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Modified Numerical and Exact Solutions of Fuzzy Wave Equation under Neutrosophic fuzzy Environment

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Abstract In this paper, a comprehensive study of the fuzzification and defuzzification of wave equation in neutrosophic fuzzy environment is presented for first time in literature including Truth (T), Indeterminacy (I), and Falsehood (F) fuzzy component. Moreover, a numerical and analytical methods that are explicit finite difference method and D'Alembert's method are reformulate, developed, and applied for solving the neutrosophic fuzzy wave equation. The neutrosophic triangular number concept was utilized for both neutrosophic numerical and exact solutions. The error bound between the neutrosophic numerical and exact solution evaluated at (α, β, γ)-cut level set is illustrated and evaluated. Finally, a numerical example is carried out and solved using both analytical and numerical techniques and the results indicate the effectiveness and feasibility of the proposed developed methods.

Keywords Neutrosophic fuzzy Wave equation, Modified finite difference method, D'Alembert's method, neutrosophic fuzzy numbers.

AMS 2010 subject classifications 35Axx , 65Mxx

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1. Introduction

Partial Differential Equations (PDEs) are essential mathematical models of systems that involve functions of several variables and their partial derivatives. PDEs are important to the description of physical phenomena such as heat conduction, fluid flow, and electromagnetic fields. The main concepts of fuzzy set and fuzzy numbers were initially presented by L. Zadeh (1965) [1]. Numerous extensions of fuzzy set theory have since been created. The fuzzy intuitionistic set is one of these significant extensions that presented by Atanassov's (1983) [2]. Intuitionistic fuzzy sets (IFS) were developed as a result of the intuitionistic set's introduction of the ideas of falsehood and degree of non-membership. In contrast to traditional fuzzy sets, this novel method offered a more effective means of characterizing concepts that are not yet complete. neutrosophy is a novel approach to dealing with issues involving inconsistent, ambiguous, and unclear information. The neutrosophic sets was first presented by Smarandache (2005) [3] as an extension of IFS. Neutrosophy is a novel approach to dealing with issues involving inconsistent, ambiguous, and unclear information. As expanding of the intuitionistic fuzzy set, Smarandache presented the neutrosophic sets in 2005. The non-standard interval - $]0, 1[+$. has been defined as the grade or degree of membership of Truth components (T), Indeterminate components (I), and False components (F) in the neutrosophic set. The neutrosophic set theory's non-standard intervals fit in nicely with the philosophical idea. However, in practice, it is impossible to place the data in the non-standard interval when dealing with real-world scientific and engineering issues. Wang et al. (2010) [4] used the standard or classical form of the unit interval between $[0, 1]$ to create single-valued neutrosophic sets in order to solve these problems. Furthermore,

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some scholars have established single-valued neutrosophic number [5, 6, 7]. Likewise, Ye (2015) [8] presented neutrosophic trapezoidal numbers and talked about how they are used in decision-making. Since then, this method has sparked extensive research on a wide range of practical issues. The main difference between the neutrosophic model and intuitionistic fuzzy models is that the neutrosophic approach includes an additional indeterminacy component. This allows the model to describe situations where the data is incomplete or unclear, such as wave propagation with uncertain measurements or partially known parameters [9].

D'Alembert's method is an analytical technique used to obtain exact solutions of the wave equation by transforming the partial differential equation into a function of traveling waves. This approach expresses the solution as a combination of two wave functions moving in opposite directions along the spatial axis. It is particularly effective for solving the one-dimensional wave equation under specific initial-boundary conditions. However, many practical problems lack closed-form solutions, necessitating numerical approaches. The finite difference method (FDM) is a widely used numerical approach that discretizes the domain into a grid and approximates derivatives using difference equations. FDM provides an efficient and stable way to approximate PDE solutions, particularly for complex geometries and boundary conditions. One of the key instruments for simulating the uncertainty (fuzzy) in specific quantities for some real-life phenomena is the Fuzzy Neutrosophic Differential Equations (FNDEs). They have basic uses in many fields, including biology, physics, engineering, and chemistry. FNDEs are typically difficult or impossible to solve exactly. As a result, many mathematicians have turned to numerical or approximation techniques. The finite difference methods (FDMs) are one of the most popular numerical techniques due to its ease of use and broad applicability. The FNDE is numerically solved using the FDM. The non-homogeneous first-order fuzzy differential equation involving neutrosophic initial conditions is solved by Parikh et al. (2022) [9]. The solution process makes use of triangular neutrosophic numbers. Recently, Kamal et. al. (2023) [10] utilized a developed FDM for solving differential equation of 2^{nd} order under neutrosophic boundary and initial conditions.

The neutrosophic model is more general than classical and intuitionistic fuzzy models because it includes an indeterminacy component. This is useful in wave problems with unclear or incomplete data, such as uncertain measurements or boundary conditions. Therefore, applying classical methods in this framework helps model more realistic situations. The Fuzzy wave equation is used to model some real-life phenomena in physics, engineering, and signal processing. the neutrosophic fuzzy numbers (NFNs) can be helpful when we are not completely sure or confirmed about certain or specific factors in dealing with fuzzy wave equations, which facilitates the selection of the most suitable numerical approaches and parameters to solve this type of equations (Smarandache, 2005) [11]. Our review of the literature indicates that there seems no attempts have been made to solve neutrosophic fuzzy wave equations (NFWEs) based on modified exact and numerical methods. Therefore, in this article, an analytical and numerical methods is developed and applied for solving the neutrosophic fuzzy wave equation (NFWE) under neutrosophic initial and boundary conditions.

2. Wave Equation Neutrosophic Fuzzy Environment:

In this section, the NFWE is investigated under Neutrosophic initial-boundary conditions [12]

$$\frac{\partial^2 \tilde{v}(\eta,\tau)}{\partial \tau^2} = \tilde{C}(\eta,\tau)\frac{\partial^2 \tilde{v}(\eta,\tau)}{\partial \eta^2} + \widetilde{m}(x,\tau), \quad 0 < \eta < l, \tau > 0$$
$$\tilde{v}(\eta,0) = \widetilde{f}_1, \frac{\partial \tilde{v}}{\partial \tau}(\eta,0) = \widetilde{f}_2, \tilde{v}(l,\tau) = \tilde{z}(\tau), \quad \tilde{v}(0,\tau) = \tilde{g}(\tau) \tag{1}$$

Such that $\tilde{v}(\eta,\tau)$ is consider as a functions in neutrosophic fuzzy domain, $\frac{\partial^2 \tilde{v}(\eta,\tau)}{\partial \tau^2}$ and $\frac{\partial^2 \tilde{v}(\eta,\tau)}{\partial \eta^2}$ are a fuzzy Hukuhara neutrosophic for time t and space x derivative. Also, the $\widetilde{m}(x,\tau), \tilde{C}(\eta,\tau)$ are functions in neutrosophic fuzzy domain. $\tilde{v}(\eta,0), \frac{\partial v}{\partial \tau}(\eta,0)$ are initial conditions in neutrosophic fuzzy domain while $\tilde{v}(0,\tau), \tilde{v}(l,\tau)$ are boundary condition neutrosophic fuzzy domain. Furthermore, the fuzzy functions $\tilde{C}, \tilde{m}$, and $\widetilde{f}_1, \widetilde{f}_2$ in Eq. (1) are defined as following [11]:

$$\begin{cases} \tilde{C}(\eta,\tau) = \tilde{\theta}_1 b_1(\eta,\tau) \\ \widetilde{m}(\eta,\tau) = \tilde{\theta}_2 b_2(\eta,\tau) \\ \widetilde{f}_1(\eta) = \tilde{\theta}_3 b_3(\eta) \\ \widetilde{f}_2(\eta) = \tilde{\theta}_4 b_4(\eta) \end{cases} \tag{2}$$

Such that b_1, b_2, b_3 and b_4 represent the classical functions of variables η, τ where $\tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3$ and $\tilde{\theta}_4$ denoted as a neutrosophic fuzzy convex normalized number. Let, the NFWE in Equation. (1) is defuzzified through the α, β, γ-cut approach for all $0 \leq \alpha + \beta + \gamma \leq 3$ that described in [6] under the single parametric form as the following

$$[\tilde{v}(\eta,\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{v_T}(\eta,\tau;\alpha), \overline{v_T}(\eta,\tau;\alpha)\right], \left[\underline{v_I}(\eta,\tau;\beta), \overline{v_I}(x,t;\beta)\right], \left[\underline{v_F}(\eta,\tau;\gamma), \overline{v_F}(\eta,\tau;\gamma)\right] \right\}$$

$$\left[\frac{\partial^2 \tilde{v}(\eta,\tau)}{\partial \tau}\right]_{\alpha,\beta,\gamma} \tag{3}$$

$$= \left\{ \left[\frac{\partial^2 v_T(\eta,\tau;\alpha)}{\partial \tau^2}, \frac{\partial^2 \overline{v_T}(\eta,\tau;\alpha)}{\partial \tau^2}\right], \left[\frac{\partial^2 v_I(\eta,\tau;\beta)}{\partial \tau^2}, \frac{\partial^2 \overline{v_I}(\eta,\tau;\beta)}{\partial \tau^2}\right], \left[\frac{\partial^2 \underline{v_F}(\eta,\tau;\gamma)}{\partial \tau^2}, \frac{\partial^2 \overline{v_F}(\eta,\tau;\gamma)}{\partial \tau^2}\right] \right\} \tag{4}$$

$$\left[\frac{\partial^2 \tilde{v}(\eta,\tau)}{\partial x^2}\right]_{\alpha,\beta,\gamma}$$

$$= \left\{ \left[\frac{\partial^2 \underline{v_T}(\eta,\tau;\alpha)}{\partial \eta^2}, \frac{\partial^2 \overline{v_T}(\eta,\tau;\alpha))}{\partial \eta^2}\right], \left[\frac{\partial^2 \underline{v_I}(\eta,\tau;\beta)}{\partial \eta^2}, \frac{\partial^2 \overline{v_I}(x,t;\beta)}{\partial \eta^2}\right], \left[\frac{\partial^2 \underline{v_F}(\eta,\tau;\gamma)}{\partial \eta^2}, \frac{\partial^2 \overline{v_F}(\eta,\tau;\gamma)}{\partial \eta^2}\right] \right\} \tag{5}$$

$$[\tilde{C}(\eta,\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{C_T}(\eta,\tau;\alpha),\ \overline{C_T}(\eta,\tau;\alpha)\right], \left[\underline{C_I}(\eta,\tau;\beta),\ \overline{C_I}(\eta,\tau;\beta)\right], \left[\underline{C_F}(\eta,\tau;\gamma),\ \overline{C_F}(\eta,\tau;\gamma)\right] \right\} \tag{6}$$

$$[\widetilde{m}(\eta,\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{m_T}(\eta,\tau;\alpha), \overline{m_T}(\eta,\tau;\alpha)\right], \left[\underline{m_I}(\eta,\tau;\beta), \overline{m_I}(\eta,\tau;\beta)\right], \left[\underline{m_F}(\eta,\tau;\gamma), \overline{m_F}(\eta,\tau;\gamma)\right] \right\} \tag{7}$$

$$[\tilde{v}(\eta,0)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{v_T}(\eta,0;\alpha), \overline{v_T}(\eta,0;\alpha)\right], \left[\underline{v_I}(\eta,0;\beta), \overline{v_I}(\eta,0;\beta)\right], \left[\underline{v_F}(\eta,0;\gamma), \overline{v_F}(\eta,0;\gamma)\right] \right\} \tag{8}$$

$$[\tilde{v}(0,\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{v_T}(0,\tau;\alpha), \overline{v_T}(0,t;\alpha)\right], \left[\underline{v_I}(0,\tau;\beta), \overline{v_I}(0,\tau;\beta)\right], \left[\underline{v_F}(0,\tau;\gamma), \overline{v_F}(0,\tau;\gamma)\right] \right\} \tag{9}$$

$$[\tilde{v}(l,\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{v_T}(l,\tau;\alpha), \overline{v_T}(l,\tau;\alpha)\right], \left[\underline{v_I}(l,\tau;\beta), \overline{v_I}(l,\tau;\beta)\right], \left[\underline{v_F}(l,\tau;\gamma), \overline{v_F}(l,\tau;\gamma)\right] \right\} \tag{10}$$

$$[\tilde{f}(\eta)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{f_T}(\eta;\alpha), \overline{f_T}(\eta;\alpha)\right], \left[\underline{f_I}(\eta;\beta), \overline{f_I}(\eta;\beta)\right], \left[\underline{f_F}(\eta;\gamma), \overline{f_F}(\eta;\gamma)\right] \right\} \tag{11}$$

$$\begin{cases} [\tilde{g}(\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{g_T}(\tau;\alpha), \overline{g_T}(\tau;\alpha)\right], \left[\underline{g_I}(\tau;\beta), \overline{g_I}(\tau;\beta)\right], \left[\underline{g_F}(\tau;\gamma), \overline{g_F}(\tau;\gamma)\right] \right\} \\ [\tilde{z}(\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[\underline{z_T}(\tau;\alpha), \overline{z_T}(\tau;\alpha)\right], \left[\underline{z_I}(\tau;\beta), \overline{z_I}(\tau;\beta)\right], \left[\underline{z_F}(\tau;\gamma), \overline{z_F}(\tau;\gamma)\right] \right\} \end{cases} \tag{12}$$

Where

$$\begin{cases} [\tilde{C}(\eta,\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[[\underline{\theta_T}(\alpha)_1, \overline{\theta_T}(\alpha)_1] b_1(x,t)\right],\ \left[[\underline{\theta_I}(\alpha)_1, \overline{\theta_I}(\alpha)_1] b_1(x,t)\right],\ \left[[\underline{\theta_F}(\alpha)_1, \overline{\theta_F}(\alpha)_1] b_1(\eta,\tau)\right] \right\} \\ [\tilde{m}(\eta,\tau)]_{\alpha,\beta,\gamma} = \left\{ \left[[\underline{\theta_T}(\alpha)_2, \overline{\theta_T}(\alpha)_2] b_2(x,t)\right],\ \left[[\underline{\theta_I}(\alpha)_2, \overline{\theta_I}(\alpha)_2] b_2(x,t)\right],\ \left[[\underline{\theta_F}(\alpha)_2, \overline{\theta_F}(\alpha)_2] b_2(\eta,\tau)\right] \right\} \\ [\tilde{f}(\eta)]_{\alpha,\beta,\gamma} = \left\{ \left[[\underline{\theta_T}(\alpha)_3, \overline{\theta_T}(\alpha)_3] b_3(x,t)\right],\ \left[[\underline{\theta_I}(\alpha)_3, \overline{\theta_I}(\alpha)_3] b_3(x,t)\right],\ \left[[\underline{\theta_F}(\alpha)_3, \overline{\theta_F}(\alpha)_3] b_3(\eta,\tau)\right] \right\} \end{cases} \tag{13}$$

Where the membership function of neutrosophic function is determined by implementing Zadeh expansion principle [15, 16].

$$\begin{cases} \underline{v_T}(\eta,\tau;\alpha) = \min\{\tilde{v}(\tilde{\mu}(\alpha)) \mid \tilde{\mu}(\alpha) \in \tilde{v}(\eta,\tau;\alpha)\} \\ \overline{v_T}(\eta,\tau;\alpha) = \max\{\tilde{v}(\tilde{\mu}(\alpha)) \mid \tilde{\mu}(\alpha) \in \tilde{v}(\eta,\tau;\alpha)\} \end{cases} \tag{14}$$

$$\begin{cases} \underline{v_I}(\eta,\tau;\beta) = \min\{\tilde{v}(\tilde{\mu}(\beta)) \mid \tilde{\mu}(\beta) \in \tilde{v}(\eta,\tau;\beta)\} \\ \overline{v_I}(\eta,\tau;\beta) = \max\{\tilde{v}(\tilde{\mu}(\beta)) \mid \tilde{\mu}(\beta) \in \tilde{v}(\eta,\tau;\beta)\} \end{cases} \tag{15}$$

$$\begin{cases} \underline{v_F}(\eta,\tau;\gamma) = \min\{\tilde{v}(\tilde{\mu}(\gamma)) \mid \tilde{\mu}(\gamma) \in \tilde{v}(\eta,\tau;\gamma)\} \\ \overline{v_F}(\eta,\tau;\gamma) = \max\{\tilde{v}(\tilde{\mu}(\gamma)) \mid \tilde{\mu}(\gamma) \in \tilde{v}(\eta,\tau;\gamma)\} \end{cases} \tag{16}$$

The Eq. (1) where $\tau > 0, 0 < \eta \leq l$ and $\alpha, \beta, \gamma \in [0,1]$ are rewritten and reformulate to get the general formula of NFWE as following:

$$\begin{cases} \dfrac{\partial^2 \underline{v_T}(\eta,\tau)}{\partial\tau^2} = \left[\underline{\theta_T}(\alpha)_1\right] b_1(\eta,\tau)\dfrac{\partial^2 \underline{v_T}(\eta,\tau;\alpha)}{\partial\eta^2} + \left[\underline{\theta_T}(\alpha)_2\right] b_2(\eta,\tau) \\ \underline{v_T}(\eta,0;\alpha) = \left[\underline{\theta_T}(\alpha)_3\right] b_3(\eta,\tau) \end{cases} \tag{17}$$

$$\begin{cases} \dfrac{\partial^2 \overline{v_T}(\eta,\tau)}{\partial\tau^2} = \left[\overline{\theta_T}(\alpha)_1\right] b_1(\eta,\tau)\dfrac{\partial^2 \overline{v_T}(\eta,\tau;\alpha)}{\partial\eta^2} + \left[\overline{\theta_T}(\alpha)_2\right] b_2(\eta,\tau) \\ \overline{v_T}(\eta,0;\alpha) = \left[\overline{\theta_T}(\alpha)_3\right] b_3(\eta,\tau) \\ \overline{v_T}(0,\tau;\alpha) = \bar{g}(\tau,\alpha), \quad \overline{v_T}(l,\tau;\alpha) = \bar{z}(\tau,\alpha) \end{cases} \tag{18}$$

$$\begin{cases} \dfrac{\partial^2 \underline{v_I}(\eta,\tau)}{\partial\tau^2} = \left[\underline{\theta_I}(\beta)_1\right] b_1(\eta,\tau)\dfrac{\partial^2 \underline{v_I}(\eta,\tau;\beta)}{\partial\eta^2} + \left[\underline{\theta_I}(\beta)_2\right] b_2(\eta,\tau) \\ \underline{v_I}(\eta,0;\beta) = \left[\underline{\theta_I}(\beta)_3\right] b_3(\eta,\tau) \\ \underline{v_I}(0,\tau;\beta) = \underline{g}(\tau,\beta), \quad \underline{v_I}(l,\tau;\beta) = \underline{z}(\tau,\beta) \end{cases} \tag{19}$$

$$\begin{cases} \dfrac{\partial^2 \overline{v_I}(\eta,\tau)}{\partial\tau^2} = \left[\overline{\theta_I}(\beta)_1\right] b_1(\eta,\tau)\dfrac{\partial^2 \overline{v_I}(\eta,\tau;\beta)}{\partial\eta^2} + \left[\overline{\theta_I}(\beta)_2\right] b_2(\eta,\tau) \\ \overline{v_I}(\eta,0;\beta) = \left[\overline{\theta_I}(\beta)_3\right] b_3(\eta,\tau) \\ \overline{v_I}(0,\tau;\beta) = \bar{g}(\tau,\beta), \quad \overline{v_I}(l,\tau;\beta) = \bar{z}(\tau,\beta) \end{cases} \tag{20}$$

$$\begin{cases} \dfrac{\partial^2 \underline{v_F}(\eta,\tau)}{\partial\tau^2} = \left[\underline{\theta_F}(\gamma)_1\right] b_1(\eta,\tau)\dfrac{\partial^2 \underline{v_F}(\eta,\tau;\gamma)}{\partial\eta^2} + \left[\underline{\theta_F}(\gamma)_2\right] b_2(\eta,\tau) \\ \underline{v_F}(\eta,0;\gamma) = \left[\underline{\theta_F}(\gamma)_3\right] b_3(\eta,\tau) \\ \underline{v_F}(0,\tau;\gamma) = \underline{g}(\tau,\gamma), \quad \underline{v_F}(l,\tau;\gamma) = \underline{z}(\tau,\gamma) \end{cases} \tag{21}$$

$$\begin{cases} \dfrac{\partial^2 \overline{v_F}(\eta,\tau)}{\partial\tau^2} = \left[\overline{\theta_F}(\gamma)_1\right] b_1(\eta,\tau)\dfrac{\partial^2 \overline{v_F}(\eta,\tau;\gamma)}{\partial\eta^2} + \left[\overline{\theta_F}(\gamma)_2\right] b_2(\eta,\tau) \\ \overline{v_F}(\eta,0;\gamma) = \left[\overline{\theta_F}(\gamma)_3\right] b_3(\eta,\tau) \\ \overline{v_F}(0,\tau;\gamma) = \bar{g}(\tau,\gamma), \quad \overline{v_F}(l,\tau;\gamma) = \bar{z}(\tau,\gamma) \end{cases} \tag{22}$$

The lower neutrosophic and upper neutrosophic terms of truth values (T) of membership for the general formula of NFWE are given by Eqs. (17) and (18). Additionally, the lower neutrosophic and upper neutrosophic terms of indeterminate values (I) for the membership function are presented by Eqs. (19) and (20), whereas the neutrosophic lower and neutrosophic upper terms of membership of false values (F) for the general formula of NFWE are presented by Eqs. (21) and (22).

3. Solution of NFWE by Modified Finite Difference Method

In this section, a modified finite difference method known as center time center space (CTCS) scheme [12] is redefined and applied to solve NFWE with single parametric representation, in which the time and spatial derivatives are approximated by central difference schemes.

The time derivatives $\frac{\partial^2 \widetilde{v_T}(\eta,\tau;\alpha)}{\partial\tau^2}, \frac{\partial^2 \widetilde{v_I}(\eta,\tau;\alpha)}{\partial\tau^2}, \frac{\partial^2 \widetilde{v_F}(\eta,\tau;\alpha)}{\partial\tau^2}$ are discretizes as follows:

$$\left(\frac{\partial^2 \tilde{v}(\eta,\tau;\alpha)}{\partial\tau^2}\right)_T = \begin{cases} \dfrac{\left(\underline{v_i^{n+1}}(\eta,\tau;\alpha)\right)_T - 2\left(\underline{v_i^{n}}(\eta,\tau;\alpha)\right)_T + \left(\underline{v_i^{n-1}}(\eta,\tau;\alpha)\right)_T}{(\Delta\tau)^2} \\ \dfrac{\left(\overline{v_i^{n+1}}(\eta,\tau;\alpha)\right)_T - 2\left(\overline{v_i^{n}}(\eta,\tau;\alpha)\right)_T + \left(\overline{v_i^{n-1}}(\eta,\tau;\alpha)\right)_T}{(\Delta\tau)^2} \end{cases} \tag{23}$$

$$\left(\frac{\partial^2 \tilde{v}(\eta,\tau;\beta)}{\partial\tau^2}\right)_I = \begin{cases} \dfrac{\left(\underline{v_i^{n+1}}(\eta,\tau;\beta)\right)_I - 2\left(\underline{v_i^{n}}(\eta,\tau;\beta)\right)_I + \left(\underline{v_i^{n-1}}(\eta,\tau;\beta)\right)_I}{(\Delta\tau)^2} \\ \dfrac{\left(\overline{v_i^{n+1}}(\eta,\tau;\beta)\right)_I - 2\left(\overline{v_i^{n}}(\eta,\tau;\beta)\right)_I + \left(\overline{v_i^{n-1}}(\eta,\tau;\beta)\right)_I}{(\Delta\tau)^2} \end{cases} \tag{24}$$

$$\left(\frac{\partial^2 \tilde{v}(\eta,\tau;\gamma)}{\partial\tau^2}\right)_F = \begin{cases} \dfrac{\left(\underline{v_i^{n+1}}(\eta,\tau;\gamma)\right)_F - 2\left(\underline{v_i^{n}}(\eta,\tau;\gamma)\right)_F + \left(\underline{v_i^{n-1}}(\eta,\tau;\gamma)\right)_F}{(\Delta\tau)^2} \\ \dfrac{\left(\overline{v_i^{n+1}}(\eta,\tau;\gamma)\right)_F - 2\left(\overline{v_i^{n}}(\eta,\tau;\gamma)\right)_F + \left(\overline{v_i^{n-1}}(\eta,\tau;\gamma)\right)_F}{(\Delta\tau)^2} \end{cases} \tag{25}$$

The spatial derivatives $\left(\frac{\partial^2 \tilde{v}(\eta,\tau;\alpha)}{\partial\tau^2}\right)_T, \left(\frac{\partial^2 \tilde{v}(\eta,\tau;\beta)}{\partial\tau^2}\right)_I, \left(\frac{\partial^2 \tilde{v}(\eta,\tau;\gamma)}{\partial\tau^2}\right)_F$ are discretizes as follows:

$$\left(\frac{\partial^2 \tilde{v}(\eta,\tau;\alpha)}{\partial\eta^2}\right)_T = \begin{cases} \dfrac{\left(\underline{v_{i+1}^{n}}(\eta,\tau;\alpha)\right)_T - 2\left(\underline{v_i^{n}}(\eta,\tau;\alpha)\right)_T + \left(\underline{v_{i-1}^{n}}(\eta,\tau;\alpha)\right)_T}{(\Delta\eta)^2} \\ \dfrac{\left(\overline{v_{i+1}^{n}}(\eta,\tau;\alpha)\right)_T - 2\left(\overline{v_i^{n}}(\eta,\tau;\alpha)\right)_T + \left(\overline{v_{i-1}^{n}}(\eta,\tau;\alpha)\right)_T}{(\Delta\eta)^2} \end{cases} \tag{26}$$

$$\left(\frac{\partial^2 \tilde{v}(\eta,\tau;\beta)}{\partial\eta^2}\right)_I = \begin{cases} \dfrac{\left(\underline{v_{i+1}^{n}}(\eta,\tau;\beta)\right)_I - 2\left(\underline{v_i^{n}}(\eta,\tau;\beta)\right)_I + \left(\underline{v_{i-1}^{n}}(\eta,\tau;\beta)\right)_I}{(\Delta\eta)^2} \\ \dfrac{\left(\overline{v_{i+1}^{n}}(\eta,\tau;\beta)\right)_I - 2\left(\overline{v_i^{n}}(\eta,\tau;\beta)\right)_I + \left(\overline{v_{i-1}^{n}}(\eta,\tau;\beta)\right)_I}{(\Delta\eta)^2} \end{cases} \tag{27}$$

$$\left(\frac{\partial^2 \tilde{v}(\eta,\tau;\gamma)}{\partial \eta^2}\right)_F = \begin{cases} \dfrac{\left(\underline{v_{i+1}^{n}}(\eta,\tau;\gamma)\right)_F - 2\left(\underline{v_{i}^{n}}(\eta,\tau;\gamma)\right)_F + \left(\underline{v_{i-1}^{n}}(\eta,\tau;\gamma)\right)_F}{(\Delta\eta)^2} \\ \dfrac{\left(\overline{v_{i+1}^{n}}(\eta,\tau;\gamma)\right)_F - 2\left(\overline{v_{i}^{n}}(\eta,\tau;\gamma)\right)_F + \left(\overline{v_{i-1}^{n}}(\eta,\tau;\gamma)\right)_F}{(\Delta\eta)^2} \end{cases} \tag{28}$$

That the left side to column i and the right side to row n. Now by substituting the Eqs. (23-28) into Eqs. (17-22) to obtain:

latex

$$\begin{aligned} &\frac{\left(\underline{v_{i}^{n+1}}(\eta,\tau;\alpha)\right)_T - 2\left(\underline{v_{i}^{n}}(\eta,\tau;\alpha)\right)_T + \left(\underline{v_{i}^{n-1}}(\eta,\tau;\alpha)\right)_T}{(\Delta\tau)^2} \\ &= \left(\underline{C}(\eta,\tau;\alpha)\right)_T \frac{\left(\underline{v_{i+1}^{n}}(\eta,\tau;\alpha)\right)_T - 2\left(\underline{v_{i}^{n}}(\eta,\tau;\alpha)\right)_T + \left(\underline{v_{i-1}^{n}}(\eta,\tau;\alpha)\right)_T}{(\Delta\eta)^2} + \left(\underline{m}(\eta,\tau;\alpha)\right)_T \end{aligned} \tag{29}$$

$$\begin{aligned} &\frac{\left(\overline{v_{i}^{n+1}}(\eta,\tau;\alpha)\right)_T - 2\left(\overline{v_{i}^{n}}(\eta,\tau;\alpha)\right)_T + \left(\overline{v_{i}^{n-1}}(\eta,\tau;\alpha)\right)_T}{(\Delta\tau)^2} \\ &= \left(\overline{C}(\eta,\tau;\alpha)\right)_T \frac{\left(\overline{v_{i+1}^{n}}(\eta,\tau;\alpha)\right)_T - 2\left(\overline{v_{i}^{n}}(\eta,\tau;\alpha)\right)_T + \left(\overline{v_{i-1}^{n}}(\eta,\tau;\alpha)\right)_T}{(\Delta\eta)^2} + \left(\overline{m}(\eta,\tau;\alpha)\right)_T \end{aligned} \tag{30}$$

$$\begin{aligned} &\frac{\left(\underline{v_{i}^{n+1}}(\eta,\tau;\beta)\right)_I - 2\left(\underline{v_{i}^{n}}(\eta,\tau;\beta)\right)_I + \left(\underline{v_{i}^{n-1}}(\eta,\tau;\beta)\right)_I}{(\Delta\tau)^2} \\ &= \left(\underline{C}(\eta,\tau;\beta)\right)_I \frac{\left(\underline{v_{i+1}^{n}}(\eta,\tau;\beta)\right)_I - 2\left(\underline{v_{i}^{n}}(\eta,\tau;\beta)\right)_I + \left(\underline{v_{i-1}^{n}}(\eta,\tau;\beta)\right)_I}{(\Delta\eta)^2} + \left(m(\eta,\tau;\beta)\right)_I \end{aligned} \tag{31}$$

$$\begin{aligned} &\frac{\left(\overline{v_{i}^{n+1}}(\eta,\tau;\beta)\right)_I - 2\left(\overline{v_{i}^{n}}(\eta,\tau;\beta)\right)_I + \left(\overline{v_{i}^{n-1}}(\eta,\tau;\beta)\right)_I}{(\Delta\tau)^2} \\ &= \left(\overline{C}(\eta,\tau;\beta)\right)_I \frac{\left(\overline{v_{i+1}^{n}}(\eta,\tau;\beta)\right)_I - 2\left(\overline{v_{i}^{n}}(\eta,\tau;\beta)\right)_I + \left(\overline{v_{i-1}^{n}}(\eta,\tau;\beta)\right)_I}{(\Delta\eta)^2} + \left(\overline{m}(\eta,\tau;\beta)\right)_I \end{aligned} \tag{32}$$

$$\begin{aligned} &\frac{\left(\underline{v_{i}^{n+1}}(\eta,\tau;\gamma)\right)_F - 2\left(\underline{v_{i}^{n}}(\eta,\tau;\gamma)\right)_F + \left(\underline{v_{i}^{n-1}}(\eta,\tau;\gamma)\right)_F}{(\Delta\tau)^2} \\ &= \left(\underline{C}(\eta,\tau;\gamma)\right)_F \frac{\left(\underline{v_{i+1}^{n}}(\eta,\tau;\gamma)\right)_F - 2\left(\underline{v_{i}^{n}}(\eta,\tau;\gamma)\right)_F + \left(\underline{v_{i-1}^{n}}(\eta,\tau;\gamma)\right)_F}{(\Delta\eta)^2} + \left(\underline{m}(\eta,\tau;\gamma)\right)_F \end{aligned} \tag{33}$$

$$\begin{aligned} &\frac{\left(\overline{v_{i}^{n+1}}(\eta,\tau;\gamma)\right)_F - 2\left(\overline{v_{i}^{n}}(\eta,\tau;\gamma)\right)_F + \left(\overline{v_{i}^{n-1}}(\eta,\tau;\gamma)\right)_F}{(\Delta\tau)^2} \\ &= \left(\overline{C}(\eta,\tau;\gamma)\right)_F \frac{\left(\overline{v_{i+1}^{n}}(\eta,\tau;\gamma)\right)_F - 2\left(\overline{v_{i}^{n}}(\eta,\tau;\gamma)\right)_F + \left(\overline{v_{i-1}^{n}}(\eta,\tau;\gamma)\right)_F}{(\Delta\eta)^2} + \left(\overline{m}(\eta,\tau;\gamma)\right)_F \end{aligned} \tag{34}$$

Now let $\tilde{S} = \frac{\tilde{C}(\eta,\tau)\Delta\tau}{\Delta\eta}$, and from Eq. (29-34) to get the following for $\alpha, \beta, \gamma \in [0, 1]$

$$\begin{cases} \left(\underline{v_i^{n+1}}(\eta,\tau;\alpha)\right)_T = S^2\left(\underline{v_{i+1}^{n}}(\eta,\tau;\alpha)\right)_T + 2(1-S^2)\left(\underline{v_i^{n}}(\eta,\tau;\alpha)\right)_T \\ \qquad + S^2\left(\underline{v_{i-1}^{n}}(\eta,\tau;\alpha)\right)_T - \left(\underline{v_i^{n-1}}(\eta,\tau;\alpha)\right)_T + (\underline{m}(\eta,\tau,\alpha))_T, \\ \left(\overline{v_i^{n+1}}(\eta,\tau;\alpha)\right)_T = S^2\left(\overline{v_{i+1}^{n}}(\eta,\tau;\alpha)\right)_T + 2(1-S^2)\left(\overline{v_i^{n}}(\eta,\tau;\alpha)\right)_T \\ \qquad + S^2\left(\overline{v_{i-1}^{n}}(\eta,\tau;\alpha)\right)_T - \left(\overline{v_i^{n-1}}(\eta,\tau;\alpha)\right)_T + (\bar{m}(\eta,\tau,\alpha))_T \end{cases} \tag{35}$$

$$\begin{cases} \left(\underline{v_i^{n+1}}(\eta,\tau;\beta)\right)_I = S^2\left(\underline{v_{i+1}^{n}}(\eta,\tau;\beta)\right)_I + 2(1-S^2)\left(\underline{v_i^{n}}(\eta,\tau;\beta)\right)_I \\ \qquad + S^2\left(\underline{v_{i-1}^{n}}(\eta,\tau;\beta)\right)_I - \left(\underline{v_i^{n-1}}(\eta,\tau;\beta)\right)_I + (\underline{m}(\eta,\tau,\beta))_I, \\ \left(\overline{v_i^{n+1}}(\eta,\tau;\beta)\right)_I = S^2\left(\overline{v_{i+1}^{n}}(\eta,\tau;\beta)\right)_I + 2(1-S^2)\left(\overline{v_i^{n}}(\eta,\tau;\beta)\right)_I \\ \qquad + S^2\left(\overline{v_{i-1}^{n}}(\eta,\tau;\beta)\right)_I - \left(\overline{v_i^{n-1}}(\eta,\tau;\beta)\right)_I + (\bar{m}(\eta,\tau,\beta))_I \end{cases} \tag{36}$$

$$\begin{cases} \left(\underline{v_i^{n+1}}(\eta,\tau;\gamma)\right)_F = S^2\left(\underline{v_{i+1}^{n}}(\eta,\tau;\gamma)\right)_F + 2(1-S^2)\left(\underline{v_i^{n}}(\eta,\tau;\gamma)\right)_F \\ \qquad + S^2\left(\underline{v_{i-1}^{n}}(\eta,\tau;\gamma)\right)_F - \left(\underline{v_i^{n-1}}(\eta,\tau;\gamma)\right)_F + (\underline{m}(\eta,\tau,\gamma))_F, \\ \left(\overline{v_i^{n+1}}(\eta,\tau;\gamma)\right)_F = S^2\left(\overline{v_{i+1}^{n}}(\eta,\tau;\gamma)\right)_F + 2(1-S^2)\left(\overline{v_i^{n}}(\eta,\tau;\gamma)\right)_F \\ \qquad + S^2\left(\overline{v_{i-1}^{n}}(\eta,\tau;\gamma)\right)_F - \left(\overline{v_i^{n-1}}(\eta,\tau;\gamma)\right)_F + (\bar{m}(\eta,\tau,\gamma))_F \end{cases} \tag{37}$$

The Eq. (35-37) are represent the general formula the implicit FDM for solving NFWE For each spatial grid point, Eq. (35-37) are evaluated to yield linear equations. At the end of each time level, a system of linear equations is obtained. This system is then solved to obtain the values $\tilde{v}(\eta,\tau)_{\alpha,\beta,\gamma}$ for that particular time level [13]. Furthermore, in this scheme, neutrosophic arithmetic is applied separately to the truth (T), indeterminacy (I), and falsehood (F) components at each (α,β,γ)-cut level. The finite difference method is used for each component individually, and the lower and upper bounds are computed by applying the same update formula to the corresponding lower and upper values. This ensures that the neutrosophic structure is preserved during the computation.

As regards the stability condition, the proposed neutrosophic fuzzy finite difference scheme is conditionally stable according to the Von Neumann stability analysis, provided that the CFL condition $\frac{C(\eta,\tau)\Delta\tau}{\Delta\eta} = s \leq 1$ holds for all $(\alpha'\beta'\gamma)$-cut levels.

4. D'Alembert Method for solving NFWE

In the section, the D'Alembert method is being generalized in order to solve the NFWE under the A generalization neutrosophic approximation. The classical D'Alembert solution is being generalized to consider the truth (T), indeterminacy (I), and falsehood (F) components of the neutrosophic fuzzy framework. The method provides a solution by the analytical expression of the wave equation in terms of forward and backward traveling waves so that the propagation of uncertainty can be represented well throughout the domain [14].

Consider the NFWE:

$$\tilde{v}_{\tau\tau}(\eta,\tau) = c^2\tilde{v}_{\eta\eta}(\eta,\tau), \quad 0<\eta<1,\ \tau>0 \tag{38}$$

$$\begin{aligned} \text{ICs:}\quad & \tilde{v}(\eta,0)=\tilde{f}(\eta)=\Big((f(\eta))_T,\,(f(\eta))_I,\,(f(\eta))_F\Big)\\ & \tilde{v}(\eta,0)=\tilde{g}(\eta)=\Big((g(\eta))_T,\,(g(\eta))_I,\,(g(\eta))_F\Big)\end{aligned} \tag{39}$$

The D'Alembert derivation is carried out component-wise for the truth (T), indeterminacy (I), and falsehood (F) parts after introducing the canonical coordinates $\phi=\eta-c\tau$ and $\xi=\eta+c\tau$; hence, each neutrosophic component is transformed and integrated independently at the (α,β,γ)-cut level as the following

Step (1): (Replacing (η,τ) by new canonical coordinates $((\varphi,\xi))$

Let $\varphi=\eta-c\tau,\xi=\eta+c\tau$

The simple application of the chain rule gives

$$[\tilde{v}_\eta]_{\alpha,\beta,\gamma}=\begin{cases}\big[(\underline{v}_\varphi(\alpha))_T+(\underline{v}_\xi(\alpha))_T\big],\big[(\overline{v}_\varphi(\alpha))_T+(\overline{v}_\xi(\alpha))_T\big]\\ \big[(\underline{v}_\varphi(\beta))_I+(\underline{v}_\xi(\beta))_I\big],\big[(\overline{v}_\varphi(\beta))_I+(\overline{v}_\xi(\beta))_I\big]\\ \big[(\underline{v}_\varphi(\gamma))_F+(\underline{v}_\xi(\gamma))_F\big],\big[(\overline{v}_\varphi(\gamma))_F+(\overline{v}_\xi(\gamma))_F\big]\end{cases}$$

$$[\tilde{v}_\tau]_{\alpha,\beta,\gamma}=\begin{cases}\big[c\big((\underline{v}_\varphi(\alpha))_T-(\underline{v}_\xi(\alpha))_T\big)\big],\big[c\big((\overline{v}_\varphi(\alpha))_T-(\overline{v}_\xi(\alpha))_T\big)\big]\\ \big[c\big((\underline{v}_\varphi(\beta))_I-(\underline{v}_\xi(\beta))_I\big)\big],\big[c\big((\overline{v}_\varphi(\beta))_I-(\overline{v}_\xi(\beta))_I\big)\big]\\ \big[c\big((\underline{v}_\varphi(\gamma))_F-(\underline{v}_\xi(\gamma))_F\big)\big],\big[c\big((\overline{v}_\varphi(\gamma))_F-(\overline{v}_\xi(\gamma))_F\big)\big]\end{cases}$$

$$[\tilde{v}_{\eta\eta}]_{\alpha,\beta,\gamma}=\begin{cases}\big[(\underline{v}_{\varphi\varphi}(\alpha))_T+2(\underline{v}_{\varphi\xi}(\alpha))_T+(\underline{v}_{\xi\xi}(\alpha))_T\big],\big[(\overline{v}_{\varphi\varphi}(\alpha))_T+2(\overline{v}_{\varphi\xi}(\alpha))_T+(\overline{v}_{\xi\xi}(\alpha))_T\big]\\ \big[(\underline{v}_{\varphi\varphi}(\beta))_I+2(\underline{v}_{\varphi\xi}(\beta))_I+(\underline{v}_{\xi\xi}(\beta))_I\big],\big[(\overline{v}_{\varphi\varphi}(\beta))_I+2(\overline{v}_{\varphi\xi}(\beta))_I+(\overline{v}_{\xi\xi}(\beta))_I\big]\\ \big[(\underline{v}_{\varphi\varphi}(\gamma))_F+2(\underline{v}_{\varphi\xi}(\gamma))_F+(\underline{v}_{\xi\xi}(\gamma))_F\big],\big[(\overline{v}_{\varphi\varphi}(\gamma))_F+2(\overline{v}_{\varphi\xi}(\gamma))_F+(\overline{v}_{\xi\xi}(\gamma))_F\big]\end{cases}$$

$$[\tilde{v}_{\tau\tau}]_{\alpha,\beta,\gamma}=\begin{cases}\big[c^2\big((\underline{v}_{\varphi\varphi}(\alpha))_T-2(\underline{v}_{\varphi\xi}(\alpha))_T+(\underline{v}_{\xi\xi}(\alpha))_T\big)\big],\big[c^2\big((\overline{v}_{\varphi\varphi}(\alpha))_T-2(\overline{v}_{\varphi\xi}(\alpha))_T+(\overline{v}_{\xi\xi}(\alpha))_T\big)\big]\\ \big[c^2\big((\underline{v}_{\varphi\varphi}(\beta))_I-2(\underline{v}_{\varphi\xi}(\beta))_I+(\underline{v}_{\xi\xi}(\beta))_I\big)\big],\big[c^2\big((\overline{v}_{\varphi\varphi}(\beta))_I-2(\overline{v}_{\varphi\xi}(\beta))_I+(\overline{v}_{\xi\xi}(\beta))_I\big)\big]\\ \big[c^2\big((\underline{v}_{\varphi\varphi}(\gamma))_F-2(\underline{v}_{\varphi\xi}(\gamma))_F+(\underline{v}_{\xi\xi}(\gamma))_F\big)\big],\big[c^2\big((\overline{v}_{\varphi\varphi}(\gamma))_F-2(\overline{v}_{\varphi\xi}(\gamma))_F+(\overline{v}_{\xi\xi}(\gamma))_F\big)\big]\end{cases}$$

Substituting the expressions for $\tilde{v}_{\eta\eta}$ into the wave equation (38), we have

$$\begin{gathered}\begin{cases}c^2\Big[(\underline{v_{\varphi\varphi}}(\alpha))_T-2(\underline{v_{\varphi\xi}}(\alpha))_T+(\underline{v_{\xi\xi}}(\alpha))_T\Big]=c^2\Big[(\underline{v_{\varphi\varphi}}(\alpha))_T+2(\underline{v_{\varphi\xi}}(\alpha))_T+(\underline{v_{\xi\xi}}(\alpha))_T\Big]\\ c^2\Big[(\overline{v_{\varphi\varphi}}(\alpha))_T-2(\overline{v_{\varphi\xi}}(\alpha))_T+(\overline{v_{\xi\xi}}(\alpha))_T\Big]=c^2\Big[(\overline{v_{\varphi\varphi}}(\alpha))_T+2(\overline{v_{\varphi\xi}}(\alpha))_T+(\overline{v_{\xi\xi}}(\alpha))_T\Big]\end{cases}\\ \begin{cases}c^2\Big[(\underline{v_{\varphi\varphi}}(\beta))_I-2(\underline{v_{\varphi\xi}}(\beta))_I+(\underline{v_{\xi\xi}}(\beta))_I\Big]=c^2\Big[(\underline{v_{\varphi\varphi}}(\beta))_I+2(\underline{v_{\varphi\xi}}(\beta))_I+(\underline{v_{\xi\xi}}(\beta))_I\Big]\\ c^2\Big[(\overline{v_{\varphi\varphi}}(\beta))_I-2(\overline{v_{\varphi\xi}}(\beta))_I+(\overline{v_{\xi\xi}}(\beta))_I\Big]=c^2\Big[(\overline{v_{\varphi\varphi}}(\beta))_I+2(\overline{v_{\varphi\xi}}(\beta))_I+(\overline{v_{\xi\xi}}(\beta))_I\Big]\end{cases}\\ \begin{cases}c^2\Big[(\underline{v_{\varphi\varphi}}(\gamma))_F-2(\underline{v_{\varphi\xi}}(\gamma))_F+(\underline{v_{\xi\xi}}(\gamma))_F\Big]=c^2\Big[(\overline{v_{\varphi\varphi}}(\gamma))_F+2(\overline{v_{\varphi\xi}}(\gamma))_F+(\overline{v_{\xi\xi}}(\gamma))_F\Big]\\ c^2\Big[(\overline{v_{\varphi\varphi}}(\gamma))_F-2(\overline{v_{\varphi\xi}}(\gamma))_F+(\overline{v_{\xi\xi}}(\gamma))_F\Big]=c^2\Big[(\overline{v_{\varphi\varphi}}(\gamma))_F+2(\overline{v_{\varphi\xi}}(\gamma))_F+(\overline{v_{\xi\xi}}(\gamma))_F\Big]\end{cases}\\ \begin{cases}-4c^2\,(\underline{v_{\varphi\xi}}(\alpha))_T=0\quad\rightarrow\quad(\underline{v_{\varphi\xi}}(\alpha))_T=0\\ -4c^2\,(\underline{v_{\varphi\xi}}(\beta))_I=0\quad\rightarrow\quad(\underline{v_{\varphi\xi}}(\beta))_I=0\\ -4c^2\,(\underline{v_{\varphi\xi}}(\gamma))_F=0\quad\rightarrow\quad(\underline{v_{\varphi\xi}}(\gamma))_F=0\end{cases}\end{gathered}\tag{40}$$

Step (2): (solving the transformed equation)
This can be done by integrate the equation with respect to (φ) and then with respect to (η), we gives

$$\tilde{v}_\xi(\varphi,\xi)=\begin{cases}(\underline{\emptyset_\xi}(\alpha))_T, & (\overline{\emptyset_\xi}(\alpha))_T\\ (\underline{\emptyset_\xi}(\beta))_I, & (\overline{\emptyset_\xi}(\beta))_I\\ (\underline{\emptyset_\xi}(\gamma))_F, & (\overline{\emptyset_\xi}(\gamma))_F\end{cases}$$

and secondly, integration with respect to (η) gives

$$[\tilde{v}_\xi(\varphi,\xi)]_{\alpha,\beta,\gamma}=\begin{cases}\left[(\underline{\varphi(\xi)}(\alpha))_T+(\underline{\psi(\varphi)}(\alpha))_T\right],\left[(\overline{\varphi(\xi)}(\alpha))_T+(\overline{\psi(\varphi)}(\alpha))_T\right]\\ \left[(\underline{\varphi(\xi)}(\beta))_I+(\underline{\psi(\varphi)}(\beta))_I\right],\left[(\overline{\varphi(\xi)}(\beta))_I+(\overline{\psi(\varphi)}(\beta))_I\right]\\ \left[(\underline{\varphi(\xi)}(\gamma))_F+(\underline{\psi(\varphi)}(\gamma))_F\right],\left[(\overline{\varphi(\xi)}(\gamma))_F+(\overline{\psi(\varphi)}(\gamma))_F\right]\end{cases} \tag{41}$$

Step (3): (Transforming back to the original coordinates (x,t)
To find the general solution to $[\tilde{v}_{\tau\tau}]_{\alpha,\beta,\gamma}=c^2\,[\tilde{v}_{\eta\eta}]_{\alpha,\beta,\gamma}$, we substitute $\varphi=\eta-c\tau,\xi=\eta+c\tau$ into $[\tilde{v}(\varphi,\xi)]_{\alpha,\beta,\gamma}=[\emptyset_\xi]_{\alpha,\beta,\gamma}+[\psi_\xi]_{\alpha,\beta,\gamma}$ to get

$$[\tilde{v}(x,t)]_{\alpha,\beta,\gamma}=\begin{cases}\left[(\underline{\varphi(\eta+c\tau)}(\alpha))_T+(\underline{\psi(\eta-c\tau)}(\alpha))_T\right],\left[(\overline{\varphi(\eta+c\tau)}(\alpha))_T+(\overline{\psi(\eta-c\tau)}(\alpha))_T\right]\\ \left[(\underline{\varphi(\eta+c\tau)}(\beta))_I+(\underline{\psi(\eta-c\tau)}(\beta))_I\right],\left[(\overline{\varphi(\eta+c\tau)}(\beta))_I+(\overline{\psi(\eta-c\tau)}(\beta))_I\right]\\ \left[(\underline{\varphi(\eta+c\tau)}(\gamma))_F+(\underline{\psi(\eta-c\tau)}(\gamma))_F\right],\left[(\overline{\varphi(\eta+c\tau)}(\gamma))_F+(\overline{\psi(\eta-c\tau)}(\gamma))_F\right]\end{cases} \tag{42}$$

Step (4): (substituting the general solution into the ICs)

$$[\tilde{v}_\tau(\eta,\tau)]_{\alpha,\beta,\gamma}=\begin{cases}-c(\underline{\varphi'(\eta+c\tau)}(\alpha))_T+c(\underline{\psi'(\eta-c\tau)}(\alpha))_T,\\ -c(\overline{\varphi'(\eta+c\tau)}(\alpha))_T+c(\overline{\psi'(\eta-c\tau)}(\alpha))_T;\\ -c(\underline{\varphi'(\eta+c\tau)}(\beta))_I+c(\underline{\psi'(\eta-c\tau)}(\beta))_I,\\ -c(\overline{\varphi'(\eta+c\tau)}(\beta))_I+c(\overline{\psi'(\eta-c\tau)}(\beta))_I;\\ -c(\underline{\varphi'(\eta+c\tau)}(\gamma))_F+c(\underline{\psi'(\eta-c\tau)}(\gamma))_F,\\ -c(\overline{\varphi'(\eta+c\tau)}(\gamma))_F+c(\overline{\psi'(\eta-c\tau)}(\gamma))_F\end{cases}$$

$$[\tilde{v}(\eta,\tau)]_{\alpha,\beta,\gamma}=\begin{cases}(\underline{\varnothing(x)}(\alpha))_T+(\underline{\psi(x)}(\alpha))_T,\\ (\overline{\varnothing(x)}(\alpha))_T+(\overline{\psi(x)}(\alpha))_T;\\ (\underline{\varnothing(x)}(\beta))_I+(\underline{\psi(x)}(\beta))_I,\\ (\overline{\varnothing(x)}(\beta))_I+(\overline{\psi(x)}(\beta))_I;\\ (\underline{\varnothing(x)}(\gamma))_F+(\underline{\psi(x)}(\gamma))_F,\\ (\overline{\varnothing(x)}(\gamma))_F+(\overline{\psi(x)}(\gamma))_F\end{cases}$$

$$[\tilde{v}_\tau(\eta,0)]_{\alpha,\beta,\gamma}=\begin{cases}-c(\underline{\varphi'(\eta)}(\alpha))_T+c(\underline{\psi'(\eta)}(\alpha))_T,\\ -c(\overline{\varphi'(\eta)}(\alpha))_T+c(\overline{\psi'(\eta)}(\alpha))_T;\\ -c(\underline{\varphi'(\eta)}(\beta))_I+c(\underline{\psi'(\eta)}(\beta))_I,\\ -c(\overline{\varphi'(\eta)}(\beta))_I+c(\overline{\psi'(\eta)}(\beta))_I;\\ -c(\underline{\varphi'(\eta)}(\gamma))_F+c(\underline{\psi'(\eta)}(\gamma))_F,\\ -c(\overline{\varphi'(\eta)}(\gamma))_F+c(\overline{\psi'(\eta)}(\gamma))_F\end{cases}$$

$$[\tilde{v}(\eta,0)]_{\alpha,\beta,\gamma}=\begin{cases}(\underline{f(\eta)}(\alpha))_T=(\underline{\phi(\eta)}(\alpha))_T+(\underline{\psi(\eta)}(\alpha))_T,\\ (\overline{f(\eta)}(\alpha))_T=(\overline{\phi(\eta)}(\alpha))_T+(\overline{\psi(\eta)}(\alpha))_T;\\ (\underline{f(\eta)}(\beta))_I=(\underline{\phi(\eta)}(\beta))_I+(\underline{\psi(\eta)}(\beta))_I,\\ (\overline{f(\eta)}(\beta))_I=(\overline{\phi(\eta)}(\beta))_I+(\overline{\psi(\eta)}(\beta))_I;\\ (\underline{f(\eta)}(\gamma))_F=(\underline{\phi(\eta)}(\gamma))_F+(\underline{\psi(\eta)}(\gamma))_F,\\ (\overline{f(\eta)}(\gamma))_F=(\overline{\phi(\eta)}(\gamma))_F+(\overline{\psi(\eta)}(\gamma))_F\end{cases}$$

$$[\tilde{v}_\tau(\eta,0)]_{\alpha,\beta,\gamma}=\begin{cases}(\underline{g(\eta)}(\alpha))_T=-c(\underline{\varphi'(\eta)}(\alpha))_T+c(\underline{\psi'(\eta)}(\alpha))_T,\\ (\overline{g(\eta)}(\alpha))_T=-c(\overline{\varphi'(\eta)}(\alpha))_T+c(\overline{\psi'(\eta)}(\alpha))_T;\\ (\underline{g(\eta)}(\beta))_I=-c(\underline{\varphi'(\eta)}(\beta))_I+c(\underline{\psi'(\eta)}(\beta))_I,\\ (\overline{g(\eta)}(\beta))_I=-c(\overline{\varphi'(\eta)}(\beta))_I+c(\overline{\psi'(\eta)}(\beta))_I;\\ (\underline{g(\eta)}(\gamma))_F=-c(\underline{\varphi'(\eta)}(\gamma))_F+c(\underline{\psi'(\eta)}(\gamma))_F,\\ (\overline{g(\eta)}(\gamma))_F=-c(\overline{\varphi'(\eta)}(\gamma))_F+c(\overline{\psi'(\eta)}(\gamma))_F\end{cases}$$

Integrate both side of equation (43) from η_0 to η to find $\emptyset(\eta)$ and $\psi(\eta)$, we get

$$\begin{cases}-c\left(\underline{\varphi'(\eta)}(\alpha)\right)_T+c\left(\underline{\psi'(\eta)}(\alpha)\right)_T=\int_{\eta_0}^{\eta}\left(\underline{g(\varphi)}(\alpha)\right)_T d\varphi+k,\\ -c\left(\overline{\varphi'(\eta)}(\alpha)\right)_T+c\left(\overline{\psi'(\eta)}(\alpha)\right)_T=\int_{\eta_0}^{\eta}\left(\overline{g(\varphi)}(\alpha)\right)_T d\varphi+k;\\ -c\left(\underline{\varphi'(\eta)}(\beta)\right)_I+c\left(\underline{\psi'(\eta)}(\beta)\right)_I=\int_{\eta_0}^{\eta}\left(\underline{g(\varphi)}(\beta)\right)_I d\varphi+k,\\ -c\left(\overline{\varphi'(\eta)}(\beta)\right)_I+c\left(\overline{\psi'(\eta)}(\beta)\right)_I=\int_{\eta_0}^{\eta}\left(\overline{g(\varphi)}(\beta)\right)_I d\varphi+k;\\ -c\left(\underline{\varphi'(\eta)}(\gamma)\right)_F+c\left(\underline{\psi'(\eta)}(\gamma)\right)_F=\int_{\eta_0}^{\eta}\left(\underline{g(\varphi)}(\gamma)\right)_F d\varphi+k,\\ -c\left(\overline{\varphi'(\eta)}(\gamma)\right)_F+c\left(\overline{\psi'(\eta)}(\gamma)\right)_F=\int_{\eta_0}^{\eta}\left(\overline{g(\varphi)}(\gamma)\right)_F d\varphi+k\end{cases} \tag{44}$$

From equation (42) and (44), we have

$$\begin{cases} (\underline{\varnothing(\eta)}(\alpha))_T = \frac{1}{2}(\underline{f(\eta)}(\alpha))_T + \frac{1}{2c}\int_{\eta_0}^{\eta}(\underline{g(\varphi)}(\alpha))_T\,d\varphi - \frac{k}{2c}, \\ (\overline{\varnothing(\eta)}(\alpha))_T = \frac{1}{2}(\overline{f(\eta)}(\alpha))_T + \frac{1}{2c}\int_{\eta_0}^{\eta}(\overline{g(\varphi)}(\alpha))_T\,d\varphi - \frac{k}{2c}; \\ (\underline{\varnothing(\eta)}(\beta))_I = \frac{1}{2}(\underline{f(\eta)}(\beta))_I + \frac{1}{2c}\int_{\eta_0}^{\eta}(\underline{g(\varphi)}(\beta))_I\,d\varphi - \frac{k}{2c}, \\ (\overline{\varnothing(\eta)}(\beta))_I = \frac{1}{2}(\overline{f(\eta)}(\beta))_I + \frac{1}{2c}\int_{\eta_0}^{\eta}(\overline{g(\varphi)}(\beta))_I\,d\varphi - \frac{k}{2c}; \\ (\underline{\varnothing(\eta)}(\gamma))_F = \frac{1}{2}(\underline{f(\eta)}(\gamma))_F + \frac{1}{2c}\int_{\eta_0}^{\eta}(\underline{g(\varphi)}(\gamma))_F\,d\varphi - \frac{k}{2c}, \\ (\overline{\varnothing(\eta)}(\gamma))_F = \frac{1}{2}(\overline{f(\eta)}(\gamma))_F + \frac{1}{2c}\int_{\eta_0}^{\eta}(\overline{g(\varphi)}(\gamma))_F\,d\varphi - \frac{k}{2c} \end{cases}$$

$$\begin{cases} (\underline{\psi(\eta)}(\alpha))_T = \frac{1}{2}(\underline{f(\eta)}(\alpha))_T - \frac{1}{2c}\int_{\eta_0}^{\eta}(\underline{g(\varphi)}(\alpha))_T\,d\varphi + \frac{k}{2c}, \\ (\overline{\psi(\eta)}(\alpha))_T = \frac{1}{2}(\overline{f(\eta)}(\alpha))_T - \frac{1}{2c}\int_{\eta_0}^{\eta}(\overline{g(\varphi)}(\alpha))_T\,d\varphi - \frac{k}{2c}; \\ (\underline{\psi(\eta)}(\beta))_I = \frac{1}{2}(\underline{f(\eta)}(\beta))_I - \frac{1}{2c}\int_{\eta_0}^{\eta}(\underline{g(\varphi)}(\beta))_I\,d\varphi + \frac{k}{2c}, \\ (\overline{\psi(\eta)}(\beta))_I = \frac{1}{2}(\overline{f(\eta)}(\beta))_I - \frac{1}{2c}\int_{\eta_0}^{\eta}(\overline{g(\varphi)}(\beta))_I\,d\varphi - \frac{k}{2c}; \\ (\underline{\psi(\eta)}(\gamma))_F = \frac{1}{2}(\underline{f(\eta)}(\gamma))_F - \frac{1}{2c}\int_{\eta_0}^{\eta}(\underline{g(\varphi)}(\gamma))_F\,d\varphi + \frac{k}{2c}, \\ (\overline{\psi(\eta)}(\gamma))_F = \frac{1}{2}(\overline{f(\eta)}(\gamma))_F - \frac{1}{2c}\int_{\eta_0}^{\eta}(\overline{g(\varphi)}(\gamma))_F\,d\varphi - \frac{k}{2c} \end{cases}$$

and, the solution to our problem is:
$\varphi = \eta - c\tau, \xi = \eta + c\tau$
$[\tilde{v}(\eta,\tau)]_{\alpha,\beta,\gamma}$

$$\begin{cases} \frac{1}{2}\Big[(\underline{f(\eta-c\tau)}(\alpha))_T + (\underline{f(\eta+c\tau)}(\alpha))_T\Big] + \frac{1}{2}\int_{\eta-c\tau}^{\eta+c\tau} (\underline{g(\varphi)}(\alpha))_T \, d\varphi, \\ \frac{1}{2}\Big[(\overline{f(\eta-c\tau)}(\alpha))_T + (\overline{f(\eta+c\tau)}(\alpha))_T\Big] + \frac{1}{2}\int_{\eta-c\tau}^{\eta+c\tau} (\overline{g(\varphi)}(\alpha))_T \, d\varphi; \\ \\ \frac{1}{2}\Big[(\underline{f(\eta-c\tau)}(\beta))_I + (\underline{f(\eta+c\tau)}(\beta))_I\Big] + \frac{1}{2}\int_{\eta-c\tau}^{\eta+c\tau} (\underline{g(\varphi)}(\beta))_I \, d\varphi, \\ \frac{1}{2}\Big[(\overline{f(\eta-c\tau)}(\beta))_I + (\overline{f(\eta+c\tau)}(\beta))_I\Big] + \frac{1}{2}\int_{\eta-c\tau}^{\eta+c\tau} (\overline{g(\varphi)}(\beta))_I \, d\varphi; \\ \\ \frac{1}{2}\Big[(\underline{f(\eta-c\tau)}(\gamma))_F + (\underline{f(\eta+c\tau)}(\gamma))_F\Big] + \frac{1}{2}\int_{\eta-c\tau}^{\eta+c\tau} (\underline{g(\varphi)}(\gamma))_F \, d\varphi, \\ \frac{1}{2}\Big[(\overline{f(\eta-c\tau)}(\gamma))_F + (\overline{f(\eta+c\tau)}(\gamma))_F\Big] + \frac{1}{2}\int_{\eta-c\tau}^{\eta+c\tau} (\overline{g(\varphi)}(\gamma))_F \, d\varphi \end{cases} \tag{45}$$

5. Numerical Experiment

Consider the NWLE:

$$\frac{\partial^2 \tilde{v}(\eta,\tau)}{\partial^2 \tau} = \frac{\partial^2 \tilde{v}(\eta,\tau)}{\partial \eta^2}, 0 < \eta < 1 \text{ and } 0 \le \tau < \infty$$
$$\tilde{v}(\eta,0) = \tilde{\varnothing}(\alpha,\beta,\gamma)\sin\eta, \tilde{v}_\tau(\eta,0) = 0 \tag{46}$$

Conditional on the fuzzy neutrosophic boundary conditions $\tilde{v}(0,\tau) = \tilde{v}(1,\tau) = 0$. where in α,β,γ-cut level for neutrosophic fuzzy numbers is define as following:

$$\widetilde{\emptyset}(\alpha,\beta,\gamma) = \Big\{[\underline{\emptyset}(\alpha),\ \overline{\emptyset}(\alpha)], [\underline{\emptyset}(\beta),\ \overline{\emptyset}(\beta)], [\underline{\emptyset}(\gamma),\ \overline{\emptyset}(\gamma)]\Big\} = \Big\{[\alpha,\ 2-\alpha], [1-0.5\beta,\ 1+0.5\beta], [1-0.25\gamma,\ 1+0.25\gamma]\Big\}$$

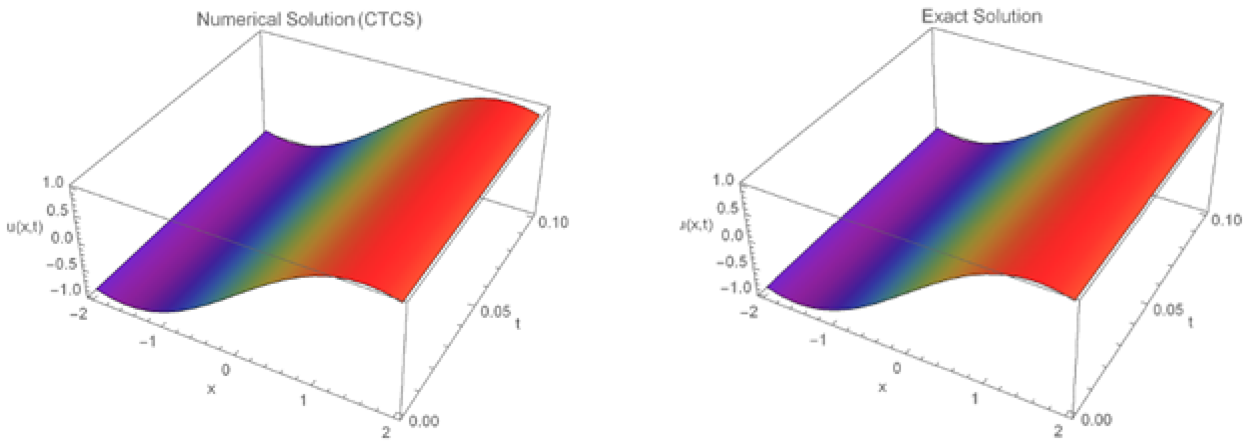
Now, using the method that explained in section 4, the neutrosophic fuzzy exact solution is given by:

$$\begin{cases} \underline{V}(\eta,\tau;\alpha) = \underline{\emptyset}(\alpha)\sin(\eta)\cos(c\tau) \\ \overline{V}(\eta,\tau;\alpha) = \overline{\emptyset}(\alpha)\sin(\eta)\cos(c\tau) \\ \underline{V}(\eta,\tau;\beta) = \underline{\emptyset}(\beta)\sin(\eta)\cos(c\tau) \\ \overline{V}(\eta,\tau;\beta) = \overline{\emptyset}(\beta)\sin(\eta)\cos(c\tau) \\ \underline{V}(\eta,\tau;\gamma) = \underline{\emptyset}(\gamma)\sin(\eta)\cos(c\tau) \\ \overline{V}(\eta,\tau;\gamma) = \overline{\emptyset}(\gamma)\sin(\eta)\cos(c\tau) \end{cases} \tag{48}$$

At $\Delta\eta = \Delta\tau = 0.1$,we get the neutrosophic fuzzy numerical and exact results as the follows:

$$0.6, \tau = 0.05$$

$$0.6, \tau = 0.05), \beta \in [0,1]$$



(a) Neutrosophic fuzzy numerical solution (b) Neutrosophic fuzzy exact solution

Figure 1. Neutrosophic fuzzy numerical (a) and neutrosophic fuzzy exact solution (b) of Eq. (38) at $\eta = 0.6$, $\tau = 0.05$.

Table 1. Lower and Upper Solution for Truth values (T) of Eq.(46) by comparison ($\eta = 0.6, \tau = 0.05$), $\alpha \in [0, 1]$

	Lower neutrosophic fuzzy solution			Upper neutrosophic fuzzy solution		
α	Numerical solution	Exact solution	Absolute error	Numerical solution	Exact solution	Absolute error
0	0	0	0	1.12906	1.12923	1.7×10^{-4}
0.2	0.112906	0.112923	1.7×10^{-5}	1.01615	1.01631	1.6×10^{-4}
0.4	0.225812	0.225846	3.4×10^{-5}	0.903248	0.903383	1.35×10^{-4}
0.6	0.338718	0.338769	5.1×10^{-5}	0.790342	0.79046	1.18×10^{-4}
0.8	0.451624	0.451691	6.7×10^{-5}	0.677436	0.677537	1.01×10^{-4}
1	0.56453	0.564614	8.4×10^{-5}	0.56453	0.564614	8.4×10^{-5}

Table 2: Lower and Upper Solution for Indeterminacy values (I) of Eq. (46) by comparison ($\eta = 0.6$, $\tau = 0.05$)

	Lower neutrosophic fuzzy solution			Upper neutrosophic fuzzy solution		
β	Numerical solution	Exact solution	Absolute error	Numerical solution	Exact solution	Absolute error
0	0.56453	0.564614	8.4×10^{-5}	0.56453	0.564614	8.4×10^{-5}
0.2	0.508077	0.508153	7.6×10^{-5}	0.620983	0.621076	9.3×10^{-5}
0.4	0.451624	0.451691	6.7×10^{-5}	0.677436	0.677537	1.01×10^{-4}
0.6	0.395171	0.395230	5.9×10^{-5}	0.733889	0.733999	1.1×10^{-4}
0.8	0.338718	0.338769	5.1×10^{-5}	0.790342	0.790450	1.08×10^{-4}
1.0	0.282265	0.282307	4.2×10^{-5}	0.846795	0.846921	1.26×10^{-4}

Table 3. Lower and Upper Solution for False values ($\mathbf{F}$) of Eq. ($\mathbf{46}$) by comparison ($\eta = \mathbf{0.6}, \tau = \mathbf{0.05}$) ,

	Lower neutrosophic fuzzy solution			Upper neutrosophic fuzzy solution		
γ	Numerical solution	Exact solution	Absolute error	Numerical solution	Exact solution	Absolute error
0	0.56453	0.564614	8.4×10^{-5}	0.56453	0.564614	8.4×10^{-5}
0.2	0.536303	0.536384	8.1×10^{-5}	0.592756	0.592845	8.9×10^{-5}
0.4	0.508077	0.508153	7.6×10^{-5}	0.620983	0.621076	9.3×10^{-5}
0.6	0.47985	0.479922	7.2×10^{-5}	0.649209	0.649306	$9,7 \times 10^{-5}$
0.8	0.451624	0.451691	6.7×10^{-5}	0.677436	0.677537	1.01×10^{-4}
1	0.423397	0.423461	6.4×10^{-5}	0.705662	0.705768	1.06×10^{-4}

in Tables 1,2,3 and Fig. 1,2, The results obtained of solving NFWE indicates that the presented neutrosophic numerical FDMs approach and neutrosophic exact solutions at $\tau = 0.25, \eta = 0.5$ for $0 \leq \alpha + \beta + \gamma \leq 3$ are

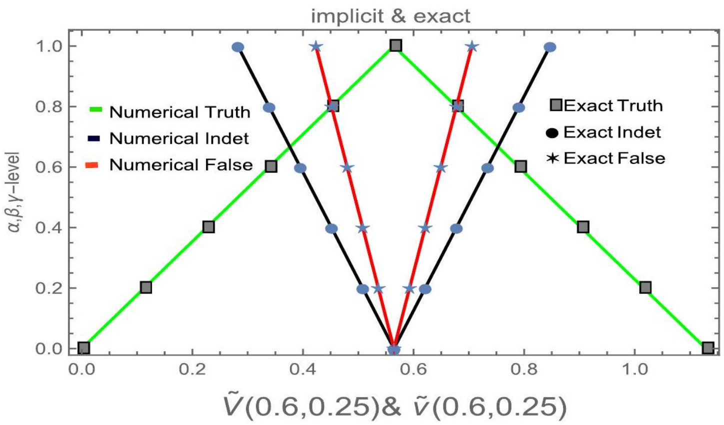


Figure 2. Neutrosophic fuzzy numerical and exact fuzzy solution of Eq. (38) at $\eta = 0.6, \tau = 0.05$ and for different values of $\alpha; \beta; \gamma \in [0, 1]$.

attaining the triangular neutrosophic fuzzy number and thus be content with the properties of neutrosophic fuzzy number. We considered three types of uncertainty that are truth (α), indeterminacy (β), and falsity (γ) where each affecting the solution differently. As α increased, the uncertainty decreased and the solution became more precise. For β and γ, the range of the solution expanded as the uncertainty grew. In all cases, the numerical results closely matched the exact ones with very small errors and this showing that our proposed methods is accurate and reliable for handling NFWE under neutrosophic initial-boundary conditions.

From a physical point of view, the indeterminacy (I) and falsehood (F) components represent the degree of uncertainty in the system. An increase in the width of these bands indicates higher uncertainty in the wave propagation, such as unclear measurements or partially known parameters, while narrower bands correspond to more reliable and precise behavior of the wave.

6. Conclusions

In this study, the fuzzification and defuzzification of the wave equation in neutrosophic fuzzy environment is presented in detail including truth (T), indeterminacy (I), and falsehood (F) fuzzy component. The CTCS scheme is reformulated and implemented to solve the neutrosophic fuzzy wave equation. Alongside the numerical approach, the Method of D'Alembert was employed to derive analytical neutrosophic fuzzy solutions under certain conditions, providing a valuable comparison and validation for the numerical results. The neutrosophic triangular number is utilized for both neutrosophic numerical and exact solutions at (α, β, γ)-cut approach. A numerical experiment is given to clarify and explain the proposed methods. It was found that the obtained results show that proposed methods accurate and reliable for solving NFWE.

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