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On Modification of the Distribution of Number of Conceptions in an Equilibrium Birth Process

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Abstract
The equilibrium birth process assumes that, at the start of the observation period, not all females are exposed to conception, as some may already be pregnant or experiencing postpartum amenorrhea. However, existing models of the equilibrium birth process do not explicitly incorporate the underlying causes of this temporary sterility. The present paper proposes a modified probability model for the number of conceptions in an equilibrium birth process, incorporating the condition that, at the start of the observational period, some females are temporarily sterile due to pregnancy or postpartum amenorrhea. The proposed model is illustrated using four observed birth distributions. It enables estimation of the proportion of temporarily sterile females at the start of the observational period and permits separate estimation of the proportions experiencing temporary sterility due to pregnancy leading to complete conception, pregnancy resulting in foetal loss, and postpartum amenorrhea following either complete or incomplete conception. Furthermore, indirect estimates of the non-susceptibility periods associated with the two types of pregnancy terminations and the probability of a complete conception were obtained for females corresponding to these distributions.

Keywords Conception, Equilibrium Birth Process, NFHS-5, Postpartum amenorrhea, Probability Model, Temporary Sterility.

AMS 2010 subject classifications 60J80, 60K05, 62P10, 92D25

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1. Introduction

Over the past few decades, various probability models [2, 3, 4, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21, 23, 24, 25, 27, 28, 29, 30] have been derived to describe the variation in the number of conceptions during an observational period of length T, typically starting from the time of marriage or the beginning of a female's sexual life. Despite differences in derivation and underlying assumptions, these models share a common premise that fecund females are continuously exposed to the risk of conception throughout the entire observational period. In practice, however, continuous exposure to conception risk is often interrupted by biological, behavioural, or social factors, potentially limiting the realism and applicability of existing models. In particular, circumstances may arise in which certain females are temporarily sterile at the beginning of the observational period, for example, due to premarital pregnancy. Pathak [6] addressed this limitation by extending the model proposed by Singh [23, 24] by incorporating a proportion of fecund females to remain unexposed to the risk of conception for a heterogeneous duration at the start of the observational period.

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Furthermore, when data on conceptions or births are collected to estimate fertility parameters for any segment of females' reproductive life, the fertility status of females at the beginning of the observational period is generally unknown. This uncertainty arises due to the possibility that a female may be fecund, sterile, or temporarily sterile at the beginning of the observational period. Consequently, the observed sequence of conceptions or births is influenced not only by the underlying birth process but also by the unobserved initial fertility status. Under such circumstances, it is appropriate to introduce the concept of an equilibrium birth process. When a sequence of conceptions or births is observed long after marriage, the resulting distribution of births is referred to as an equilibrium birth process [7, 8, 18, 19, 22]. Singh and Yadava [18] proposed a probability model for the number of conceptions in an equilibrium birth process by considering both complete and incomplete conceptions. A conception is considered complete if it results in a live birth; otherwise, it is classified as incomplete [16]. Subsequently, Singh *et al.* [22] extended this model by incorporating k types of pregnancy outcomes. In their study, complete conceptions were further classified into two categories: those in which the child dies within one year of birth and those in which the child survives beyond one year. Thus, considering two categories of complete conception along with incomplete conception (i.e., $k = 3$ pregnancy outcomes), Singh *et al.* [22] applied the extended model to the same birth distribution data used by Singh and Yadava [18] and obtained a better fit.

Probability models previously derived in the context of an equilibrium birth process [18, 22] are based solely on the assumption that the fertility status of females at the beginning of the observational period is unknown. However, these models do not explicitly account for or distinguish between the various possible fertility statuses of females at the beginning of the observational period. It is evident that if the observational period begins long after marriage, the temporary sterility of fecund females at the start of this period cannot be ignored, as they may be pregnant or in a state of postpartum amenorrhea at the beginning of the observational period.

In the present paper, an attempt has been made to derive a probability model for the number of conceptions in an equilibrium birth process by considering both complete and incomplete conceptions. The present model assumes that during an observational period of length T, the cohort of females consists of two groups: (a) those who are biologically mature for conception throughout the period, referred to as fecund females, and (b) those who are sterile throughout the period. Furthermore, the fecund females are divided into two subgroups: (a) those who are exposed to the risk of conception throughout the entire observational period, and (b) those who are not exposed to the risk of conception at the beginning of the observational period, which is further divided according to the reason for sterility at the start of the observational period, viz., those who are pregnant with certainty that the outcome of the pregnancy will result in a complete conception and those who are either pregnant with certainty that the outcome will result in foetal loss (i.e., incomplete conception) or in the state of postpartum amenorrhea (PPA) associated with either complete or incomplete conception. The other assumptions are similar to those of Singh and Yadava [18].

The proposed model is applied to the observed distribution of the number of births in urban areas of Bihar as obtained from the National Family and Health Survey-5 (NFHS-5), 2019-21 [5], as well as to the birth distributions previously considered by Singh and Yadava [18] and Singh *et al.* [22]. The model enables the estimation of the proportion of females who are temporarily sterile at the beginning of the observational period. Furthermore, it allows separate estimation of the proportions of females experiencing temporary sterility due to pregnancies resulting in complete conception, and those due to pregnancies ending in foetal loss or postpartum amenorrhea associated with either complete or incomplete conception. Additionally, the proposed model provides indirect estimates of the non-susceptibility periods associated with the two types of pregnancy terminations and the probability of complete conception for the females corresponding to these distributions.

2. The Model and Its Derivation in Detail

A probability model describing the variation in the number of conceptions to a female during the observational period $(T_0, T_0 + T)$ of length T, where T_0 is a point significantly distant from the time of marriage, is derived under the following assumptions:

1. The females considered under the study have led a married life throughout the period of observation.

2. The outcome of a conception may be either complete or incomplete. Let θ denote the probability of a complete conception. Thus, $(1-\theta)$ denotes the probability that a conception is incomplete. Suppose h_1 and h_2 are the non-susceptibility periods associated with incomplete and complete conceptions, respectively, where $h_1 < h_2$.
3. The waiting time between the $(r-1)^{\text{th}}$ and r^{th} $(r > 1)$ conceptions follows a displaced exponential distribution, depending on whether the outcome of the $(r-1)^{\text{th}}$ conception is complete or incomplete, with probability density function

$$g_i(t) = \lambda e^{-\lambda(t-ih_2-(1-i)h_1)}, \qquad t > ih_2 + (1-i)h_1, \quad \lambda > 0$$

where

$$i = \begin{cases} 1, & \text{if the } (r-1)^{\text{th}} \text{ conception is complete} \\ 0, & \text{if the } (r-1)^{\text{th}} \text{ conception is incomplete} \end{cases}$$

Here, λ denotes the risk of conception.
4. The cohort of females consists of two groups:
   (a) those who are biologically mature for conception throughout the observational period of length T, referred to as *fecund females*; and
   (b) those who are sterile throughout the observational period.

   Suppose the respective proportions of females in each groups are α and $(1-\alpha)$.
5. The fecund females are divided into two subgroups:
   (a) those who are exposed to the risk of conception throughout the entire observational period of length T, referred to as *Group A* females for our purposes; and
   (b) those who are not exposed to the risk of conception at the beginning of the observational period, referred to as *Group B* females.

   Let β and $(1-\beta)$ denote the proportions of Group A and Group B females, respectively.
6. Group B females are further divided based on the reason for sterility at the start of the observational period:
   (a) those who are pregnant with certainty that the outcome of the pregnancy will result in a complete conception, referred to as *Group B_1* females; and
   (b) those who are either pregnant with certainty that the outcome will result in foetal loss or are in the state of postpartum amenorrhea (PPA) associated with either a complete or an incomplete conception, referred to as *Group B_2* females.

   Let β_1 and $(1-\beta_1)$ denote the proportions of Group B_1 and Group B_2 females, respectively.
7. Group B_1 and Group B_2 females are not exposed to the risk of conception for the first u and v units of time, respectively, from the start of the observational period.

The first three assumptions are similar to those of Singh and Yadava [18]. The classification of the female cohort described in assumptions (4)–(6) is illustrated in Figure 1.

Following Singh and Yadava [18], and under assumption (2), the total number of conceptions for Group A females during an observational period of length T cannot exceed n_A, where

$$n_A = \begin{cases} \dfrac{T}{h_1}, & \text{if } \dfrac{T}{h_1} \text{ is an integer} \\ \left\lfloor \dfrac{T}{h_1} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

Similarly, under assumptions (2) and (7), the total number of conceptions for Group B_1 and Group B_2 females during an observational period of length T cannot exceed n_{B_1} and n_{B_2}, respectively, where

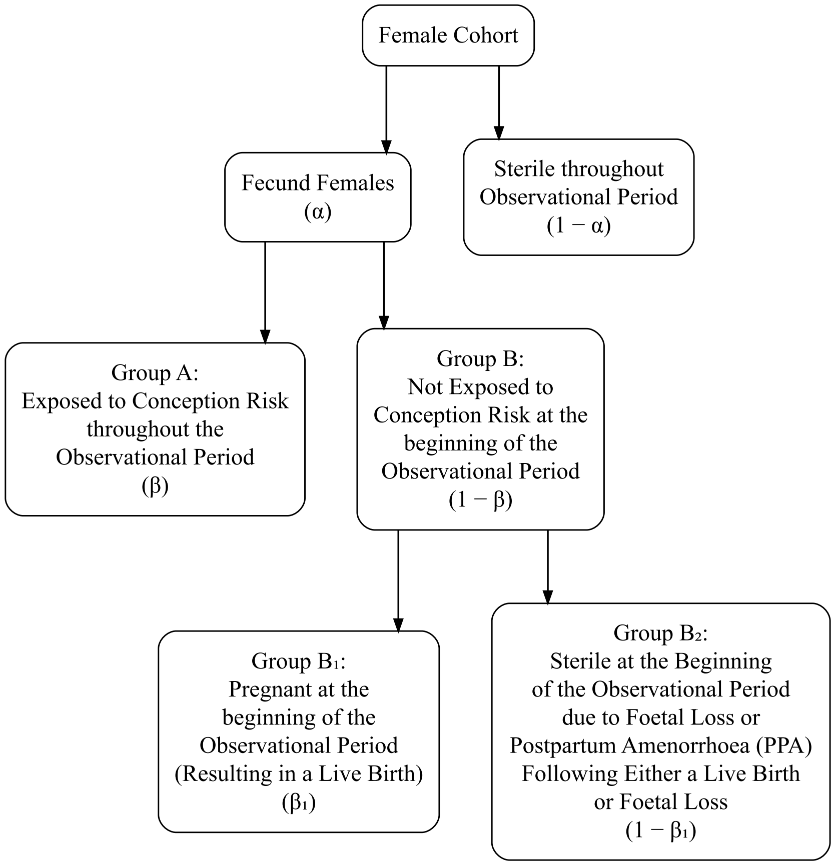


Figure 1. Flow diagram illustrating the classification of females under the proposed model

$$n_{B_1} = \begin{cases} \dfrac{T-u}{h_1}, & \text{if } \dfrac{T-u}{h_1} \text{ is an integer} \\ \left\lfloor \dfrac{T-u}{h_1} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

and

$$n_{B_2} = \begin{cases} \dfrac{T-v}{h_1}, & \text{if } \dfrac{T-v}{h_1} \text{ is an integer} \\ \left\lfloor \dfrac{T-v}{h_1} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

with

$$0 < u < h_2', \qquad 0 < v < h_1'$$

Here, h_1' and h_2' represent the upper bounds of u and v, respectively, and may vary from female to female. Obviously,

$$n_{B_1} < n_{B_2} < n_A$$

and $\lfloor w \rfloor$ denotes the greatest integer that does not exceed w.

Let $X(T)$ represent the number of conceptions to a female during the observational period (T_0, T_0+T), where T_0 is a point significantly distant from the time of marriage. Then, under assumptions (1)–(7), the probability distribution of $X(T)$ is expressed as follows:

$$\begin{aligned} P\big(X(T)=0\big) &= 1-\alpha+\alpha\beta\left\{1-G_0(T)+G_1(T)\right\} \\ &\quad + \alpha(1-\beta)(1-\beta_1)\left\{1-\int_0^{h_1'}\frac{H_0'(T)}{h_1'}\,dv+\int_0^{h_1'}\frac{H_1'(T)}{h_1'}\,dv\right\} \end{aligned} \tag{1}$$

$$\begin{aligned} P\big(X(T)=1\big) &= \alpha\beta\left\{G_0(T)-2G_1(T)+G_2(T)\right\} \\ &\quad + \alpha(1-\beta)\beta_1\left\{1-\int_0^{h_2'}\frac{H_0(T)}{h_2'}\,du+\int_0^{h_2'}\frac{H_1(T)}{h_2'}\,du\right\} \\ &\quad + \alpha(1-\beta)(1-\beta_1)\left\{\int_0^{h_1'}\frac{H_0'(T)}{h_1'}\,dv-2\int_0^{h_1'}\frac{H_1'(T)}{h_1'}\,dv+\int_0^{h_1'}\frac{H_2'(T)}{h_1'}\,dv\right\} \end{aligned} \tag{2}$$

$$\begin{aligned} P\big(X(T)=r\big) &= \alpha\beta\left\{G_{r-1}(T)-2G_r(T)+G_{r+1}(T)\right\} \\ &\quad + \alpha(1-\beta)\beta_1\left\{\int_0^{h_2'}\frac{H_{r-2}(T)}{h_2'}\,du-2\int_0^{h_2'}\frac{H_{r-1}(T)}{h_2'}\,du+\int_0^{h_2'}\frac{H_r(T)}{h_2'}\,du\right\} \\ &\quad + \alpha(1-\beta)(1-\beta_1)\left\{\int_0^{h_1'}\frac{H_{r-1}'(T)}{h_1'}\,dv-2\int_0^{h_1'}\frac{H_r'(T)}{h_1'}\,dv+\int_0^{h_1'}\frac{H_{r+1}'(T)}{h_1'}\,dv\right\}, \\ &\quad r=2,3,\ldots,n-2 \end{aligned} \tag{3}$$

$$\begin{aligned} P\big(X(T)=n-1\big) &= \alpha\beta\left\{G_{n-2}(T)-2G_{n-1}(T)\right\} \\ &\quad + \alpha(1-\beta)\beta_1\left\{\int_0^{h_2'}\frac{H_{n-3}(T)}{h_2'}\,du-2\int_0^{h_2'}\frac{H_{n-2}(T)}{h_2'}\,du+\int_0^{h_2'}\frac{H_{n-1}(T)}{h_2'}\,du\right\} \\ &\quad + \alpha(1-\beta)(1-\beta_1)\left\{\int_0^{h_1'}\frac{H_{n-2}'(T)}{h_1'}\,dv-2\int_0^{h_1'}\frac{H_{n-1}'(T)}{h_1'}\,dv\right\} \end{aligned} \tag{4}$$

$$\begin{aligned} P\big(X(T)=n\big) &= \alpha\beta G_{n-1}(T) \\ &\quad + \alpha(1-\beta)\beta_1\left\{\int_0^{h_2'}\frac{H_{n-2}(T)}{h_2'}\,du-2\int_0^{h_2'}\frac{H_{n-1}(T)}{h_2'}\,du\right\} \\ &\quad + \alpha(1-\beta)(1-\beta_1)\int_0^{h_1'}\frac{H_{n-1}'(T)}{h_1'}\,dv \end{aligned} \tag{5}$$

where,

$$\begin{aligned} G_r(T) &= \sum_{j=0}^{l_r}\binom{r}{j}\theta^j(1-\theta)^{r-j}\left[\frac{T\lambda}{1+\lambda\{(1-\theta)h_1+\theta h_2\}}-r\right. \\ &\quad \left. +\frac{1}{1+\lambda\{(1-\theta)h_1+\theta h_2\}}\sum_{s=0}^{r-1}\sum_{k=0}^{s}e^{-\lambda(T-jh_2-(r-j)h_1)}\frac{\{\lambda(T-jh_2-(r-j)h_1)\}^k}{k!}\right] \end{aligned} \tag{6}$$

$$
\begin{aligned}
H_r(T) = \sum_{j=0}^{l_r} \binom{r}{j} \theta^j (1-\theta)^{r-j} \Bigg[& \frac{(T-u)\lambda}{1+\lambda\{(1-\theta)h_1+\theta h_2\}} - r \\
& + \frac{1}{1+\lambda\{(1-\theta)h_1+\theta h_2\}} \sum_{s=0}^{r-1} \sum_{k=0}^{s} e^{-\lambda(T-u-jh_2-(r-j)h_1)} \frac{\{\lambda(T-u-jh_2-(r-j)h_1)\}^k}{k!} \Bigg]
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
H_r'(T) = \sum_{j=0}^{l_r} \binom{r}{j} \theta^j (1-\theta)^{r-j} \Bigg[& \frac{(T-v)\lambda}{1+\lambda\{(1-\theta)h_1+\theta h_2\}} - r \\
& + \frac{1}{1+\lambda\{(1-\theta)h_1+\theta h_2\}} \sum_{s=0}^{r-1} \sum_{k=0}^{s} e^{-\lambda(T-v-jh_2-(r-j)h_1)} \frac{\{\lambda(T-v-jh_2-(r-j)h_1)\}^k}{k!} \Bigg]
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
l_r &= \min\left\{r,\, \frac{T-rh_1}{h_2-h_1}\right\} \\
h_1' &= \min\{h_1,\, T-l_rh_2-(r-l_r)h_1\} \\
h_2' &= \min\{h_2,\, T-l_rh_2-(r-l_r)h_1\}
\end{aligned} \tag{9}
$$

Proof.

Under assumption (5), fecund females are divided into two groups, viz., Group A and Group B. Furthermore, Group B females are classified into two subgroups based on the reason for sterility at the start of the observational period, viz., Group B_1 and Group B_2. Under assumptions (1), (2), and (3), Singh and Yadava [18] derived a probability model for the number of conceptions occurring during an observational period of length T among Group A females in an equilibrium birth process. For completeness, the derivation of the probability model for Group A females is provided below.

Let T_1 denote the time to the first conception from T_0, and let T_r $(r > 1)$ represent the time interval between the $(r-1)^{\text{th}}$ and r^{th} conceptions. Accordingly, the total time to the r^{th} conception is expressed as

$$T_1 + T_2 + \cdots + T_r$$

As previously stated, if T_0 is a distant point from the time of marriage, the birth process becomes an equilibrium birth process, wherein the intervals $T_2, T_3, \ldots$ are independently and identically distributed, while T_1 follows a different distribution.

Under assumptions (2) and (3), the probability density function (p.d.f.) of T_r $(r > 1)$ is given by

$$f(t) = \theta g_1(t) + (1-\theta) g_0(t) \tag{10}$$

or equivalently,

$$f(t) = \sum_{j=0}^{1} \binom{1}{j} \theta^j (1-\theta)^{1-j} \lambda e^{-\lambda(t-jh_2-(1-j)h_1)} I_{(jh_2+(1-j)h_1,\infty)}(t), \qquad \lambda > 0 \tag{11}$$

Here, $j = 0, 1$ denotes the number of complete conceptions.

Therefore, the p.d.f. of T_1 is expressed as

$$f_1(t) = \frac{1-F(t)}{\mu} \tag{12}$$

where $F(t)$ and μ denote the cumulative distribution function and the mean, respectively, corresponding to the p.d.f. $f(t)$. Specifically,

$$F(t)=\sum_{j=0}^{1}\binom{1}{j}\theta^j(1-\theta)^{1-j}\left(1-e^{-\lambda(t-jh_2-(1-j)h_1)}\right)I_{(jh_2+(1-j)h_1,\infty)}(t) \tag{13}$$

and

$$\mu=\frac{1+\lambda\{(1-\theta)h_1+\theta h_2\}}{\lambda} \tag{14}$$

Let $\phi(s)$ denote the Laplace transform of $f(t)$. Thus, the Laplace transform of $f_1(t)$ is given by

$$\frac{1-\phi(s)}{\mu s}$$

Since $T_1, T_2, \ldots, T_r$ are mutually independent, the Laplace transform of $T_1+T_2+\cdots+T_r$ is given by

$$\frac{1-\phi(s)}{\mu s}\{\phi(s)\}^{r-1}=\frac{\{\phi(s)\}^{r-1}}{\mu s}-\frac{\{\phi(s)\}^{r}}{\mu s} \tag{15}$$

Using the inverse Laplace transform of the above expression, the probability density function of $T_1+T_2+\cdots+T_r$, denoted by $f'_r(t)$, is given by

$$f'_r(t)=\frac{1}{\mu}F^{(r-1)}(t)-\frac{1}{\mu}F^{(r)}(t) \tag{16}$$

where $F^{(r)}(t)$ represents the r-fold convolution of $F(t)$ with itself. The expression for $F^{(r)}(t)$ is obtained as follows:

$$F^{(r)}(t)=\sum_{j=0}^{l_r}\binom{r}{j}\theta^j(1-\theta)^{r-j}\left[1-e^{-\lambda(t-jh_2-(r-j)h_1)}\sum_{s=0}^{r-1}\frac{\{\lambda(t-jh_2-(r-j)h_1)\}^s}{s!}\right] I_{(jh_2+(r-j)h_1,\infty)}(t) \tag{17}$$

Here, $j=0,1,2,\ldots,l_r$ denotes the number of complete conceptions, where

$$l_r=\min\left[r,\ \frac{T-rh_1}{h_2-h_1}\right]$$

which represents the maximum number of complete conceptions [16, 18].

Now let $X(T)$ represent the number of conceptions to a female during the observational period (T_0, T_0+T). The distribution function of $X(T)$ can therefore be derived from the p.d.f. of $T_1+T_2+\cdots+T_r$ as follows:

$$\begin{aligned}\phi_{X(T)}(r)&=P\left[\text{At most } r \text{ conceptions occur in the interval } (T_0,T_0+T)\right]\\&=P\left[\text{The } (r+1)^{\text{th}} \text{ conception occurs after an observational period of } T \text{ years}\right]\\&=\int_T^\infty f'_{r+1}(t)\,dt\\&=1-\int_0^T f'_{r+1}(t)\,dt\end{aligned} \tag{18}$$

Therefore, the probability that $X(T)=r$ $(r>0)$ conceptions occur during the observational period (T_0, T_0+T) is given by

$$\begin{aligned}
P(X(T)=r) &= \phi_{X(T)}(r) - \phi_{X(T)}(r-1) \\
&= \int_0^T f'_r(t)\,dt - \int_0^T f'_{r+1}(t)\,dt \\
&= \int_0^T \left\{\frac{1}{\mu}F^{(r-1)}(t) - \frac{1}{\mu}F^{(r)}(t)\right\} dt - \int_0^T \left\{\frac{1}{\mu}F^{(r)}(t) - \frac{1}{\mu}F^{(r+1)}(t)\right\} dt \\
&= \frac{1}{\mu}\int_0^T F^{(r-1)}(t)\,dt - \frac{2}{\mu}\int_0^T F^{(r)}(t)\,dt + \frac{1}{\mu}\int_0^T F^{(r+1)}(t)\,dt \\
&= G_{r-1}(T) - 2G_r(T) + G_{r+1}(T) \qquad (19)
\end{aligned}$$

where

$$G_{r-1}(T) = \frac{1}{\mu}\int_0^T F^{(r-1)}(t)\,dt, \qquad G_r(T) = \frac{1}{\mu}\int_0^T F^{(r)}(t)\,dt$$

and

$$G_{r+1}(T) = \frac{1}{\mu}\int_0^T F^{(r+1)}(t)\,dt$$

We know that

$$\begin{aligned}
P(X(T)=0) &= \phi_{X(T)}(0) \\
&= P[\text{At most zero conception occurs in the interval } (T_0, T_0+T)] \\
&= P[\text{No conception occurs in the interval } (T_0, T_0+T)] \\
&= P[\text{The first conception occurs after an observational period of } T \text{ years}] \\
&= \int_T^\infty f'_1(t)\,dt \\
&= \int_T^\infty f_1(t)\,dt
\end{aligned}$$

since

$$\begin{aligned}
f'_1(t) &= \frac{1}{\mu}F^{(1-1)}(t) - \frac{1}{\mu}F^{(1)}(t) \\
&= \frac{1}{\mu} - \frac{F(t)}{\mu} \\
&= \frac{1-F(t)}{\mu} \\
&= f_1(t)
\end{aligned}$$

Hence,

$$
\begin{aligned}
P(X(T)=0) &= 1-\int_0^T f_1(t)\,dt \\
&= 1-\int_0^T \frac{1-F(t)}{\mu}\,dt \\
&= 1-\frac{1}{\mu}\int_0^T dt+\frac{1}{\mu}\int_0^T F(t)\,dt \\
&= 1-G_0(T)+G_1(T)
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
G_0(T) &= \frac{1}{\mu}\int_0^T dt \\
&= \frac{T}{\mu} \\
&= \frac{\lambda T}{1+\lambda\left\{(1-\theta)h_1+\theta h_2\right\}}
\end{aligned} \tag{21}
$$

And

$$
\begin{aligned}
G_1(T) &= \frac{1}{\mu}\int_0^T F(t)\,dt \\
&= \sum_{j=0}^{1}\binom{1}{j}\theta^j(1-\theta)^{1-j}\frac{1}{\mu}\int_{jh_2+(1-j)h_1}^{T}\left[1-e^{-\lambda(t-jh_2-(1-j)h_1)}\right]dt \\
&= \sum_{j=0}^{1}\binom{1}{j}\theta^j(1-\theta)^{1-j}\left[\frac{T\lambda}{1+\lambda\{(1-\theta)h_1+\theta h_2\}}-1+\frac{e^{-\lambda(T-jh_2-(1-j)h_1)}}{1+\lambda\{(1-\theta)h_1+\theta h_2\}}\right]
\end{aligned} \tag{22}
$$

From equation (19), it follows that

$$
P(X(T)=1)=G_0(T)-2G_1(T)+G_2(T) \tag{23}
$$

Next, we derive an expression for $G_2(T)$ as follows.

$$
\begin{aligned}
G_2(T) &= \frac{1}{\mu}\int_0^T F^{(2)}(t)\,dt \\
&= \sum_{j=0}^{l_2}\binom{2}{j}\theta^j(1-\theta)^{2-j}\frac{1}{\mu}\int_{jh_2+(2-j)h_1}^{T}\left[1-e^{-\lambda(t-jh_2-(2-j)h_1)}\left\{1+\lambda\left(t-jh_2-(2-j)h_1\right)\right\}\right]dt \\
&= \sum_{j=0}^{l_2}\binom{2}{j}\theta^j(1-\theta)^{2-j}\left[\frac{T\lambda}{1+\lambda\{(1-\theta)h_1+\theta h_2\}}-2\right. \\
&\quad \left.+\frac{1}{1+\lambda\{(1-\theta)h_1+\theta h_2\}}\sum_{s=0}^{1}\sum_{k=0}^{s}e^{-\lambda(T-jh_2-(2-j)h_1)}\frac{\left\{\lambda\left(T-jh_2-(2-j)h_1\right)\right\}^k}{k!}\right]
\end{aligned} \tag{24}
$$

Following a similar approach, we readily obtain

$$G_r(T) = \sum_{j=0}^{l_r} \binom{r}{j} \theta^j (1-\theta)^{r-j} \Bigg[\frac{T\lambda}{1+\lambda\{(1-\theta)h_1+\theta h_2\}} - r + \frac{1}{1+\lambda\{(1-\theta)h_1+\theta h_2\}} \sum_{s=0}^{r-1} \sum_{k=0}^{s} e^{-\lambda(T-jh_2-(r-j)h_1)} \frac{\{\lambda\,(T-jh_2-(r-j)h_1)\}^k}{k!} \Bigg] \tag{25}$$

Clearly,

$$G_r(T) = 0 \quad \text{if} \quad T - l_r h_2 - (r-l_r)h_1 \leq 0$$

This directly implies that

$$G_n(T) = 0$$

since

$$T - l_n h_2 - (n-l_n)h_1 \leq 0$$

As the maximum number of conceptions, denoted by n, is chosen such that $l_n = 0$, it follows that

$$n = n_A = \frac{T}{h_1} \quad \text{or} \quad \left\lfloor \frac{T}{h_1} \right\rfloor + 1$$

This directly ensures that

$$T - l_n h_2 - (n-l_n)h_1 \leq 0$$

Alternatively, if n is predetermined, then

$$l_n = \min\left[n, \frac{T-nh_1}{h_2-h_1}\right]$$

is such that

$$T - l_n h_2 - (n-l_n)h_1 = 0$$

Equivalently, we have

$$G_{n+1}(T) = 0$$

Therefore, the probability model for the number of conceptions among Group A females in an equilibrium birth process is given by

$$P_0 = 1 - G_0(T) + G_1(T) \tag{26}$$
$$P_r = G_{r-1}(T) - 2G_r(T) + G_{r+1}(T) \tag{27}$$
$$P_{n-1} = G_{n-2}(T) - 2G_{n-1}(T) \tag{28}$$
$$P_n = G_{n-1}(T) \tag{29}$$

Here, $P_r,\ r = 0, 1, \ldots, n$, denotes the probability of having exactly r births during an observational period of length T among females in Group A.

Now, we derive the probability model for the number of conceptions among Group B_1 females. Given the assumption that, at the start of the observational period, Group B_1 females are pregnant with certainty that the outcome of the pregnancy will result in a complete conception, and are not exposed to the risk of conception for the first u units of time.

If u is considered a known constant, then the probability model for the number of conceptions among Group B_1 females over the observational period $(T_0, T_0 + T)$ is analogous to that of Group A females over the observational period

$$(T_0, T_0 + (T - u))$$

Therefore, the probability of r conceptions ($r > 0$) occurring within an observational period of length $(T - u)$, i.e., the probability that $X(T - u) = r$ conceptions occur during the observational period

$$(T_0, T_0 + (T - u))$$

is given by

$$\begin{aligned} P(X(T-u) = r) &= \phi_{X(T-u)}(r) - \phi_{X(T-u)}(r-1) \\ &= \frac{1}{\mu}\int_0^T F^{(r-1)}(t-u)\,dt - \frac{2}{\mu}\int_0^T F^{(r)}(t-u)\,dt + \frac{1}{\mu}\int_0^T F^{(r+1)}(t-u)\,dt \\ &= H_{r-1}(T) - 2H_r(T) + H_{r+1}(T) \end{aligned} \tag{30}$$

Similarly,

$$P(X(T-u) = 0) = 1 - H_0(T) + H_1(T) \tag{31}$$
$$P(X(T-u) = n-1) = H_{n-2}(T) - 2H_{n-1}(T) \tag{32}$$
$$P(X(T-u) = n) = H_{n-1}(T) \tag{33}$$

where

$$\begin{aligned} H_r(T) = \sum_{j=0}^{l_r} \binom{r}{j} \theta^j (1-\theta)^{r-j} \Bigg[&\frac{(T-u)\lambda}{1 + \lambda\{(1-\theta)h_1 + \theta h_2\}} - r \\ &+ \frac{1}{1 + \lambda\{(1-\theta)h_1 + \theta h_2\}} \sum_{s=0}^{r-1}\sum_{k=0}^{s} e^{-\lambda(T-u-jh_2-(r-j)h_1)} \frac{\{\lambda(T-u-jh_2-(r-j)h_1)\}^k}{k!} \Bigg] \end{aligned} \tag{34}$$

It is obvious that the maximum value of u (say, h_2') depends on the month of pregnancy of the female at the start of the observational period. Unfortunately, this is not directly observable and varies from female to female. However, the value of h_2' can be obtained indirectly, as it is directly associated with the maximum number of complete conceptions that a female may experience during an observational period of length T.

Let r be the number of conceptions experienced by a female during an observational period of length T. Then, the total (maximum) possible number of complete conceptions is given by

$$l_r = \min\left[r, \frac{T - rh_1}{h_2 - h_1}\right]$$

Therefore, if a female from Group B_1 experiences r conceptions, of which l_r conceptions are complete during the observational period of length T, then she must have to complete her first pregnancy, resulting in a complete conception, within h_2' units of time. Clearly, h_2, which represent the non-susceptibility period associated with a complete conception and is a combination of gestation and postpartum amenorrhea, is the maximum value of h_2'. However, it is evident that

$$T > l_r h_2 + (r - l_r)h_1$$

and therefore it follows that

$$h_2' = \min\left[h_2,\, T - l_r h_2 - (r - l_r)h_1\right]$$

For example, during an observational period of $T = 5$ years, let us consider $h_1 = 0.5$ years (4 months of gestation and 2 months of postpartum amenorrhea or 6 months of Postpartum amenorrhea) and $h_2 = 1.75$ years (9 months of gestation and 12 months of postpartum amenorrhea). Under these conditions, a female would experience four complete conceptions if her first pregnancy concluded within 0.75 years (9 months). Thus, if we assume there is at least 6 months of postpartum amenorrhea associated with a complete conception, the female must be at least 6 months pregnant at the start of the observational period. It should be noted that the non-susceptibility period associated with the fourth complete conception is considered to be 0.75 years (9 months), while the remaining 1 year of postpartum amenorrhea is excluded from consideration in the case of last possible complete conception the female may experience within the observational period.

Hence, h_2' is a function of both r and l_r. Accordingly, for females with fixed values of r and l_r, we assume that u follows a uniform distribution over $(0, h_2')$. Since h_2' depends on r and l_r, the range of this uniform distribution varies across females. Therefore, using equations (30)–(33), together with the approach of Pathak [6], the probability model for the number of conceptions in an equilibrium birth process for Group B_1 females is obtained as follows:

$$P_{10} = 1 - \int_0^{h_2'} \frac{H_0(T)}{h_2'}\, du + \int_0^{h_2'} \frac{H_1(T)}{h_2'}\, du \tag{35}$$

$$P_{1r} = \int_0^{h_2'} \frac{H_{r-1}(T)}{h_2'}\, du - 2\int_0^{h_2'} \frac{H_r(T)}{h_2'}\, du + \int_0^{h_2'} \frac{H_{r+1}(T)}{h_2'}\, du \quad , r = 1, 2, \ldots, n-2 \tag{36}$$

$$P_{1(n-1)} = \int_0^{h_2'} \frac{H_{n-2}(T)}{h_2'}\, du - 2\int_0^{h_2'} \frac{H_{n-1}(T)}{h_2'}\, du \tag{37}$$

$$P_{1n} = \int_0^{h_2'} \frac{H_{n-1}(T)}{h_2'}\, du \tag{38}$$

Here, $P_{1r},\ r = 0, 1, \ldots, n$, denotes the probability of having exactly r births during an observational period of length T among females in Group B_1.

It is also assumed that, at the start of the observational period, Group B_2 females are either pregnant with certainty that the outcome will result in foetal loss, or are in the state of postpartum amenorrhea (PPA) associated with either a complete or an incomplete conception, and are therefore not exposed to the risk of conception for the first v units of time. Similarly, as explained above, the maximum value of v (say, h_1') is given by

$$h_1' = \min\left[h_1,\, T - l_r h_2 - (r - l_r)h_1\right]$$

Here, h_1 represents either the non-susceptibility period associated with an incomplete conception, consisting of the gestation period and postpartum amenorrhea, or the postpartum amenorrhea period associated with a complete conception.

Therefore, in a similar manner, the probability model for the number of conceptions in an equilibrium birth process among Group B_2 females can be written as follows:

$$P_{20} = 1 - \int_0^{h_1'} \frac{H_0'(T)}{h_1'}\,dv + \int_0^{h_1'} \frac{H_1'(T)}{h_1'}\,dv \tag{39}$$

$$P_{2r} = \int_0^{h_1'} \frac{H_{r-1}'(T)}{h_1'}\,dv - 2\int_0^{h_1'} \frac{H_r'(T)}{h_1'}\,dv + \int_0^{h_1'} \frac{H_{r+1}'(T)}{h_1'}\,dv \quad ,r = 1, 2, \ldots, n-2 \tag{40}$$

$$P_{2(n-1)} = \int_0^{h_1'} \frac{H_{n-2}'(T)}{h_1'}\,dv - 2\int_0^{h_1'} \frac{H_{n-1}'(T)}{h_1'}\,dv \tag{41}$$

$$P_{2n} = \int_0^{h_1'} \frac{H_{n-1}'(T)}{h_1'}\,dv \tag{42}$$

Here, $P_{2r},\ r = 0, 1, \ldots, n$, denotes the probability of having exactly r births during an observational period of length T among females in Group B_2.

And,

$$H_r'(T) = \sum_{j=0}^{l_r} \binom{r}{j} \theta^j (1-\theta)^{r-j} \Bigg[\frac{(T-v)\lambda}{1+\lambda\{(1-\theta)h_1+\theta h_2\}} - r + \frac{1}{1+\lambda\{(1-\theta)h_1+\theta h_2\}} \sum_{s=0}^{r-1}\sum_{k=0}^{s} e^{-\lambda(T-v-jh_2-(r-j)h_1)} \frac{\{\lambda(T-v-jh_2-(r-j)h_1)\}^k}{k!} \Bigg] \tag{43}$$

It is important to note that the females in Group B_1 had already experienced one complete conception at the beginning of the observational period. Therefore, for these females, the probability of having no births during the remaining exposure period (i.e., excluding the period of non-susceptibility associated with the pregnancy that resulted in a complete conception at the start of the observational period) is equivalent to the probability of having one birth during the entire observational period, and so on. Given this assumption for Group B_1 females, and under assumptions (1)–(7), the probability model for the number of conceptions in an equilibrium birth process can be derived from equations (26)–(29) and equations (35)–(42), and is presented in equations (1)–(5).

For $\alpha = \beta = 1$, the probability model given in equations (1)–(5) reduces to the one proposed by Singh and Yadava [18].

3. Application and Estimation of Parameters

The estimation of all seven parameters h_1, h_2, θ, λ, α, β, and β_1 associated with the proposed model presents considerable complexity. Singh and Yadava [18] and Singh *et al.* [22] obtained estimates of certain parameters of their model by considering other parameters as known constants and applied their model to two sets of observed distributions of the number of births taken from the *Demographic Survey of Varanasi (Rural), 1969–1970*, where the distribution of births to eligible (fecund) females aged 25-29 over the last 5 and 7 years was observed.

To illustrate the applicability of the proposed model, we analyze two contemporary distributions of the number of births among females aged 25–29 years during the last 5 years of the observational period, obtained from the National Family Health Survey-5 (NFHS-5), 2019–21, for the urban areas of Bihar [5]. Bihar has been selected because, according to the NFHS-5, it recorded the highest mean number of births among all Indian states in both urban and rural areas. In addition to the NFHS-5 data, the proposed model is also applied to the observed distributions of the number of births among eligible (fecund) females aged 25–29 years during the last 5 and 7 years of the observational period, as considered by Singh and Yadava [18] and Singh *et al.* [22]. Table 1 presents the distribution of the number of births among females aged 25–29 years, irrespective of age at marriage, in urban Bihar, whereas Table 2 presents the corresponding distribution for females aged 25–29 years who married between the ages of 16 and 20 years. In addition, Tables 3 and 4 present the observed distributions of the number of births during the last 5 and 7 years of the observational period, respectively, based on the data considered by Singh and Yadava [18] and Singh *et al.* [22].

To proceed with the analysis, it is necessary to estimate the parameters of the proposed model using the data presented in Tables 1 – 4. However, the simultaneous estimation of all model parameters is analytically challenging. Therefore, following the approach employed by Singh and Yadava [18] and Singh *et al.* [22], some parameters are estimated by treating the remaining parameters as known constants. Singh *et al.* [22] estimated the probability of incomplete conception as 0.15 and 0.09 for fecund females over observational periods of 5 and 7 years, respectively. Additionally, Singh and Yadava [18] assumed a non-susceptibility period of 1 year for incomplete conceptions and 1.75 years for complete conceptions. Contrastingly, Singh *et al.* [22] defined the non-susceptibility period as 0.5 years for females whose pregnancies resulted in incomplete conception, and as 1 year and 1.75 years for females with complete conceptions culminating in either infant death or survival beyond one year, respectively.

Although this approach considerably simplifies parameter estimation, it is important to note that, for any observed birth distribution, assuming fixed values of the non-susceptibility period associated with incomplete conception (h_1), the non-susceptibility period associated with complete conception (h_2), and the probability of complete conception (θ), without a detailed assessment of the reproductive characteristics of the study population, may lead to biased estimates of the model parameters λ, α, β, and β_1. Furthermore, these parameters are likely to exhibit variation across individuals owing to biological and demographic heterogeneity. Therefore, it is important to assess the extent of such variation to determine whether the population of females may reasonably be treated as homogeneous with respect to these characteristics. Previous studies have indicated that, within a relatively narrow female age group, the underlying fertility characteristics may be regarded as approximately homogeneous. Accordingly, the present paper assumes that these characteristics remain constant across all females. Nevertheless, selecting appropriate values of h_1, h_2, and θ remains a challenging task. To address this issue, several plausible combinations of these parameters are considered. For each combination, the ML estimates of λ, α, β, and β_1 are obtained, and the corresponding expected frequencies are compared with the observed frequencies. The suitability of the assumed values of h_1, h_2, and θ is assessed based on the goodness-of-fit of the expected frequencies to the observed frequencies.

Nevertheless, Singh and Yadava [18] and Singh *et al.* [22] assumed fixed values of h_1, h_2, and θ based on the prevailing demographic and biological evidence. Treating these parameters as known constants, they estimated the remaining parameters and obtained satisfactory agreement between the expected and observed birth distributions. These findings suggest that, for the populations considered in those studies, the assumed values of h_1, h_2, and θ were reasonable. The results also indicate that the female populations corresponding to those birth distributions could be regarded as approximately homogeneous with respect to the fertility parameters of the model.

In this paper, a range of values for h_1, h_2, and θ was considered for all the birth distributions presented in Tables 1 – 4. For each combination of these assumed values, the remaining model parameters were estimated, and the corresponding goodness-of-fit was evaluated. The combinations that provided satisfactory agreement between the expected and observed distributions were regarded as indirect estimates of h_1, h_2, and θ. Previous studies have shown that, when θ is sufficiently large, variations in h_2 have a greater influence on the estimates of the remaining model parameters than comparable variations in h_1 within the proposed model [16]. Accordingly, the parameter values were chosen as

$$h_2 = (1.00,\ 1.05,\ 1.10,\ \ldots,\ 2.00), \qquad h_1 = (0.4,\ 0.5,\ 0.6), \qquad \theta = (0.80,\ 0.85,\ 0.90)$$

Even after assuming h_1, h_2, and θ to be known constants, the simultaneous estimation of the remaining parameters λ, α, β, and β_1 remains a complicated task. As noted earlier, Singh and Yadava [18] and Singh *et al.* [22] considered the observed distribution of the number of births obtained from the *Demographic Survey of Varanasi (Rural), 1969–1970*, in which births were recorded exclusively for eligible (i.e., fecund) females aged 25–29 years. Consequently, the parameter α, which represents the proportion of biologically mature females, was not included in their model. Therefore, for the birth distributions presented in Tables 3 and 4, it is reasonable to assume $\alpha = 1$ in our proposed model. In contrast, for the datasets presented in Tables 1 and 2, there is no available information regarding the parameter α; hence, it is necessary to estimate this parameter. Singh and Bhattacharya [16] demonstrated that the variation of α is only minimally influenced by variations in the other fertility parameters. Consequently, the variation in the values of the parameters λ, β, and β_1 does not significantly affect the value of α. This property provides a substantial advantage in estimating the parameters of our proposed model. It is noted that our proposed model is an extension of the model proposed by Singh and Yadava [18].

By assuming $\beta = 1$, in our proposed model, it simplifies to the model proposed by Singh and Yadava [18] with the addition of the parameter α, which represents the proportion of fecund females, since β_1 becomes zero when $\beta = 1$. Within this framework, we can obtain the ML estimates of α and λ for specified values of β, β_1, h_1, h_2, and θ by maximizing the following likelihood function:

$$L(\alpha, \lambda \mid \beta, \beta_1, h_1, h_2, \theta) = \frac{N!}{\prod_{r=0}^{n} N_r!} \prod_{r=0}^{n} \{P(X(T) = r)\}^{N_r} \tag{44}$$

where N_r $(r = 0, 1, 2, \ldots, n)$ denotes the observed frequency of females having exactly r births during an observational period of length T, such that

$$N = \sum_{r=0}^{n} N_r$$

The ML estimates of α and λ are obtained by solving the following likelihood equations:

$$\left(\frac{\partial \log L(\alpha, \lambda)}{\partial \alpha}, \frac{\partial \log L(\alpha, \lambda)}{\partial \lambda} \right) = (0, 0) \tag{45}$$

Equivalently, the asymptotic covariance matrix, $\mathbf{V}$, of the ML estimators is given by

$$\mathbf{V} = \begin{pmatrix} V(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha}, \hat{\lambda}) \\ \mathrm{Cov}(\hat{\alpha}, \hat{\lambda}) & V(\hat{\lambda}) \end{pmatrix} = \mathbf{I}^{-1}, \tag{46}$$

where

$$\mathbf{I} = \begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\lambda} \\ I_{\alpha\lambda} & I_{\lambda\lambda} \end{pmatrix} \Bigg|_{\alpha=\hat{\alpha},\, \lambda=\hat{\lambda}} \tag{47}$$

is the Fisher information matrix evaluated at the ML estimates.
The elements of the Fisher information matrix are

$$I_{\alpha\alpha} = -E\left[\frac{\partial^2}{\partial \alpha^2} \log L(\alpha, \lambda \mid \beta, \beta_1, h_1, h_2, \theta) \right] \tag{48}$$

$$I_{\alpha\lambda} = -E\left[\frac{\partial^2}{\partial \alpha\, \partial \lambda} \log L(\alpha, \lambda \mid \beta, \beta_1, h_1, h_2, \theta) \right] \tag{49}$$

$$I_{\lambda\lambda} = -E\left[\frac{\partial^2}{\partial \lambda^2} \log L(\alpha, \lambda \mid \beta, \beta_1, h_1, h_2, \theta) \right] \tag{50}$$

After determining the ML estimates of α and λ, we proceed by treating the estimate of α as known and use it to estimate the remaining parameters of the proposed model.

The above methodology was applied to all the birth distributions presented in Tables 1 – 4. Using the above approach, the parameter α was also estimated for the birth distributions presented in Tables 3 and 4 to verify the assumption that the surveyed populations consisted exclusively of eligible (i.e., fecund) females aged 25–29 years. For each birth distribution presented in Tables 1 – 4, the ML estimates of α and λ were obtained for every combination of h_1, h_2, and θ using equation (45). Figures 2–5 present the ML estimates of α for the birth distributions presented in Tables 1 – 4, respectively. The figures indicate that the estimated value of α decreases only marginally as h_2 increases from 1.0 to 2.0 years. On average, the estimated value of α decreases by approximately 0.0953, 0.0932, 0.0178, and 0.0016 as h_2 increases from 1.0 to 2.0 for the birth distributions presented in Tables 1 – 4, respectively, suggesting that the estimated value of α is only moderately sensitive to changes in h_2. Furthermore, the estimated values of α exhibit only negligible variation and show a slight decreasing trend with increasing values of h_1 and θ. Figures 4 and 5 show that, for the birth distributions presented in Tables 3 and 4, the estimated value of α remains approximately equal to 1.00 across all combinations of h_1, h_2, and θ. These findings provide further evidence that the surveyed populations corresponding to Tables 3 and 4 consisted exclusively of eligible (i.e., fecund) females aged 25–29 years. Accordingly, the ML estimates of α were subsequently treated as known constants and were used, together with the specified combinations of h_1, h_2, and θ, to estimate β, β_1, and λ under the proposed model. Now, considering the known values of the parameters α, θ, h_1, and h_2, we can obtain the ML estimates of the remaining parameters β, β_1, and λ. The likelihood function can be written as follows:

$$L(\beta,\beta_1,\lambda \mid \alpha,\theta,h_1,h_2) = \frac{N!}{\prod_{r=0}^{n} N_r!}\prod_{r=0}^{n}\left\{P\big(X(T)=r\big)\right\}^{N_r} \tag{51}$$

Suppose $(\hat{\beta},\hat{\beta}_1,\hat{\lambda})$ denote the ML estimates of (β,β_1,λ); then $(\hat{\beta},\hat{\beta}_1,\hat{\lambda})$ are the solutions of the following equations:

$$\left(\frac{\partial \log L(\beta,\beta_1,\lambda)}{\partial\beta}, \frac{\partial \log L(\beta,\beta_1,\lambda)}{\partial\beta_1}, \frac{\partial \log L(\beta,\beta_1,\lambda)}{\partial\lambda}\right) = (0,0,0) \tag{52}$$

Furthermore, the asymptotic covariance matrix, $\mathbf{W}$, of ML estimators is given by

$$\mathbf{W} = \begin{bmatrix} V(\hat{\beta}) & \mathrm{Cov}(\hat{\beta},\hat{\beta}_1) & \mathrm{Cov}(\hat{\beta},\hat{\lambda}) \\ \mathrm{Cov}(\hat{\beta},\hat{\beta}_1) & V(\hat{\beta}_1) & \mathrm{Cov}(\hat{\beta}_1,\hat{\lambda}) \\ \mathrm{Cov}(\hat{\beta},\hat{\lambda}) & \mathrm{Cov}(\hat{\beta}_1,\hat{\lambda}) & V(\hat{\lambda}) \end{bmatrix} = \mathbf{I}^{-1}, \tag{53}$$

where

$$\mathbf{I} = \begin{bmatrix} I_{\beta\beta} & I_{\beta\beta_1} & I_{\beta\lambda} \\ I_{\beta\beta_1} & I_{\beta_1\beta_1} & I_{\beta_1\lambda} \\ I_{\beta\lambda} & I_{\beta_1\lambda} & I_{\lambda\lambda} \end{bmatrix}\Bigg|_{\beta=\hat{\beta},\,\beta_1=\hat{\beta}_1,\,\lambda=\hat{\lambda}} \tag{54}$$

is the Fisher information matrix evaluated at the ML estimates.

The elements of the Fisher information matrix are given by

$$I_{\beta\beta} = -E\left[\frac{\partial^2}{\partial\beta^2}\log L(\beta,\beta_1,\lambda \mid \alpha,\theta,h_1,h_2)\right] \tag{55}$$

$$I_{\beta\beta_1} = -E\left[\frac{\partial^2}{\partial\beta\,\partial\beta_1}\log L(\beta,\beta_1,\lambda \mid \alpha,\theta,h_1,h_2)\right] \tag{56}$$

$$I_{\beta\lambda} = -E\left[\frac{\partial^2}{\partial\beta\,\partial\lambda}\log L(\beta,\beta_1,\lambda \mid \alpha,\theta,h_1,h_2)\right] \tag{57}$$

$$I_{\beta_1\beta_1} = -E\left[\frac{\partial^2}{\partial\beta_1^2}\log L(\beta,\beta_1,\lambda \mid \alpha,\theta,h_1,h_2)\right] \tag{58}$$

$$I_{\beta_1\lambda} = -E\left[\frac{\partial^2}{\partial\beta_1\,\partial\lambda}\log L(\beta,\beta_1,\lambda \mid \alpha,\theta,h_1,h_2)\right] \tag{59}$$

$$I_{\lambda\lambda} = -E\left[\frac{\partial^2}{\partial\lambda^2}\log L(\beta,\beta_1,\lambda \mid \alpha,\theta,h_1,h_2)\right] \tag{60}$$

Subsequently, using equation (52), the ML estimates of β, β_1, and λ were obtained for each birth distribution presented in Tables 1 – 4 by treating α, h_1, h_2, and θ as known constants. The corresponding estimates of β, β_1, and λ for all combinations of h_1, h_2, and θ are presented in Figures 2–5. It is evident from Figures 2–5 that, for fixed values of h_1 and θ, the estimates of β and β_1 decrease, whereas the estimate of λ increases, with increasing h_2. The decrease in the estimate of β indicates that, as the period of non-susceptibility associated with a complete conception becomes longer, a smaller proportion of fecund females remain continuously exposed to the risk of conception throughout the observational period. Consequently, a larger proportion of fecund females belong to Group B, comprising those who are temporarily non-susceptible at the beginning of the observational period. Furthermore, the decrease in the estimate of β_1 can be attributed to the increase in the duration of postpartum amenorrhoea associated with a complete conception. As h_2 increases, females who have experienced a complete conception remain in the postpartum non-susceptible state for a longer period and are therefore more likely to be classified into Group B_2 at the beginning of the observational period. Since the duration of pregnancy remains unchanged, the number of females observed in Group B_1 at the beginning of the observational period does not increase correspondingly. Consequently, the proportion of Group B_2 increases relative to that of Group B_1, resulting in a decrease in the estimate of β_1. The corresponding increase in the estimate of λ indicates that the smaller proportion of females who remain exposed to the risk of conception throughout the observational period experience a higher risk of conception. This finding is consistent with that of Singh and Bhattacharya [16], who demonstrated that, as h_1 and h_2 increase, the proportion of fecund females remaining exposed to the risk of conception during the observational period decreases slightly, whereas the corresponding conception rate among those females increases. Furthermore, Figures 2–5 show that, for fixed values of h_1 and h_2, the estimates of β, β_1, and λ decrease slightly with increasing θ. Likewise, for fixed values of h_2 and θ, these estimates also decrease slightly with increasing h_1. These results indicate that variations in the probability of complete conception and the duration of non-susceptibility associated with an incomplete conception have only a limited influence on the estimates of the remaining biological parameters of the model.

After obtaining the ML estimates of the parameters α, β, β_1, and λ for each combination of h_1, h_2, and θ, the adequacy of the proposed model was assessed by evaluating its fit to the observed birth distributions. The expected frequencies were computed using the corresponding ML estimates, and the goodness of fit was evaluated using the chi-square statistic. Figures 6–9 display the variation in the chi-square statistic with h_2 for different combinations of h_1 and θ. The table provided in the middle-right corner of each figure identifies, for every combination of (h_1, θ), the value of h_2 at which the chi-square statistic attains its minimum. Furthermore, the left panels of Figures 6–9 show the variation in the chi-square statistic with h_2 for fixed values of h_1, with separate curves corresponding to different values of θ, whereas the right panels show the corresponding variation for fixed values of θ, with separate curves corresponding to different values of h_1. In all four figures, the curves are nearly superimposed, indicating that moderate variations in h_1 and θ have only a negligible influence on the chi-square statistic for a fixed value of h_2. The chi-square statistic initially decreases with increasing h_2, attains a minimum at a value that depends on the birth distribution under consideration, and subsequently increases with further increases in h_2. This pattern indicates that the fit of the proposed model is primarily influenced by the choice of h_2, whereas the effects of h_1 and θ remain comparatively small over the ranges considered.

Moreover, the inset tables accompanying the left-hand panels of Figure 6 show that, for each fixed value of h_1, the minimum chi-square statistic occurs as θ decreases and h_2 increases. Considering this trend together with the observed U-shaped pattern of the chi-square curves, it may be inferred that the chi-square statistic initially decreases with decreasing θ and increasing h_2, attains a minimum value, and subsequently increases. Similarly, the inset tables accompanying the right-hand panels show that, for each fixed value of θ, the minimum chi-square statistic occurs as h_1 increases and h_2 decreases, indicating that the chi-square statistic decreases up to a particular combination of h_1 and h_2, reaches a minimum, and subsequently increases. Among all parameter combinations considered, the smallest chi-square statistic is 0.022146, which is obtained at $h_1 = 0.6$, $h_2 = 1.20$, and $\theta = 0.80$. These values may therefore be regarded as the indirect estimates of the non-susceptibility periods associated with incomplete and complete conceptions and the probability of complete conception, respectively. Since the minimum occurs at the boundary of the parameter space considered, it is plausible that a slightly larger value of h_1, a slightly smaller value of h_2, and a slightly smaller value of θ may yield an even smaller chi-square statistic. Accordingly, Column 5 of Table 1 presents the expected frequencies obtained using the ML estimates corresponding to $h_1 = 0.7$, $h_2 = 1.15$, and $\theta = 0.75$, for which the chi-square statistic increases to 0.095512481.

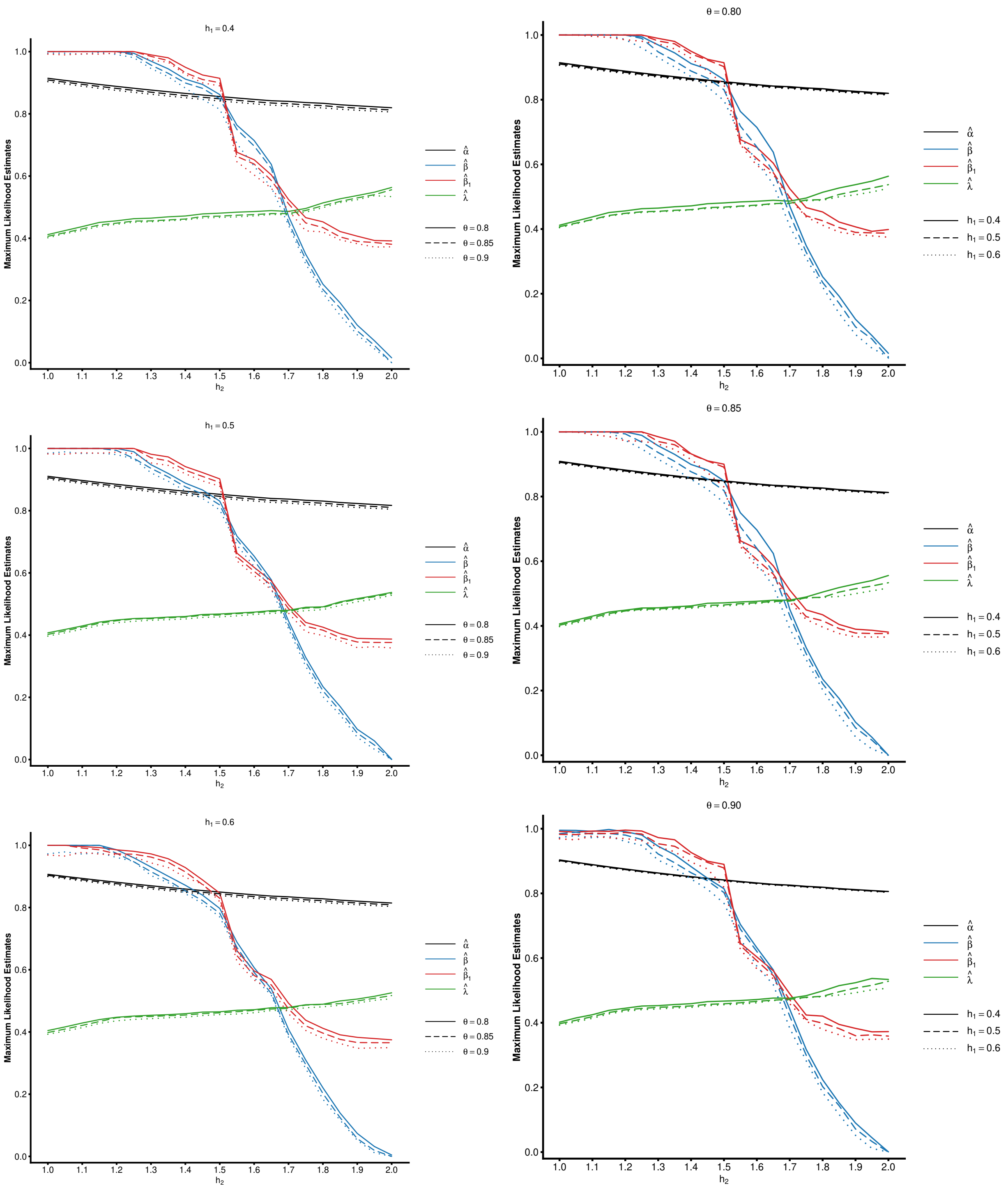


Figure 2. Maximum likelihood estimates of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\beta}_1$, and $\hat{\lambda}$ plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of urban females aged 25–29 years (irrespective of age at marriage) in Bihar.

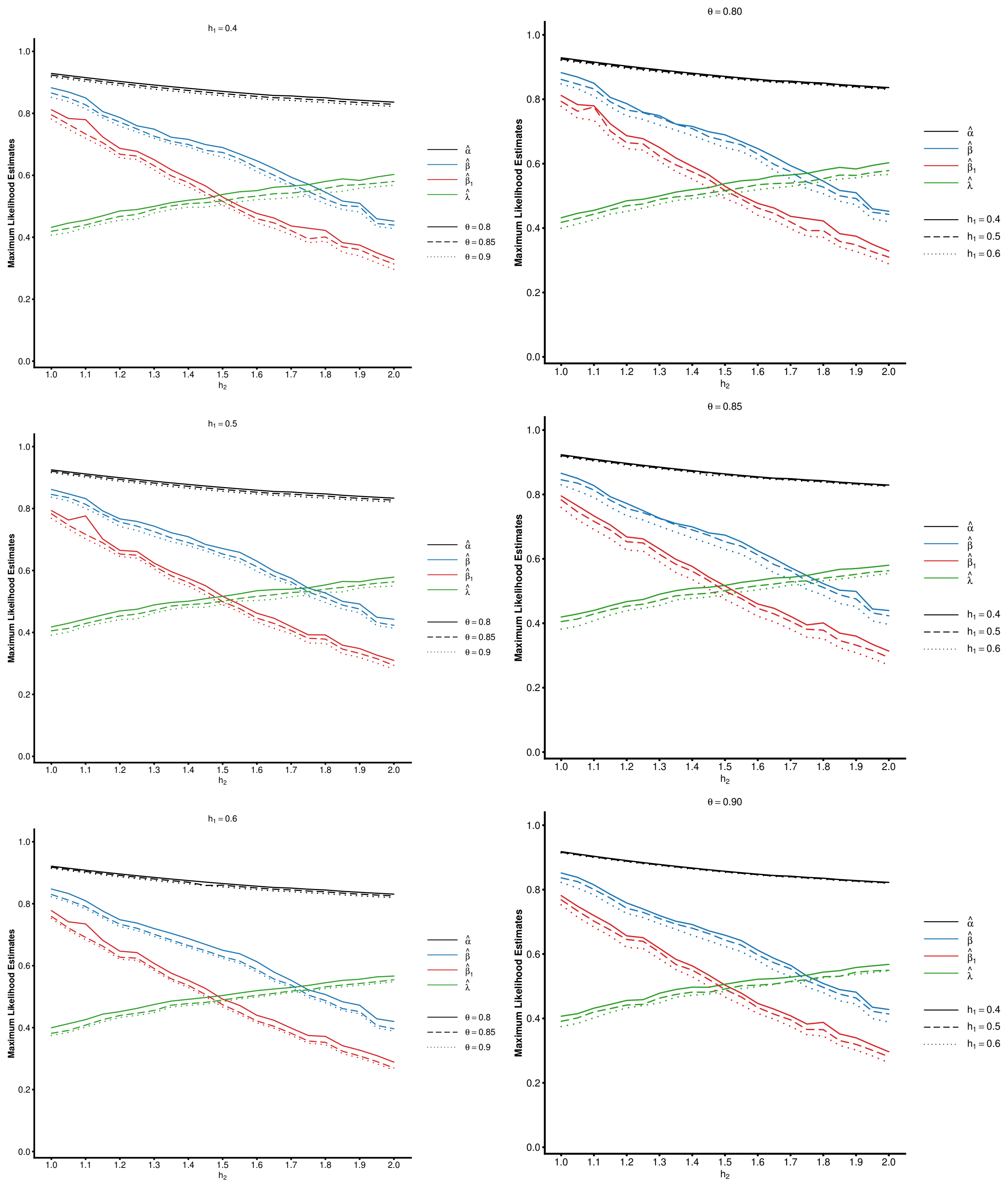


Figure 3. Maximum likelihood estimates of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\beta}_1$, and $\hat{\lambda}$ plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of urban females aged 25–29 years with age at marriage 16–20 years in Bihar.

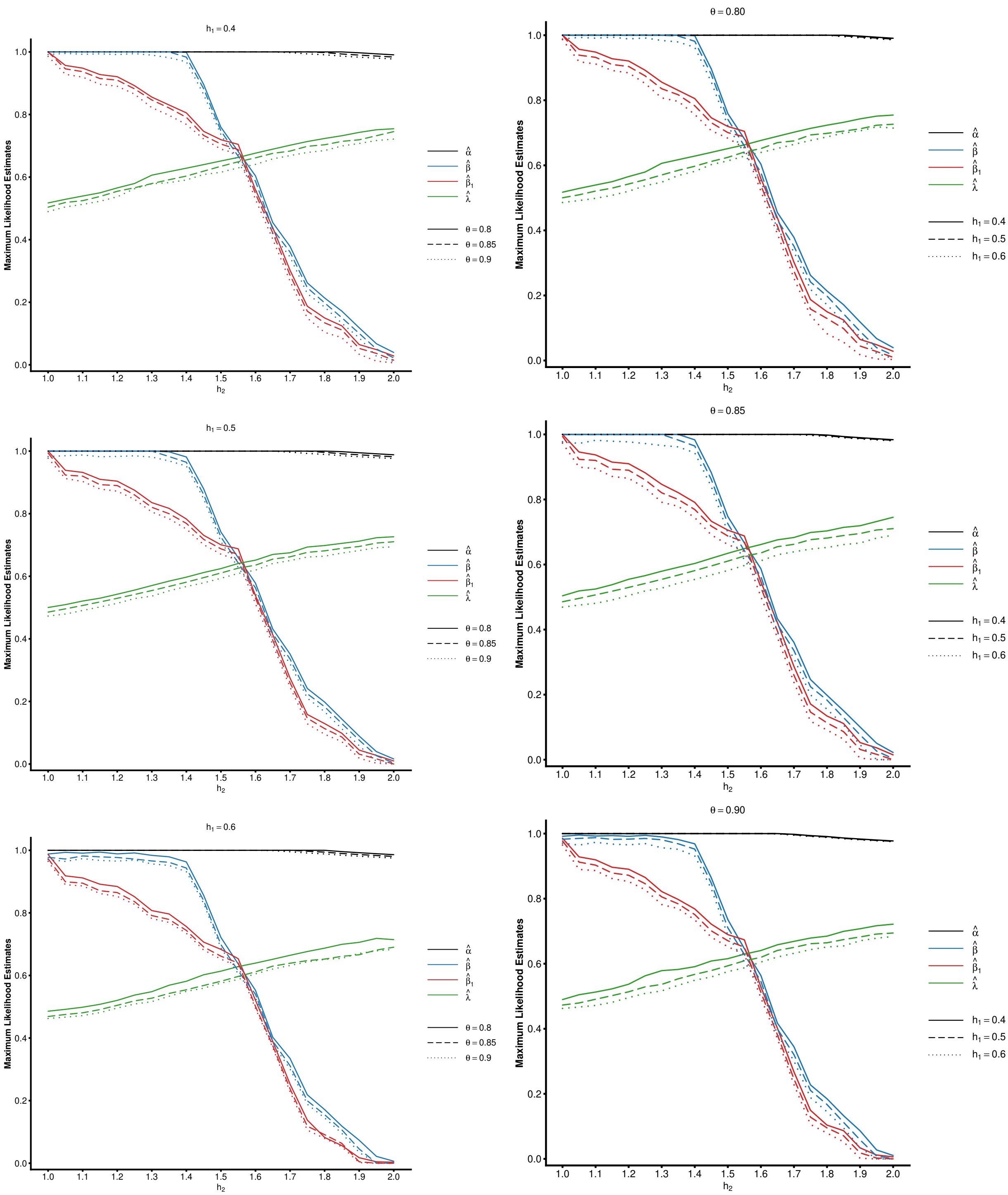


Figure 4. Maximum likelihood estimates of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\beta}_1$, and $\hat{\lambda}$ plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of females aged 25–29 years over the last 5 years.

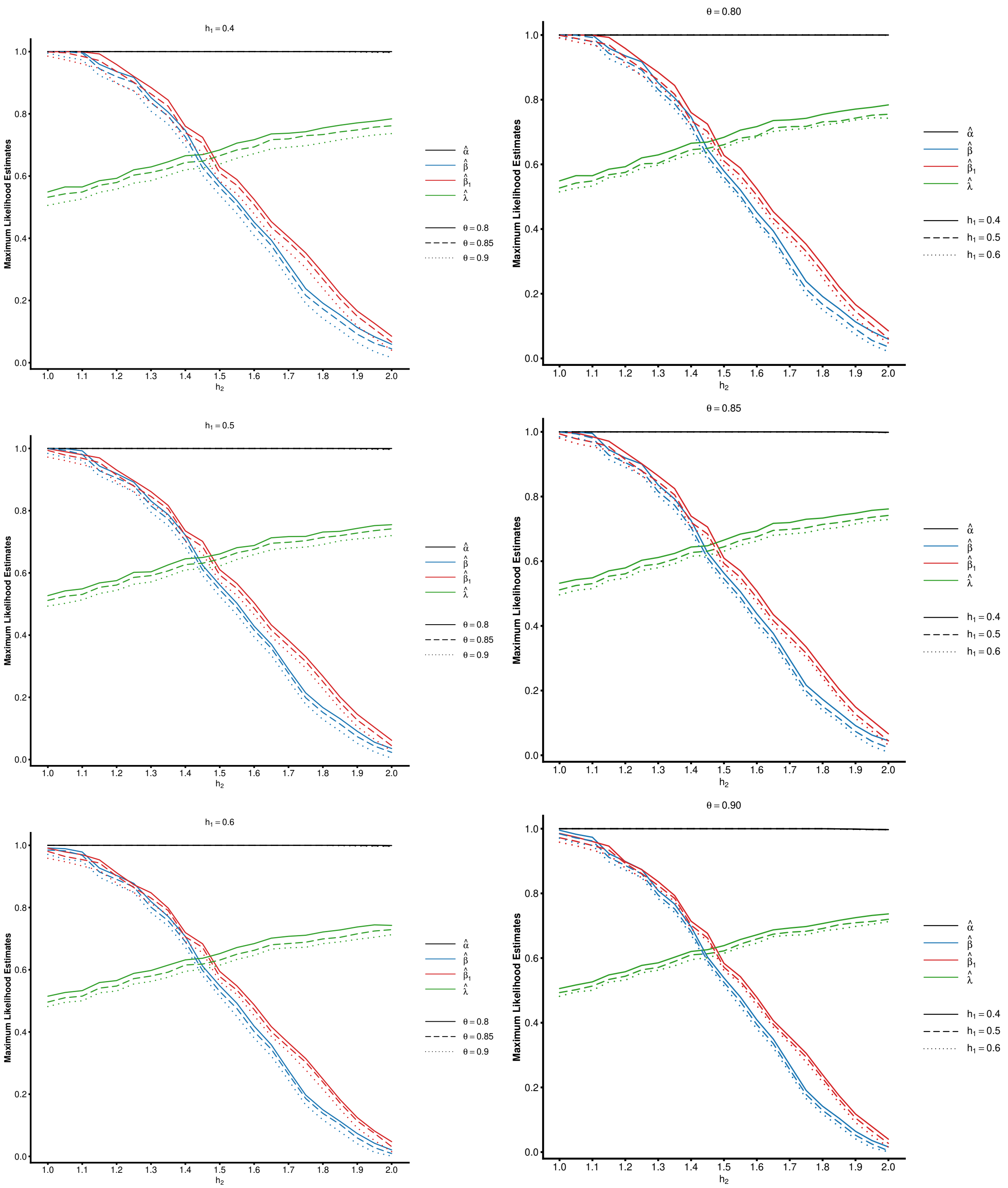


Figure 5. Maximum likelihood estimates of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\beta}_1$, and $\hat{\lambda}$ plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of females aged 25–29 years over the last 7 years.

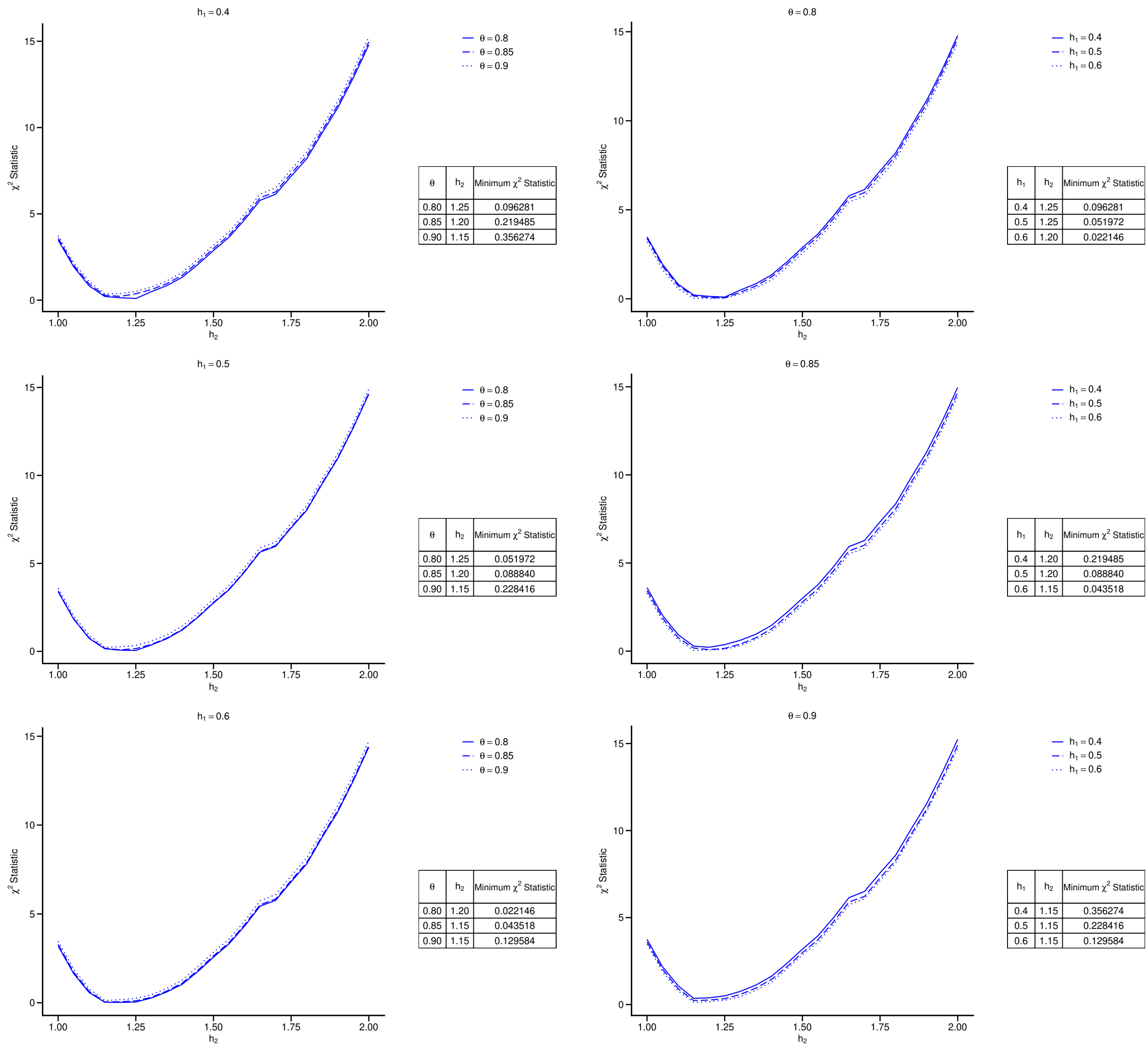


Figure 6. Chi-square statistic plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of urban females aged 25–29 years (irrespective of age at marriage) in Bihar.

These results suggest that females aged 25–29 years corresponding to the observed birth distribution of urban Bihar, irrespective of age at marriage, experience approximately 7.2 months of non-susceptibility following an incomplete conception and about 14.4 months following a complete conception, while the probability of a conception resulting in a live birth is approximately 0.80. A similar pattern is observed in Figure 7, where the smallest chi-square statistic among all parameter combinations is 0.0134820, corresponding to $h_1 = 0.6$, $h_2 = 1.15$, and $\theta = 0.80$. As in the previous case, the minimum occurs at the boundary of the parameter space considered, suggesting that a slightly larger value of h_1, a slightly smaller value of h_2, and a slightly smaller value of θ may yield an even smaller chi-square statistic. Accordingly, Column 5 of Table 2 presents the expected frequencies obtained using the ML estimates corresponding to $h_1 = 0.7$, $h_2 = 1.10$, and $\theta = 0.75$, for which the chi-square statistic increases to 0.0354. These results indicate that females aged 25–29 years in urban Bihar whose age at marriage is 16–20 years experience approximately 7.2 months of non-susceptibility following an incomplete conception and about 13.8 months following a complete conception, while the probability that a conception results in a live birth is

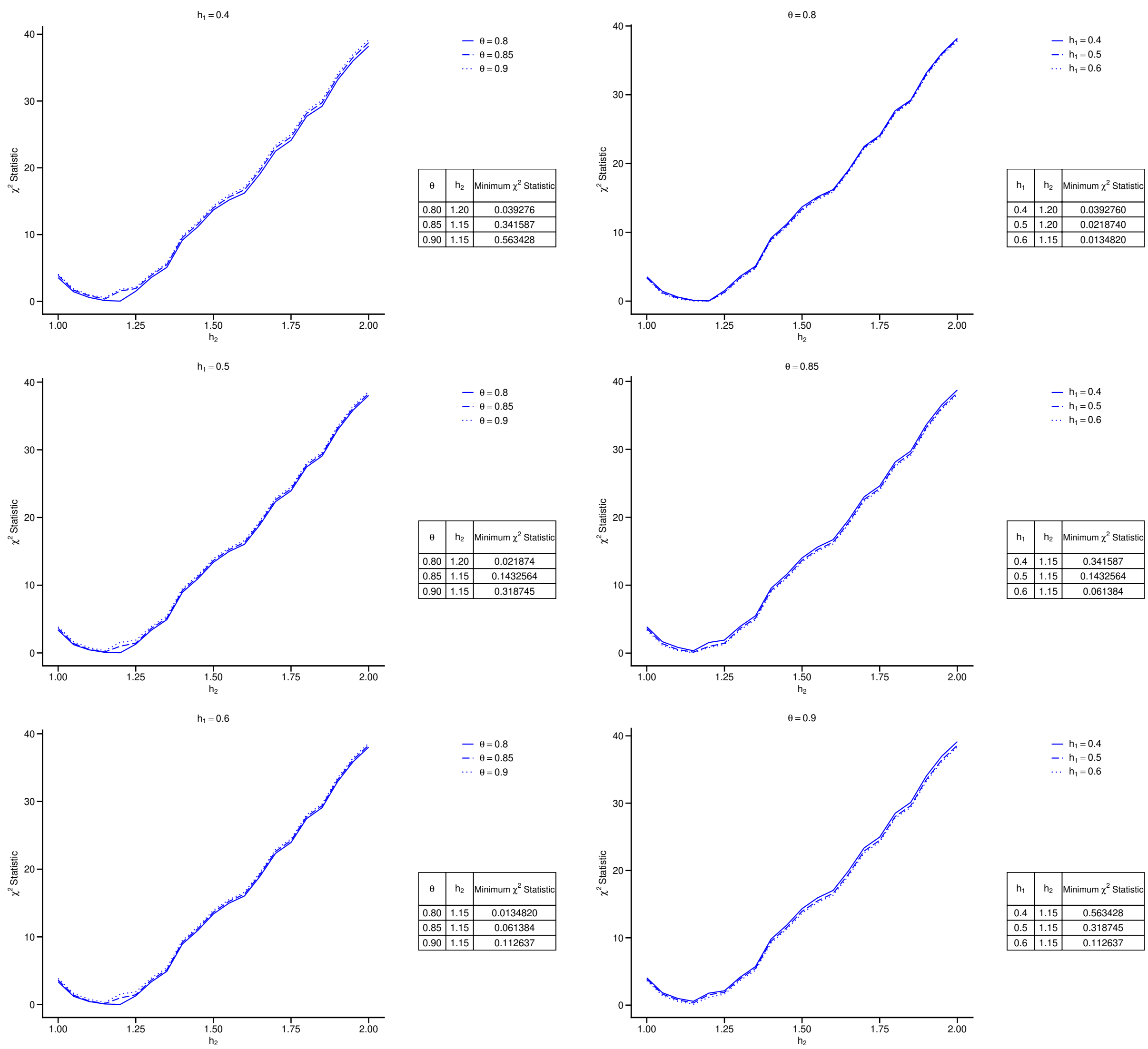


Figure 7. Chi-square statistic plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of urban females aged 25–29 years with age at marriage 16–20 years in Bihar.

approximately 0.80. Sharma *et al.* [26] estimated the mean duration of postpartum amenorrhea associated with complete conceptions, that is, live births, using life table techniques based on data from the NFHS-2 (1998–1999) for several major states in India. For Bihar, they estimated this duration to be approximately 13.44 months (1.12 years). Our indirect estimate of the non-susceptibility period associated with live births is consistent with the corresponding estimate reported by Sharma *et al.* [26], thereby providing further evidence for the adequacy of the proposed model. It should also be noted that, although NFHS-5 reports a live birth proportion of approximately 0.872, a direct comparison is not appropriate because the estimated parameter θ represents the probability of a conception ending in a live birth, whereas the NFHS-5 estimate is the observed proportion of live births in the population. Furthermore, survey data may be affected by under-reporting of fetal losses, including miscarriages and induced abortions, which may partly account for the observed difference.

Furthermore, Figure 8 shows that the smallest chi-square statistic among all parameter combinations is 0.082516, corresponding to $h_1 = 0.6$, $h_2 = 1.70$, and $\theta = 0.80$. It is evident from Figure 8 that the value of h_2 corresponding

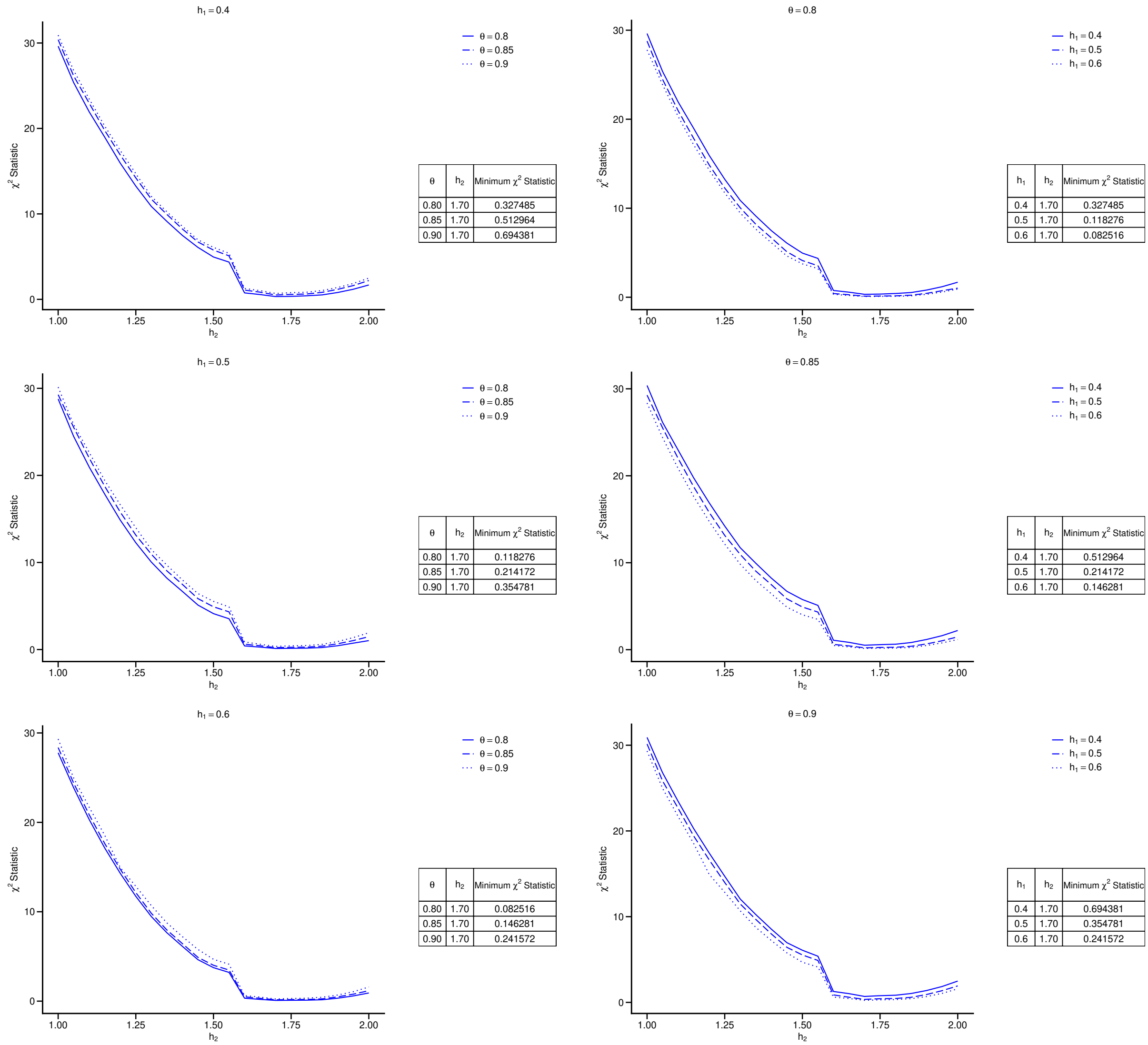


Figure 8. Chi-square statistic plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of females aged 25–29 years over the last 5 years.

to the minimum chi-square statistic remains constant at 1.70 over the range of h_1 and θ considered. However, the overall pattern in the location of the minimum chi-square statistic is consistent with that observed in Figures 6 and 7. Therefore, it is reasonable to expect that a slightly larger value of h_1, together with a slightly smaller value of θ, while keeping $h_2 = 1.70$, may yield an even smaller chi-square statistic. Accordingly, Column 6 of Table 3 presents the expected frequencies obtained using the ML estimates corresponding to $h_1 = 0.7$, $h_2 = 1.70$, and $\theta = 0.75$, for which the chi-square statistic is further reduced to 0.038128. Since this parameter combination yields a smaller chi-square statistic, a further extrapolation in the same direction was considered. Accordingly, Column 7 of Table 3 presents the expected frequencies obtained using the ML estimates corresponding to $h_1 = 0.8$, $h_2 = 1.70$, and $\theta = 0.70$, for which the chi-square statistic increases to 0.20253. These results indicate that females aged 25–29 years represented by the observed birth distribution in Table 3 experience approximately 8.4 months of non-susceptibility following an incomplete conception and about 20.4 months following a complete conception, while the probability that a conception results in a live birth is approximately 0.75.

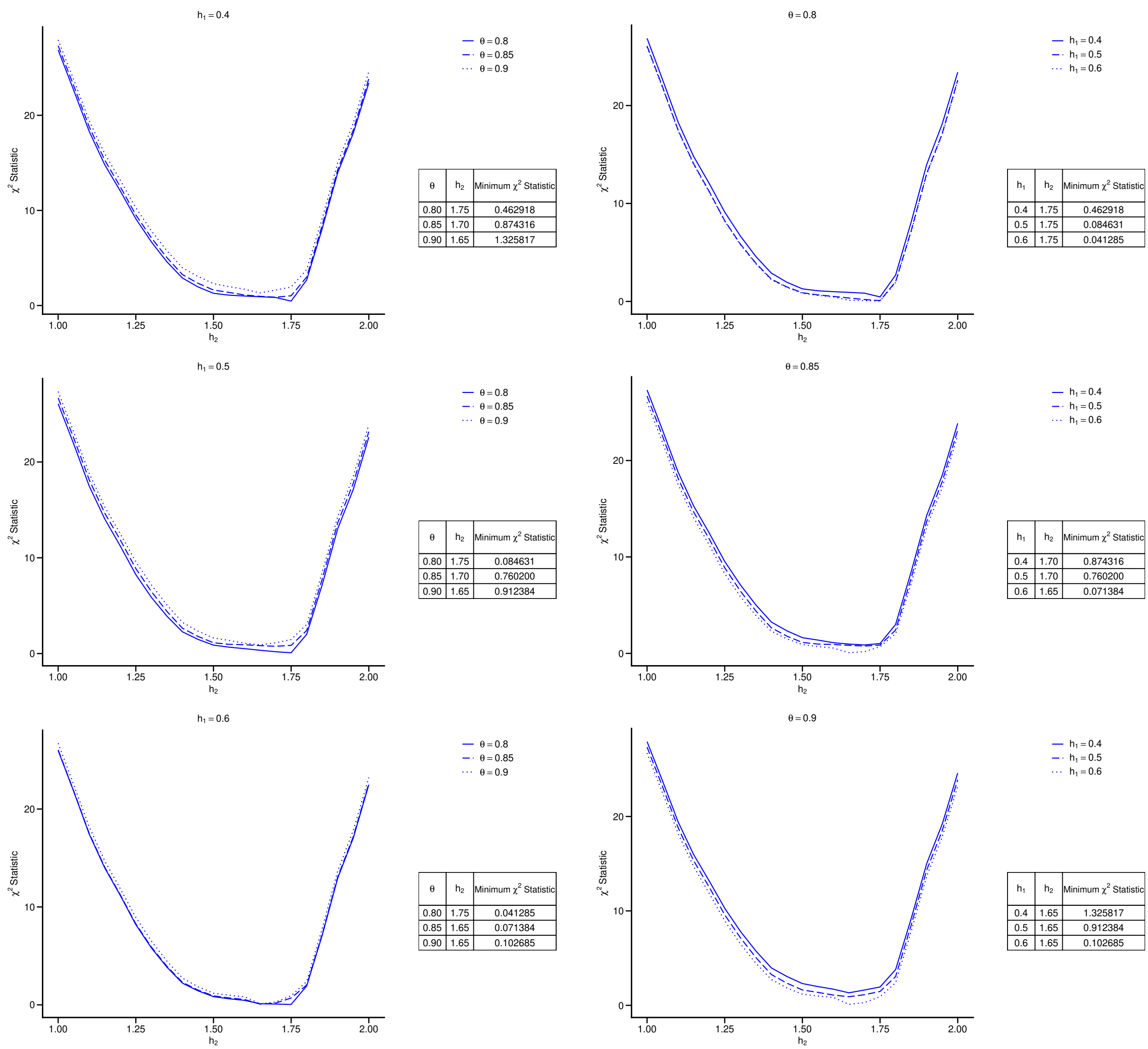


Figure 9. Chi-square statistic plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of females aged 25–29 years over the last 7 years.

A similar pattern is observed in Figure 9, where the smallest chi-square statistic among all parameter combinations is 0.041285, corresponding to $h_1 = 0.6$, $h_2 = 1.75$, and $\theta = 0.80$. Figure 9 further shows that, for the fixed value of $\theta = 0.80$, the value of h_2 corresponding to the minimum chi-square statistic remains constant at 1.75 for all values of h_1. For $\theta = 0.85$, however, the corresponding value of h_2 decreases from 1.70 to 1.65 as h_1 increases, whereas for $\theta = 0.90$, it remains constant at 1.65 for all values of h_1. Since the smallest chi-square statistic is attained at $\theta = 0.80$, a further extrapolation was carried out by increasing h_1 and decreasing θ while keeping h_2 fixed at 1.75. Accordingly, Column 6 of Table 4 presents the expected frequencies obtained using the ML estimates corresponding to $h_1 = 0.7$, $h_2 = 1.75$, and $\theta = 0.75$, for which the chi-square statistic increases to 0.30259. These results indicate that females aged 25–29 years represented by the observed birth distribution in Table 4 experience approximately 7.2 months of non-susceptibility following an incomplete conception and about 21 months following a complete conception, while the probability that a conception results in a live birth is approximately 0.80. Furthermore, the Akaike Information Criterion (AIC) and the Bayesian Information Criterion

Table 1. Distribution of the observed and expected number of births among females aged 25–29 years during the last 5 years of marriage, irrespective of age at marriage, in urban Bihar (NFHS-5, 2019–21)

Number of Births	Observed Number of Females	Expected Number of Females	Expected Number of Females	Expected Number of Females
		$\begin{pmatrix} h_1 = 0.60 \\ h_2 = 1.20 \\ \theta = 0.80 \end{pmatrix}$	$\begin{pmatrix} h_1 = 0.60 \\ h_2 = 1.20 \\ \theta = 0.80 \end{pmatrix}$	$\begin{pmatrix} h_1 = 0.70 \\ h_2 = 1.15 \\ \theta = 0.80 \end{pmatrix}$
(1)	(2)	(3)	(4)	(5)
0	310	310.714	308.522	310.195
1	475	475.811	476.392	473.186
2	461	460.809	459.428	462.255
3	136	135.023	135.525	137.230
4	11	10.643	13.133	10.134
Total	1393	1393	1393	1393
		$\hat{\alpha} = 0.8813$	$\hat{\alpha} = 0.8813$	$\hat{\alpha} = 0.8874$
		$\hat{\lambda} = 0.4473$	$\hat{\lambda} = 0.4464$	$\hat{\lambda} = 0.4398$
		$\hat{\beta} = 0.9857$	—	$\hat{\beta} = 0.9999$
		$\hat{\beta}_1 = 0.9855$	—	$\hat{\beta}_1 = 0.9927$
Chi-square		0.022146	0.364623	0.095512481
d.f.		1	2	1
p-value		0.881699	0.833342	0.757283
Log-Likelihood		-1856.501274	-1857.517735	-1856.642300
AIC		3719.00255	3719.03547	3719.28460
BIC		3734.72019	3729.51390	3735.002245
$V(\hat{\alpha})$		—	0.000220793	—
$V(\hat{\lambda})$		0.000153420	0.000206188	0.000168950
$V(\hat{\beta})$		0.001483720	—	0.001562310
$V(\hat{\beta}_1)$		0.001157490	—	0.001084760
Cov$(\hat{\alpha}, \hat{\lambda})$		—	-0.000112311	—
Cov$(\hat{\beta}, \hat{\beta}_1)$		0.000865510	—	0.000878500
Cov$(\hat{\beta}, \hat{\lambda})$		-0.000241900	—	-0.000254300
Cov$(\hat{\beta}_1, \hat{\lambda})$		-0.000176000	—	-0.000184000
Corr$(\hat{\alpha}, \hat{\lambda})$		—	-0.526379	—
Corr$(\hat{\beta}, \hat{\beta}_1)$		0.66044627	—	0.67482526
Corr$(\hat{\beta}, \hat{\lambda})$		-0.50692890	—	-0.49497545
Corr$(\hat{\beta}_1, \hat{\lambda})$		-0.41755560	—	-0.42980543

(BIC) were computed for different combinations of h_1, h_2, and θ. Figures 10–13 show the plots of the AIC and BIC against h_2 for different combinations of h_1 and θ. The tables included in the middle right corner of each figure identify, for every combination of (h_1, θ), the value of h_2 at which the AIC and BIC attain their minimum values. It is observed that the AIC and BIC yield identical results, which are consistent with the results of the chi-square goodness-of-fit test presented in Figures 6–9.

Therefore, the parameter combinations

$$(h_1, h_2, \theta) = (0.60, 1.20, 0.80),\ (0.60, 1.15, 0.80),\ (0.70, 1.70, 0.75),\ \text{and}\ (0.60, 1.75, 0.80)$$

are considered the indirect estimates of h_1, h_2, and θ corresponding to the observed birth distributions presented in Tables 1–4, respectively.

Furthermore, Column 3 of Table 1 presents the expected frequencies under the proposed model obtained using the ML estimates $\hat{\alpha} = 0.8813$, $\hat{\beta} = 0.9857$, $\hat{\beta}_1 = 0.9855$, and $\hat{\lambda} = 0.4473$, corresponding to $h_1 = 0.6$, $h_2 = 1.20$, and $\theta = 0.80$. The corresponding chi-square statistic, p-value, AIC, and BIC are also presented in Column 3. It is noteworthy that the estimates of $\hat{\beta}$ and $\hat{\beta}_1$ are very close to one. This suggests that the proposed model may be overparameterized for the observed birth distribution presented in Table 1. Therefore, we also considered a reduced form of the proposed model with $\beta = 1$. As discussed earlier, this reduced model is equivalent to the

Table 2. Distribution of the observed and expected number of births among females aged 25–29 years during the last 5 years of marriage, whose age at marriage was 16–20 years, in urban Bihar (NFHS-5, 2019–21)

Number of Births	Observed Number of Females	Expected Number of Females $\begin{pmatrix} h_1 = 0.60 \\ h_2 = 1.15 \\ \theta = 0.80 \end{pmatrix}$	Expected Number of Females $\begin{pmatrix} h_1 = 0.60 \\ h_2 = 1.15 \\ \theta = 0.80 \end{pmatrix}$	Expected Number of Females $\begin{pmatrix} h_1 = 0.70 \\ h_2 = 1.10 \\ \theta = 0.75 \end{pmatrix}$
(1)	(2)	(3)	(4)	(5)
0	103	103.998	103.000	102.766
1	177	176.501	173.117	178.331
2	161	160.749	167.586	161.079
3	54	53.852	52.8637	53.025
4	6	5.900	4.43318	5.799
Total	501	501	501	501
		$\hat{\alpha} = 0.9020$	$\hat{\alpha} = 0.9020$	$\hat{\alpha} = 0.9118$
		$\hat{\lambda} = 0.4436$	$\hat{\lambda} = 0.4424$	$\hat{\lambda} = 0.4388$
		$\hat{\beta} = 0.7764$	—	$\hat{\beta} = 0.8215$
		$\hat{\beta}_1 = 0.6819$	—	$\hat{\beta}_1 = 0.7375$
Chi-square		0.013482	0.984482	0.0354
d.f.		1	2	1
p-value		0.907564	0.611255	0.85076
Log-Likelihood		-672.852741	-677.2333901	-673.092741
AIC		1351.705482	1358.46678	1352.1855
BIC		1364.3553	1366.89999	1364.835
$V(\hat{\alpha})$		—	0.000594297	—
$V(\hat{\lambda})$		0.00028765	0.000538453	0.00030127
$V(\hat{\beta})$		0.00184567	—	0.00191234
$V(\hat{\beta}_1)$		0.00245678	—	0.00238951
Cov$(\hat{\alpha}, \hat{\lambda})$		—	-0.000306094	—
Cov$(\hat{\beta}, \hat{\beta}_1)$		0.00130394	—	0.0012783
Cov$(\hat{\beta}, \hat{\lambda})$		-0.00037331	—	-0.0003992
Cov$(\hat{\beta}_1, \hat{\lambda})$		-0.00036885	—	-0.0003835
Corr$(\hat{\alpha}, \hat{\lambda})$		—	-0.541102	—
Corr$(\hat{\beta}, \hat{\beta}_1)$		0.612346793	—	0.597992598
Corr$(\hat{\beta}, \hat{\lambda})$		-0.512342529	—	-0.525932531
Corr$(\hat{\beta}_1, \hat{\lambda})$		-0.438767513	—	-0.451994342

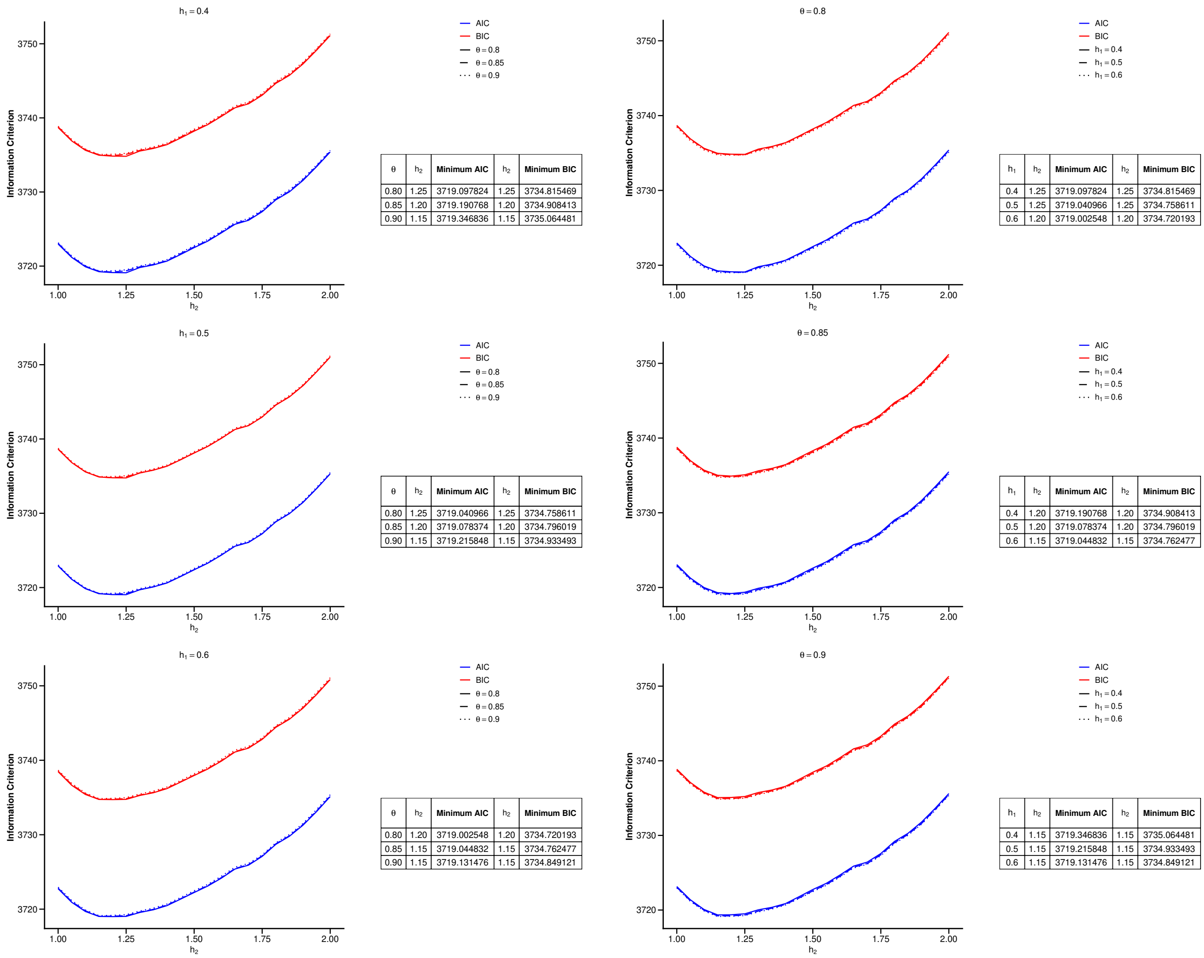
model proposed by Singh and Yadava [18], with the inclusion of the parameter α. Using the same values of $h_1 = 0.60$, $h_2 = 1.20$, and $\theta = 0.80$, the ML estimates under the reduced model are $\hat{\alpha} = 0.8813$ and $\hat{\lambda} = 0.4464$. It may be noted that these estimates had already been obtained in the estimation procedure for β, β_1, and λ in our proposed model. The corresponding expected frequencies are presented in Column 4 of Table 1. A comparison of the chi-square statistic, p-value, AIC, and BIC indicates that the proposed model provides a marginally better fit to the observed distribution than the reduced model with $\beta = 1$. However, since the differences in these goodness-of-fit measures are very small, the evidence also suggests that the proposed model may be overparameterized for this particular birth distribution. It is also important to note that the observed distribution presented in Table 1 includes females aged 25–29 years irrespective of their age at marriage. Consequently, this distribution may be less suitable for assessing the performance of the proposed model. Considering this limitation, we restricted the observed birth distribution in Table 1 to females aged 25–29 years whose age at marriage was 16–20 years. The resulting distribution of births is presented in Table 2. Column 3 of Table 2 presents the expected frequencies under the proposed model obtained using the ML estimates $\hat{\alpha} = 0.9020$, $\hat{\beta} = 0.7764$, $\hat{\beta}_1 = 0.6819$, and $\hat{\lambda} = 0.4436$, corresponding to $(h_1, h_2, \theta) = (0.60, 1.15, 0.80)$. The associated chi-square statistic, p-value, AIC, and BIC are also presented in Column 3. Unlike the estimates in Table 1, the estimates of $\hat{\beta}$ and $\hat{\beta}_1$ are substantially smaller than one, indicating that the additional parameters contribute significantly to the proposed model. To further

Table 3. Distribution of observed and expected number of births to fecund females aged 25–29 years over the last 5 years

Number of Births	Observed Number of Females	Singh *et al.* [22]	Singh and Yadava [18]	Proposed Model			
		Expected Number of Females	Expected Number of Females	Expected Number of Females $\begin{pmatrix} h_1 = 0.60 \\ h_2 = 1.70 \\ \theta = 0.80 \end{pmatrix}$	Expected Number of Females $\begin{pmatrix} h_1 = 0.70 \\ h_2 = 1.70 \\ \theta = 0.75 \end{pmatrix}$	Expected Number of Females $\begin{pmatrix} h_1 = 0.80 \\ h_2 = 1.70 \\ \theta = 0.70 \end{pmatrix}$	Expected Number of Females $\begin{pmatrix} h_1 = 0.70 \\ h_2 = 1.70 \\ \theta = 0.75 \end{pmatrix}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	20	22.8	25.8	21.234	20.282	19.139	20.736
1	155	150.6	150.4	155.191	155.973	155.347	153.458
2	200	199.7	189.2	198.889	199.617	199.441	200.998
3	45	49.2	54.9	44.614	44.702	46.725	42.012
4 and above	5	2.7	4.7	5.072	4.656	4.348	7.796
Total	425	425	425	425	425	425	425
		$\hat{\lambda} = 0.65$	$\hat{\lambda} = 0.605$	$\hat{\alpha} = 1$	$\hat{\alpha} = 1$	$\hat{\alpha} = 1$	$\hat{\alpha} = 1$
		$\hat{\theta}_1 = 0.15$	—	$\hat{\lambda} = 0.6644$	$\hat{\lambda} = 0.6651$	$\hat{\lambda} = 0.6613$	$\hat{\lambda} = 0.6529$
				$\hat{\beta} = 0.3348$	$\hat{\beta} = 0.3349$	$\hat{\beta} = 0.3218$	—
				$\hat{\beta}_1 = 0.2528$	$\hat{\beta}_1 = 0.2488$	$\hat{\beta}_1 = 0.2369$	—
Chi-square		2.7907	3.8654	0.082516	0.038128	0.20253	1.26186
d.f.		2	3	1	1	1	2
p-value		0.2477	0.2764	0.773916	0.845186	0.652686	0.532097
Log-Likelihood		—	—	-491.672	-490.256	-492.356	-495.5464
AIC		—	—	989.344282	986.512	990.712	995.0928
BIC		—	—	1001.50055	998.6683	1002.868	1003.197
$V(\hat{\alpha})$		—	—	—	—	—	0.000212447
$V(\hat{\lambda})$		—	—	0.00085691	0.00091374	0.00096852	0.00113096
$V(\hat{\beta})$		—	—	0.00176453	0.00182347	0.00189582	—
$V(\hat{\beta}_1)$		—	—	0.00193826	0.00201258	0.00208649	—
Cov$(\hat{\alpha}, \hat{\lambda})$		—	—	—	—	—	-0.00026602
Cov$(\hat{\beta}, \hat{\beta}_1)$		—	—	0.00105844	0.00109192	0.00111764	—
Cov$(\hat{\beta}, \hat{\lambda})$		—	—	-0.00051974	-0.00054768	-0.00057631	—
Cov$(\hat{\beta}_1, \hat{\lambda})$		—	—	-0.00048728	-0.00051986	-0.00054817	—
Corr$(\hat{\alpha}, \hat{\lambda})$		—	—	—	—	—	-0.542713
Corr$(\hat{\beta}, \hat{\beta}_1)$		—	—	0.572328937	0.569986977	0.561946839	—
Corr$(\hat{\beta}, \hat{\lambda})$		—	—	-0.42267252	-0.424293376	-0.425307762	—
Corr$(\hat{\beta}_1, \hat{\lambda})$		—	—	-0.378098509	-0.383352683	-0.385614018	—

Table 4. Distribution of observed and expected number of births to fecund females aged 25–29 years over the last 7 years

2*Number of Births	2*Observed Number of Females	Singh *et al.* [22]	Singh and Yadava [18]	Proposed Model		
		Expected Number of Females	Expected Number of Females	Expected Number of Females	Expected Number of Females	Expected Number of Females
				$\begin{pmatrix} h_1 = 0.60 \\ h_2 = 1.75 \\ \theta = 0.80 \end{pmatrix}$	$\begin{pmatrix} h_1 = 0.70 \\ h_2 = 1.75 \\ \theta = 0.75 \end{pmatrix}$	$\begin{pmatrix} h_1 = 0.60 \\ h_2 = 1.75 \\ \theta = 0.80 \end{pmatrix}$
(1)	(2)	(3)	(4)	(5)	(6)	(7)
0	3	4.3	6.9	3.359	3.931	4.4527
1	58	52.8	61.7	57.790	56.069	53.5321
2	166	171.8	162.7	165.702	166.519	171.552
3	151	155.0	135.8	150.917	150.792	149.244
4 and above	34	28.1	44.9	34.232	34.689	33.2193
Total	412	412	412	412	412	412
		$\hat{\lambda} = 0.705$	$\hat{\lambda} = 0.620$	$\hat{\alpha} = 1$	$\hat{\alpha} = 1$	$\hat{\alpha} = 1$
		$\hat{\theta}_1 = 0.09$	—	$\hat{\lambda} = 0.7108$	$\hat{\lambda} = 0.7037$	$\hat{\lambda} = 0.7017$
				$\hat{\beta} = 0.1973$	$\hat{\beta} = 0.1899$	—
				$\hat{\beta}_1 = 0.3146$	$\hat{\beta}_1 = 0.3009$	—
Chi-square		2.4429	6.8406	0.041285	0.30259	1.06554
d.f.		2	3	1	1	2
p-value		0.2948	0.0772	0.838989	0.582263	0.586977
Log-Likelihood		—	—	-514.015862	-515.913586	-518.478
AIC		—	—	1034.03172	1037.827	1040.956
BIC		—	—	1046.09479	1049.8902	1048.998
$V(\hat{\alpha})$		—	—	—	—	0.0000422949
$V(\hat{\lambda})$		—	—	0.00043762	0.00048137	0.000714784
$V(\hat{\beta})$		—	—	0.00048231	0.00051384	—
$V(\hat{\beta}_1)$		—	—	0.00071894	0.00075492	—
$\mathrm{Cov}(\hat{\alpha}, \hat{\lambda})$		—	—	—	—	-0.0000473939
$\mathrm{Cov}(\hat{\beta}, \hat{\beta}_1)$		—	—	0.00033755	0.00036727	—
$\mathrm{Cov}(\hat{\beta}, \hat{\lambda})$		—	—	-0.0002105	-0.0002247	—
$\mathrm{Cov}(\hat{\beta}_1, \hat{\lambda})$		—	—	-0.0001847	-0.0001968	—
$\mathrm{Corr}(\hat{\alpha}, \hat{\lambda})$		—	—	—	—	-0.272578
$\mathrm{Corr}(\hat{\beta}, \hat{\beta}_1)$		—	—	0.57322966	0.5896863	—
$\mathrm{Corr}(\hat{\beta}, \hat{\lambda})$		—	—	-0.458141	-0.451703	—
$\mathrm{Corr}(\hat{\beta}_1, \hat{\lambda})$		—	—	-0.3293207	—	—



θ	h_2	Minimum AIC	h_2	Minimum BIC
0.80	1.25	3719.097824	1.25	3734.815469
0.85	1.20	3719.190768	1.20	3734.908413
0.90	1.15	3719.346836	1.15	3735.064481

h_1	h_2	Minimum AIC	h_2	Minimum BIC
0.4	1.25	3719.097824	1.25	3734.815469
0.5	1.25	3719.040966	1.25	3734.758611
0.6	1.20	3719.002548	1.20	3734.720193

θ	h_2	Minimum AIC	h_2	Minimum BIC
0.80	1.25	3719.040966	1.25	3734.758611
0.85	1.20	3719.078374	1.20	3734.796019
0.90	1.15	3719.215848	1.15	3734.933493

h_1	h_2	Minimum AIC	h_2	Minimum BIC
0.4	1.20	3719.190768	1.20	3734.908413
0.5	1.20	3719.078374	1.20	3734.796019
0.6	1.15	3719.044832	1.15	3734.762477

θ	h_2	Minimum AIC	h_2	Minimum BIC
0.80	1.20	3719.002548	1.20	3734.720193
0.85	1.15	3719.044832	1.15	3734.762477
0.90	1.15	3719.131476	1.15	3734.849121

h_1	h_2	Minimum AIC	h_2	Minimum BIC
0.4	1.15	3719.346836	1.15	3735.064481
0.5	1.15	3719.215848	1.15	3734.933493
0.6	1.15	3719.131476	1.15	3734.849121

Figure 10. AIC and BIC plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of urban females aged 25–29 years (irrespective of age at marriage) in Bihar.

evaluate the contribution of these parameters, we again considered the reduced model with $\beta = 1$. Using the same values of $(h_1, h_2, \theta) = (0.60, 1.15, 0.80)$, the ML estimates under the reduced model are $\hat{\alpha} = 0.9020$ and $\hat{\lambda} = 0.4424$. The expected frequencies obtained from the reduced model are presented in Column 4 of Table 2. A comparison of the chi-square statistic, p-value, AIC, and BIC indicates that the proposed model provides a substantially better fit to the observed distribution than the reduced model with $\beta = 1$. This suggests that the proposed model is not overparameterized for the equilibrium distribution of births presented in Table 2. Moreover, from another perspective, for certain birth distributions, estimates of β and β_1 approximately equal to 1 indicate that the proposed model is effectively overparameterized with respect to these parameters. Nevertheless, the proposed model provides a fit that is virtually identical to that of the corresponding reduced model. This suggests that the estimation procedure can identify situations in which the additional parameters do not contribute meaningfully to the model fit, thereby allowing the proposed model to reduce naturally to its simpler form. Such an adaptive property may be regarded as a practical advantage of the proposed model.

Furthermore, Column 6 of Table 3 presents the expected frequencies under the proposed model obtained using the ML estimates $\hat{\alpha} = 1.00$, $\hat{\beta} = 0.3349$, $\hat{\beta}_1 = 0.2488$, and $\hat{\lambda} = 0.6651$, corresponding to $(h_1, h_2, \theta) = (0.70, 1.70, 0.75)$, whereas Column 8 presents the expected frequencies under the reduced model with $\beta = 1$, obtained using the ML estimates $\hat{\alpha} = 1.00$ and $\hat{\lambda} = 0.6529$ for the same values of (h_1, h_2, θ). Similarly, Column 5

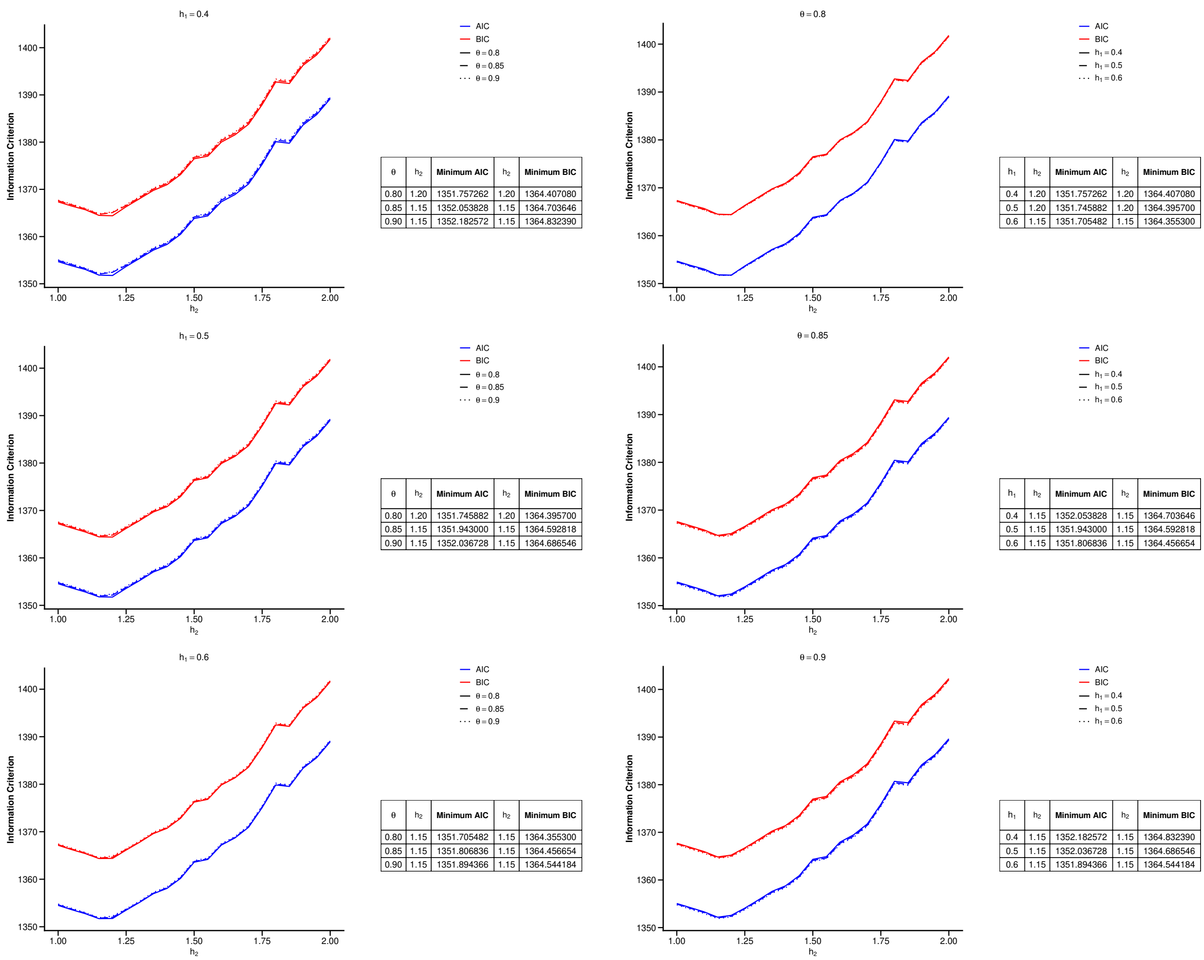
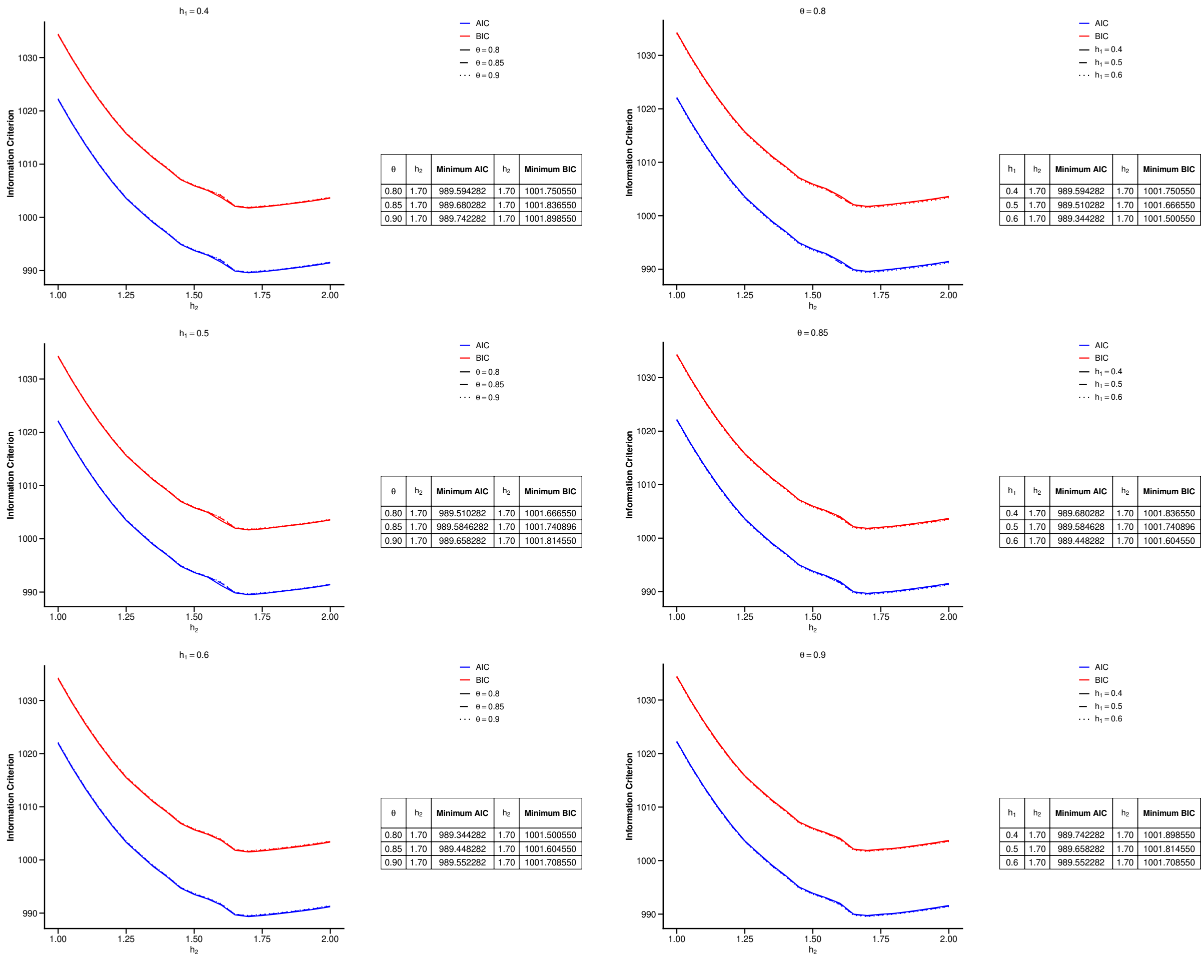


Figure 11. AIC and BIC plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of urban females aged 25–29 years with age at marriage 16–20 years in Bihar.

of Table 4 presents the expected frequencies under the proposed model obtained using the ML estimates $\hat{\alpha} = 1.00$, $\hat{\beta} = 0.1973$, $\hat{\beta}_1 = 0.3146$, and $\hat{\lambda} = 0.7108$, corresponding to $(h_1, h_2, \theta) = (0.60, 1.75, 0.80)$, whereas Column 7 presents the expected frequencies under the reduced model with $\beta = 1$, obtained using the ML estimates $\hat{\alpha} = 1.00$ and $\hat{\lambda} = 0.7017$ for the same values of (h_1, h_2, θ). A comparison of the associated chi-square statistics, p-values, AIC, and BIC indicates that the proposed model provides a substantially better fit than the reduced model with $\beta = 1$ for the birth distributions considered by Singh and Yadava [18] and Singh *et al.* [22]. Moreover, the p-values of the chi-square test reported in Columns (3), (4), and (6) of Table 3 and Columns (3), (4), and (5) of Table 4 indicate that the proposed model provides a better fit to both observed birth distributions than the models proposed by Singh and Yadava [18] and Singh *et al.* [22].

The asymptotic covariance matrices of the ML estimators under the proposed model and the reduced model with $\beta = 1$ were obtained for each observed birth distribution and are presented in the corresponding columns of Tables 1 – 4. Across all four birth distributions, the estimated variances of the ML estimators are relatively small, indicating that the parameter estimates are obtained with high precision. As discussed earlier, the expected frequencies corresponding to the ML estimates that provide the best fit under the proposed model are presented in Column 3 of Tables 1 and 2, Column 6 of Table 3, and Column 5 of Table 4. For the birth distribution presented in Table 1, it is seen from Column 3 that the corresponding 95% confidence intervals for $\hat{\lambda}$, $\hat{\beta}$, and



θ	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.80	1.70	989.594282	1.70	1001.750550
0.85	1.70	989.680282	1.70	1001.836550
0.90	1.70	989.742282	1.70	1001.898550

h_1	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.4	1.70	989.594282	1.70	1001.750550
0.5	1.70	989.510282	1.70	1001.666550
0.6	1.70	989.344282	1.70	1001.500550

θ	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.80	1.70	989.510282	1.70	1001.666550
0.85	1.70	989.5846282	1.70	1001.740896
0.90	1.70	989.658282	1.70	1001.814550

h_1	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.4	1.70	989.680282	1.70	1001.836550
0.5	1.70	989.584628	1.70	1001.740896
0.6	1.70	989.448282	1.70	1001.604550

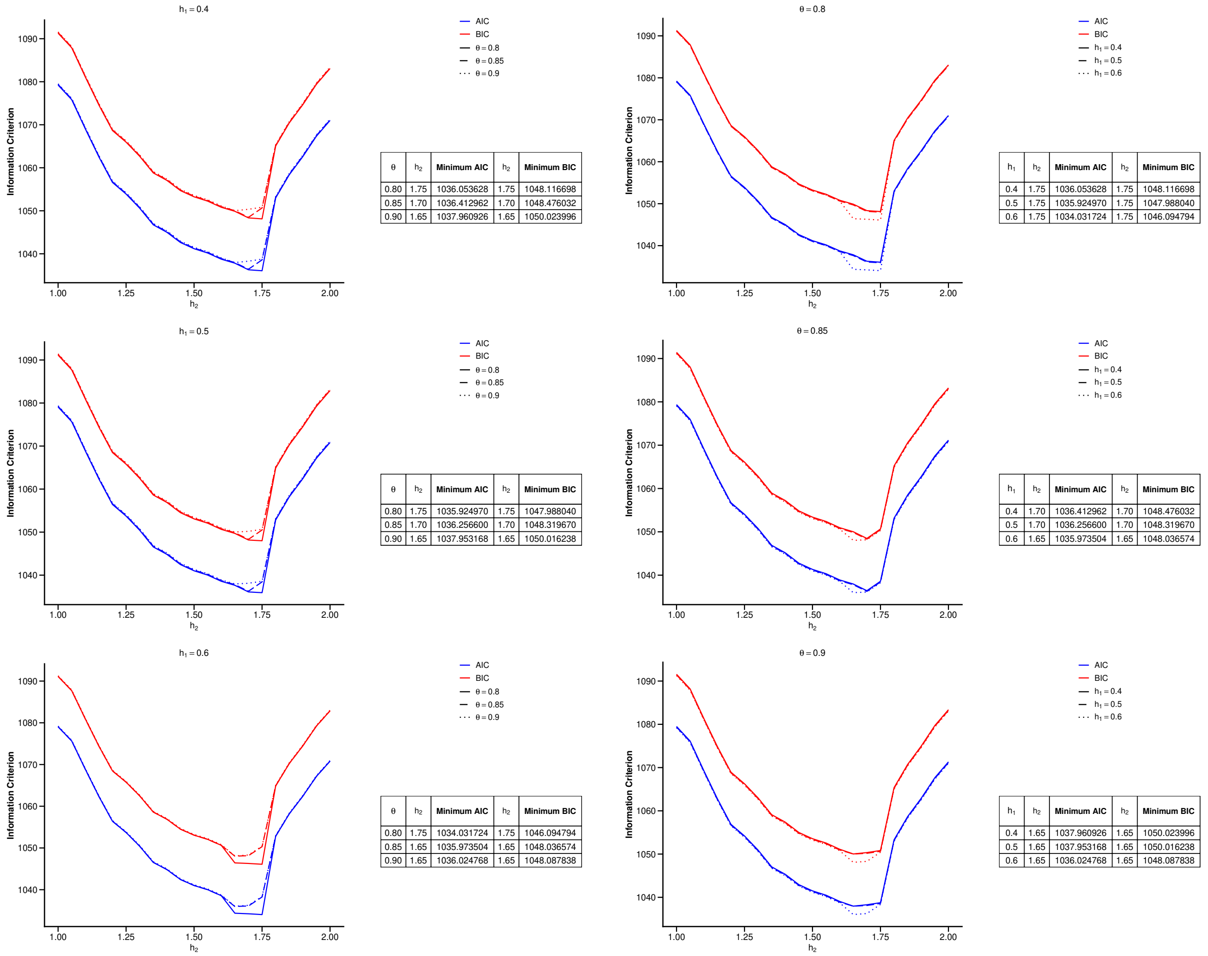
θ	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.80	1.70	989.344282	1.70	1001.500550
0.85	1.70	989.448282	1.70	1001.604550
0.90	1.70	989.552282	1.70	1001.708550

h_1	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.4	1.70	989.742282	1.70	1001.898550
0.5	1.70	989.658282	1.70	1001.814550
0.6	1.70	989.552282	1.70	1001.708550

Figure 12. AIC and BIC plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of females aged 25–29 years over the last 5 years.

$\hat{\beta}_1$ are $(0.4230, 0.4716)$, $(0.9102, 1.0612)$, and $(0.9188, 1.0522)$, respectively. It is noteworthy that the upper confidence limits for $\hat{\beta}$ and $\hat{\beta}_1$ exceed their theoretical upper bound of 1. As discussed previously, this occurs because the proposed model is overparameterized for the birth distribution presented in Table 1, causing the asymptotic confidence intervals to extend beyond the admissible parameter space. For the birth distribution presented in Table 2, it is seen from Column 3 that the corresponding 95% confidence intervals for $\hat{\lambda}$, $\hat{\beta}$, and $\hat{\beta}_1$ are $(0.4103, 0.4768)$, $(0.6922, 0.8606)$, and $(0.5847, 0.7790)$, respectively. Similarly, for the birth distribution presented in Table 3, it is seen from Column 6 that the corresponding 95% confidence intervals for $\hat{\lambda}$, $\hat{\beta}$, and $\hat{\beta}_1$ are $(0.6058, 0.7243)$, $(0.2512, 0.4186)$, and $(0.1609, 0.3367)$, respectively. Likewise, for the birth distribution presented in Table 4, it is seen from Column 5 that the corresponding 95% confidence intervals for $\hat{\lambda}$, $\hat{\beta}$, and $\hat{\beta}_1$ are $(0.6697, 0.7518)$, $(0.1542, 0.2403)$, and $(0.2620, 0.3671)$, respectively.

Furthermore, the confidence intervals for $\hat{\alpha}$ are obtained from the reduced model with $\beta = 1$, since, under the considered estimation procedure, the ML estimate of α is identical under both the proposed and the reduced models. As noted earlier, the expected frequencies under the reduced model corresponding to the ML estimates based on the indirect estimates of h_1, h_2, and θ are presented in Column 4 of Tables 1 and 2, Column 8 of Table 3, and Column 7 of Table 4. As shown in Column 4 of Tables 1 and 2, the corresponding 95% confidence intervals for $\hat{\alpha}$ are $(0.8521, 0.9104)$ and $(0.8542, 0.9498)$, respectively. As mentioned above, the ML estimates of α for



θ	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.80	1.75	1036.053628	1.75	1048.116698
0.85	1.70	1036.412962	1.70	1048.476032
0.90	1.65	1037.960926	1.65	1050.023996

h_1	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.4	1.75	1036.053628	1.75	1048.116698
0.5	1.75	1035.924970	1.75	1047.988040
0.6	1.75	1034.031724	1.75	1046.094794

θ	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.80	1.75	1035.924970	1.75	1047.988040
0.85	1.70	1036.256600	1.70	1048.319670
0.90	1.65	1037.953168	1.65	1050.016238

h_1	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.4	1.70	1036.412962	1.70	1048.476032
0.5	1.70	1036.256600	1.70	1048.319670
0.6	1.65	1035.973504	1.65	1048.036574

θ	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.80	1.75	1034.031724	1.75	1046.094794
0.85	1.65	1035.973504	1.65	1048.036574
0.90	1.65	1036.024768	1.65	1048.087838

h_1	h_2	**Minimum AIC**	h_2	**Minimum BIC**
0.4	1.65	1037.960926	1.65	1050.023996
0.5	1.65	1037.953168	1.65	1050.016238
0.6	1.65	1036.024768	1.65	1048.087838

Figure 13. AIC and BIC plotted against h_2 for different combinations of h_1 and θ, based on the birth distribution of females aged 25–29 years over the last 7 years.

the birth distributions presented in Tables 3 and 4 are computed only for verification purposes. Consequently, α is overparameterized under both the proposed and the reduced models. Therefore, the corresponding asymptotic 95% confidence intervals for $\hat{\alpha}$ extend beyond the admissible parameter space, with the upper confidence limits exceeding the theoretical upper bound of 1, analogous to the behaviour observed for $\hat{\beta}$ and $\hat{\beta}_1$ in Table 1. Moreover, the estimated correlation coefficients show a moderately strong positive association between $\hat{\beta}$ and $\hat{\beta}_1$ for all four birth distributions. In contrast, $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\beta}_1$ each exhibit moderate negative correlations with $\hat{\lambda}$.

The estimated risk of conception under the proposed model is very similar to that obtained under the reduced model with $\beta = 1$ across all datasets considered. For the contemporary birth distribution of urban Bihar females aged 25–29 years irrespective of age at marriage (Table 1), the estimated risk of conception is 0.4473 under the proposed model and 0.4464 under the reduced model. Likewise, for females in the same age group whose age at marriage is 16–20 years (Table 2), the corresponding estimates are 0.4436 and 0.4424, respectively. For the historical birth distributions based on the observational period of 5 and 7 years (Tables 3 and 4), the proposed model gives estimates of 0.6651 and 0.7108, respectively, compared with 0.6529 and 0.7017 under the reduced model. These estimates are in close agreement with those obtained by Singh *et al.* [22], whereas the corresponding estimates obtained by Singh and Yadava [18] are marginally lower. The close agreement between the estimates obtained from the proposed and reduced models indicates that the estimated risk of conception is largely unaffected by the inclusion of the additional parameters. Furthermore, Tables 1 and 2 suggest that age at marriage has a negligible effect on the estimated risk of conception. In contrast, Tables 3 and 4 indicate a modest increase in the estimated risk of conception for birth distributions based on an observational period of 7 years compared with those based on an observational period of 5 years. It is also evident that the estimated risk of conception for the contemporary urban Bihar datasets (Tables 1 and 2) is substantially lower than that estimated from the historical birth distributions (Tables 3 and 4).

Furthermore, as noted earlier, the proposed model is overparameterized by the parameters β and β_1 for the birth distribution presented in Table 1. Nevertheless, the corresponding parameter estimates remain demographically interpretable. The estimate of α indicates that approximately 88.13% of females are biologically mature for conception during the last 5 years of the observational period. Among these fecund females, the estimate of β shows that approximately $100(1-\beta)\% = 1.43\%$ are not exposed to the risk of conception at the beginning of the observational period (Group B females). The estimate of β_1 further indicates that, within this subgroup, approximately 98.55% are unexposed solely because of pregnancy leading to a complete conception (Group B_1 females), whereas the remaining 1.45% are temporarily sterile due to either pregnancy ending in foetal loss or postpartum amenorrhea associated with complete or incomplete conception (Group B_2 females). Consequently, the estimated proportion of females who remain continuously fecund throughout the 5 years of the observational period is $\alpha\beta = 0.8687$, indicating that approximately 86.87% of females are continuously exposed to the risk of conception during this period. Likewise, the estimated proportion of females who are unexposed to the risk of conception at the beginning of the observational period solely because of pregnancy leading to a complete conception is $\alpha(1-\beta)\beta_1 = 0.0124$, corresponding to approximately 1.24% of all females. It is evident that the estimated proportion of females who are temporarily sterile at the beginning of the observational period is very small. This suggests that the birth distribution in Table 1 is unlikely to represent an equilibrium birth process because age at marriage is not taken into account, whereas the equilibrium birth process assumes that the beginning of the observational period is sufficiently distant from the time of marriage. This limitation is addressed by the birth distribution presented in Table 2. The estimate of α indicates that approximately 90.2% of females are biologically mature for conception during the last 5 years of the observational period. Among these fecund females, the estimate of β shows that approximately $100(1-\beta)\% = 22.36\%$ belong to Group B. The estimate of β_1 further indicates that, within Group B, approximately 68.19% belong to Group B_1, whereas the remaining 31.81% belong to Group B_2. Consequently, the estimated proportion of females who remain continuously fecund throughout the 5 years of the observational period is $\alpha\beta = 0.7003$, indicating that approximately 70.03% of females are continuously exposed to the risk of conception during this period. Likewise, the estimated proportion of females who belong to Group B_1 is $\alpha(1-\beta)\beta_1 = 0.1375$, corresponding to approximately 13.75% of all females.

Furthermore, the estimate of α is 1 for the birth distributions presented in Tables 3 and 4, as these distributions include only fecund females. Based on the estimates of β and β_1 for the birth distribution in Table 3, approximately $100(1-\beta)\% = 66.51\%$ of females belong to Group B. Among these Group B females, approximately 24.88% belong to Group B_1, whereas the remaining 75.12% belong to Group B_2. Consequently, the estimated proportion of females who remain continuously fecund throughout the 5 years of the observational period is $\alpha\beta = 0.3349$, indicating that approximately 33.49% of females are continuously exposed to the risk of conception during this period. Likewise, the estimated proportion of females belonging to Group B_1 is $\alpha(1-\beta)\beta_1 = 0.1655$, corresponding to approximately 16.55% of all females. For the birth distribution presented in Table 4, corresponding to the last 7 years of the observational period, the estimate of β indicates that approximately $100(1-\beta)\% = 80.27\%$ of females belong to Group B. Among these Group B females, approximately 31.46% belong to Group B_1, whereas the remaining 68.54% belong to Group B_2. Consequently, the estimated proportion of females who remain continuously fecund throughout the 7 years of the observational period is $\alpha\beta = 0.1973$, indicating that approximately 19.73% of females are continuously exposed to the risk of conception during this period. Likewise, the estimated proportion of females belonging to Group B_1 is $\alpha(1-\beta)\beta_1 = 0.2525$, corresponding to approximately 25.25% of all females.

4. Discussion and Conclusions

The present paper proposes a modified probability model for the number of conceptions in an equilibrium birth process, extending the model of Singh and Yadava [18] by explicitly accounting for the proportion of females who may experience temporary sterility at the start of the observational period due to pregnancy or postpartum amenorrhea. It is important to note that the proposed model is derived under the assumptions of a constant non-susceptibility period associated with the two types of pregnancy terminations, a constant risk of conception, and a constant probability of a complete conception. However, empirical studies indicate that these factors are age-dependent and may vary across females according to parity and marriage duration, and may also vary with age for the same female [1]. Although these parameters are age-dependent, differences in their values between two nearby ages observed within a narrower observational period are generally expected to be small [17]. Nevertheless, the existing literature does not specify the age interval or observational period that may reasonably be regarded as sufficiently narrow for this approximation to hold. Therefore, rather than assuming homogeneity over a particular age interval and its corresponding observational period, the present paper evaluates this assumption empirically. Specifically, if the female population is sufficiently homogeneous with respect to these parameters over the age interval and the corresponding observational period under consideration, there should exist a set of constant parameter values that provides a satisfactory fit to the observed birth distribution. Conversely, if substantial heterogeneity exists, no single set of constant parameter values would be expected to adequately describe the observed birth distribution.

In the present paper, the non-susceptibility periods associated with the two types of pregnancy terminations and the probability of a complete conception were estimated indirectly by evaluating a range of plausible values for these parameters. For each parameter combination, ML estimates of the remaining model parameters were obtained, and the corresponding goodness of fit to the observed birth distribution was assessed. The parameter combination corresponding to the minimum chi-square statistic was regarded as the indirect estimate of these parameters. For all the observed birth distributions considered, the minimum chi-square statistics were non-significant, indicating that the proposed model provides an adequate fit to the observed birth distributions. These findings suggest that the female populations considered are sufficiently homogeneous with respect to the parameters of the proposed model over the age interval and observational period under consideration. It is noteworthy that the minimum chi-square statistics were generally larger for birth distributions observed over an observational period of 7 years than for those observed over an observational period of 5 years, suggesting a greater degree of heterogeneity as the length of the observational period increases. If substantial heterogeneity were present, no single combination of constant parameter values would be expected to provide an adequate fit to the observed birth distribution. Nevertheless, the non-significant minimum chi-square statistics obtained for the birth distributions observed over a period of 7 years indicate that the assumption of homogeneity remains reasonable for the female populations considered.

It should be acknowledged, however, that the assumption of homogeneity may not hold when a broader age group of females is observed, irrespective of whether the observational period is shorter or longer, or when a narrower age group is observed over a relatively longer observational period. Under such circumstances, the female population may become heterogeneous with respect to the non-susceptibility periods associated with the two types of pregnancy terminations, the risk of conception, and the probability of complete conception. To overcome this limitation, the proposed model may be generalized by incorporating variability in these parameters across females, thereby providing a more generalizable framework suitable for heterogeneous populations. The present paper does not consider such extensions because they would substantially increase the complexity of the model and its estimation procedure. The development of such generalized models therefore remains an important direction for future research.

In conclusion, the proposed model addresses a major limitation of earlier models [18, 19, 22] by explicitly accounting for the proportion of fecund females who are temporarily sterile at the start of the observational period due to pregnancy or postpartum amenorrhea. The model is primarily applicable to, and provides valuable insights into, fertility behaviour among female populations in which the fertility parameters remain approximately constant over the age group and the corresponding observational period. Such conditions are more likely to be satisfied when birth distributions are observed over a relatively narrow age range and a relatively short observational period.

Availability of data and materials

The datasets used in the present paper are obtained from the National Family Health Survey-5 (NFHS-5), conducted during 2019-21. These data are publicly accessible upon request through the Demographic and Health Surveys (DHS) Program website: https://dhsprogram.com/data/available-datasets.cfm

Competing interests

The authors have no competing interests to declare.

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