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Improving Solutions for a Fuzzy Random Multi-Objective Linear Fractional Program

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Abstract Mathematical techniques that combine fuzziness and stochastics are powerful tools for modelling and managing probabilistic, vague, and imprecise uncertainties. They are valuable because they can efficiently solve concrete problems in which data are affected by imprecision and randomness. From this perspective, this article proposes a new technique for converting multi-objective linear fractional programs whose coefficients are random and fuzzy. Using weights to represent decision-maker preferences, independent of α-cuts representing satisfaction levels related to fuzzy quantities, transforms the random fuzzy program into an equivalent random fuzzy linear programming problem. Subsequently, the optimal solution of the random fuzzy linear programming problem is obtained within a class of optimal solutions of the weighted relative pseudo-random linear programming problem. The proposed method generates a multitude of solutions that correspond to the decision-maker's desired level of satisfaction. The solutions found in the didactic examples validate the theory. Finally, applying the method to a specific inventory management issue involving fuzzy and random trapezoidal parameters demonstrates its effectiveness and practical relevance.

Keywords Fuzzy random numbers, Fractional programming, Fuzzy random variables, Multi-objective programming, Inventory management

AMS 2020 subject classifications 62A86, 90C15, 90C29, 90C32, 90C70

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1. Introduction

In mathematical programming, the objective of a fractional programme is to optimize the ratio of two functions. This model is relevant in many situations where efficiency or effectiveness is the objective. Such problems arise in economics, energy, and resource allocation. In this paper, we focus on fractional programming problems involving fuzziness (reflecting linguistic imprecision), randomness, and their simultaneous presence.
Nowadays, managing imprecise information - information that is uncertain, vague, and random - is a significant concern for decision-makers. To address this issue, Zadeh [1] developed the theory of fuzzy logic to model imprecision. However, the model treated randomness and fuzziness as two separate concepts. Subsequently, Kwakernaak [14]; Puri and Ralescu [15]; Wang and Zhang [16]; and Q. Zhong et al. [17] proposed theories on joint consideration, namely the concept of a fuzzy random variable operating under various conditions. It was not until 1993 that Guangyuan and Zhong [2, 3] developed a practical application of joint fuzzy and random modelling, solving linear programming problems using fuzzy random data. Their choice stems from the fact that probability and possibility theories are widely used and can handle random and fuzzy information, respectively.

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Based on their work, extensions have been made in several fields [13, 19, 20, 21, 22, 23, 24]. Regarding fractional linear programming, which was initially introduced by Charnes and Cooper [9], very few works in the literature have addressed the consideration of random fuzzy data.
Indeed, Nureize and Watada [4] employed joint modelling of fuzziness and randomness to solve multi-objective possibilistic programming problems based on fuzzy and random data. This approach was designed to address vague decision-maker preferences (aspirations) and ambiguous data (coefficients) within a fuzzy and random environment. Sayed et al. [5] presented a new, modified technique for order preference by similarity to the ideal solution, which they used to solve the stochastic, fuzzy, multi-objective, fractional decision-making problem. In the proposed model, the coefficients and scalars of the fractional objectives are fuzzy. Their proposed approach offers a new way of solving the problem that does not involve approximation or alteration of its nature.

Iskander [6] considers a programming problem involving linear fractional fuzzy objectives and stochastic fuzzy constraints. The coefficients and scalars in the linear fractional objectives, as well as the coefficients on the left-hand side of the limitations, can be trapezoidal or triangular fuzzy numbers. The variables on the right-hand side of the constraints are treated as independent random variables with known distribution functions. He proposes a modified possibility programming approach within the framework of chance constraints to transform the proposed programme into a deterministic equivalent problem with multiple linear objectives in cases of either overshoot or strict overshoot possibility. M. S. Osman et al. [10] considered a fuzzy objective programming approach to solving the fractional linear multiobjective programming problem involving fuzzy stochastic uncertainty. This approach employs α-cuts and the chance-constrained technique to transform the stochastic problem into a deterministic one.

Nasseri and Bavandi [11] transformed a stochastic fuzzy linear fractional programming problem into an equivalent deterministic multi-objective linear fractional programming problem. To find a solution, they converted the fuzzy multi-objective linear programming problem into a single-objective linear programming problem.

To better leverage the interplay between fuzziness and randomness, Khalifa et al. [7] recently studied a multi-objective fractional linear programming problem with random fuzzy coefficients and fuzzy pseudo-random decision variables. In their study, they converted the multi-objective random fuzzy fractional linear programming model into a single-objective fuzzy linear programming (FLP) model. They then demonstrated that a random optimal solution to an FLP problem can be derived from a set of optimal solutions to a relative pseudo-random linear programming (LP) model. Therefore, their theorems demonstrate that a random fuzzy optimal solution to a fuzzy pseudo-random LP problem can be obtained by combining a series of random optimal solutions to relative pseudo-random LP problems.

Building on previous work, Kumar et al. [8] developed a new approach to determining the acceptable range of objective values for a multi-objective fuzzy stochastic fractional linear programming problem. Their method involves constructing an expectation model based on the mean of the fuzzy random variable. To determine the level of satisfaction of decision-makers, the properties of the fuzzy set are applied to the objective function. The multi-objective fuzzy stochastic linear programming problem is then transformed into an equivalent deterministic form using the probability constraint programming method.

Upon analysing all these studies, we found that, in many cases, the solution space was limited. This offered decision-makers a single or limited set of feasible solutions, without accounting for their satisfaction levels or preferences. Furthermore, other studies failed to provide decision-makers with flexible means of modifying the conclusion based on their risk tolerance or degree of satisfaction. Additionally, some studies did not provide solutions with sufficiently wide confidence intervals to adequately capture uncertainty. Therefore, this work seeks to address these limitations and make the solutions practically applicable by combining the decision-maker's preferences and addressing both fuzzy and random uncertainty.

To achieve the results presented in this work, we first modified Dinkelbach's [12] and Guzel's [18] theorems to account for the decision-maker's preferences. This enabled us to convert the multi-objective fractional fuzzy linear random optimization problem into a stochastic multi-objective fuzzy linear programming problem. Next, we converted the multi-objective fuzzy random linear programming problem into a class of weighted relative pseudo-random multi-objective fuzzy linear programming problems. Finally, we introduced α-cut comparison elements to convert each class of fuzzy linear problems into a weighted, relative, pseudo-random, deterministic linear problem, depending on the decision-maker's satisfaction levels (α) and preferences (λ).

The results of this work provide decision-makers with a more comprehensive and reliable tool for situations that require optimizing outcomes amid ambiguity and uncertainty. Examples of such situations include inventory management, portfolio optimization, and wireless sensor networks. Unlike other methods, our approach offers a broader range of acceptable solutions that can be adapted to decision-makers' levels of satisfaction and preferences.

To present our results more clearly, we will first describe the fundamental concepts necessary for understanding our work in Section 2. Section 3 will show our main results. This section presents preliminary findings that will inform the theory we propose. Section 4 will present a conclusion and future directions.

2. Preliminaries

This section provides an overview of the fundamental concepts of fuzzy numbers, fuzzy random variables, and stochastic fuzzy fractional multi-objective linear programming.

2.1. Fundamentals of fuzzy set

Definition 2.1. [1]
Let X be a universe of discourse whose elements are denoted by x. A fuzzy set $\tilde{A}$ in X is formally defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)), x \in X\}, \tag{2.1}$$

where $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$, denotes the membership function of $\tilde{A}$ which assigns to each element $x \in X$ a degree of membership in the fuzzy set $\tilde{A}$.

Definition 2.2. [1]
A fuzzy set $\tilde{A}$ is said to be normalized if there exists at least one element $x \in X$ such that :

$$\mu_{\tilde{A}}(x) = 1. \tag{2.2}$$

Definition 2.3. [1]
An α-level subset (or α-cut) of a fuzzy set $\tilde{A}$, with $\alpha \in (0,1]$ denoted by $\tilde{A}^\alpha$ is defined as follows:

$$\tilde{A}^\alpha = \{x \in X, \mu_{\tilde{A}^\alpha}(x) \geq \alpha\}. \tag{2.3}$$

It is worth noting that the α-coupe, $\tilde{A}^\alpha$, is a closed interval expressed as follows:

$$\tilde{A}^\alpha = [a^L_\alpha, a^U_\alpha]$$

where $a^L_\alpha < a^U_\alpha$.

Definition 2.4. [1]
A fuzzy set $\tilde{A}$ of X is a convex fuzzy set if and only if $\forall x_1, x_2 \in X$ and $\lambda \in [0,1]$, we have:

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min\left(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right). \tag{2.4}$$

Definition 2.5. [25]
Let $\tilde{A}$ and $\tilde{B}$ be fuzzy sets of α-cuts, respectively $\tilde{A}^\alpha = [a^L_\alpha, a^U_\alpha]$ and $\tilde{B}^\alpha = [b^L_\alpha, b^U_\alpha]$.
For $\alpha \in (0, 1]$, we have the following relations.

1. Multiplication by a scalar:

$$k\tilde{A}^\alpha \simeq \begin{cases} [ka^L_\alpha, ka^U_\alpha] \text{ if } & k \geq 0, \\ [ka^U_\alpha, ka^L_\alpha] \text{ if } & k \leq 0. \end{cases} \tag{2.5}$$

2. Sum :
$$\tilde{A^\alpha} \oplus \tilde{B^\alpha} \tilde{=} \left[a^L_\alpha + b^L_\alpha, a^U_\alpha + b^U_\alpha\right], \tag{2.6}$$
3. Subtraction:
$$\tilde{A^\alpha} \ominus \tilde{B^\alpha} \tilde{=} \tilde{A^\alpha} \oplus (-1)\tilde{B^\alpha} = \left[a^L_\alpha - b^U_\alpha, a^U_\alpha - b^L_\alpha\right]. \tag{2.7}$$
4. Multiplication: Given $\tilde{A}$ and $\tilde{B}$ in $\mathbb{R}^+$, we have:
$$\tilde{A^\alpha} \odot \tilde{B^\alpha} \tilde{=} \left[\min(a^L_\alpha.b^L_\alpha, a^L_\alpha.b^U_\alpha, a^U_\alpha.b^L_\alpha, a^U_\alpha.b^U_\alpha), \max(a^L_\alpha.b^L_\alpha, a^L_\alpha.b^U_\alpha, a^U_\alpha.b^L_\alpha, a^U_\alpha.b^U_\alpha)\right], \tag{2.8}$$
5. Inverse:
$$\tilde{B}^{-1} \tilde{=} \left[\frac{1}{b^U_\alpha}, \frac{1}{b^L_\alpha}\right], with \quad b^L_\alpha, b^U_\alpha \neq 0. \tag{2.9}$$
6. Division:
$$\tilde{A^\alpha} \oslash \tilde{B^\alpha} \tilde{=} \left[a^L_\alpha, a^U_\alpha\right] \odot \left[\frac{1}{b^U_\alpha}, \frac{1}{b^L_\alpha}\right] = \left[\frac{a^L_\alpha}{b^U_\alpha}; \frac{a^U_\alpha}{b^L_\alpha}\right]; \quad b^L_\alpha, b^U_\alpha \neq 0. \tag{2.10}$$
7. Maximum ($\vee$) :
$$\vee_{i\in l}\left[(a_i)^L_\alpha, (a_i)^U_\alpha\right] \tilde{=} \left[\vee_{i\in l}(a_i)^L_\alpha, \vee_{i\in l}(a_i)^U_\alpha\right], \vee_{i\in l} a^L_\alpha < +\infty, \tag{2.11}$$
where I is the set of indices.
8. Minimum ($\wedge$):
$$\wedge_{i\in l}[(a_i)^L_\alpha, (a_i)^U_\alpha] \tilde{=} [\wedge_{i\in l}(a_i)^L_\alpha, \wedge_{i\in l}(a_i)^U_\alpha], \wedge_{i\in l}(a_i)^L_\alpha > -\infty, \tag{2.12}$$
where I is the set of indices.
9. Order relation ($\preceq$):
$$\tilde{A}^\alpha \preceq \tilde{B}^\alpha \Leftrightarrow a^L_\alpha \leq b^L_\alpha \; and \; a^U_\alpha \leq b^U_\alpha. \tag{2.13}$$

Definition 2.6. [25]
A fuzzy number is a normalized and convex fuzzy set $\tilde{A}$ of $\mathbb{R}$ whose membership function is piecewise continuous.

Definition 2.7. [25] A fuzzy set $\tilde{A}$ is said to be non-negative if $\mu_{\tilde{A}}(x) = 0$, $\forall x < 0$.

Definition 2.8. [25]
A fuzzy number $\tilde{a} = (a_1/a_2/a_3/a_4)$ is a trapezoidal fuzzy number if its membership function is defined by:
$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if} \quad a_1 < x \leq a_2, \\ 1 & \text{if} \quad a_2 \leq x \leq a_3, \\ \dfrac{a_4 - x}{a_4 - a_3} & \text{if} \quad a_3 \leq x < a_4, \\ 0 & \text{otherwise.} \end{cases}$$

The α-cut of $\tilde{a} = (a_1/a_2/a_3/a_4)$ is defined as follows:
$$\tilde{A}_\alpha = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4], \quad \forall \alpha \in [0, 1]. \tag{2.14}$$

2.2. Fuzzy random variables

Let (ψ, $\mathcal{A}$, P) be a probability measure space.

Let $F_0(\mathbb{R})$ be the space of all compact fuzzy numbers.

Definition 2.9. [7, 17]
An application $\tilde{A} : \psi \longrightarrow F_0(\mathbb{R})$ is said to be a fuzzy random variable on (ψ, A) if and only if:
$$\tilde{A}_\alpha(u) = (\tilde{A}(u))_\alpha = \{u \in \mathbb{R}/A(u)(x) \geq \alpha\} = [a^L_\alpha(u), a^U_\alpha(u)], \forall \alpha \in (0, 1]. \tag{2.15}$$
It is an interval on (ψ, A), that is to say that $a^L_\alpha(u)$ and $a^U_\alpha(u)$ are two random variables on (ψ, A).

The fuzzy random variable $\tilde{A}$ is said to be discrete if ψ is an enumerable set.

Furthermore, the concepts of a random fuzzy set and the concept of a random fuzzy variable are equivalent for functions with values belonging to $F_c(\mathbb{R})$ (the set of convex elements of $F_0(\mathbb{R}^n)$.

For the rest of this work, we will consider $FR(\psi)$ to be the set of all fuzzy variables on (ψ, A).

2.3. Generalities on multi-objective fuzzy random linear fractional programming

A multi-objective linear fractional programming problem with fuzzy random variables is formulated as suit [7] :

$$\begin{cases} \max \tilde{Z}_k(\tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X}) = \dfrac{\mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X})}{\mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})} = \dfrac{\tilde{C}_k^T \odot \tilde{X} \oplus \tilde{C}_k^0}{\tilde{D}_k^T \odot \tilde{X} \oplus \tilde{D}_k^0}, \quad \forall k = 1, 2, \cdots, K \\ S.t.: \\ \tilde{\Omega}(\tilde{A}, \tilde{B}, \tilde{X}) = \{\tilde{X} \in \mathbb{R}^n : \tilde{A} \odot \tilde{X} \leq \tilde{B}, \tilde{X} \geq 0\}, \end{cases} \tag{2.16}$$

where

- $\tilde{A}$ is a matrix of dimension $m \times n$, and$\tilde{B} = (\tilde{b}_i)_{m\times 1} \in \mathbb{R}^m$;
- $\tilde{C} = (\tilde{c}_{jk})_{1\times n} \in \mathbb{R}^n, \quad \tilde{D} = (\tilde{d}_{jk})_{1\times n} \in \mathbb{R}^n, \quad \tilde{X} = (\tilde{x}_j) \in \mathbb{R}^n, \quad \tilde{C}_k^0 \in \mathbb{R}, \quad \tilde{D}_k^0 \in \mathbb{R}$;
- $\left(\tilde{a}_{ij}, \quad \tilde{b}_i, \quad \tilde{c}_{jk}, \quad \tilde{d}_{jk}, \quad \tilde{c}_k^0, \quad \tilde{d}_k^0\right) \in \left[F_0(\mathbb{R})\right]^6, \quad \forall i = 1, \cdots, m, \quad \forall j = 1, \cdots, n, \quad \forall k = 1, \cdots, K$;
- $\mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}) = \tilde{D}_k^T \odot \tilde{X} \oplus \tilde{D}_k^0 > 0, \quad \forall k = 1, \cdots, K, \quad \forall \tilde{X} \in \tilde{\Omega}$.

In this case, the coefficients of the objective function, the constraints, and the decision variables are all fuzzy random variables.

Remark 2.10. [17]
$\tilde{X} \geq 0$ means that $(\tilde{x_j})_\alpha \geq 0, \quad \forall j = 1, 2, \cdots, n$ and $\forall \alpha \in (0, 1]$.
The problem (2.16) can be rewritten in the following form:
Find X such that, $\forall k$:

$$\frac{\tilde{C}_k^T \odot X \oplus \tilde{C}_k^0}{\tilde{D}_k^T \odot \tilde{X} \oplus \tilde{D}_k^0} = \max_{X \in \tilde{\Omega}} \frac{\tilde{C}_k^T \odot X \oplus \tilde{C}_k^0}{\tilde{D}_k^T \odot X \oplus \tilde{D}_k^0}. \tag{2.17}$$

The following definition will be useful when we have to defuzzify a fuzzy multi-objective linear fractional programming problem with random variables.

Definition 2.11. [7, 17]
The α-level set of $\left(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X}\right)$ is defined as the ordinary set $\left(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X}\right)_\alpha$ for which the degree of its membership function exceeds the level α.

Based on the concept of α-level, the problem (2.17) can be rewritten in non-fuzzy form as follows:

$$\begin{cases} \dfrac{\mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X})}{\mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})} = \dfrac{C_k^T X + C_k^0}{D_k^T X + D_k^0} = \max\limits_{X \in \Omega} \dfrac{C_k^T X + C_k^0}{D_k^T X + D_k^0}, \quad \forall k = 1, 2, \cdots, K, \\ (A, B, C, D, C^0, D^0, X) \in (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X})_\alpha, \\ \Omega(A, B, X) = \{X \in \mathbb{R}^n : AX \leq B, X \geq 0\}. \end{cases} \tag{2.18}$$

We have the following optimality conditions:

Definition 2.12. [7, 17]
$\tilde{X}^\circ \in \tilde{\Omega}(\tilde{A}, \tilde{B}, \tilde{X})$ is said to be an efficient fuzzy pseudo-random solution of the problem (2.17) if and only if

$$\mathcal{Z}_k(\tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X}) \leq \mathcal{Z}_k(\tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X}^\circ) \text{ and}$$
$$\mathcal{Z}_j(\tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X}) < \mathcal{Z}_j(\tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X}^\circ) \text{ for at least one } j.$$

Definition 2.13. [7, 17]
$X^\circ \in \Omega(A, B, X)$ is called an α-efficient solution of the problem (2.18) if and only if there does not exist another $X \in \Omega(A, B, X)$, $(A, B, C, D, C^0, D^0, X) \in (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{C}^0, \tilde{D}^0, \tilde{X})_\alpha$ such that:

$\mathcal{Z}_k(C, D, C^0, D^0, X^\circ) \leq \mathcal{Z}_k(C, D, C^0, D^0, X)$ and $\mathcal{Z}_k(C, D, C^0, D^0, X^\circ) < \mathcal{Z}_j(C, D, C^0, D^0, X)$ for at least one j.

The problem (2.17) reduces to the following fuzzy random linear programming problem:

$$\max_{\tilde{X} \in \tilde{\Omega}} \left\{ \sum_{k=1}^{K} (\mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})) \right\}, \tag{2.19}$$

$$with \quad \tilde{\mathcal{Z}}_k^* = \frac{\mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}^*)}{\mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}^*)} = \max_{\tilde{X} \in \tilde{\Omega}} \frac{\tilde{C}_k^T \odot \tilde{X} \oplus \tilde{C}_k^0}{\tilde{D}_k^T \odot \tilde{X} \oplus \tilde{D}_k^0}, \forall k = 1, 2, \cdots, K.$$

We will now introduce the following lemmas.

Lemma 2.14 (See in [7, 17])
Let $\tilde{A} \geq 0$, that is $(\tilde{a}_{ij})_\alpha \geq 0$, $\forall \alpha \in (0, 1]$. If $\tilde{X} \in \tilde{\Omega}$, then $X_\alpha^L \in \tilde{\Omega}_\alpha^L$ and $X_\alpha^U \in \tilde{\Omega}_\alpha^U$, $\forall \alpha \in (0, 1)$.

Lemma 2.15 (See in [7, 17])
Let $\tilde{A} \leq 0$, that is to say $(\tilde{a}_{ij})_\alpha \leq 0$, $\forall \alpha \in (0, 1]$. If $\tilde{X} \in \tilde{\Omega}$, then $X_\alpha^L \in \tilde{\Omega}_\alpha^U$ and $X_\alpha^U \in \tilde{\Omega}_\alpha^L$, $\forall \alpha \in (0, 1)$.

Lemma 2.16 (See in [7, 17])
Let $A_\alpha^L \leq 0$ and $A_\alpha^U \geq 0$, that is to say $(a_{ij})_\alpha^L \leq 0$ and $(a_{ij})_\alpha^U \geq 0$, $\forall \alpha \in (0, 1)$. If $\tilde{X} \in \tilde{\Omega}$, then $X_\alpha^U \in \tilde{\Omega}_\alpha(L, U)$, $\forall \alpha \in (0, 1)$.

Definition 2.17. [17]
Let $\{g_t : t \in l\} \subset F_0(\mathbb{R})$, $\alpha \in (0, 1]$.

1. $\wedge_{t \in l} g_t$ is defined by $h \in F_0(\mathbb{R})$ such that $h_\alpha = \wedge_{t \in l}(g_t)_\alpha$.
2. $\vee_{t \in l} g_t$ is defined by $h \in F_0(\mathbb{R})$ such that $h_\alpha = \vee_{t \in l}(g_t)_\alpha$.

Lemma 2.18
[17]
Let $g, h \in F_0(\mathbb{R})$, then for all $\alpha \in (0, 1]$, we obtain:

$$(g * h)_\alpha = g_\alpha * h_\alpha \text{ where } * \text{ is an algebraic operation such as } " + "," - ","/",".$$

Definition 2.19.
Suppose that $*$ is an algebraic operation on $\mathbb{R}$, $\tilde{A}, \tilde{B} \in F_0(\mathbb{R})$. The algebraic operation $*$ on $FR(\psi)$ is defined as follows:

$$(\tilde{A} * \tilde{B})(u) = \tilde{A}(u) * \tilde{B}(u), \forall u \in \psi.$$

Remark 2.20. [17]
$\tilde{A} \leq \tilde{B} \Leftrightarrow \tilde{A}(u) \leq \tilde{B}(u), \forall u \in \psi$, where $\tilde{A}, \tilde{B} \in FR(\psi)$.
If $\{\tilde{A}_i : i \in l\} \subset FR(\psi)$, where l is the set of indices, then

$$\left(=_{i \in l} \tilde{A}_i\right)(u) ==_{i \in l} \tilde{A}_i(u), \forall u \in \psi.$$

3. Methodology and Results

In this section, we propose a generalization of the method of Khalifa et al.[7]. To do this, we modify Dinkelbach's theorem[12] by inserting normalized weights.

3.1. Preliminary results

Let Ω be a compact and convex subset of $\mathbb{R}^n$. Let $\mathcal{P}(C, C^0, X)$ and $\mathcal{Q}(D, D^0, X)$ be real-valued functions $X \in \Omega$ and continuous. Furthermore, the following assumption is made:

$$\mathcal{Q}(D, D^0, X) > 0, \quad \forall X \in \Omega.$$

The following theorem is a modified version of Dinkelbach's theorem [12], which allows preferences to be given to the numerator and denominator of a ratio.

Theorem 3.1 (Weighted parametric optimality conditions)
Let $\Omega \subset \mathbb{R}^n$ be a compact and convex set.

Let $\mathcal{P}(C, C^0, \cdot)$ and $\mathcal{Q}(D, D^0, \cdot)$ be continuous real-valued functions on Ω, with $\mathcal{Q}(D, D^0, X) > 0$ for all $X \in \Omega$.

Let $\lambda_1, \lambda_2 \in [0, 1]$ be such that $\lambda_1 + \lambda_2 = 1$, $\lambda_1 \leq \lambda_2$, and further suppose that $\mathcal{Z}^* \geq 0$. The following assertions are verified.

(I) Necessary and sufficient condition (Dinkelbach's theorem).

$$\mathcal{Z}^* = \frac{\mathcal{P}(C, C^0, X^*)}{\mathcal{Q}(D, D^0, X^*)} = \max_{X \in \Omega} \frac{\mathcal{P}(C, C^0, X)}{\mathcal{Q}(D, D^0, X)} \tag{3.1}$$

if and only if

$$F(Z^*) := \max_{X \in \Omega} \big\{ \mathcal{P}(C, C^0, X) - \mathcal{Z}^* \, \mathcal{Q}(D, D^0, X) \big\} = 0, \tag{3.2}$$

the maximum being reached at X^*.

(II) Weighted necessary condition.
If $Z^* = \max_{X \in \Omega} \mathcal{P}/\mathcal{Q}$, then for all $\lambda_1, \lambda_2 \in [0, 1]$, with $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \leq \lambda_2$:

$$\lambda_1 \, \mathcal{P}(C, C^0, X) - \lambda_2 \, \mathcal{Z}^* \, \mathcal{Q}(D, D^0, X) \ \leq \ 0, \qquad \forall \, X \in \Omega, \tag{3.3}$$

$$\lambda_1 \, \mathcal{P}(C, C^0, X^*) - \lambda_2 \, \mathcal{Z}^* \, \mathcal{Q}(D, D^0, X^*) \ = \ (\lambda_1 - \lambda_2) \, \mathcal{Z}^* \, \mathcal{Q}(D, D^0, X^*) \ \leq \ 0. \tag{3.4}$$

In particular, the value $(\lambda_1 - \lambda_2) \, \mathcal{Z}^* \, \mathcal{Q}(D, D^0, X^*)$ is zero if and only if $\lambda_1 = \lambda_2 = \frac{1}{2}$ or $\mathcal{Z}^* = 0$.

(III) Sufficient weighted condition.
Case $\lambda_1 > 0$.
If there exists $X^* \in \Omega$ and $Z^* \geq 0$ such that

$$\lambda_1 \, \mathcal{P}(C, C^0, X^*) - \lambda_2 \, \mathcal{Z}^* \, \mathcal{Q}(D, D^0, X^*) = 0 \tag{3.5}$$

and

$$\lambda_1 \, \mathcal{P}(C, C^0, X) - \lambda_2 \, \mathcal{Z}^* \, \mathcal{Q}(D, D^0, X) \leq 0, \qquad \forall \, X \in \Omega, \tag{3.6}$$

then X^* is an optimal solution of the fractional program (3.1), with optimal value

$$\frac{\mathcal{P}(C, C^0, X^*)}{\mathcal{Q}(D, D^0, X^*)} = \frac{\lambda_2}{\lambda_1} \, Z^*. \tag{3.7}$$

When $\lambda_1 = \lambda_2 = \frac{1}{2}$, we find $Z^* = \mathcal{P}(C, C^0, X^*) \, / \, \mathcal{Q}(D, D^0, X^*)$.

Case $\lambda_1 = 0$ (i.e. $\lambda_2 = 1$). Condition (3.5) reduces to

$$-Z^* Q(D, D^0, X^*) = 0.$$

Since $Q(D, D^0, X^*) > 0$ by assumption, this forces $Z^* = 0$. Condition (3.6) then becomes

$$-Z^* Q(D, D^0, X) = 0 \leq 0, \quad \forall X \in \Omega,$$

which is trivially satisfied and imposes no constraint on the ratio $P(X)/Q(X)$. Therefore, conditions (3.5)–(3.6) carry no optimality information when $\lambda_1 = 0$: the value of P/Q is left unconstrained, and X^* cannot be certified as an optimal solution of problem (3.1) from these conditions alone.

In this degenerate case, one must either:

(a) revert to the classical Dinkelbach criterion (Part I), which remains valid independently of λ_1 and λ_2, or
(b) select any $\lambda_1 \in (0, \frac{1}{2}]$, for which the sufficient condition (3.5)–(3.1) applies without restriction.

Proof
To simplify the notation, we will write $\mathcal{P}(X)$, $\mathcal{Q}(X)$ instead of $\mathcal{P}(C, C^0, X)$, $\mathcal{Q}(D, D^0, X)$.

Proof of (I).
($\Rightarrow$) Suppose that $\mathcal{Z}^* = \max_{X \in \Omega} \mathcal{P}(X)/\mathcal{Q}(X)$.
For all $X \in \Omega$, we have $\mathcal{P}(X)/\mathcal{Q}(X) \leq \mathcal{Z}^*$, or $\mathcal{P}(X) - \mathcal{Z}^* \mathcal{Q}(X) \leq 0$. At X^*, the equality $\mathcal{P}(X^*)/\mathcal{Q}(X^*) = \mathcal{Z}^*$ gives $\mathcal{P}(X^*) = \mathcal{Z}^* \mathcal{Q}(X^*)$, therefore
$\mathcal{P}(X^*) - \mathcal{Z}^* \mathcal{Q}(X^*) = 0$. It follows that
$\mathcal{F}(Z^*) = \max_{X \in \Omega} \{\mathcal{P}(X) - \mathcal{Z}^* \mathcal{Q}(X)\} = 0$, the maximum being reached at X^*.
($\Leftarrow$) Suppose that $F(Z^*) = 0$, with the maximum being reached at X^*.
For all $X \in \Omega$: $\mathcal{P}(X) - \mathcal{Z}^* \mathcal{Q}(X) \leq 0$, hence $\mathcal{P}(X)/\mathcal{Q}(X) \leq \mathcal{Z}^*$.
On the other hand, $\mathcal{P}(X^*) - \mathcal{Z}^* \mathcal{Q}(X^*) = 0$ gives $\mathcal{P}(X^*)/\mathcal{Q}(X^*) = \mathcal{Z}^*$.
So $\mathcal{Z}^* = \max_{X \in \Omega} \mathcal{P}(X)/\mathcal{Q}(X)$.

Proof of (II).
Suppose $\mathcal{Z}^* = \max_{X \in \Omega} \mathcal{P}/\mathcal{Q} \geq 0$. For all $X \in \Omega$, we have $\mathcal{P}(X)/\mathcal{Q}(X) \leq \mathcal{Z}^*$, i.e. $\mathcal{P}(X) \leq \mathcal{Z}^* \mathcal{Q}(X)$. Since $\lambda_1 \geq 0$, $\lambda_1 \leq \lambda_2$, $\mathcal{Z}^* \geq 0$ and $\mathcal{Q}(X) > 0$:

$$\lambda_1 \mathcal{P}(X) \;\leq\; \lambda_1 \mathcal{Z}^* \mathcal{Q}(X) \;\leq\; \lambda_2 \mathcal{Z}^* \mathcal{Q}(X),$$

hence $\lambda_1 \mathcal{P}(X) - \lambda_2 \mathcal{Z}^* \mathcal{Q}(X) \leq 0$, which establishes (3.3).
In X^*, we have $\mathcal{P}(X^*) = \mathcal{Z}^* \mathcal{Q}(X^*)$, therefore:

$$\begin{aligned} \lambda_1 \mathcal{P}(X^*) - \lambda_2 \mathcal{Z}^* \mathcal{Q}(X^*) &= \lambda_1 Z^* \mathcal{Q}(X^*) - \lambda_2 \mathcal{Z}^* \mathcal{Q}(X^*) \\ &= (\lambda_1 - \lambda_2) \mathcal{Z}^* \mathcal{Q}(X^*), \end{aligned}$$

which is indeed ≤ 0 since $\lambda_1 \leq \lambda_2$, $\mathcal{Z}^* \geq 0$ and $\mathcal{Q}(X^*) > 0$.
This establishes (3.4).
The expression $(\lambda_1 - \lambda_2)\mathcal{Z}^* \mathcal{Q}(X^*)$ is zero if and only if $\lambda_1 = \lambda_2$ or $\mathcal{Z}^* = 0$.

Remark 3.2. The value $(\lambda_1 - \lambda_2)\mathcal{Z}^* \mathcal{Q}(X^*)$ is not necessarily the maximum of the functional $X \mapsto \lambda_1 \mathcal{P}(X) - \lambda_2 Z^* \mathcal{Q}(X)$ on Ω: other points $X \in \Omega$ can give a value within $\big((\lambda_1 - \lambda_2)\mathcal{Z}^* \mathcal{Q}(X^*),\, 0\big]$, since only the upper bound (3.3) is guaranteed.

Proof of (III)— Case $\lambda_1 > 0$.
Suppose that the conditions (3.5)–(3.6) are satisfied with $\lambda_1 > 0$. From (3.5):

$$\lambda_1 \mathcal{P}(X^*) = \lambda_2 \mathcal{Z}^* \mathcal{Q}(X^*) \implies \frac{\mathcal{P}(X^*)}{\mathcal{Q}(X^*)} = \frac{\lambda_2}{\lambda_1} \mathcal{Z}^*.$$

From (3.6), for all $X \in \Omega$: $\lambda_1 \mathcal{P}(X) \leq \lambda_2 \mathcal{Z}^* \mathcal{Q}(X)$, i.e.

$$\frac{\mathcal{P}(X)}{\mathcal{Q}(X)} \leq \frac{\lambda_2}{\lambda_1} \mathcal{Z}^* = \frac{\mathcal{P}(X^*)}{\mathcal{Q}(X^*)}.$$

Therefore, X^* maximizes P/Q over Ω, with an optimal value of $(\lambda_2/\lambda_1)\,\mathcal{Z}^*$, which establishes (3.7).

Proof of (III) — Case $\lambda_1 = 0$.
With $\lambda_1 = 0$ and $\lambda_2 = 1$, condition (3.5) becomes $-Z^* Q(X^*) = 0$. Since $Q(X^*) > 0$, we obtain $Z^* = 0$. Condition (3.6) then reads $0 \leq 0$ for all $X \in \Omega$, which holds trivially regardless of the value of $P(X)/Q(X)$. Hence no information on the optimality of X^* for problem (3.7) can be derived from (3.5)–(3.6) in this case, and the sufficient condition (III) is vacuous when $\lambda_1 = 0$.

□

Note that $\mathcal{Z}^*$ may not be unique. Furthermore, the theorem remains valid if we replace " max" with " min".

The theorem below is a modification of Guzel's theorem [18].

Theorem 3.3
$\overline{X}$ is an efficient solution to the multi-objective linear fractional programming problem if and only if $\forall k = 1, 2, \cdots, K$, for any λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$, with $\lambda_1 \leq \lambda_2$, and further suppose that $\mathcal{Z}_k^* \geq 0$ for all $k = 1, \ldots, K$. $\overline{X}$ is the optimal solution to the following problem:

$$\max_{X \in \Omega} \left\{ \sum_{k=1}^{K} (\lambda_1 \mathcal{P}_k(C, C^0, X) - \lambda_2 \mathcal{Z}_k^* \mathcal{Q}_k(D, D^0, X)) \right\},$$

with $\mathcal{Z}_k^* = \dfrac{\mathcal{P}_k(C, C^0, X^*)}{\mathcal{Q}_k(D, D^0, X^*)} = \max\limits_{X \in \Omega} \left\{ \dfrac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)} \right\}$.

Therefore, the multi-objective linear fractional programming problem is reduced to the linear programming problem.

Proof

Let X^* and $\mathcal{Z}^*$ denote respectively the point of global maximum (ideal point) and the values of each objective function of the multi-objective linear fractional programming problem, and let $\lambda_1, \lambda_2 \in [0, 1]$ be such that $\lambda_1 + \lambda_2 = 1$. We have

$$\mathcal{Z}_k^* = \mathcal{Z}_k^*(X^*) = \frac{\mathcal{P}_k(C, C^0, X^*)}{\mathcal{Q}_k(D, D^0, X^*)} = \max_{X \in \Omega} \left\{ \frac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)} \right\}, \forall k = 1, 2, \cdots, K. \tag{3.8}$$

So

$$\frac{\mathcal{P}_k(C, C^0, X^*)}{\mathcal{Q}_k(D, D^0, X^*)} \geq \frac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)}, \quad \forall X \in \Omega.$$

Since $\lambda_1 \leq \lambda_2$ and $\dfrac{\mathcal{P}_k(C, C^0, X^*)}{\mathcal{Q}_k(D, D^0, X^*)} \geq \dfrac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)}$, for all $X \in \Omega$ (from equation (3.8)) it follows directly that

$$\lambda_2 \frac{\mathcal{P}_k(C, C^0, X^*)}{\mathcal{Q}_k(D, D^0, X^*)} \geq \lambda_1 \frac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)}$$

$$\Rightarrow \lambda_2 \mathcal{Q}_k(D, D^0, X) \frac{\mathcal{P}_k(C, C^0, X^*)}{\mathcal{Q}_k(D, D^0, X^*)} \geq \lambda_1 \mathcal{P}_k(C, C^0, X), \forall X \in \Omega, \forall k = 1, \cdots, K.$$

Which leads to:
$\lambda_1\mathcal{P}_k(C,C^0,X)-\lambda_2\mathcal{Q}_k(D,D^0,X)\dfrac{\mathcal{P}_k(C,C^0,X^*)}{\mathcal{Q}_k(D,D^0,X^*)}\leq 0,\ \forall k=1,2,\cdots,K$ and $\forall X\in\Omega$. Hence

$$\lambda_1\mathcal{P}_k(C,C^0,X)-\lambda_2\mathcal{Z}_k^*\mathcal{Q}_k(D,D^0,X)\leq 0, \forall k=1,2,\cdots,K \quad and \quad \forall X\in\Omega. \tag{3.9}$$

Let $\overline{X}$ be an optimal solution of the problem:

$$\max\left\{\sum_{k=1}^{K}(\lambda_1\mathcal{P}_k(C,C^0,X)-\lambda_2 Z_k^*\mathcal{Q}_k(D,D^0,X)\Big|X\in\Omega)\right\}$$
$$\text{where}\quad \max\left\{\sum_{k=1}^{K}(\lambda_1\mathcal{P}_k(C,C^0,X)-\lambda_2\frac{\mathcal{P}_k(C,C^0,X^*)}{\mathcal{Q}_k(D,D^0,X^*)}\mathcal{Q}_k(D,D^0,X)\Big|X\in\Omega)\right\}.$$

From the equation (3.9), we have:

$$\begin{aligned}\sum_{k=1}^{K}(\lambda_1\mathcal{P}_k(C,C^0,X)-\lambda_2\mathcal{Z}_k^*\mathcal{Q}_k(D,D^0,X) &\leq \sum_{k=1}^{K}(\lambda_1\mathcal{P}_k(C,C^0,\overline{X})-\lambda_2\mathcal{Z}_k^*\mathcal{Q}_k(D,D^0,\overline{X}),\\ &\leq \sum_{k=1}^{K}\max\left\{\lambda_1\mathcal{P}_k(C,C^0,X)-\lambda_2\mathcal{Z}_k^*\mathcal{Q}_k(D,D^0,X)\right\},\\ &\leq 0,\quad \forall X\in\Omega,\quad \lambda_1,\ \text{and}\ \lambda_2\in[0,1],\ \text{such as}\ \lambda_1+\lambda_2=1.\end{aligned}$$

From these inequalities, we obtain:

$\lambda_1\mathcal{P}_k(C,C^0,X)-\lambda_2\mathcal{Z}_k^*\mathcal{Q}_k(D,D^0,X)\leq\lambda_1\mathcal{P}_k(C,C^0,\overline{X})-\lambda_2\mathcal{Z}_k^*\mathcal{Q}_k(D,D^0,\overline{X})\leq 0,\quad \forall k=1,\cdots,K,\quad \lambda_1,$ and $\lambda_2\in[0,1]$ such as $\lambda_1+\lambda_2=1$.

So

$$\left[\lambda_1\frac{\mathcal{P}_k(C,C^0,X)}{\mathcal{Q}_k(D,D^0,X)}-\lambda_2\mathcal{Z}_k^*\right]\mathcal{Q}_k(D,D^0,X)\leq\left[\lambda_1\frac{\mathcal{P}_k(C,C^0,\overline{X})}{\mathcal{Q}_k(D,D^0,\overline{X})}-\lambda_2\mathcal{Z}_k^*\right]\mathcal{Q}_k(D,D^0,\overline{X}),$$
$$\forall k=1,\cdots,K,\quad \lambda_1,\lambda_2\in[0,1]\ \text{such as}\ \lambda_1+\lambda_2=1.$$

Consequently,

$$\left[\lambda_1\frac{\mathcal{P}_k(C,C^0,X)}{\mathcal{Q}_k(D,D^0,X)}-\lambda_2\mathcal{Z}_k^*\right]\leq\frac{\mathcal{Q}_k(D,D^0,\overline{X})}{\mathcal{Q}_k(D,D^0,X)}\left[\lambda_1\frac{\mathcal{P}_k(C,C^0,\overline{X})}{\mathcal{Q}_k(D,D^0,\overline{X})}-\lambda_2\mathcal{Z}_k^*\right], \tag{3.10}$$
$$\forall k=1,\cdots,K,\quad \lambda_1,\text{and}\lambda_2\in[0,1]\ \text{such as}\ \lambda_1+\lambda_2=1. \tag{3.11}$$

Using Theorem 3.1, part (II), the quantities

$$\left[\lambda_1\frac{\mathcal{P}_k(C,C^0,X)}{\mathcal{Q}_k(D,D^0,X)}-\lambda_2\mathcal{Z}_k^*\right]\ \text{and}\ \left[\lambda_1\frac{\mathcal{P}_k(C,C^0,\overline{X})}{\mathcal{Q}_k(D,D^0,\overline{X})}-\lambda_2\mathcal{Z}_k^*\right]$$

are non-positive, $\forall k=1,\cdots,K,$ and $\lambda_1,\lambda_2\in[0,1]$ such as $\lambda_1+\lambda_2=1$, since $\mathcal{Z}_k^*=\max\limits_{X\in\Omega}\left\{\dfrac{\mathcal{P}_k(C,C^0,X)}{\mathcal{Q}_k(D,D^0,X)}\right\}$.

The necessary condition $\forall k=1,\cdots,K,\ \dfrac{\mathcal{Q}_k(D,D^0,\overline{X})}{\mathcal{Q}_k(D,D^0,X)}\geq 1$ implies:

$\dfrac{\mathcal{Q}_k(D,D^0,\overline{X})}{\mathcal{Q}_k(D,D^0,X)}\left[\lambda_1\dfrac{\mathcal{P}_k(C,C^0,\overline{X})}{\mathcal{Q}_k(D,D^0,\overline{X})}-\lambda_2\mathcal{Z}_k^*\right]\leq\left[\lambda_1\dfrac{\mathcal{P}_k(C,C^0,\overline{X})}{\mathcal{Q}_k(D,D^0,\overline{X})}-\lambda_2\mathcal{Z}_k^*\right],\ \forall k=1,...,K,\quad \lambda_1,$ and $\lambda_2\in[0,1]$

such as $\lambda_1 + \lambda_2 = 1$.

According to equation (3.10), we have:

$$\lambda_1 \frac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)} - \lambda_2 \mathcal{Z}_k^* \leq \lambda_1 \frac{\mathcal{P}_k(C, C^0, \overline{X})}{\mathcal{Q}_k(D, D^0, \overline{X})} - \lambda_2 \mathcal{Z}_k^*$$
$$\Longrightarrow \lambda_1 \frac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)} \leq \lambda_1 \frac{\mathcal{P}_k(C, C^0, \overline{X})}{\mathcal{Q}_k(D, D^0, \overline{X})}$$
$$\Longrightarrow \frac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)} \leq \frac{\mathcal{P}_k(C, C^0, \overline{X})}{\mathcal{Q}_k(D, D^0, \overline{X})}$$
$$\Longrightarrow \mathcal{Z}_k(X) \leq \mathcal{Z}_k(\overline{X}), \forall k = 1, 2, \cdots, K.$$

Hence, $\overline{X}$ is an efficient solution to the multi-objective linear fractional programming problem.

Now suppose that $\overline{X}$ is not an efficient solution to the multi-objective linear fractional programming problem. Then there exists an $X \in \Omega$ such that:

$\dfrac{\mathcal{P}_k(C, C^0, \overline{X})}{\mathcal{Q}_k(D, D^0, \overline{X})} \leq \dfrac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)}, \forall k = 1, 2, \cdots, K$ and $\dfrac{\mathcal{P}_j(C, C^0, \overline{X})}{\mathcal{Q}_j(D, D^0, \overline{X})} < \dfrac{\mathcal{P}_j(C, C^0, X)}{\mathcal{Q}_j(D, D^0, X)}$ for at least one $j \in \{1, ..., K\}$.

Then, using equation (3.8), $\mathcal{Z}_k^* \geq 0$, and for $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \leq \lambda_2$, we have:

$\lambda_1 \dfrac{\mathcal{P}_k(C, C^0, \overline{X})}{\mathcal{Q}_k(D, D^0, \overline{X})} \leq \lambda_1 \dfrac{\mathcal{P}_k(C, C^0, X)}{\mathcal{Q}_k(D, D^0, X)} \leq \lambda_2 \dfrac{\mathcal{P}_k(C, C^0, X^*)}{\mathcal{Q}_k(D, D^0, X^*)} = \lambda_2 \mathcal{Z}_k^*$.
Which leads to $\lambda_1 \mathcal{P}_k(C, C^0, \overline{X}) - \lambda_2 \mathcal{Z}_k^* \mathcal{Q}_k(D, D^0, \overline{X}) \leq \lambda_1 \mathcal{P}_k(C, C^0, X) - \lambda_2 \mathcal{Z}_k^* \mathcal{Q}_k(D, D^0, X)$, $\forall k \in 1, \cdots, K$, and
$\lambda_1 \dfrac{\mathcal{P}_j(C, C^0, \overline{X})}{\mathcal{Q}_j(D, D^0, \overline{X})} < \lambda_1 \dfrac{\mathcal{P}_j(C, C^0, X)}{\mathcal{Q}_j(D, D^0, X)} < \lambda_2 \dfrac{\mathcal{P}_j(C, C^0, X^*)}{\mathcal{Q}_j(D, D^0, X^*)} = \lambda_2 \mathcal{Z}_j^*$ for at least one $j \in \{1, ..., K\}$.
We have: $\lambda_1 \mathcal{P}_j(C, C^0, \overline{X}) - \lambda_2 \mathcal{Z}_j^* \mathcal{Q}_j(D, D^0, \overline{X}) < \lambda_1 \mathcal{P}_j(C, C^0, X) - \lambda_2 \mathcal{Z}_j^* \mathcal{Q}_j(D, D^0, X)$, for at least one $j \in \{1, ..., K\}$.
By summing the K inequalities, we have
$\sum_{k=1}^{K} \lambda_1 \mathcal{P}_k(C, C^0, \overline{X}) - \lambda_2 \mathcal{Z}_k^* \mathcal{Q}_k(D, D^0, \overline{X}) \leq \sum_{k=1}^{K} \lambda_1 \mathcal{P}_k(C, C^0, X) - \lambda_2 \mathcal{Z}_k^* \mathcal{Q}_k(D, D^0, X)\ \forall X \in \Omega$. This contradicts the fact that $\overline{X}$ is an optimal solution to the problem:

$$\max \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C, C^0, X) - \lambda_2 \mathcal{Z}_k^* \mathcal{Q}_k(D, D^0, X) \right) | X \in \Omega) \right\}.$$

□

3.2. Fundamental Results

The multi-objective linear fractional programming problem reduces to the following linear program:
For $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \leq \lambda_2$, we have:

$$\max \left\{ \sum_{k=1}^{K} (\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})) \right\}, \tilde{X} \in \tilde{\Omega} \tag{3.12}$$

$$where \quad \tilde{Z}_k^* = \frac{\mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}^*)}{\mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}^*)} = \max_{\tilde{X}\in\tilde{\Omega}} \frac{\tilde{C}_k^T \odot \tilde{X} \oplus \tilde{C}_k^0}{\tilde{D}_k^T \odot \tilde{X} \oplus \tilde{D}_k^0}, \forall k = 1, 2, \cdots, K.$$

Considering fuzzy random variables, we have the following modified relative problems:
For $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \leq \lambda_2$, we have:

$$\max_{X\in\tilde{\Omega}_\alpha^L} \left\{ \sum_{k=1}^{K} (\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha)) \right\}, \tag{3.13}$$

$$\max_{X\in\tilde{\Omega}_\alpha^U} \left\{ \sum_{k=1}^{K} (\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \lambda_2 \tilde{Z}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha)) \right\}, \tag{3.14}$$

$$\max_{X\in\tilde{\Omega}_\alpha^U} \left\{ \sum_{k=1}^{K} (\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha)) \right\}, \tag{3.15}$$

$$\max_{X\in\tilde{\Omega}_\alpha^L} \left\{ \sum_{k=1}^{K} (\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha)) \right\}, \tag{3.16}$$

$$\max_{X\in\tilde{\Omega}_\alpha^{(L,U)}} \left\{ \sum_{k=1}^{K} (\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha)) \right\}, \tag{3.17}$$

$$\max_{X\in\tilde{\Omega}_\alpha^{(L,U)}} \left\{ \sum_{k=1}^{K} (\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha)) \right\}, \tag{3.18}$$

where $\tilde{\mathcal{Z}}_k^*$ denotes the different ideal values of each of the objective functions. The proof of the relations (3.13)-(3.18) will be done using the following theorems:

Theorem 3.4
Assuming that $\tilde{A} \geq 0, \quad \tilde{C} \leq 0, \quad \tilde{D} \geq 0, \quad \tilde{\Omega}_\alpha^L \subset \{X_\alpha^L : \tilde{X} \in \tilde{\Omega}\}$ and $\tilde{\Omega}_\alpha^U \subset \{X_\alpha^U : \tilde{X} \in \tilde{\Omega}\}$, with $\alpha \in (0, 1]$.
Let $\lambda_1, \lambda_2 \in [0, 1]$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \leq \lambda_2$.
If $\tilde{X}^\circ$ is the pseudo-random optimal solution of the problem (3.12), then:

1. $X_\alpha^{\circ^L}$ is a pseudo-random optimal solution of the problem (3.16),
2. $X_\alpha^{\circ^U}$ is a pseudo-random optimal solution of the problem (3.15) and
3. (a) $\mathcal{Z}_\alpha^L = \max_{X\in\tilde{\Omega}_\alpha^U} \left\{ \sum_{k=1}^{K} \Big(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha)\Big) \right\}$,

   (b) $\mathcal{Z}_\alpha^U = \max_{X\in\tilde{\Omega}_\alpha^L} \left\{ \sum_{k=1}^{K} \Big(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha)\Big) \right\}$,

   with $\tilde{\mathcal{Z}} = \max_{\tilde{X}\in\tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})\right) \right\}$.

In what follows, we will set: $\tilde{\mathcal{Z}}_\alpha = [\mathcal{Z}_\alpha^L, \mathcal{Z}_\alpha^U]$.

Proof

Suppose that $\tilde{X}^{\circ}$ is a random optimal solution of the problem (3.12), then

$$\tilde{X}^{\circ} \in \tilde{\Omega} \text{ and } \mathcal{Z}_k(\tilde{X}^{\circ}) = \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}^{\circ}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}^{\circ})\right) \right\}$$
$$= \max_{\tilde{X} \in \tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})\right) \right\}.$$

Using Lemma 2.14, we have:

$$X^{\circ^L} \in \tilde{\Omega}_\alpha^L,\ X^{\circ^U} \in \tilde{\Omega}_\alpha^U,\ \tilde{\Omega}_\alpha^L = \{X_\alpha^L : \tilde{X} \in \tilde{\Omega}\} \text{ and } \tilde{\Omega}_\alpha^U = \{X_\alpha^U : \tilde{X} \in \tilde{\Omega}\}.$$

Considering $\tilde{C} \leq 0, \quad \tilde{D} \geq 0$, Definition 2.9 and Lemma 2.18, we have:

$$\begin{aligned}
\mathcal{Z}_k(\tilde{X}^{\circ}) &= \left\{ \sum_{k=1}^{K} \Big(\lambda_1 \mathcal{P}_k([C_\alpha^L, C_\alpha^U], \tilde{C}^0, [X_\alpha^{\circ^L}, X_\alpha^{\circ^U}]) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k([D_\alpha^L, D_\alpha^U], \tilde{D}^0, [X_\alpha^{\circ^L}, X_\alpha^{\circ^U}])\Big) \right\} \\
&= \Bigg[\sum_{k=1}^{K} \Big(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha^{\circ^U}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha^{\circ^U})\Big), \\
&\quad \sum_{k=1}^{K} \Big(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha^{\circ^L}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha^{\circ^L})\Big) \Bigg] \\
&= \Bigg[\sum_{k=1}^{K} \sum_{j=1}^{n} \left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^L, (c_k^0)_\alpha^L, (x_j^{\circ})^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^U, (d_j^0)_\alpha^U, (x_j^{\circ})_\alpha^U)\right), \\
&\quad \sum_{k=1}^{K} \sum_{j=1}^{n} \left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^U, (c_k^0)_\alpha^U, (x_j^{\circ})^L) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^L, (d_j^0)_\alpha^L, (x_j^{\circ})_\alpha^L)\right) \Bigg] \\
&= \sum_{k=1}^{K} \sum_{j=1}^{n} \Big[\left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^L, (c_k^0)_\alpha^L, (x_j^{\circ})^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^U, (d_j^0)_\alpha^U, (x_j^{\circ})_\alpha^U)\right), \\
&\quad \left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^U, (c_k^0)_\alpha^U, (x_j^{\circ})^L) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^L, (d_j^0)_\alpha^L, (x_j^{\circ})_\alpha^L)\right) \Big] \\
&= \sum_{k=1}^{K} \sum_{j=1}^{n} \Big[\left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^L, (c_k^0)_\alpha^L) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^U, (d_j^0)_\alpha^U)\right), \\
&\quad \left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^U, (c_k^0)_\alpha^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^L, (d_j^0)_\alpha^L)\right) \Big] \left[(x_j^{\circ})_\alpha^L, (x_j^{\circ})_\alpha^U\right] \\
&= \sum_{k=1}^{K} \sum_{j=1}^{n} \left(\lambda_1 \mathcal{P}_k((\tilde{c}_j)_\alpha, (\tilde{c}_k^0)_\alpha, (\tilde{x}_j^{\circ})_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((\tilde{d}_j)_\alpha, (\tilde{d}_k^0)_\alpha, (x_j^{\circ})_\alpha)\right) \\
&= \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}^{\circ}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}^{\circ})\right)_\alpha. \qquad (\star)
\end{aligned}$$

Using Definitions 2.9, 2.5, 2.19 and Lemma 2.18, we have:

$$\mathcal{Z}_k(\tilde{X}^{\circ}) = \left(\max_{\tilde{X} \in \tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{Z}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})\right) \right\} \right)_\alpha,$$
$$= \max_{\tilde{X} \in \tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})\right) \right\}_\alpha$$
$$= \max_{\tilde{X} \in \tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k([C_\alpha^L, C_\alpha^U], \tilde{C}^0, [X_\alpha^L, X_\alpha^U]) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k([D_\alpha^L, D_\alpha^U], \tilde{D}^0, [X_\alpha^L, X_\alpha^U])\right) \right\}_\alpha$$

$$= \max_{\tilde{X}\in\tilde{\Omega}} \sum_{k=1}^{K}\sum_{j=1}^{n} \left(\lambda_1 \mathcal{P}_k((\tilde{c}_j)_\alpha, (\tilde{c}_k^0)_\alpha, (\tilde{x}_j)_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((\tilde{d}_j)_\alpha, (\tilde{d}_k^0)_\alpha, (x_j)_\alpha)\right)$$

$$= \max_{\tilde{X}\in\tilde{\Omega}} \left(\sum_{k=1}^{K}\sum_{j=1}^{n} \left[\left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^L, (c_k^0)_\alpha^L) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^U, (d_j^0)_\alpha^U)\right) , \right.\right.$$
$$\left.\left. \left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^U, (c_k^0)_\alpha^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^L, (d_j^0)_\alpha^L)\right)\right] \left[(x_j)_\alpha^L, (x_j)_\alpha^U\right]\right)$$

$$= \max_{\tilde{X}\in\tilde{\Omega}} \sum_{k=1}^{K} \left[\left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha^U)\right) , \right.$$
$$\left. \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha^L) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha^L)\right)\right]$$

$$= \left[\max_{X\in\tilde{\Omega}_\alpha^U} \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha)\right), \right.$$
$$\left. \max_{X\in\tilde{\Omega}_\alpha^L} \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha)\right)\right]. \qquad (\star\star)$$

Then, $\forall\alpha \in (0,1]$, $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$ with $\lambda_1 \leq \lambda_2$, and relations $(\star)$ and $(\star\star)$, we have:

$$\sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha^{\circ^L}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha^{\circ^L})\right) = \max_{X\in\tilde{\Omega}_\alpha^L} \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \right.$$
$$\left. \lambda_2 \tilde{Z}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha)\right).$$

Which corresponds to the equation(3.16). And

$$\sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha^{\circ^U}) - \lambda_2 \tilde{Z}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha^{\circ^U})\right) = \max_{X\in\tilde{\Omega}_\alpha^U} \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \right.$$
$$\left. \lambda_2 \tilde{Z}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha)\right)$$

corresponds to the equation (3.15). □

Theorem 3.5
Suppose that $\tilde{A} \geq 0$, $\tilde{C} \geq 0$, $\tilde{D} \leq 0$, $\tilde{\Omega}_\alpha^L \subset \{X_\alpha^L : \tilde{X} \in \tilde{\Omega}\}$ and $\tilde{\Omega}_\alpha^U \subset \{X_\alpha^U : \tilde{X} \in \tilde{\Omega}\}$, with $\forall\alpha \in (0,1]$, $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \leq \lambda_2$,
If $\tilde{X}^\circ$ is the pseudo-random optimal solution of the problem (3.12), then we have:

1. $X_\alpha^{\circ^L}$ is a pseudo-random optimal solution of the problem (3.13),
2. $X_\alpha^{\circ^U}$ is a pseudo-random optimal solution of the problem (3.14) and
3. (a) $\mathcal{Z}_\alpha^L = \max_{X\in\tilde{\Omega}_\alpha^L} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha)\right) \right\}$,
   (b) $\mathcal{Z}_\alpha^U = \max_{X\in\tilde{\Omega}_\alpha^U} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \lambda_2 \tilde{Z}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha)\right) \right\}$, with
   $$\tilde{\mathcal{Z}} = \max_{\tilde{X}\in\tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X})) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X})\right) \right\}.$$

Proof
Suppose that $\tilde{X}^\circ$ is a random optimal solution to the problem (3.12), then

$$\tilde{X}^\circ \in \tilde{\Omega} \text{ and } \mathcal{Z}_k(\tilde{X}^\circ) = \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}^\circ) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}^\circ) \right) \right\}$$
$$= \max_{\tilde{X} \in \tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}) \right) \right\}.$$

Using Lemma 2.14, we have $X^{\circ^L} \in \tilde{\Omega}^L_\alpha,\ X^{\circ^U} \in \tilde{\Omega}^U_\alpha,\ \tilde{\Omega}^L_\alpha = \{X^L_\alpha : \tilde{X} \in \tilde{\Omega}\}$ *and* $\tilde{\Omega}^U_\alpha = \{X^U_\alpha : \tilde{X} \in \tilde{\Omega}\}$.
Considering $\tilde{C} \geq 0$, $\tilde{D} \leq 0$, Definition 2.9 and Lemma 2.18 , we have:

$$\mathcal{Z}_k(\tilde{X}^\circ) = \sum_{k=1}^{K} \Big(\lambda_1 \mathcal{P}_k([C^L_\alpha, C^U_\alpha], \tilde{C}^0, [X^{\circ^L}_\alpha, X^{\circ^U}_\alpha]) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k([D^L_\alpha, D^U_\alpha], \tilde{D}^0, [X^{\circ^L}_\alpha, X^{\circ^U}_\alpha]) \Big)$$
$$= \Bigg[\sum_{k=1}^{K} \Big(\lambda_1 \mathcal{P}_k(C^L_\alpha, C^{0^L}_\alpha, X^{\circ^L}_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D^U_\alpha, D^{0^U}_\alpha, X^{\circ^L}_\alpha) \Big),$$
$$\sum_{k=1}^{K} \Big(\lambda_1 \mathcal{P}_k(C^U_\alpha, C^{0^U}_\alpha, X^{\circ^U}_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D^L_\alpha, D^{0^L}_\alpha, X^{\circ^U}_\alpha) \Big) \Bigg]$$
$$= \Bigg[\sum_{k=1}^{K} \sum_{j=1}^{n} \left(\lambda_1 \mathcal{P}_k((c_j)^L_\alpha, (c^0_k)^L_\alpha, (x^\circ_j)^L) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)^U_\alpha, (d^0_j)^U_\alpha, (x^\circ_j)^L_\alpha) \right),$$
$$\sum_{k=1}^{K} \sum_{j=1}^{n} \left(\lambda_1 \mathcal{P}_k((c_j)^U_\alpha, (c^0_k)^U_\alpha, (x^\circ_j)^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)^L_\alpha, (d^0_j)^L_\alpha, (x^\circ_j)^U_\alpha) \right) \Bigg]$$
$$= \sum_{k=1}^{K} \sum_{j=1}^{n} \Big[\left(\lambda_1 \mathcal{P}_k((c_j)^L_\alpha, (c^0_k)^L_\alpha, (x^\circ_j)^L) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)^U_\alpha, (d^0_j)^U_\alpha, (x^\circ_j)^L_\alpha) \right),$$
$$\left(\lambda_1 \mathcal{P}_k((c_j)^U_\alpha, (c^0_k)^U_\alpha, (x^\circ_j)^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)^L_\alpha, (d^0_j)^L_\alpha, (x^\circ_j)^U_\alpha) \right) \Big]$$
$$= \sum_{k=1}^{K} \sum_{j=1}^{n} \Big[\left(\lambda_1 \mathcal{P}_k((c_j)^L_\alpha, (c^0_k)^L_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)^U_\alpha, (d^0_j)^U_\alpha) \right),$$
$$\left(\lambda_1 \mathcal{P}_k((c_j)^U_\alpha, (c^0_k)^U_\alpha) - \lambda_2 \tilde{Z}_k^* \mathcal{Q}_k((dj)^L_\alpha, (d^0_j)^L_\alpha) \right) \Big] \left[(x^\circ_j)^L_\alpha, (x^\circ_j)^U_\alpha \right]$$
$$= \sum_{k=1}^{K} \sum_{j=1}^{n} \left(\lambda_1 \mathcal{P}_k((\tilde{c}_j)_\alpha, (\tilde{c}^0_k)_\alpha, (\tilde{x}^\circ_j)_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((\tilde{d}_j)_\alpha, (\tilde{d}^0_k)_\alpha, (x^\circ_j)_\alpha) \right)$$
$$= \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}^\circ) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}^\circ) \right)_\alpha. \qquad (\star)$$

Using Definitions 2.9, 2.5, 2.19 and Lemma 2.18, we have:

$$\mathcal{Z}_k(\tilde{X}^\circ) = \left(\max_{\tilde{X} \in \tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}) \right) \right\} \right)_\alpha,$$
$$= \max_{\tilde{X} \in \tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}) \right) \right\}_\alpha$$

$$= \max_{\tilde{X}\in\tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k([C_\alpha^L, C_\alpha^U], \tilde{C}^0, [X_\alpha^L, X_\alpha^U]) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k([D_\alpha^L, D_\alpha^U], \tilde{D}^0, [X_\alpha^L, X_\alpha^U]) \right) \right\}_\alpha$$

$$= \max_{\tilde{X}\in\tilde{\Omega}} \sum_{k=1}^{K} \sum_{j=1}^{n} \left(\lambda_1 \mathcal{P}_k((\tilde{c}_j)_\alpha, (\tilde{c}_k^0)_\alpha, (\tilde{x}_j)_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((\tilde{d}_j)_\alpha, (\tilde{d}_k^0)_\alpha, (x_j)_\alpha) \right)$$

$$= \max_{\tilde{X}\in\tilde{\Omega}} \left(\sum_{k=1}^{K} \sum_{j=1}^{n} \left[\left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^L, (c_k^0)_\alpha^L) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^U, (d_j^0)_\alpha^U) \right), \right. \right.$$

$$\left. \left. \left(\lambda_1 \mathcal{P}_k((c_j)_\alpha^U, (c_k^0)_\alpha^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k((dj)_\alpha^L, (d_j^0)_\alpha^L) \right) \right] \left[(x_j)_\alpha^L, (x_j)_\alpha^U \right] \right)$$

$$= \max_{\tilde{X}\in\tilde{\Omega}} \sum_{k=1}^{K} \left[\left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha^L) - \lambda_2 \tilde{Z}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha^L) \right), \right.$$

$$\left. \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha^U) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha^U) \right) \right]$$

$$= \left[\max_{X\in\tilde{\Omega}_\alpha^L} \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha) \right), \right.$$

$$\left. \max_{X\in\tilde{\Omega}_\alpha^U} \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha) \right) \right]. \qquad (\star\star)$$

Then, for $\alpha \in (0,1]$, $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$ with $\lambda_1 \leq \lambda_2$ and relations $(\star)$ and $(\star\star)$ we have:

$$\sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha^{\circ^U}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha^{\circ^U}) \right) = \max_{X\in\tilde{\Omega}_\alpha^U} \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^U, C_\alpha^{0^U}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^L, D_\alpha^{0^L}, X_\alpha) \right)$$

which corresponds to the equation (3.14) and

$$\sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha^{\circ^L}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha^{\circ^L}) \right) = \max_{X\in\tilde{\Omega}_\alpha^L} \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(C_\alpha^L, C_\alpha^{0^L}, X_\alpha) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(D_\alpha^U, D_\alpha^{0^U}, X_\alpha) \right)$$

which corresponds to the equation (3.13). □

Theorem 3.6
Suppose that $A_\alpha^L \leq 0, \quad A_\alpha^U \geq 0, \quad C_\alpha^L \leq 0, \quad C_\alpha^U \geq 0, \quad D_\alpha^U \geq 0, \quad D_\alpha^L \leq 0$ and $\tilde{\Omega}_\alpha^{(L,U)} \subset \{X_\alpha^U : \tilde{X} \in \tilde{\Omega}\}$, with $\alpha \in (0,1]$, $\lambda_1, \lambda_2 \in [0,1]$ such as $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \leq \lambda_2$.
If $\tilde{X}^\circ$ is a pseudo-random optimal solution of the problem (3.12), then for all $\alpha \in (0,1]$, $X_\alpha^{\circ^U}$ is a pseudo-random optimal solution of the problem (3.17) and (3.18).

Proof
Suppose that $\tilde{X}^\circ$ is a random optimal solution of the problem (3.12), then

$$\tilde{X}^\circ \in \tilde{\Omega} \text{ and } \mathcal{Z}_k(\tilde{X}^\circ) = \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}^\circ) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}^\circ) \right) \right\}$$
$$= \max_{\tilde{X}\in\tilde{\Omega}} \left\{ \sum_{k=1}^{K} \left(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{\mathcal{Z}}_k^* \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}) \right) \right\}.$$

Using Lemma 2.16, we have

$$X^{\circ^U} \in \tilde{\Omega}^{L,U}_{\alpha}, \ \forall \alpha \in (0,1].$$

Considering $A^L_\alpha \leq 0, \quad A^U_\alpha \geq 0, \quad C^L_\alpha \leq 0, \quad C^U_\alpha \geq 0, \quad D^U_\alpha \geq 0, \quad D^L_\alpha \leq 0$, the Definition 2.9 and the Lemma 2.18, we have:

$$\begin{aligned}
\mathcal{Z}_k(\tilde{X}^\circ) &= \left\{\sum_{k=1}^{K}\Big(\lambda_1 \mathcal{P}_k([C^L_\alpha, C^U_\alpha], \tilde{C}^0, [X^{\circ^L}_\alpha, X^{\circ^U}_\alpha]) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k([D^L_\alpha, D^U_\alpha], \tilde{D}^0, [X^{\circ^L}_\alpha, X^{\circ^U}_\alpha])\Big)\right\} \\
&= \left[\sum_{k=1}^{K}\Big(\lambda_1 \mathcal{P}_k(C^L_\alpha, C^{0^L}_\alpha, X^{\circ^U}_\alpha) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k(D^U_\alpha, D^{0^U}_\alpha, X^{\circ^U}_\alpha)\Big),\right. \\
&\left.\sum_{k=1}^{K}\Big(\lambda_1 \mathcal{P}_k(C^U_\alpha, C^{0^U}_\alpha, X^{\circ^U}_\alpha) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k(D^L_\alpha, D^{0^L}_\alpha, X^{\circ^U}_\alpha)\Big)\right] \\
&= \left[\sum_{k=1}^{K}\sum_{j=1}^{n}\big(\lambda_1 \mathcal{P}_k((c_j)^L_\alpha, (c^0_k)^L_\alpha, (x^\circ_j)^U) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k((dj)^U_\alpha, (d^0_j)^U_\alpha, (x^\circ_j)^U_\alpha)\big),\right. \\
&\left.\sum_{k=1}^{K}\sum_{j=1}^{n}\big(\lambda_1 \mathcal{P}_k((c_j)^U_\alpha, (c^0_k)^U_\alpha, (x^\circ_j)^U) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k((dj)^L_\alpha, (d^0_j)^L_\alpha, (x^\circ_j)^U_\alpha)\big)\right] \\
&= \sum_{k=1}^{K}\sum_{j=1}^{n}\big[\big(\lambda_1 \mathcal{P}_k((c_j)^L_\alpha, (c^0_k)^L_\alpha, (x^\circ_j)^U) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k((dj)^U_\alpha, (d^0_j)^U_\alpha, (x^\circ_j)^U_\alpha)\big), \\
&\big(\lambda_1 \mathcal{P}_k((c_j)^U_\alpha, (c^0_k)^U_\alpha, (x^\circ_j)^U) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k((dj)^L_\alpha, (d^0_j)^L_\alpha, (x^\circ_j)^U_\alpha)\big)\big] \\
&= \sum_{k=1}^{K}\sum_{j=1}^{n}\big[\big(\lambda_1 \mathcal{P}_k((c_j)^L_\alpha, (c^0_k)^L_\alpha) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k((dj)^U_\alpha, (d^0_j)^U_\alpha)\big), \\
&\big(\lambda_1 \mathcal{P}_k((c_j)^U_\alpha, (c^0_k)^U_\alpha) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k((dj)^L_\alpha, (d^0_j)^L_\alpha)\big)\big][(x^\circ_j)^L_\alpha, (x^\circ_j)^U_\alpha] \\
&= \sum_{k=1}^{K}\sum_{j=1}^{n}\big(\lambda_1 \mathcal{P}_k((\tilde{c}_j)_\alpha, (\tilde{c}^0_k)_\alpha, (\tilde{x}^\circ_j)_\alpha) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k((\tilde{d}_j)_\alpha, (\tilde{d}^0_k)_\alpha, (x^\circ_j)_\alpha)\big) \\
&= \sum_{k=1}^{K}\big(\lambda_1 \mathcal{P}_k(\tilde{C}, \tilde{C}^0, \tilde{X}^\circ) - \lambda_2 \tilde{\mathcal{Z}}^*_k \mathcal{Q}_k(\tilde{D}, \tilde{D}^0, \tilde{X}^\circ)\big)_\alpha. \quad (\star)
\end{aligned}$$

Using Definitions 2.9, 2.5, 2.19 and Lemma 2.18, we have:

$$\mathcal{Z}_k(\tilde{X}^{\circ}) = \left(\max_{\tilde{X}\in\tilde{\Omega}}\left\{\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k(\tilde{C},\tilde{C}^0,\tilde{X}) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(\tilde{D},\tilde{D}^0,\tilde{X})\right)\right\}\right)_\alpha,$$
$$= \max_{\tilde{X}\in\tilde{\Omega}}\left\{\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k(\tilde{C},\tilde{C}^0,\tilde{X}) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(\tilde{D},\tilde{D}^0,\tilde{X})\right)\right\}_\alpha$$
$$= \max_{\tilde{X}\in\tilde{\Omega}}\left\{\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k([C_\alpha^L,C_\alpha^U],\tilde{C}^0,[X_\alpha^L,X_\alpha^U]) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k([D_\alpha^L,D_\alpha^U],\tilde{D}^0,[X_\alpha^L,X_\alpha^U])\right)\right\}_\alpha$$
$$= \max_{\tilde{X}\in\tilde{\Omega}}\sum_{k=1}^{K}\sum_{j=1}^{n}\left(\lambda_1\mathcal{P}_k((\tilde{c}_j)_\alpha,(\tilde{c}_k^0)_\alpha,(\tilde{x}_j)_\alpha) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k((\tilde{d}_j)_\alpha,(\tilde{d}_k^0)_\alpha,(x_j)_\alpha)\right)$$
$$= \max_{\tilde{X}\in\tilde{\Omega}}\left(\sum_{k=1}^{K}\sum_{j=1}^{n}\left[\left(\lambda_1\mathcal{P}_k((c_j)_\alpha^L,(c_k^0)_\alpha^L) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k((dj)_\alpha^U,(d_j^0)_\alpha^U)\right),\right.\right.$$
$$\left.\left.\left(\lambda_1\mathcal{P}_k((c_j)_\alpha^U,(c_k^0)_\alpha^U) - \lambda_2\tilde{Z}_k^*\mathcal{Q}_k((dj)_\alpha^L,(d_j^0)_\alpha^L)\right)\right]\left[(x_j)_\alpha^L,(x_j)_\alpha^U\right]\right)$$
$$= \max_{\tilde{X}\in\tilde{\Omega}}\sum_{k=1}^{K}\left[\left(\lambda_1\mathcal{P}_k(C_\alpha^L,C_\alpha^{0^L},X_\alpha^U) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(D_\alpha^U,D_\alpha^{0^U},X_\alpha^U)\right),\right.$$
$$\left.\left(\lambda_1\mathcal{P}_k(C_\alpha^U,C_\alpha^{0^U},X_\alpha^U) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(D_\alpha^L,D_\alpha^{0^L},X_\alpha^U)\right)\right]$$
$$= \left[\max_{X\in\tilde{\Omega}_\alpha^{(L,U)}}\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k(C_\alpha^L,C_\alpha^{0^L},X_\alpha) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(D_\alpha^U,D_\alpha^{0^U},X_\alpha)\right),\right.$$
$$\left.\max_{X\in\tilde{\Omega}_\alpha^{(L,U)}}\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k(C_\alpha^U,C_\alpha^{0^U},X_\alpha) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(D_\alpha^L,D_\alpha^{0^L},X_\alpha)\right)\right]. \qquad (\star\star)$$

Then, for $\alpha \in (0,1]$, $\lambda_1, \lambda_2 \in [0,1]$ such that $\lambda_1 + \lambda_2 = 1$ with $\lambda_1 \leq \lambda_2$, and relations $(\star)$, $(\star\star)$, we have:

$$\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k(C_\alpha^U,C_\alpha^{0^U},X_\alpha^{\circ^U}) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(D_\alpha^L,D_\alpha^{0^L},X_\alpha^{\circ^U})\right) = \max_{X\in\tilde{\Omega}_\alpha^{(L,U)}}\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k(C_\alpha^U,C_\alpha^{0^U},X_\alpha) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(D_\alpha^L,D_\alpha^{0^L},X_\alpha)\right)$$

which corresponds to the equation (3.18), and

$$\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k(C_\alpha^L,C_\alpha^{0^L},X_\alpha^{\circ^L}) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(D_\alpha^U,D_\alpha^{0^U},X_\alpha^{\circ^L})\right) = \max_{X\in\tilde{\Omega}_\alpha^{(L,U)}}\sum_{k=1}^{K}\left(\lambda_1\mathcal{P}_k(C_\alpha^L,C_\alpha^{0^L},X_\alpha) - \lambda_2\tilde{\mathcal{Z}}_k^*\mathcal{Q}_k(D_\alpha^U,D_\alpha^{0^U},X_\alpha)\right)$$

which corresponds to the equation (3.17). □

Remark 3.7. By varying the signs of $\tilde{A}, \tilde{C}$ and $\tilde{D}$, we obtain similar conclusions.

3.2.1. Handling General Sign Configurations Theorems 3.5–3.6 address three canonical sign configurations of the coefficient matrices $\tilde{A}$, $\tilde{C}$, and $\tilde{D}$. In practice, problems with mixed or non-standard sign patterns may arise. The following three strategies extend the applicability of the proposed framework to such cases.

Strategy 3.8 (Coefficient Decomposition)**.** Any fuzzy random coefficient $\tilde{a}_{ij}$ whose α-cuts contain values of mixed sign can be decomposed as

$$\tilde{a}_{ij} \; = \; \tilde{a}_{ij}^{+} \ominus \tilde{a}_{ij}^{-},$$

where $\tilde{a}_{ij}^{+} = \max(\tilde{a}_{ij}, 0)$ and $\tilde{a}_{ij}^{-} = \max(-\tilde{a}_{ij}, 0)$ are non-negative fuzzy numbers. Substituting this decomposition reduces the problem to a sign configuration compatible with one of the three cases covered by Theorems 3.4–3.6.

Algorithm 1 Solution Framework for Multi-Objective Fuzzy Random Linear Fractional Programming

Require: Fuzzy random problem data: $\tilde{A}$, $\tilde{B}$, $\{\tilde{C}_k, \tilde{D}_k, \tilde{C}_k^0, \tilde{D}_k^0\}_{k=1}^{K}$
Require: Parameter grids: $\mathcal{A} \subset (0,1]$ for α-levels, $\mathcal{L} \subset [0,1]^2$ for weights
Ensure: Solution family $\mathcal{S} = \{(\alpha, \lambda_1, \lambda_2, \tilde{X}^*)\}$

Stage I: Fractional-to-Linear Transformation
for $k = 1, \ldots, K$ **do**
  Solve: $\tilde{Z}_k^* = \max_{\tilde{X} \in \tilde{\Omega}} \frac{\tilde{C}_k^T \odot \tilde{X} \oplus \tilde{C}_k^0}{\tilde{D}_k^T \odot \tilde{X} \oplus \tilde{D}_k^0}$ ▷ Compute ideal values
end for

Stage II: Preference-Weighted Formulation
Transform to: $\max_{\tilde{X} \in \tilde{\Omega}} \sum_{k=1}^{K} \left(\lambda_1 P_k(\tilde{C}, \tilde{C}^0, \tilde{X}) - \lambda_2 \tilde{Z}_k^* Q_k(\tilde{D}, \tilde{D}^0, \tilde{X})\right)$

Stage III: Defuzzification and Solution
Initialize solution set: $\mathcal{S} \leftarrow \emptyset$
for each $\alpha \in \mathcal{A}$ **do**
  for each $(\lambda_1, \lambda_2) \in \mathcal{L}$ with $\lambda_1 + \lambda_2 = 1$, $\lambda_1 \leq \lambda_2$ **do**
    Apply α-cuts: $[\tilde{A}_\alpha^L, \tilde{A}_\alpha^U] \leftarrow \text{AlphaCut}(\tilde{A}, \alpha)$
    Similarly for $\tilde{B}, \tilde{C}_k, \tilde{D}_k, \tilde{C}_k^0, \tilde{D}_k^0$
    Determine coefficient signs: $\sigma \leftarrow \text{SignConfig}(\tilde{A}, \tilde{C}, \tilde{D})$
    if $\sigma = (\tilde{A} \geq 0, \tilde{C} \leq 0, \tilde{D} \geq 0)$ **then** ▷ Theorem 3.4
      $X_{L,\alpha}^* \leftarrow \arg\max_{X \in \Omega_\alpha^L} \sum_k \left(\lambda_1 P_k(C_\alpha^U, C_\alpha^{0U}, X) - \lambda_2 Z_k^* Q_k(D_\alpha^L, D_\alpha^{0L}, X)\right)$
      $X_{U,\alpha}^* \leftarrow \arg\max_{X \in \Omega_\alpha^U} \sum_k \left(\lambda_1 P_k(C_\alpha^L, C_\alpha^{0L}, X) - \lambda_2 Z_k^* Q_k(D_\alpha^U, D_\alpha^{0U}, X)\right)$
    else if $\sigma = (\tilde{A} \geq 0, \tilde{C} \geq 0, \tilde{D} \leq 0)$ **then** ▷ Theorem 3.5
      $X_{L,\alpha}^* \leftarrow \arg\max_{X \in \Omega_\alpha^L} \sum_k \left(\lambda_1 P_k(C_\alpha^L, C_\alpha^{0L}, X) - \lambda_2 Z_k^* Q_k(D_\alpha^U, D_\alpha^{0U}, X)\right)$
      $X_{U,\alpha}^* \leftarrow \arg\max_{X \in \Omega_\alpha^U} \sum_k \left(\lambda_1 P_k(C_\alpha^U, C_\alpha^{0U}, X) - \lambda_2 Z_k^* Q_k(D_\alpha^L, D_\alpha^{0L}, X)\right)$
    else if $\sigma =$ mixed signs **then** ▷ Theorem 3.6
      Apply Theorem 3.6 formulation (Equations 3.17-3.18)
    end if
    $\tilde{X}^*(\alpha, \lambda_1, \lambda_2) \leftarrow [X_{L,\alpha}^*, X_{U,\alpha}^*]$
    $\mathcal{S} \leftarrow \mathcal{S} \cup \{(\alpha, \lambda_1, \lambda_2, \tilde{X}^*)\}$
  end for
end for
return $\mathcal{S}$ ▷ Complete solution family

Strategy 3.9 (Variable Substitution)**.** If the objective coefficient matrix $\tilde{C}$ does not satisfy the sign condition required by the applicable theorem, introduce the substitution $x_j = x_j^+ - x_j^-$, with $x_j^+, x_j^- \geq 0$, for each decision variable x_j. This reformulation restores the non-negativity of the relevant coefficient products without altering the feasible region $\tilde{\Omega}$.

Strategy 3.10 (Conservative Interval-Arithmetic Reformulation)**.** When neither Strategy 3.8 nor Strategy 3.9 is directly applicable, a robust reformulation can be employed: for each $\alpha \in (0,1]$, replace every fuzzy coefficient

$\tilde{c}_{jk}$ by the interval $\big[\min(\tilde{c}_{jk})_\alpha,\ \max(\tilde{c}_{jk})_\alpha\big]$ and solve the resulting deterministic interval linear program. This approach is *conservative*: the resulting solution intervals are guaranteed to contain the true fuzzy-optimal interval, but may be wider. A penalty-based (Big-M) variant can further approximate the interval-valued objective, at the cost of introducing a parameter M whose calibration may influence solution quality.

Remark 3.11. Strategies 3.8–3.10 collectively ensure that the methodology remains applicable beyond the specific sign configurations of Theorems 3.5–3.6, thus extending the practical scope of the proposed framework. The development of unified theorems that remove sign restrictions entirely–possibly leveraging generalized α-cut comparison operators–remains a priority direction for future research.

3.3. Algorithmic Framework

To enhance reproducibility and practical implementation, we provide a detailed algorithmic description of our solution methodology. The complete framework consists of three main stages, as outlined in Algorithm 1.

3.3.1. Computational Complexity Let n denote the number of decision variables, m the number of constraints, K the number of objectives, $|\mathcal{A}|$ the number of α-levels, and $|\mathcal{L}|$ the number of preference weight pairs. The time complexity of Algorithm 1 decomposes naturally across its three stages.

- **Stage I** solves K independent fractional programs to obtain the ideal values $\tilde{Z}_k^*$, requiring
$$\mathcal{O}\big(K \cdot T_{\mathrm{LP}}(n,m)\big)$$
time, where $T_{\mathrm{LP}}(n,m)$ denotes the complexity of solving a single linear program.
- **Stage III** solves at most $2 \cdot |A| \cdot |L|$ linear programs (two per $(\alpha, \lambda_1, \lambda_2)$ combination, corresponding to the lower- and upper-bound subproblems), requiring
$$\mathcal{O}\big(|A| \cdot |L| \cdot T_{\mathrm{LP}}(n,m)\big)$$
time.

The total complexity is therefore
$$\mathcal{O}\Big(\big(K + |A| \cdot |L|\big) \cdot T_{\mathrm{LP}}(n,m)\Big), \tag{3.19}$$
where $T_{\mathrm{LP}}(n,m) = \mathcal{O}(n^3)$ for the simplex method or $\mathcal{O}(n^{3.5})$ for interior-point methods. Since in practice $|A| \cdot |L| \gg K$, the dominant term is $\mathcal{O}(|A| \cdot |L| \cdot n^3)$.

3.3.2. Implementation Guidelines For practical implementation, we recommend:

- **Parameter selection:** Use $\mathcal{A} = \{0.3, 0.5, 0.7, 1.0\}$ for exploratory analysis, or $\mathcal{A} = \{0.1, 0.2, \ldots, 1.0\}$ for high-precision applications. For preference weights, $\mathcal{L} = \{(\lambda, 1-\lambda) : \lambda = 0, 0.1, \ldots, 0.5\}$ provides good coverage of the Pareto frontier.
- **LP solvers:** Standard commercial solvers (Gurobi, CPLEX) or open-source alternatives (GLPK, COIN-OR) can be employed. For large-scale problems, interior-point methods generally outperform simplex.
- **Numerical stability:** Scale problem coefficients to avoid numerical instability. Use tolerance $\epsilon = 10^{-6}$ for feasibility checks and optimality conditions.
- **Fuzzy arithmetic:** Implement interval arithmetic for α-cuts. For trapezoidal fuzzy numbers (a_1, a_2, a_3, a_4), the α-cut is computed as:
$$[\tilde{a}]_\alpha = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4] \tag{3.20}$$

3.3.3. Dominance Relations for Interval-Valued Fuzzy Random Solutions To compare solutions in the fuzzy random setting where objective values are characterized by interval-valued α-cuts, we introduce the following dominance relations, which extend classical Pareto dominance to the interval-valued context.

Definition 3.12 (Dominance for Interval-Valued Solutions)**.** Let $\tilde{X}^1$ and $\tilde{X}^2$ be two solutions to problem (2.16), with their respective objective values given by the α-cuts:

$$\tilde{Z}^1_\alpha = [Z^{1,L}_\alpha, Z^{1,U}_\alpha] \quad \text{and} \quad \tilde{Z}^2_\alpha = [Z^{2,L}_\alpha, Z^{2,U}_\alpha], \quad \forall \alpha \in (0,1]. \tag{3.21}$$

We define three types of dominance relationships between interval-valued fuzzy random solutions:

1. **Strong Dominance** [26, 27]: $\tilde{X}^1$ strongly dominates $\tilde{X}^2$ (denoted $\tilde{X}^1 \succ_s \tilde{X}^2$) if and only if:

$$Z^{1,L}_\alpha \geq Z^{2,L}_\alpha \quad \text{and} \quad Z^{1,U}_\alpha \geq Z^{2,U}_\alpha, \quad \forall \alpha \in (0,1], \tag{3.22}$$

with at least one strict inequality for some $\alpha \in (0,1]$.
2. **Intermediate Dominance** [28, 29]: $\tilde{X}^1$ intermediately dominates $\tilde{X}^2$ (denoted $\tilde{X}^1 \succ_i \tilde{X}^2$) if and only if:

$$Z^{1,L}_\alpha \geq Z^{2,L}_\alpha \quad \text{or} \quad Z^{1,U}_\alpha \geq Z^{2,U}_\alpha, \quad \forall \alpha \in (0,1], \tag{3.23}$$

but not necessarily both simultaneously. This implies that the solution is superior in at least one dimension of uncertainty.
3. **Weak Dominance** [30, 31]: $\tilde{X}^1$ weakly dominates $\tilde{X}^2$ (denoted $\tilde{X}^1 \succ_w \tilde{X}^2$) if and only if:

$$\frac{Z^{1,L}_\alpha + Z^{1,U}_\alpha}{2} \geq \frac{Z^{2,L}_\alpha + Z^{2,U}_\alpha}{2}, \quad \forall \alpha \in (0,1]. \tag{3.24}$$

This criterion compares the midpoints of the intervals, representing the expected values under uniform uncertainty.

Remark 3.13. For multi-objective problems with K objectives, these dominance relations extend naturally by requiring that the dominance condition holds for all objectives $k = 1, \ldots, K$. Specifically, $\tilde{X}^1$ dominates $\tilde{X}^2$ (in any of the three senses defined above) if the respective dominance relation is satisfied for all $k = 1, \ldots, K$, with at least one objective exhibiting strict dominance for at least one $\alpha \in (0,1]$.

Remark 3.14 (Relationship to Classical Definitions)**.** The proposed dominance relations are consistent with and extend the efficient solution concept given in Definition 2.12. When $\alpha = 1$ (complete certainty), strong dominance reduces to the classical Pareto dominance in deterministic multi-objective optimization [32]. The intermediate and weak dominance concepts provide practical decision-making criteria when decision-makers have different risk attitudes or when complete strong dominance is too restrictive in the presence of significant uncertainty.

Remark 3.15 (Justification and Interpretation)**.** These dominance definitions are well-grounded in fuzzy set theory and interval arithmetic [28, 25]:

- **Strong dominance** guarantees that solution $\tilde{X}^1$ is better than $\tilde{X}^2$ in all possible scenarios (both lower and upper bounds), representing the most conservative criterion.
- **Intermediate dominance** reflects improvement in at least one dimension of uncertainty, which is acceptable in practical decision contexts where partial improvement is valuable [30, 29].
- **Weak dominance** is particularly useful when decision-makers focus on central tendency rather than worst-case or best-case scenarios, aligning with expected value approaches in stochastic optimization [33].

3.4. Application: Solving didactic example

We take here an example taken from [7].

$$\begin{cases} \max \tilde{\mathcal{Z}}_1 = \dfrac{(4/7/10/12)x_1 + (8/10/14/15)x_2 + (2.5/4.5/7.5/11.5)x_3 + (2/3/4/6)}{(10/14/20/22)x_1 + (20/3/27/29)x_2 + (18/20/25/28)x_3 + (5/10/18/20)} \\ \max \tilde{\mathcal{Z}}_2 = \dfrac{(18/20/24/28)x_1 + (16/18/25/30)x_2 + (12/14/19/25)x_3 + (1/3/6/10)}{(14/16/19/23)x_1 + (18/21/25/27)x_2 + (15/20/25/30)x_3 + (10/15/20/25)} \\ S.t.: \\ (10/17/19/25)x_1 + (14/16/22/24)x_2 + (20/25/27/30)x_3 \leq (25/35/40/55) \\ (0.01/0.03/0.07/0.09)x_1 + (0.03/0.05/0.08/0.1)x_2 + (0.01/0.02/0.06/0.07)x_3 \leq (0.3/0.5/0.9/1.0) \\ (4/6/10/13)x_1 + (0/5/10/15)x_2 + (8/11/14/20)x_3 \leq (10/20/30/40) \\ x_1, x_2, x_3 \in \mathbb{R}^+. \end{cases} \tag{3.25}$$

Before presenting the numerical results, we verify the sign conditions for problem (3.25) to identify the applicable theorem (Theorem 3.4, 3.5, or 3.6) and ensure reproducibility.

- **Constraint matrix** $\tilde{A}$**.** All components of $\tilde{A}$ (e.g., $(10/17/19/25)$, $(14/16/22/24)$, $(20/25/27/30)$, $(0.01/0.03/0.07/0.09)$, $(4/6/10/13)$) have strictly positive α-cut bounds for all $\alpha \in (0, 1]$. Hence $\tilde{A} \geq 0$.
- **Numerator coefficients** $\tilde{C}$**.** The numerator vectors $\tilde{C}_k$ (e.g., $(4/7/10/12)$, $(8/10/14/15)$, $(18/20/24/28)$, $(16/18/25/30)$) have non-negative components in both objectives. Hence $\tilde{C} \geq 0$.
- **Denominator coefficients** $\tilde{D}$**.** The denominator vectors $\tilde{D}_k$ (e.g., $(10/14/20/22)$, $(14/16/19/23)$) have non-negative components. Hence $\tilde{D} \geq 0$ as stated.

Since both objectives are maximisation problems with $\tilde{A} \geq 0$, $\tilde{C} \geq 0$, and $\tilde{D} \geq 0$, the direct application of Theorem 3.5 (which requires $\tilde{D} \leq 0$) calls for a standard reformulation: maximising $\tilde{Z}_k$ is equivalent to minimising $-\tilde{Z}_k$, for which the effective denominator becomes $-\tilde{D}_k \leq 0$. This places problem (3.25) within the scope of **Theorem 3.5** ($\tilde{A} \geq 0$, $\tilde{C} \geq 0$, $\tilde{D} \leq 0$ in the minimisation reformulation). Accordingly, the optimal α-cut bounds Z^L_α and Z^U_α are computed via the pair of linear programs (3.13)–(3.14).

Changing λ_1 in the interval $[0, 1]$, we obtain the following results:

Table 1. Summary of the results of the proposed method

Results of the proposed method for $\alpha = 0.6$				
Weight	$\tilde{\mathcal{Z}}_\alpha$	$(\tilde{x_1})_\alpha$	$(\tilde{x_2})_\alpha$	$(\tilde{x_3})_\alpha$
$\lambda = 0$	$[-28.42, -15.52]$	0	0	0
$\lambda = 0.1$	$[-25.1, -12.72]$	0	0	0
$\lambda = 0.2$	$[-21.78, -9.93]$	0	0	0
$\lambda = 0.3$	$[-18.46, -7.14]$	0	0	0
$\lambda = 0.4$	$[-8.10, 2.02]$	$[2.14953271, 2.18309859]$	0	0
$\lambda = 0.5$	$[2.81, 16.99]$	$[2.14953271, 2.18309859]$	0	0
$\lambda = 0.6$	$[13.74, 31.17]$	$[2.14953271, 2.18309859]$	0	0
$\lambda = 0.7$	$[24.67, 46.94]$	$[2.14953271, 2.18309859]$	0	0
$\lambda = 0.8$	$[35.60, 63.11]$	$[0, 2.14953271]$	$[0, 2.03947368]$	0
$\lambda = 0.9$	$[46.53, 79.97]$	$[0, 2.14953271]$	$[0, 2.03947368]$	0
$\lambda = 1$	$[58.06, 96.83]$	0	$[0, 2.01754386]$	0

Using the method of Khalifa et al. [7], the solution to the problem (3.25) for $\alpha = 0.6$ is given by

Table 2. Results of the method of Khalifa et al.[7]

Results of Khalifa et al.[7] method for $\alpha = 0.6$			
$\tilde{Z}_\alpha$	$(\tilde{x_1})_\alpha$	$(\tilde{x_2})_\alpha$	$(\tilde{x_3})_\alpha$
$[-23.6295, 33.99]$	$[0, 2.18309859]$	0	0

Quantitative Comparison

To provide rigorous, objective evidence supporting the dominance claims, we introduce the following two performance metrics, which allow interval-valued solutions to be compared on a dimensionally consistent basis.

Definition 3.16 (Bound Improvement)**.** Let $\tilde{Z}^{\text{ours}}_\alpha = [Z^{L,\text{ours}}_\alpha, Z^{U,\text{ours}}_\alpha]$ and $\tilde{Z}^{\text{ref}}_\alpha = [Z^{L,\text{ref}}_\alpha, Z^{U,\text{ref}}_\alpha]$ be two interval-valued objective values for a given $\alpha \in (0, 1]$. The **lower bound improvement** δ^L and **upper bound improvement** δ^U are defined as:

$$\delta^L = \frac{Z^{L,\text{ours}}_\alpha - Z^{L,\text{ref}}_\alpha}{\left|Z^{L,\text{ref}}_\alpha\right|} \times 100\%, \qquad \delta^U = \frac{Z^{U,\text{ours}}_\alpha - Z^{U,\text{ref}}_\alpha}{\left|Z^{U,\text{ref}}_\alpha\right|} \times 100\%. \tag{3.26}$$

A positive δ^L (resp. δ^U) indicates that our solution yields a strictly higher lower bound (resp. upper bound), meaning an improvement in the worst-case (resp. best-case) objective value.

Definition 3.17 (Uncertainty Reduction)**.** The **interval width** $W_\alpha = Z^U_\alpha - Z^L_\alpha$ measures the degree of uncertainty in the solution. The relative uncertainty reduction is:

$$\Delta W = \frac{W^{\text{ours}}_\alpha - W^{\text{ref}}_\alpha}{W^{\text{ref}}_\alpha} \times 100\%. \tag{3.27}$$

A negative ΔW indicates that our solution is associated with less uncertainty than the reference.

Applying Definitions 3.16 and 3.17 to the results of Tables 1 and 2, the reference solution of Khalifa et al. [7] for $\alpha = 0.6$ is:

$$\tilde{Z}^{\text{ref}}_{\alpha=0.6} = [-23.6295,\ 33.99], \quad W_{\text{ref}} = 57.62, \quad M_{\text{ref}} = 5.18. \tag{3.28}$$

Table 3 reports, for each value of $\lambda_1 \in [0, 1]$, the proposed solution, the improvements of the lower and upper bounds (δ^L, δ^U), the reduction of uncertainty (ΔW), and the type of dominance (Definition 3.12).

Table 3. Quantitative comparison for the didactic example ($\alpha = 0.6$). Reference: $\tilde{Z}_{\text{ref}} = [-23.63, 33.99]$; $W_{\text{ref}} = 57.62$. Improvements δ^L, δ^U, ΔW are computed according to Definitions 3.16 and 3.17.

λ_1	Z^L_α	Z^U_α	δ^L (%)	δ^U (%)	W_α	ΔW (%)	Dominance
0.0	−28.42	−15.52	−20.27	−145.66	12.90	−77.61	None
0.1	−25.10	−12.72	−6.22	−137.42	12.38	−78.51	None
0.2	−21.78	−9.93	+7.83	−129.21	11.85	−79.43	Intermediate
0.3	−18.46	−7.14	+21.88	−121.01	11.32	−80.35	Intermediate
0.4	−8.10	2.02	+65.72	−94.06	10.12	−82.44	Intermediate
0.5	2.81	16.99	+111.89	−50.01	14.18	−75.39	Intermediate
0.6	13.74	31.17	+158.15	−8.30	17.43	−69.75	Intermediate
0.7	24.67	46.94	+204.40	+38.10	22.27	−61.35	**Strong**
0.8	35.60	63.11	+250.66	+85.67	27.51	−52.26	**Strong**
0.9	46.53	79.97	+296.91	+135.28	33.44	−41.96	**Strong**
1.0	58.06	96.83	+345.71	+184.88	38.77	−32.71	**Strong**

The results in Table 3 allow us to draw the following objective conclusions.

Strong Pareto dominance for $\lambda_1 \in [0.7, 1.0]$. For all four values $\lambda_1 \in \{0.7, 0.8, 0.9, 1.0\}$, our method simultaneously improves *both* the lower and upper bounds relative to the reference [7], thus establishing strong dominance in the sense of Definition 3.12. The improvements are quantified as follows:

- For $\lambda_1 = 0.7$: $\delta^L = +204.40\%$ and $\delta^U = +38.10\%$, corresponding to absolute improvements of $+48.30$ and $+12.95$ units in the lower and upper bounds, respectively. The uncertainty interval is reduced by 61.35% (from $W = 57.62$ to $W = 22.27$).
- For $\lambda_1 = 0.8$: $\delta^L = +250.66\%$ and $\delta^U = +85.67\%$, corresponding to absolute improvements of $+59.23$ and $+29.12$ units. The uncertainty is reduced by 52.26%.
- For $\lambda_1 = 0.9$: $\delta^L = +296.91\%$ and $\delta^U = +135.28\%$, corresponding to absolute improvements of $+70.16$ and $+45.98$ units. The uncertainty is reduced by 41.96%.
- For $\lambda_1 = 1.0$: $\delta^L = +345.71\%$ and $\delta^U = +184.88\%$, corresponding to absolute improvements of $+81.69$ and $+62.84$ units. The resulting interval $[58.06, 96.83]$ lies entirely above the reference interval $[-23.63, 33.99]$, confirming unambiguous Pareto superiority.

Intermediate dominance for $\lambda_1 \in [0.2, 0.6]$. For $\lambda_1 \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$, the lower bound improves significantly (by $+7.83\%$ to $+158.15\%$), while the upper bound remains below that of the reference. This constitutes intermediate dominance (Definition 3.12). Notably, at $\lambda_1 = 0.6$, the upper bound deficit is only -8.30% (-2.82 absolute units), nearly qualifying as strong dominance. In all these cases, the uncertainty interval is substantially narrower (by 69.75% to 82.44%), indicating markedly reduced uncertainty.

Flexibility advantage. In contrast to the single solution $[-23.63, 33.99]$ provided by [7], our method generates a family of $|\mathcal{A}| \times |\mathcal{L}|$ solutions parameterized by (α, λ_1). For $\alpha = 0.6$ alone, this yields 11 distinct solutions covering a wide range of risk profiles. Decision-makers seeking strong dominance can select $\lambda_1 \in [0.7, 1.0]$; those willing to accept intermediate dominance in exchange for narrower uncertainty bands can select $\lambda_1 \in [0.2, 0.6]$.

3.5. Application to a stock management problem

Consider the problem below taken from [7].

3.5.1. Problem statement

The following tables provide information about the inventory of a commercial enterprise. The inventory includes two items $(i = 1, 2)$, for which the information is summarized below:

Table 4. First article

Item 1	Possession	Purchase Price	Selling Price	Order Cost	Space Required
1	8	115	140	60	0
2	10	120	150	70	1
3	12	125	200	80	2
4	16	130	300	100	4

Table 5. Second batch of Article

Item 2	Possession	Purchase Price	Selling Price	Order Cost	Space Required
1	12	140	180	70	2
2	14	150	200	80	3
3	16	160	220	90	4
4	20	170	240	100	5

The maximum budget available for this problem is $90,000. There is storage space for 300 items. The maximum number of orders that can be placed is five, with a fixed cost of $7 per order.

3.5.2. Data analysis

The holding cost, purchase price, selling price, ordering cost, and required space for each item are approximate figures based on market conditions. For example, the purchase price for the first item is $(115/120/125/130)$, and the purchase price for the second item is $(140/150/160/170)$. The data can then be summarised as fuzzy coefficients (trapezoidal fuzzy numbers) as follows:

Table 6. Summarize the data in fuzzy form

Item	$\tilde{h}_i$	$\tilde{P}_i$	$\tilde{S}_i$	$\tilde{OC}_i$	$\tilde{f}_i$
i = 1	(8/10/12/16)	(115/120/125/130)	(140/150/200/300)	(60/70/80/100)	(0/1/2/4)
i = 2	(12/14/16/20)	(140/150/160/170)	(180/200/220/240)	(70/80/90/100)	(2/3/4/5)

The constants are defined as follows: $\beta = 80000, \quad \gamma = 1.20, \quad N_0 = 5, \quad \lambda = 7, \quad F = 300, \quad B = 90000,$
i : number of items, $i = 1, 2, ..., n$
λ : fixed cost per order
B : maximum budget available for all items
F : maximum space available for all items
N_0 : maximum number of orders placed
$\tilde{Q}_i$: quantity of item ordered (decision variable)
$\tilde{h}_i$: cost of possession per item per unit of time for the i-th item
$\tilde{P}_i$:purchase price for the i-th item
$\tilde{S}_i$: selling price for the i-th item
$\tilde{D}_i$: demand per unit of time of the i-th article
$\tilde{f}_i$: space required per unit for the i-th article
$\tilde{OC}_i$: cost of ordering the i-th item.

3.5.3. Hypotheses:

The following assumptions were taken into account when formulating the suggested model.

i. We assume a multi-item inventory model without shortages.

ii. We are considering an infinite time horizon.

iii. Zero delays taken into account

iv. The cost of possession is constant for each item.

v. Demand is inversely proportional to the selling price

$$\tilde{D} = \tilde{D}_i(S_i) = \beta \tilde{S}_i^{-\gamma}$$

where $\beta > 0$ is a scaling constant and $\gamma > 1$ is the price elasticity coefficient.
To simplify the notation, D_i and $D_i(S_i)$ can be used interchangeably in this research.
There is no discount; that is, the purchase price is constant for each item.

3.5.4. Modeling:

Based on the above assumptions, the following multi-objective fuzzy linear fractional inventory model is formulated:

$$\begin{cases} \max \tilde{\mathcal{Z}}_1 = \dfrac{\sum\limits_{i=1}^{n}(\tilde{S}_i - \tilde{P}_i)\tilde{Q}_i}{\sum\limits_{i=1}^{n}(\beta\tilde{S}_i^{-\gamma} - \tilde{Q}_i)} \\ \min \tilde{\mathcal{Z}}_2 = \dfrac{\sum\limits_{i=1}^{n}\dfrac{\tilde{h}_i\tilde{Q}_i}{2}}{\sum\limits_{i=1}^{n}\tilde{Q}_i} \end{cases} \tag{3.29}$$

$$\text{Constraint I}: \sum_{i=1}^{n}\tilde{P}_i\tilde{Q}_i \leq B,$$

$$\text{Constraint II}: \sum_{i=1}^{n}\tilde{f}_i\tilde{Q}_i \leq F,$$

$$\text{Constraint III}: \sum_{i=1}^{n}\tilde{S}_i^{\gamma}\tilde{Q}_i \geq \frac{\beta}{N_0},$$

$$\text{Constraint IV}: \tilde{Q}_i \geq \frac{\lambda\beta}{\tilde{OC}_i\tilde{S}_i^{\gamma}}, \forall i = 1, 2, \cdots, n,$$

$$\text{Constraint V}: \begin{cases} N_0 > 0, \\ \beta > 0, \\ \lambda > 0, \\ \gamma > 1, \\ \tilde{Q}_i > 0, \forall i = 1, 2, \cdots, n, \\ \tilde{OC}_i > 0, \forall i = 1, 2, \cdots, n, \end{cases} \tag{3.30}$$

where,

- $\sum\limits_{i=1}^{n}(\tilde{S}_i - \tilde{P}_i)\tilde{Q}_i$ represents profit,
- $\sum\limits_{i=1}^{n}(\beta\tilde{S}_i^{-\gamma} - \tilde{Q}_i)$ represents the quantity that is out of stock.
- $\sum\limits_{i=1}^{n}\dfrac{\tilde{h}_i\tilde{Q}_i}{2}$ represents the cost of possession,
- $\sum\limits_{i=1}^{n}\tilde{Q}_i$ represents the total quantity ordered.

In this model, the constraint

I: represents the restriction on the total budget,

II: represents the restriction on the total space,

III: represents the upper limit of the number of orders,

Remark 3.18 (Consistency of exponent signs in the objective and Constraint III)**.** The demand function $\tilde{D}_i(\tilde{S}_i) = \beta \tilde{S}_i^{-\gamma}$ involves a *negative* exponent: a higher selling price reduces demand, consistently with standard price-elasticity theory ($\gamma > 1$).

Constraint III, which bounds the maximum number of replenishment orders, is derived from the Economic Order Quantity (EOQ) relationship. For item i, the number of orders per unit time equals

$$\frac{\tilde{D}_i}{\tilde{Q}_i} = \frac{\beta \tilde{S}_i^{-\gamma}}{\tilde{Q}_i}.$$

Summing over all items and imposing the upper bound N_0 gives

$$\sum_{i=1}^{n} \frac{\beta \tilde{S}_i^{-\gamma}}{\tilde{Q}_i} \leq N_0.$$

Multiplying both sides by $\dfrac{\prod_i \tilde{S}_i^{\gamma}}{\beta}$ and rearranging yields the equivalent form

$$\sum_{i=1}^{n} \tilde{S}_i^{\gamma} \tilde{Q}_i \geq \frac{\beta}{N_0},$$

which is precisely Constraint III. The *positive* exponent γ in Constraint III therefore arises from the algebraic inversion of the demand function, and is fully consistent with the negative exponent appearing in the objective $\tilde{\mathcal{Z}}_\infty$.

IV: represents the budget constraint on the order cost,

V: represents non-negative restrictions.

3.5.5. Results obtained

For $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_2 = 1 - \lambda_1$, we have the following table of results:

Table 7. Summary table of the results obtained by the two methods.

Results of the proposed method for $\alpha = 0.6$			
Weight	$\tilde{\mathcal{Z}}_\alpha$	$(\tilde{Q}_1)_\alpha$	$(\tilde{Q}_2)_\alpha$
$\lambda = 0$	$[-6917.94, -928.99]$	$[93.28, 441.89]$	$[8.82, 13.40]$
$\lambda = 0.1$	$[-5064.70, -366.61]$	$[93.28, 441.89]$	$[8.82, 13.40]$
$\lambda = 0.2$	$[-3211.45, -1662.22]$	$[93.28, 441.89]$	$[8.82, 13.40]$
$\lambda = 0.3$	$[-2957.84, -1358.20]$	$[93.28, 441.89]$	$[8.82, 13.40]$
$\lambda = 0.4$	$[-4253.45, 495.03]$	$[93.28, 441.89]$	$[8.82, 13.40]$
$\lambda = 0.5$	$[-5542.07, 2348.28]$	$[93.28, 441.89]$	$[8.82, 13.40]$
$\lambda = 0.6$	$[-5863.48, 4201.53]$	$[31.5, 93.28]$	$[8.82, 108.11]$
$\lambda = 0.7$	$[-4097.58, 6054.77]$	$[31.5, 93.28]$	$[8.82, 108.11]$
$\lambda = 0.8$	$[-2331.68, 7908.02]$	$[31.5, 93.28]$	$[8.82, 108.11]$
$\lambda = 0.9$	$[-565.78, 9761.27]$	$[31.5, 93.28]$	$[8.82, 108.11]$
$\lambda = 1$	$[1200, 11614]$	$[31.5, 93.28]$	$[8.82, 108.11]$
Results of Khalifa et al. [7] method for $\alpha = 0.6$			
$\tilde{\mathcal{Z}}_\alpha$		$\tilde{Q}_1$	$\tilde{Q}_2$
$[-11098, 4696.56]$		$[93.28, 441.89]$	$[8.82, 13.40]$

3.5.6. Quantitative Comparison

Applying Definitions 3.16 and 3.17 to the results of Table 7, the reference solution of Khalifa et al. [7] for $\alpha = 0.6$ is as follows:

$$\tilde{Z}^{\text{ref}}_{\alpha=0.6} = [-11098,\ 4696.56], \quad W_{\text{ref}} = 15794.56, \quad M_{\text{ref}} = -3200.72. \tag{3.31}$$

Table 8 reports the complete quantitative comparison for each value of λ_1.

Table 8. Quantitative comparison for the inventory management problem ($\alpha = 0.6$). Reference: $\tilde{Z}_{\text{ref}} = [-11098,\ 4696.56]$; $W_{\text{ref}} = 15794.56$.

λ_1	Z^L_α	Z^U_α	δ^L (%)	δ^U (%)	W_α	ΔW (%)	Dominance
0.0	−6917.94	−928.99	+37.66	−119.78	5988.95	−62.08	Intermediate
0.1	−5064.70	−366.61	+54.36	−107.81	4698.09	−70.26	Intermediate
0.2	−3211.45	−1662.22	+71.06	−135.39	1549.23	−90.19	Intermediate
0.3	−2957.84	−1358.20	+73.35	−128.92	1599.64	−89.87	Intermediate
0.4	−4253.45	495.03	+61.67	−89.46	4748.48	−69.94	Intermediate
0.5	−5542.07	2348.28	+50.06	−50.00	7890.35	−50.04	Intermediate
0.6	−5863.48	4201.53	+47.17	−10.54	10065.01	−36.28	Intermediate
0.7	−4097.58	6054.77	+63.08	+28.92	10152.35	−35.72	**Strong**
0.8	−2331.68	7908.02	+78.99	+68.38	10239.70	−35.17	**Strong**
0.9	−565.78	9761.27	+94.90	+107.84	10327.05	−34.62	**Strong**
1.0	1200.00	11614.00	+110.81	+147.29	10414.00	−34.07	**Strong**

The data in Table 8 support the following objective conclusions.

Strong Pareto dominance for $\lambda_1 \in [0.7, 1.0]$**.** For all four values $\lambda_1 \in \{0.7, 0.8, 0.9, 1.0\}$, both the lower and upper bounds strictly exceed the corresponding bounds of the reference [7], establishing strong dominance (Definition 3.12):

- For $\lambda_1 = 0.7$: the lower bound improves by $\delta^L = +63.08\%$ (-4097.58 vs. -11098, an absolute gain of $+7000.42$), and the upper bound improves by $\delta^U = +28.92\%$ (6054.77 vs. 4696.56, an absolute gain of $+1358.21$). Uncertainty is reduced by 35.72%.
- For $\lambda_1 = 0.8$: $\delta^L = +78.99\%$ and $\delta^U = +68.38\%$, representing absolute gains of $+8766.32$ in the lower bound and $+3211.46$ in the upper bound. Uncertainty is reduced by 35.17%.
- For $\lambda_1 = 0.9$: $\delta^L = +94.90\%$ and $\delta^U = +107.84\%$, representing absolute gains of $+10532.22$ and $+5064.71$, respectively. The upper bound alone surpasses the reference upper bound by more than 100%. Uncertainty is reduced by 34.62%.
- For $\lambda_1 = 1.0$: $\delta^L = +110.81\%$ and $\delta^U = +147.29\%$, representing absolute gains of $+12{,}298$ and $+6{,}917.44$ units in the lower and upper bounds, respectively. Strong dominance in the sense of Definition 3.12 is established since both bounds of the proposed solution strictly exceed those of the reference method [7]:

$$Z^L_{\text{ours}} = 1200 \;>\; Z^L_{\text{ref}} = -11{,}098 \qquad \text{and} \qquad Z^U_{\text{ours}} = 11{,}614 \;>\; Z^U_{\text{ref}} = 4{,}696.56.$$

Note that strong dominance does *not* require the proposed interval to lie entirely above the reference interval (i.e., $Z^L_{\text{ours}} > Z^U_{\text{ref}}$ is a strictly stronger, non-required condition). The uncertainty interval is reduced by 34.07% (from $W_{\text{ref}} = 15{,}794.56$ to $W_{\text{ours}} = 10{,}414.00$).

Intermediate dominance for $\lambda_1 \in [0.0, 0.6]$**.** For all values $\lambda_1 \in \{0.0, 0.1, \ldots, 0.6\}$, the lower bound is strictly superior to the reference by $+37.66\%$ to $+73.35\%$, while the upper bound remains below the reference. The most notable case is $\lambda_1 = 0.6$, where the upper bound deficit is only -10.54% (4201.53 vs. 4696.56, i.e., -495.03

absolute units), while the lower bound is already $+47.17\%$ above the reference. In all these cases, the interval width is reduced by between 36.28% and 90.19% relative to the reference, reflecting substantially lower uncertainty.

Consistent uncertainty reduction. Across *all* values of λ_1, the interval width W_α^{ours} is strictly smaller than the reference $W_{\text{ref}} = 15794.56$, with reductions ranging from 34.07% (at $\lambda_1 = 1.0$) to 90.19% (at $\lambda_1 = 0.2$). This demonstrates that our method not only produces superior objective values but also yields more *precise* solutions with narrower uncertainty bands across all preference configurations.

Inflexibility of the reference method. The method of Khalifa et al. [7] yields only one solution, $[-11098, 4696.56]$, regardless of the decision-maker's risk preferences. This inflexibility creates a risk of inefficiency, as the single solution may not align with the decision-maker's actual profile. In contrast, our method generates 11 distinct Pareto-optimal solutions for $\alpha = 0.6$ alone (one for each $\lambda_1 \in \{0.0, 0.1, \ldots, 1.0\}$), enabling:

1. **Risk-averse decision-makers** to select $\lambda_1 \in [0.7, 1.0]$, achieving strong dominance with improvements of 63%–111% in worst-case performance (δ^L) and 29%–147% in best-case performance (δ^U).
2. **Uncertainty-sensitive decision-makers** to select $\lambda_1 \in [0.2, 0.3]$, achieving the narrowest uncertainty bands ($W = 1549$–1600, a reduction of $\approx 90\%$ compared to the reference) at the cost of lower upper bound performance.
3. **Balanced decision-makers** to select intermediate values such as $\lambda_1 = 0.6$, which provides a $+47.17\%$ improvement in worst-case performance and an uncertainty reduction of 36.28%, with only a 10.54% deficit in upper bound.

This flexibility substantially increases the probability of identifying an efficient solution that is both quantitatively superior and practically relevant to the specific decision context.

4. Conclusion

4.1. Summary of Contributions

We have presented four principal contributions to the field of fuzzy random optimization:

First, we extended classical results of Dinkelbach [12] and Guzel [18] by introducing normalized preference weights $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 \leq \lambda_2$. This extension, formalized in Theorems 3.1 and 3.3, enables explicit modeling of decision-maker preferences for numerator versus denominator components in fractional objectives, providing finer control over the optimization process than previously possible.

Second, we developed a systematic three-stage transformation framework that converts complex fuzzy random fractional programs into tractable deterministic linear programs. Stage I computes the ideal fractional values for each objective, thereby transforming the fractional program into an equivalent linear program. Stage II incorporates preference weights to generate a weighted linear formulation. Stage III applies α-cut-based defuzzification, yielding deterministic linear programs that can be solved by standard optimization software. This framework preserves optimality throughout the transformation chain, as rigorously established by Theorems 3.4, 3.5, and 3.6.

Third, we highlighted the independence of preference parameters (λ) and satisfaction levels (α), a critical distinction absent from prior work [7]. This independence permits decision-makers to adjust their preference structure and desired confidence levels independently, yielding a rich family of Pareto-optimal solutions that comprehensively characterizes the efficient frontier. For each combination $(\alpha, \lambda_1, \lambda_2)$, our methodology produces solutions $\tilde{X}^*(\alpha, \lambda_1, \lambda_2)$, yielding $|\mathcal{A}| \times |\mathcal{L}|$ distinct solutions compared to only $|\mathcal{A}|$ solutions in existing approaches.

Fourth, we validated our methodology through two complementary applications. The didactic example (Section 3.4) demonstrated theoretical properties and computational feasibility, while the inventory management problem (Section 3.5) illustrated practical applicability with fuzzy random trapezoidal parameters. Comparative analysis revealed that for appropriately chosen preference parameters ($\lambda \in [0.4, 1.0]$), our solutions exhibit intermediate to strong stochastic dominance over those obtained by Khalifa et al. [7], the current state-of-the-art method.

4.2. Practical Implications

The proposed methodology offers several practical advantages for decision-makers facing optimization problems under dual uncertainty:

Enhanced decision flexibility: By generating families of solutions rather than single points, our approach accommodates diverse stakeholder preferences and risk tolerances. Decision-makers can explore trade-offs among competing objectives by varying λ while maintaining consistent confidence levels through α, or vice versa.

Improved robustness: The explicit treatment of both fuzziness and randomness enables more realistic modeling of real-world uncertainty. Unlike methods that address only one form of uncertainty, our approach captures the full complexity of environments where data are simultaneously imprecise and stochastic.

Computational tractability: With total time complexity $\mathcal{O}\big((K + |A| \cdot |L|) \cdot n^3\big)$, dominated in practice by the $\mathcal{O}(|A| \cdot |L| \cdot n^3)$ term of Stage III, the algorithm scales well to moderately sized problems. For typical parameter choices ($|A| = 4$, $|L| = 6$, $K = 2$), Stage I solves 2 linear programs independently of the grid, while Stage III solves at most 48 linear programs – a total of 50 LP instances per problem, which is tractable using modern optimisation software such as Gurobi, CPLEX, or GLPK. The algorithm is furthermore highly parallelisable: the LP problems in Stage III can be solved independently for different $(\alpha, \lambda_1, \lambda_2)$ combinations, yielding a potential speedup factor of $\mathcal{O}(|A| \cdot |L|)$ with sufficient computational resources.

Broad applicability: Our framework applies to diverse domains, including inventory management (demonstrated in Section 3.5), portfolio optimization, resource allocation, production planning, and wireless sensor network design - any context where efficiency ratios must be optimized under dual uncertainty.

4.3. Limitations and Assumptions

While our methodology provides significant advances, several limitations warrant acknowledgment:

Linearity assumption: The current framework addresses linear fractional programs. Although many practical problems admit linear or piecewise-linear approximations, genuinely nonlinear fractional objectives require methodological extensions.

Single-level structure: Our approach handles multi-objective problems but assumes a single decision-making level. Multilevel or hierarchical decision structures, common in organizational settings, necessitate additional theoretical development.

Coefficient sign requirements. Theorems 3.4, 3.5, and 3.6 rest on specific sign patterns for the fuzzy coefficient matrices $\tilde{A}$, $\tilde{C}$, and $\tilde{D}$. These assumptions are not merely technical conveniences; they are structurally necessary to guarantee that the α-cut bounds of the fuzzy objective propagate correctly under interval arithmetic (Definition 2.5), and that the lower and upper components X^L_α and X^U_α of the optimal solution can be associated with the correct feasible subsets $\tilde{\Omega}^L_\alpha$ and $\tilde{\Omega}^U_\alpha$, respectively.

When these sign conditions are violated, two distinct failure modes may occur. *First*, the correspondence between fuzzy optimal solutions and their deterministic α-cut counterparts may break down: solving the lower-bound linear program (LP) does not necessarily yield the lower bound of the true fuzzy-optimal objective value, leading to potentially misleading solutions. *Second*, the monotonicity of the objective value with respect to α may fail, producing non-nested solution families across α-levels and complicating both sensitivity analysis and the interpretation of satisfaction levels.

Several approaches can mitigate these issues, as described in section 3.2.1. A *coefficient decomposition strategy* splits any mixed-sign fuzzy coefficient into its non-negative and non-positive parts, reducing the general case to one of the three theoretically supported configurations. A *variable-splitting substitution* $x_j = x_j^+ - x_j^-$ restores sign compatibility for objective coefficients without modifying the feasible region. When neither strategy is directly applicable, a *conservative interval-arithmetic reformulation* provides a tractable fallback, at the expense of potentially wider solution intervals. While these alternatives extend the applicability of our framework, they also introduce additional variables or parameters that may increase computational cost or require problem-specific tuning. The development of unified theorems that remove sign restrictions entirely - possibly leveraging generalized α-cut comparison operators [28, 29]-remains a priority direction for future research (see Section 4.4).

Fuzzy number representation: We assume trapezoidal or triangular fuzzy numbers, which may not capture all forms of imprecision. Generalizations to arbitrary fuzzy numbers or type-2 fuzzy sets could enhance modeling flexibility but would complicate α-cut operations.

Solution selection: Although we generate comprehensive solution families, selecting a single solution for implementation requires additional decision support mechanisms. While we provide a selection heuristic within Algorithm 1 (Stage III), final selection depends on context-specific considerations beyond our model's scope.

4.4. Directions for Future Research

Several promising avenues emerge from this work:

Nonlinear extensions: Extending our framework to nonlinear fuzzy random fractional programming represents a natural next step. This would require generalizing Theorems 3.4-3.6 to handle nonlinear transformations and developing specialized solution algorithms for the resulting nonlinear programs. Sequential convex programming or successive linearization techniques may prove fruitful.

Multilevel formulations: Many practical problems exhibit hierarchical decision structures (e.g., supply chain management with multiple organizational levels). Developing multilevel variants of our methodology would require integrating game-theoretic concepts with fuzzy random optimization, potentially drawing on recent advances in multilevel programming [10].

Dynamic programming: Extending to multi-period settings where decisions evolve over time would broaden applicability to dynamic resource allocation, adaptive portfolio management, and real-option valuation problems. This would require incorporating scenario trees or stochastic dynamic programming principles into our framework.

Robust optimization integration: Combining our approach with robust optimization techniques could yield solutions that are simultaneously optimal under fuzzy randomness and robust against worst-case scenarios. This hybrid methodology would appeal to highly risk-averse decision-makers.

Machine learning integration: Incorporating machine learning techniques to estimate fuzzy membership functions from historical data, or to learn decision-maker preferences from past choices, could enhance practical applicability. Inverse optimization or preference learning methods offer relevant technical foundations.

Computational enhancements: Developing specialized interior-point algorithms tailored to our transformed linear programs, exploiting problem structure (e.g., sparsity, separability), could improve computational efficiency for large-scale applications. Warm-start strategies leveraging solutions at nearby parameter values may also accelerate solution of the complete family.

Empirical validation: While our inventory management application demonstrates feasibility, more extensive empirical studies across diverse domains (finance, energy, healthcare) would establish the methodology's practical value more conclusively. Collaboration with industry practitioners would identify domain-specific adaptations and implementation challenges.

Sign-free unified theorems: A significant theoretical limitation of the present framework is its dependence on the sign conditions imposed in Theorems 3.4–3.6. Although section 3.2.1 proposes practical workarounds, a fully general theory that handles arbitrary sign configurations without decomposition or variable splitting would be more elegant and computationally efficient. Developing such unified theorems likely requires extending the α-cut comparison relations of Definition 3.12 to account for sign-indefinite interval arithmetic, possibly drawing on the generalized interval-order frameworks of [28] and [29]. We identify this as a priority direction for future work, as it would remove the most restrictive structural assumption of the current methodology and substantially broaden its applicability.

4.5. Closing Remarks

The integration of fuzzy logic and probability theory in optimization has matured considerably since Zadeh's pioneering work [1] and the foundational contributions of Guangyuan and Zhong [2, 3]. Our work advances this tradition by providing decision-makers with tools that acknowledge the dual nature of real-world uncertainty while maintaining computational tractability and theoretical rigor.

By generating rich families of solutions parameterized by both satisfaction levels and preferences, our methodology enables decision-makers to navigate complex trade-offs in uncertain environments. The theoretical foundations established by our extended Dinkelbach and Guzel theorems, together with rigorous optimality-preservation results (Theorems 3.4-3.6), ensure that solutions are not merely heuristic approximations but are provably optimal within the fuzzy random framework.

We anticipate that this work will stimulate further research in fuzzy random optimization, particularly in addressing the limitations and pursuing the extensions outlined above. The accompanying software implementation and detailed algorithmic descriptions aim to facilitate reproducibility and encourage practical adoption.

As optimization problems grow increasingly complex and data increasingly uncertain, methodologies that systematically address multiple forms of uncertainty while respecting decision-maker preferences will become indispensable. This paper represents a step toward that goal, providing both theoretical insights and practical tools for optimization under fuzziness and randomness.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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