\leq \frac{1}{h'^2}\mathbb{E}\int_{\mathbb{R}} G^2\left(\frac{y - \widehat{\psi}^{1}_{n}(X^j)}{h'}\right)dy \\
&\leq \frac{1}{h'}\int_{\mathbb{R}} G^2(z)dz
\end{aligned}$$

therefore,

$$\begin{aligned}
\mathbb{E}\left\|\widehat{f}^{(1)}_{h'} - \mathbb{E}(\widehat{f}^{(1)}_{h'})\right\|^2 &= \mathbb{E}\int_{\mathbb{R}}\left(\widehat{f}^{(1)}_{h'}(y) - \mathbb{E}\widehat{f}^{(1)}_{h'}(y)\right)^2 dy \\
&= \frac{1}{n^2}\mathbb{E}\int_{\mathbb{R}}\left(\sum_{i=1}^N \eta_{i,n}(y)\right)^2 dy \\
&= \frac{1}{n}\int_{\mathbb{R}}\mathbb{E}\left(\eta_{1,n}(y)\right)^2 dy \leq \frac{\|G\|^2}{nh'}.
\end{aligned}$$

Finally, we conclude that (16) holds with $A_2 = 2 \vee D_3$. □

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