

Implementation of the Laplace Variational Iteration Method (LT-VIM) for various types of nonlinear partial differential equations (PDEs)

Mohammed M. Azeez¹, Wafaa M. Taha¹, Issa Alkharusi², Ali F. Jameel², Walid Wannes²

¹*Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq*

²*Faculty of Education and Arts, Sohar University, Sohar, 311, Oman*

Abstract In this study, Laplace Transform Variational Iteration Method (LT-VIM) is used to obtain approximate solutions for the Airy equation, the Allen–Cahn equation, and the Sine–Gordon equation. This technique develops correction functionals in the Laplace domain, where the Lagrange multiplier is obtained directly in an algebraic form. Our numerical computations show that successive approximations lead to fast-convergence series solutions in only a few iterations. By comparing our results with the exact solutions and other existing numerical methods, we show that LT-VIM provides high accuracy and effective convergence with minimal iterations. The results obtained in this study show that LT-VIM is a reliable and efficient method for solving linear dispersive wave equations, nonlinear reaction–diffusion equations, and nonlinear wave equations, and can be successfully applied to similar linear and nonlinear models found in applied mathematics and mathematical physics.

Keywords Airy equation, Allen-Cahn equation, Sine-Gordon equation, Iterative approximation, Laplace Transform Variational Iteration Method (LT-VIM), Semi-analytical methods

AMS 2010 subject classifications 35A99, 65M99, 44A10

DOI: 10.19139/soic-2310-5070-3686

1. Introduction

Linear and nonlinear partial differential equations appear in many areas of science and engineering, including fluid flow, wave propagation, and heat transfer [1, 2, 3, 4, 5, 6]. In most practical situations, obtaining exact solutions of many PDEs, particularly nonlinear ones, is difficult or impossible. For this reason, various analytical and semi-analytical methods have been developed to construct approximate solutions [7, 8, 9].

Several methods have been used to solve nonlinear differential equations, including the Adomian Decomposition Method (ADM), Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), and Variational Iteration Method (VIM) [9, 10, 11, 12, 18]. Among them, VIM has been successfully applied to a wide range of nonlinear problems because of its straightforward iterative structure and satisfactory accuracy [13, 14, 15, 16, 19].

However, finding the Lagrange multiplier in the classical VIM method usually requires variational theory, which can make the method difficult to implement.

The LT-VIM method was developed to address this difficulty by combining the Laplace transform with the variational iteration method [21, 22, 23, 24]. In the LT-VIM method, the Lagrange multiplier is determined directly in the Laplace domain, which simplifies the iterative formulation and reduces the computational effort required to construct approximate solutions [23, 24, 25, 26, 27].

*Correspondence to: Wafaa M. Taha. Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq.

Although LT-VIM has been applied to several differential equations, the present work does not introduce a new LT-VIM algorithm. To the best of our knowledge, the combined treatment of the Airy, Allen–Cahn, and Sine–Gordon equations within a single unified LT-VIM framework has not been previously reported. These equations were selected because they represent three well-known physical models with different mathematical structures and model three physical phenomena, namely linear dispersive waves, nonlinear reaction–diffusion processes, and nonlinear wave propagation, respectively. The present study extends the use of LT-VIM to these selected problems using a unified LT-VIM procedure. The numerical solutions are compared with the corresponding exact solutions, previously published methods, graphical solutions, and convergence analysis. The results indicate that the method provides accurate approximations using only a few iterations and can be considered an effective semi-analytical technique for solving both linear and nonlinear partial differential equations.

2. Mathematical Formulation and Theoretical Framework of the Laplace Transform Variational Iteration method (LT-VIM)

To outline the procedure of the LT-VIM method, let us consider the following general partial differential equation

$$Lu(x, t) + Nu(x, t) = g(x, t), \quad (1)$$

where $L = \partial^n / \partial t^n$ is a linear differential operator of order n with respect to the time t , N denotes the nonlinear operator, and $g(x, t)$ is a known source term.

In the LT-VIM method, the first step is to apply the Laplace transform with respect to the time variable t to both sides of Eq. (1). By using standard properties of the Laplace transform and the given initial conditions, the partial differential equation is converted into an algebraic equation in the Laplace domain [20, 22, 23].

Following the principles of the variational iteration method, a correction functional can be constructed in the Laplace domain in the form [23, 25, 28]

$$U_{n+1}(x, s) = U_n(x, s) + \lambda(s) \mathcal{L}\{g(x, t) - N(u_n(x, t))\},$$

Where $n = 0, 1, 2, \dots$, and $U_n(x, s) = \mathcal{L}\{u_n(x, t)\}$ denotes the Lagrange multiplier.

In the LT-VIM method, the Lagrange multiplier is obtained algebraically in the Laplace domain as the inverse of the linear time operator. For the operator $L = \partial^n / \partial t^n$, the multiplier is obtained explicitly as [28, 29]

$$\lambda(s) = -\frac{1}{s^n}.$$

This eliminates the need for variational calculus and simplifies the iterative procedure.

Once the Lagrange multiplier is determined, the inverse Laplace transform is used iteratively to obtain a sequence of approximate solutions in the time domain. If the sequence converges, the exact solution of the original problem is given by:

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t)$$

2.1. Laplace Transform Properties

The Laplace transform is an effective tool for solving differential equations. It transforms time derivatives into algebraic terms in the Laplace domain, which simplifies the calculations required in the LT-VIM. Some basic definitions and properties of the Laplace transform used in this study are presented below.

The Laplace transform of a sufficiently smooth function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt, \quad s > 0$$

The Laplace transform satisfies the linearity property

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s),$$

where a and b are constants.

For derivatives with respect to time, the Laplace transform yields

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= sF(s) - f(0), \\ \mathcal{L}\{f''(t)\} &= s^2F(s) - sf(0) - f'(0).\end{aligned}$$

More generally, for the n -th derivative of $f(t)$, one has

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0).$$

In addition, the convolution theorem is frequently used in LT-VIM and is given by

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s),$$

Where

$$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau.$$

These classical properties allow the governing partial differential equations to be transformed into algebraic equations in the Laplace domain, greatly simplifying the identification of the Lagrange multiplier and the iterative correction functional in LT-VIM [30, 31].

2.2. Determination of the Lagrange Multiplier

In the Laplace Transform Variational Iteration Method (LT-VIM), the determination of the Lagrange multiplier is carried out directly in the Laplace domain. To this end, the Laplace transform with respect to the time variable t is first applied to the governing equation (1). By incorporating the prescribed initial conditions, the transformed equation is converted into an algebraic form in the s -domain as follows [14, 15]:

$$U(x, s) = \frac{1}{s^n} \left[\mathcal{L}\{g(x, t) - N(u(x, t))\} + \text{IC terms} \right], \quad (2)$$

where the initial-condition terms are defined by

$$\text{IC terms} = \sum_{k=0}^{n-1} s^{n-1-k} u^{(k)}(x, 0).$$

According to the variational iteration method originally proposed by He [14] a correction functional is constructed to generate successive approximations. In the LT-VIM framework, this correction functional is formulated in the Laplace domain as [15, 32]:

$$\mathcal{L}\{u_{n+1}(x, t)\} = \mathcal{L}\{u_n(x, t)\} + \lambda(s) \mathcal{L}\{g(x, t) - N(u_n(x, t))\},$$

where $\lambda(s)$ denotes the Lagrange multiplier in the Laplace domain.

From the algebraic structure of Eq. (2), the Lagrange multiplier is identified directly as the inverse of the linear time operator in the Laplace domain, yielding [32]:

$$\lambda(s) = -\frac{1}{s^n}.$$

To obtain the corresponding expression in the time domain, the inverse Laplace transform is applied, which gives [33]:

$$\lambda(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = -\frac{t^{n-1}}{(n-1)!}.$$

This direct determination of the Lagrange multiplier removes the need for variational calculus and makes the iterative procedure simpler than the classical VIM.

3. Application of LT-VIM to the Linear Airy Equation

In this section, LT-VIM is used to solve the Airy equation. This linear dispersive equation is chosen to examine the accuracy and effectiveness of the proposed method [34]. We consider the following physical problem to achieve our objective:

$$u_t + u_{xxx} = 0, \quad (3)$$

with the following initial condition:

$$u(x, 0) = \sin(x),$$

The exact solution is given by:

$$u(x, t) = \sin(x + t).$$

We apply the Laplace Transform with respect to the time variable t :

$$\mathcal{L}\{u(x, t)\} = U(x, s),$$

Applying the transform to the equation (3), and using properties of the Laplace Transform

$$U_{n+1}(x, s) = U_n(x, s) + \mathcal{L} \left\{ \lambda \left[\frac{\partial u_n(x, t)}{\partial t} + \frac{\partial^3 u_n(x, t)}{\partial x^3} \right] \right\}, \quad (4)$$

Using the identified Lagrange multiplier $\lambda(s) = -1/s$ together with the initial approximation $U_0(x, s) = \sin(x)/s$, Eq. (4) is directly simplified to Eq. (5), which is used in the subsequent computations

$$U_{n+1}(x, s) = \frac{\sin(x)}{s} - \frac{1}{s} \mathcal{L} \left\{ \frac{\partial^3 u_n(x, t)}{\partial x^3} \right\}, \quad (5)$$

$$U_{n+1}(x, s) = \frac{\sin(x)}{s} - \frac{1}{s} U_{n,xxx}(x, s). \quad (6)$$

Subsequently, the initial approximation is utilized to start the iterative process based on the LT-VIM iteration functional (6), and applying the inverse Laplace transform term-by-term, the following time-domain approximations are obtained.

$$u_0(x, t) = u(x, 0) = \sin(x)$$

$$u_1(x, t) = \sin(x) + t \cos(x)$$

$$u_2(x, t) = \sin(x) + t \cos(x) - \frac{t^2}{2!} \sin(x)$$

The three-iteration approximate solution is:

$$u_3(x, t) = \sin(x) + t \cos(x) - \frac{t^2}{2!} \sin(x) - \frac{t^3}{3!} \cos(x).$$

This is exactly the truncated Taylor expansion of the exact solution:

$$u_3(x, t) = \sin(x) \left(1 - \frac{t^2}{2!} \right) + \cos(x) \left(t - \frac{t^3}{3!} \right).$$

The LT-VIM solution for the Airy equation reconstructs the Taylor series of

$$u_{exact} = \sin(x + t) = \sin(x) \cos(t) + \cos(x) \sin(t).$$

$$u_n(x, t) = \sum_{k=0}^n \frac{t^k}{k!} \frac{d^k}{dt^k} \sin(x + t) \Big|_{t=0}$$

Table 1. Presents the numerical results for the approximate solutions obtained LT-VIM in comparison with the exact solutions of Equation (3).

x	t	Exact	LT-VIM	LT-VIM Error	VIM Error
0.1	0.1	0.198669330795061	0.198669330795062	6e-16	9e-16
0.2	0.2	0.389418342308650	0.389418342308650	0	0
0.3	0.3	0.564642473395035	0.564642473395035	0	2e-15
0.4	0.4	0.717356090889523	0.717356090889523	0	1e-15
0.5	0.5	0.841470984807897	0.841470984807896	1e-15	5e-15
0.6	0.6	0.932039085967226	0.932039085967225	1e-15	2e-15
0.7	0.7	0.985449729988460	0.985449729988461	1e-15	0
0.8	0.8	0.999573603041505	0.999573603041508	3e-15	4e-15
0.9	0.9	0.973847630878195	0.973847630878195	0	5e-15
1.0	1.0	0.909297426825682	0.909297426825682	0	2e-15

Similarly, we can obtain the remaining elements of the iteration (6) through the use of Maple Software. The 20th-order approximation solution produced by LT-VIM and the exact solution are summarized in Table 1 and Figure 1 for different values of x and t .

Table 1 compares the 20th-order LT-VIM approximation with the exact solution and the VIM errors reported in [34]. The LT-VIM errors are very small, ranging from 0 to 3×10^{-15} , which shows that the approximate values are almost identical to the exact values at the selected points. The comparison also shows that LT-VIM gives results comparable to the VIM results reported in [34].

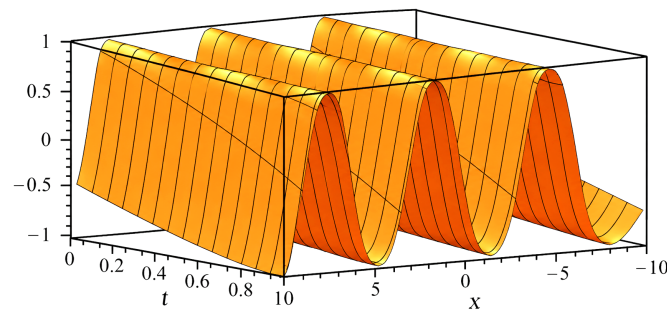


Figure 1. The 3D surface of the exact solution for Airy equation, $0 \leq t \leq 1$, $-10 \leq x \leq 10$

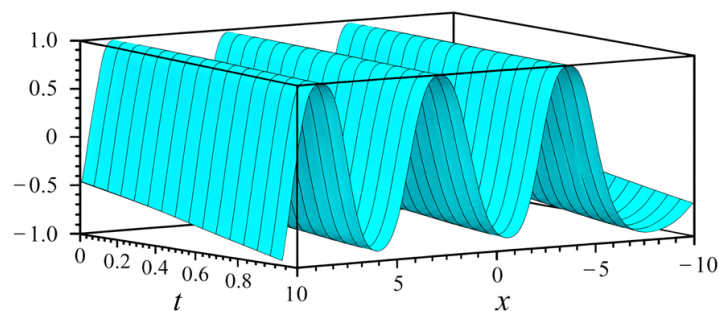


Figure 2. The 3D surface of the approximate solution for Airy equation obtained by LT-VIM, $0 \leq t \leq 1$, $-10 \leq x \leq 10$

4. Application of LT-VIM to the Nonlinear Allen-Cahn Equation

In this section, LT-VIM is used to solve the Allen–Cahn equation. This nonlinear reaction–diffusion model provides an example for studying the accuracy of the method for equations with nonlinear terms. The obtained results are compared with those reported by other semi-analytical methods [17, 35, 36].

We consider the following physical problem:

$$u_t = u_{xx} + u - u^3, \quad (7)$$

subject to the initial condition:

$$u(x, 0) = -0.5 + 0.5 \tanh(0.3536x),$$

where the exact solution for this problem is given by:

$$u(x, t) = -0.5 + 0.5 \tanh(0.3536x - 0.75t).$$

Applying the Laplace transform with respect to the time variable t yields

$$\mathcal{L}\{u(x, t)\} = U(x, s),$$

Using the initial condition $\Phi_0(x, s) = \phi(x, 0)/s$, the transformed equation (7) becomes

$$U_{n+1}(x, s) = U_n(x, s) - \frac{1}{s} \mathcal{L}\{u_{n,t} - u_{n,xx} - u_n + u_n^3\},$$

In LT-VIM, the Lagrange multiplier λ is implicitly represented by the Laplace operator $-1/s$, and its inverse transform yields a unity multiplier in the time domain, eliminating the need for explicit identification. Taking the inverse Laplace transform leads to the LT-VIM correction functional, and using convolution

$$u_{n+1}(x, t) = u_n(x, t) - \int_0^t \left[\frac{\partial u_n}{\partial \tau} - \frac{\partial^2 u_n}{\partial x^2} - u_n + u_n^3 \right] d\tau.$$

where the initial guess

$$u_0(x) = -0.5 + 0.5 \tanh(0.3536x)$$

This represents the traveling wave solution at $t = 0$.

$$u_1(x, t) = u_0(x) + t F_0(x)$$

Where

$$F_0(x) = u_{0,xx} + u_0 - u_0^3$$

Following a similar procedure, the third-order approximation is obtained as:

$$u_2(x, t) = u_0(x) + t F_0(x) + t^2 F_1(x)$$

$$u_3(x, t) = u_0(x) + t F_0(x) + t^2 F_1(x) + t^3 F_2(x)$$

The third-order approximate solution using LT-VIM is expressed as:

$$u(x, t) = u_0(x) + t F_0(x) + t^2 F_1(x) + t^3 F_2(x)$$

By substituting the series form into the Allen–Cahn equation and comparing equal powers of t , the following coefficients are obtained:

$$F_0(x) = u_{0,xx} + u_0 - u_0^3$$

$$F_1(x) = \frac{1}{2} [F_{0,xx} + F_0 - 3u_0^2 F_0] \quad \text{and} \quad F_2(x) = \frac{1}{3} [F_{1,xx} + F_1 - 3u_0^2 F_1 - 3u_0 F_0^2]$$

Similarly, the remaining components of the LT-VIM iterations were computed using Maple software. The third-order LT-VIM approximate solution is compared with the exact solution in Table 2 for various values of x and t .

Table 2. Comparison between the exact solution and the LT-VIM approximate solution for the current problem

$t = 0.001$			
x	Exact solution	LT-VIM	Absolute Error
0.1	-0.4827019063	-0.4827019062	1.0×10^{-10}
0.2	-0.4650719816	-0.4650719816	0
0.3	-0.4475289019	-0.4475289019	0
0.4	-0.4301154832	-0.4301154832	0
0.5	-0.4128732694	-0.4128732694	0
0.6	-0.3958421462	-0.3958421461	1.0×10^{-10}
0.7	-0.3790599984	-0.3790599983	1.0×10^{-10}
0.8	-0.3625623520	-0.3625623520	0
0.9	-0.3463821971	-0.3463821971	0
$t = 0.005$			
x	Exact solution	LT-VIM	Absolute Error
0.1	-0.4842002620	-0.4842002620	0
0.2	-0.4665649702	-0.4665649702	0
0.3	-0.4490128454	-0.4490128454	0
0.4	-0.4315867926	-0.4315867927	1.0×10^{-10}
0.5	-0.4143284793	-0.4143284795	2.0×10^{-10}
0.6	-0.3972779460	-0.3972779460	0
0.7	-0.3804732518	-0.3804732517	1.0×10^{-10}
0.8	-0.3639501576	-0.3639501577	1.0×10^{-10}
0.9	-0.3477418556	-0.3477418557	1.0×10^{-10}
$t = 0.01$			
x	Exact solution	LT-VIM	Absolute Error
0.1	-0.4860736029	-0.4860736029	0
0.2	-0.4684320454	-0.4684320455	1.0×10^{-10}
0.3	-0.4508690488	-0.4508690489	1.0×10^{-10}
0.4	-0.4334276258	-0.4334276260	2.0×10^{-10}
0.5	-0.4161495938	-0.4161495940	2.0×10^{-10}
0.6	-0.3990751836	-0.3990751838	2.0×10^{-10}
0.7	-0.3822426798	-0.3822426800	2.0×10^{-10}
0.8	-0.3656881012	-0.3656881013	1.0×10^{-10}
0.9	-0.3494449234	-0.3494449236	2.0×10^{-10}

Table 3. Comparison of absolute errors between LT-VIM and other numerical methods (ADJTM [17], TBS [35], and MVIA-II [36]) for the current problem

$t = 0.001$				
x	ADJTM	TBS	MVIA-II	LT-VIM
0.1	1.150E-9	2.448E-4	1.620E-5	1.000E-10
0.2	2.302E-9	2.000E-4	1.614E-5	0
0.3	3.431E-9	1.797E-4	1.604E-5	0
0.4	4.527E-9	1.594E-4	1.590E-5	0
0.5	5.578E-9	1.410E-4	1.572E-5	0
0.6	6.577E-9	1.242E-4	1.551E-5	1.000E-10
0.7	7.514E-9	1.091E-4	1.526E-5	1.000E-10
0.8	8.838E-9	9.502E-5	1.499E-5	0
0.9	9.177E-9	8.886E-5	1.468E-5	0
$t = 0.005$				
x	ADJTM	TBS	MVIA-II	LT-VIM
0.1	5.502E-9	9.687E-4	8.101E-5	0
0.2	1.126E-8	1.016E-3	8.073E-5	0
0.3	1.690E-8	9.082E-4	8.024E-5	0
0.4	2.239E-8	8.052E-4	7.956E-5	1.000E-10
0.5	2.765E-8	7.142E-4	7.869E-5	2.000E-10
0.6	3.266E-8	6.280E-4	7.764E-5	0
0.7	3.736E-8	5.520E-4	7.643E-5	1.000E-10
0.8	4.172E-8	4.824E-4	7.505E-5	1.000E-10
0.9	4.572E-8	4.725E-4	7.535E-5	1.000E-10
$t = 0.01$				
x	ADJTM	TBS	MVIA-II	LT-VIM
0.1	1.040E-8	1.606E-3	1.621E-4	0
0.2	2.194E-8	1.935E-3	1.615E-4	1.000E-10
0.3	3.327E-8	1.826E-3	1.606E-4	1.000E-10
0.4	4.427E-8	1.632E-3	1.593E-4	2.000E-10
0.5	5.458E-8	1.444E-3	1.576E-4	2.000E-10
0.6	6.491E-8	1.273E-3	1.555E-4	2.000E-10
0.7	7.437E-8	1.116E-3	1.531E-4	2.000E-10
0.8	8.316E-8	9.434E-4	1.504E-4	1.000E-10
0.9	9.121E-8	6.534E-4	1.441E-4	2.000E-10

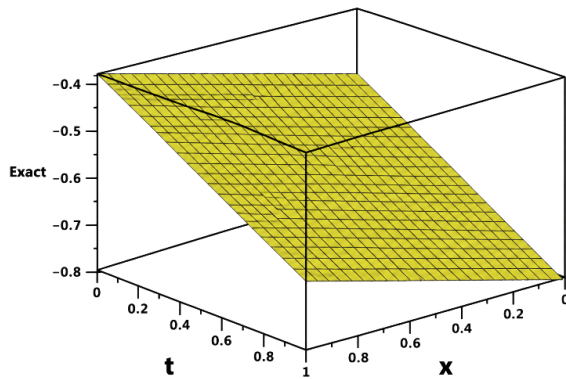


Figure 3. The 3D surface of the exact solution for Allen-Cahn equation, $0 \leq t \leq 1$, $0 \leq x \leq 1$

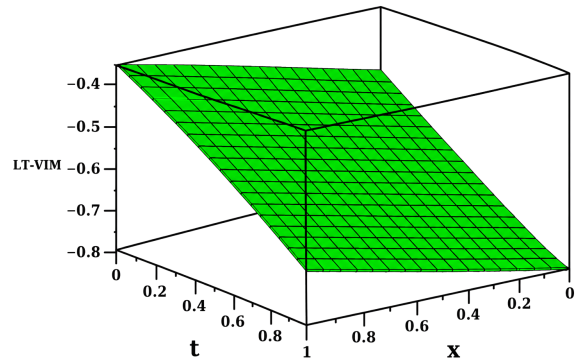


Figure 4. The 3D surface of the approximate solution for Allen-Cahn equation obtained by LT-VIM, $0 \leq t \leq 1$, $0 \leq x \leq 1$

Tables 2 and 3 summarize the numerical results for the third-order LT-VIM approximation of the Allen-Cahn equation. In Table 2, the approximate results are compared with the exact traveling-wave solution at $t = 0.001, 0.005$, and 0.01 . The absolute errors vary from 0 to 2.0×10^{-10} showing that the approximate values are very close to the exact values at the selected points. Table 3 gives a comparison with the errors reported for ADJTM [17], TBS [35], and MVIA-II [36]. For the listed values of x and t , the LT-VIM errors are lower than those of the compared methods. This shows that the third-order LT-VIM approximation gives accurate results for the Allen-Cahn equation.

5. Application of LT-VIM to the Nonlinear Sine-Gordon Equation

In this section, LT-VIM is applied to the Sine-Gordon equation. This example is used to evaluate the accuracy of the method and its ability to solve nonlinear equations. The results are compared with those obtained by the Homotopy Perturbation Method (HPM) [37].

We consider the following physical problem:

$$u_{tt} - u_{xx} + \sin(u) = 0, \quad (8)$$

Initial Conditions:

$$u(x, 0) = 0, \quad u_t(x, 0) = 4 \operatorname{sech}(x),$$

Exact solution:

$$u_{exact} = 4 \tanh^{-1}(t \operatorname{sech}(x)).$$

Applying the transform to the equation (8), and using properties of the Laplace Transform, the correction functional is constructed as:

$$U_{n+1}(x, s) = U_n(x, s) + \lambda(s) \mathcal{L}\{u_{n,tt} - u_{n,xx} + \sin(u_n)\}, \quad (9)$$

Where $\lambda(s) = -1/s^2$ is the Lagrange multiplier for second-order problems, and the Initial condition $U_0 = 4 \operatorname{sech}(x)/s^2$, the iteration formula (9) is simplified to the following form used for computation:

$$U_{n+1}(x, s) = \frac{4 \operatorname{sech}(x)}{s^2} + \frac{U_{n,xx}}{s^2} - \frac{1}{s^2} \mathcal{L}\{\sin(u_n(x, t))\}, \quad (10)$$

To handle the nonlinear term $\sin(u_n)$, the Taylor expansion is used. For the initial approximation $u_0 = 4t \operatorname{sech}(x)$, we have the Taylor expansion:

$$\sin(u_0) = u_0 - \frac{u_0^3}{6} + \frac{u_0^5}{120} - \dots$$

Subsequently, the initial approximation is utilized to start the iterative process based on the LT-VIM iteration functional (10), and applying the inverse Laplace transform term-by-term, the following time-domain approximations are obtained.

$$u_0(x, t) = 4t \operatorname{sech}(x)$$

$$u_1(x, t) = 4t \operatorname{sech}(x) - \frac{4t^3}{3} \operatorname{sech}^3(x)$$

$$u_2(x, t) = 4t \operatorname{sech}(x) - \frac{4t^3}{3} \operatorname{sech}^3(x) + \frac{4t^5}{15} \operatorname{sech}^5(x)$$

$$u_3(x, t) = 4t \operatorname{sech}(x) - \frac{4t^3}{3} \operatorname{sech}^3(x) + \frac{4t^5}{15} \operatorname{sech}^5(x) - \frac{4t^7}{105} \operatorname{sech}^7(x)$$

$$u_4(x, t) = 4t \operatorname{sech}(x) - \frac{4t^3}{3} \operatorname{sech}^3(x) + \frac{4t^5}{15} \operatorname{sech}^5(x) - \frac{4t^7}{105} \operatorname{sech}^7(x) + \frac{4t^9}{945} \operatorname{sech}^9(x)$$

Final LT-VIM Approximate Solution

$$u(x, t) \approx 4t \operatorname{sech}(x) - \frac{4t^3}{3} \operatorname{sech}^3(x) + \frac{4t^5}{15} \operatorname{sech}^5(x) - \frac{4t^7}{105} \operatorname{sech}^7(x) + \frac{4t^9}{945} \operatorname{sech}^9(x)$$

Pattern we now have a (clean LT-VIM series)

$$u_n(x, t) = 4 \sum_{k=0}^n (-1)^k \frac{2^k k!}{(2k+1)!} t^{2k+1} \operatorname{sech}^{2k+1}(x)$$

Table 4. The numerical results for the exact solution and approximate solutions obtained by LT-VIM of Sine-Gordon equation

t	$x = 2.5$		$x = 5$		$x = 7.5$		$x = 10$	
	u_{Exact}	LT-VIM	u_{Exact}	LT-VIM	u_{Exact}	LT-VIM	u_{Exact}	LT-VIM
0.1	0.065234	0.065234	0.005390	0.005390	0.000442	0.000442	0.000036	0.000036
0.2	0.130503	0.130503	0.010780	0.010780	0.000885	0.000885	0.000073	0.000073
0.3	0.195842	0.195842	0.016170	0.016170	0.001327	0.001327	0.000109	0.000109
0.4	0.261285	0.261285	0.021561	0.021561	0.001770	0.001770	0.000145	0.000145
0.5	0.326868	0.326868	0.026951	0.026951	0.002212	0.002212	0.000182	0.000182

Table 5. The numerical results for the approximate solutions obtained by HPM [37] and LT-VIM of Sine-Gordon equation

t	$x = 2.5$		$x = 5$		$x = 7.5$		$x = 10$	
	HPM	LT-VIM	HPM	LT-VIM	HPM	LT-VIM	HPM	LT-VIM
0.1	0.065222	0.065234	0.005390	0.005390	0.000442	0.000442	0.000036	0.000036
0.2	0.130411	0.130503	0.010780	0.010780	0.000885	0.000885	0.000073	0.000073
0.3	0.195529	0.195842	0.016170	0.016170	0.001327	0.001327	0.000109	0.000109
0.4	0.260544	0.261285	0.021561	0.021561	0.001770	0.001770	0.000145	0.000145
0.5	0.325418	0.326868	0.026951	0.026951	0.002212	0.002212	0.000182	0.000182

Table 4 shows that the LT-VIM approximation agrees with the exact solution to the reported six decimal places. Table 5 compares the LT-VIM results with the HPM results reported in [37]. The LT-VIM values are close to the exact values shown in Table 4, while small differences appear between LT-VIM and HPM, especially at $x = 2.5$.

6. Convergence Analysis

Theorem 1

Let us consider the following general partial differential equation

$$L(u(x, t)) + N(u(x, t)) = g(x, t)$$

Here, L and N denote the linear and nonlinear operators, respectively, while $g(x, t)$ is a known function. Assume that the nonlinear operator N satisfies the Lipschitz condition

$$\|N(u) - N(v)\| \leq k\|u - v\|, \quad 0 < k < 1$$

for all u and v in a Banach space X .

Let $\{u_n\}$ be the sequence generated by the LT-VIM iterative scheme, which can be expressed in the operator form

$$u_{n+1} = T(u_n),$$

where $T : X \rightarrow X$ denotes the LT-VIM iteration operator. Then the sequence $\{u_n\}$ converges with the unique exact solution u of the nonlinear problem.

Proof

Since the LT-VIM iteration is defined by

$$u_{n+1} = T(u_n),$$

where T denotes the LT-VIM iteration operator.

Then

$$\|u_{n+1} - u_n\| = \|T(u_n) - T(u_{n-1})\|$$

As N satisfies the Lipschitz condition, the operator T is a contraction mapping with contraction constant k . Therefore,

$$\|u_{n+1} - u_n\| \leq k\|u_n - u_{n-1}\|.$$

Applying this inequality recursively yields

$$\|u_{n+1} - u_n\| \leq k^n \|u_1 - u_0\|.$$

Because $0 < k < 1$,

$$\lim_{n \rightarrow \infty} k^n = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} \|u_{n+1} - u_n\| = 0.$$

Furthermore, for any integers $m > n$,

$$\|u_m - u_n\| \leq \sum_{j=n}^{m-1} \|u_{j+1} - u_j\| \leq \|u_1 - u_0\| \sum_{j=n}^{m-1} k^j$$

By the geometric series formula

$$\|u_m - u_n\| \leq \frac{k^n}{1 - k} \|u_1 - u_0\|$$

$k^n \rightarrow 0$ as $n \rightarrow \infty$, so

$$\lim_{m, n \rightarrow \infty} \|u_m - u_n\| = 0.$$

Thus, $\{u_n\}$ is a Cauchy sequence in the Banach space X . Because every Cauchy sequence in a Banach space converges, there exists $u \in X$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$.

By continuity of the operator T , passing to the limit in the iterative relation $u_{n+1} = T(u_n)$ gives

$$u = T(u),$$

which implies that $u(x, t)$ satisfies the original nonlinear equation. Therefore, the LT-VIM sequence converges with the exact solution $u(x, t)$. $\square$

Theorem 2

If the sequence $\{u_n\}$ generated by LT-VIM converges, then the series solution

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$$

converges in a Banach space X . Then its sum $u(x, t)$ is a solution to the nonlinear problem. That is,

$$L(u(x, t)) + N(u(x, t)) = g(x, t)$$

then, if the solution is unique, then the LT-VIM series converges with the exact solution.

Proof

Assume that the LT-VIM method generates the approximate solution in the form

$$U_m(x, t) = \sum_{n=0}^m u_n(x, t)$$

Since the series converges, $\sum_{n=0}^{\infty} u_n(x, t)$ in X , there exists a function $u(x, t) \in X$ such that

$$\lim_{m \rightarrow \infty} U_m(x, t) = \lim_{m \rightarrow \infty} \sum_{n=0}^m u_n(x, t) = u(x, t)$$

Let $R_m = L(U_m) + N(U_m) - g$. The term R_m , namely residual

Because the LT-VIM sequence is convergent, the residual tends to zero as $m \rightarrow \infty$, that is,

$$\lim_{m \rightarrow \infty} R_m = 0.$$

Taking the limit as $m \rightarrow \infty$, and using the continuity of the linear operator L and the nonlinear operator N , we obtain

$$\lim_{m \rightarrow \infty} [L(U_m) + N(U_m) - g] = 0,$$

hence proved

$$L(u) + N(u) - g = 0$$

Therefore, it is

$$L(u(x, t)) + N(u(x, t)) = g(x, t)$$

Thus, the sum of the LT-VIM series satisfies the original nonlinear system. If the nonlinear problem has a unique solution, then this limit must be the exact solution. Hence,

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$$

$\square$

Theorem 2 establishes that once the iterative sequence generated by Theorem 1 converges, the resulting LT-VIM series satisfies the original partial differential equation. Thus, Theorem 1 guarantees the convergence of the iterative process, while Theorem 2 confirms that the limit of the convergent series is the desired solution.

The Airy equation involves only a linear differential operator, whereas the Allen–Cahn and Sine–Gordon equations involve smooth nonlinear operators that are locally Lipschitz on the bounded domain considered in this study. Therefore, the assumptions of Theorem 1 are satisfied for all three equations, and Theorem 2 guarantees that the corresponding LT-VIM series converges to the exact solution. The numerical results presented in the previous sections support these theoretical results.

7. Conclusion

The Laplace Transform Variational Iteration Method (LT-VIM) was applied to the Airy, Allen–Cahn, and Sine–Gordon equations. To the best of our knowledge, this is the first study to apply LT-VIM to these three equations using a single LT-VIM procedure. The method combines the Laplace transform with the variational iteration technique and determines the Lagrange multiplier directly in the Laplace domain as $\lambda(s) = -1/s^n$, making the iterative process simpler.

The approximate solutions obtained by LT-VIM agreed well with the corresponding exact solutions and were compared with previously published methods. The numerical results and convergence analysis showed that LT-VIM converges rapidly and yields accurate approximations using only a few iterations for both linear and nonlinear partial differential equations. These results indicate that LT-VIM is an accurate semi-analytical technique for solving nonlinear evolution equations with different mathematical characteristics.

Future studies may apply the present LT-VIM framework to more complex nonlinear models, including higher-dimensional, coupled, and fractional partial differential equations.

REFERENCES

1. I. A. Ibrahim, R. A. Hameed, W. M. Taha, and M. A. Rasheed, *New path of popularized homogeneous balance method and travelling wave solutions of a nonlinear Klein-Gordon equation*, Iraqi Journal of Science, pp. 2656–2666, 2022.
2. M. A. Rasheed, R. A. Hameed, and A. N. Khalaf, *Numerical blow-up time of a one-dimensional semilinear parabolic equation with a gradient term*, Iraqi Journal of Science, vol. 64, no. 1, pp. 354–364, 2023.
3. L. C. Evans, *Partial differential equations*, 2nd ed., American Mathematical Society, 2010.
4. S. M. Eid Abdul Kadir, W. M. Taha, R. A. Hameed, and A. H. Jameel, *Efficient approximate analytical methods to solve some partial differential equations*, Iraqi Journal of Science, vol. 64, no. 9, pp. 4644–4657, 2023.
5. J. D. Logan, *A first course in partial differential equations*, 3rd ed., Springer, 2015.
6. A. M. Wazwaz, *Partial differential equations and solitary waves theory*, Springer, 2009.
7. Y. F. Patel, and M. Izadi, *Exact solutions of the generalized time-fractional order Burgers–Huxley equation with a robust semi-analytical technique*, Journal of Umm Al-Qura University for Applied Sciences, 2025.
8. S. Hussain, G. Arora, and R. Kumar, *Semi-analytical methods for solving non-linear differential equations: A review*, Birla Institute of Technology and Science, Pilani, 2023.
9. K. J. Audu, L. Linus, Y. A. Yahaya, and S. Akande, *Adomian decomposition and variational iteration methods in the context of partial differential equations*, Journal of Innovative Science and Engineering, vol. 9, no. 1, pp. 39–53, 2025.
10. D. D. Ganji, and A. Sadighi, *Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations*, International Journal of Nonlinear Sciences and Numerical Simulation, vol. 7, no. 4, pp. 411–418, 2006.
11. G. Adomian, *Solving frontier problems of physics: The decomposition method*, Kluwer Academic Publishers, 1994.
12. S. T. Mohyud-Din, and M. A. Noor, *Homotopy perturbation method for solving partial differential equations*, Zeitschrift für Naturforschung A, vol. 64a, pp. 157–170, 2009.
13. J. H. He, *Variational iteration method for delay differential equations*, Communications in Nonlinear Science and Numerical Simulation, vol. 2, no. 4, pp. 235–236, 1997.
14. J. H. He, *Variational iteration method—A kind of nonlinear analytical technique: Some examples*, International Journal of Non-Linear Mechanics, vol. 34, no. 4, pp. 699–708, 1999.
15. J. H. He, *Variational iteration method—Some recent results*, International Journal of Nonlinear Sciences and Numerical Simulation, vol. 8, no. 1, pp. 1–10, 2007.
16. J. H. He, and X. H. Wu, *Variational iteration method: New development and applications*, Computers & Mathematics with Applications, vol. 54, no. 7–8, pp. 881–894, 2007.
17. S. N. Huseen, and M. Q. Karim, *A new modification of Adomian decomposition method for solving nonlinear parabolic partial differential equations*, Iraqi Journal of Science, vol. 66, no. 6, pp. 2406–2420, 2025.
18. S. J. Liao, *Beyond perturbation: Introduction to the homotopy analysis method*, Chapman & Hall/CRC Press, 2003.

19. A. Hassan, M. Y. Adamu, A. G. Madaki, Y. Nehemiah, M. Cornelius, and U. M. Nasir, *An enhanced variational iteration method for solving ordinary and partial differential equations*, Journal of Multidisciplinary Science: MIKAILALSYS, vol. 3, no. 2, pp. 470–489, 2025.
20. M. A. Noor, and S. T. Mohyud-Din, *Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials*, International Journal of Nonlinear Sciences and Numerical Simulation, vol. 9, no. 2, pp. 141–156, 2008.
21. H. Jafari, and M. Alipour, *Laplace transform variational iteration method for solving differential equations*, Applied Mathematical Sciences, vol. 4, pp. 2673–2681, 2010.
22. E. M. A. Hilal, and T. M. Elzaki, *Solution of nonlinear partial differential equations by new Laplace variational iteration method*, Journal of Function Spaces, vol. 2014, Article 790714, 2014.
23. G. Singh, and I. Singh, *New Laplace variational iterative method for solving two-dimensional telegraph equations*, Journal of Mathematics and Computer Science, vol. 10, no. 6, pp. 2943–2954, 2020.
24. M. Nadeem, F. Li, and H. Ahmad, *Modified Laplace variational iteration method for solving fourth-order parabolic partial differential equation with variable coefficients*, Computers & Mathematics with Applications, vol. 78, no. 6, pp. 2052–2062, 2019.
25. G. Singh, I. Singh, A. M. AlDerea, A. M. Alanzi, and H. A. E.-W. Khalifa, *Solutions of (2+1)-D and (3+1)-D Burgers equations by new Laplace variational iteration technique*, Axioms, vol. 12, no. 7, p. 647, 2023.
26. G. C. Wu, *Laplace variational iteration method for nonlinear equations*, Applied Mathematics Letters, vol. 25, pp. 177–183, 2012.
27. G. C. Wu, *Two new Laplace variational iteration methods*, Applied Mathematical Modelling, vol. 37, pp. 25–31, 2013.
28. G. C. Wu, *Laplace transform overcoming principal drawbacks in application of the variational iteration method to fractional heat equations*, Thermal Science, vol. 16, no. 4, pp. 1257–1261, 2012.
29. G. C. Wu, *Challenge in the variational iteration method: A new approach to identification of the Lagrange multipliers*, Journal of King Saud University – Science, vol. 25, no. 2, pp. 175–178, 2013.
30. M. R. Spiegel, *Schaum's outline of theory and problems of Laplace transforms*, McGraw-Hill, 1965.
31. E. Kreyszig, *Advanced engineering mathematics*, 10th ed., John Wiley & Sons, 2011.
32. S. Tomar, M. Singh, K. Vajravelu, and H. Ramos, *Simplifying the variational iteration method: A new approach to obtain the Lagrange multiplier*, Mathematics and Computers in Simulation, vol. 204, pp. 640–644, 2023.
33. F. A. Al-Zubaidi, and M. A. Al-Jawary, *Optimal variational iteration method for solving nonlinear ordinary differential equations appeared in engineering and applied sciences*, Journal of Physics: Conference Series, vol. 3028, no. 1, Article 012028, 2025.
34. M. A. Shihab, W. M. Taha, R. A. Hameed, A. Jameel, and I. M. Sulaiman, *Implementation of variational iteration method for various types of linear and nonlinear partial differential equations*, International Journal of Electrical and Computer Engineering, vol. 13, no. 2, pp. 2131–2141, 2023.
35. W. K. Zahra, *Trigonometric B-spline collocation method for solving PHI-four and Allen–Cahn equations*, Mediterranean Journal of Mathematics, vol. 14, no. 3, Article 122, 2017.
36. H. Ahmad, A. R. Seadawy, T. A. Khan, and P. Thounthong, *Analytic approximate solutions for some nonlinear parabolic dynamical wave equations*, Journal of Taibah University for Science, vol. 14, no. 1, pp. 346–358, 2020.
37. L. Jin, *Analytical approach to the sine–Gordon equation using homotopy perturbation method*, International Journal of Contemporary Mathematical Sciences, vol. 4, no. 5–8, pp. 225–231, 2009.