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A Refined Conjugate Gradient Algorithm for Image Denoising

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Abstract The conjugate formula is often the major focus of research in the field of conjugate gradient approaches. The conjugate gradient approach is employed to address challenges encountered in the realm of image restoration. Through the application of a quadratic model, an innovative coefficient conjugate will be generated for the operation. The algorithms exhibit both local and global convergence along with descent characteristics. The numerical evaluations indicated that the recently formulated method significantly surpasses its predecessor. The newly established conjugate gradient approach demonstrates enhanced efficacy relative to the FR conjugate gradient methodology, which is regarded as the benchmark within the industry.

Keywords Refined Conjugate Gradient, Convergence property, Impulse Noise Reduction from Images.

AMS 2010 subject classifications 65F10, 68U10, 90C06.

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1. Introduction

Interestingly, the noisy pixel cleaning model is one of a large class of challenging optimization issues in engineering and management that may be solved and made simpler using iteration problem:

$$f_\alpha(\mu) = \sum_{(\hat{\imath},\hat{\jmath})\in \mathrm{N}} \left[\left|\mu_{\hat{\imath},\hat{\jmath}} - y_{\hat{\imath},\hat{\jmath}}\right| + \frac{\beta}{2}(S^1_{\hat{\imath},\hat{\jmath}} + S^2_{\hat{\imath},\hat{\jmath}}) \right]. \tag{1}$$

where $S^1_{\hat{\imath},\hat{\jmath}} = 2 \sum_{(\hat{\imath},\hat{\jmath})\in \mathrm{P}_{\hat{\imath},\mathrm{j}} \cap \mathrm{N}^c} \phi_\alpha(\mu_{\hat{\imath},\hat{\jmath}} - y_{\hat{\imath},\hat{\jmath}})$, $S^2_{\hat{\imath},\hat{\jmath}} = \sum_{(\hat{\imath},\hat{\jmath})\in \mathrm{P}_{\hat{\imath},\mathrm{j}} \cap \mathrm{N}} \phi_\alpha(\mu_{\hat{\imath},\hat{\jmath}} - y_{\hat{\imath},\hat{\jmath}})$. In terms of image restoration, $y_{\hat{\imath},\hat{\jmath}}$ is the value of the picture's pixel at position $(\hat{\imath},\hat{\jmath})$, $\varphi_\alpha = \sqrt{\alpha + \mu^2}$ is the edge-preserving potential function with parameter $\alpha > 0$, and β is the parameter when the conditions are satisfied. $\mu_{\hat{\imath},\hat{\jmath}} = \left[\mu_{\hat{\imath},\hat{\jmath}}\right]_{(\hat{\imath},\hat{\jmath})\in \mathrm{N}}$ represents the restored values of the entire number of noisy pixels, $|\mathrm{N}|$ [10, 16, 30]. This functional form is particularly advantageous in image restoration, as it enables the approximation of non-smooth penalties (such as those derived from total variation norms) by smooth, differentiable functions that are computationally more tractable. The principal objective of the second stage, therefore, is to remove the non-smooth component of $f_\omega(\mu)$ and obtain a smooth, continuous approximation suitable for optimization via gradient-based methods. In this context, we initially revisit the widely recognized unconstrained optimization problem characterized by:

$$f_\alpha(\mu) = \sum_{(\hat{\imath},\hat{\jmath})\in \mathrm{N}} \left[(2 \times S^1_{\hat{\imath},\hat{\jmath}} + S^2_{\hat{\imath},\hat{\jmath}}) \right]. \tag{2}$$

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The smooth function is $f_\alpha(\mu) : R^n \to R$. The Conjugate Gradient (CG) technique generally uses the following recursive computational framework to derive the iterates:

$$\mu_{k+1} = \mu_k + \alpha_k \text{\d{d}}_k. \tag{3}$$

where α_k uses a suitable line search technique to get the step length. Refined from the denominator is:

$$\alpha_k = -\frac{\dot{\mathrm{g}}_k^T \text{\d{d}}_k}{\text{\d{d}}_k^T Q \text{\d{d}}_k}. \tag{4}$$

See [27]. In iterative methods, the step length α_k is typically selected to conform to specific requirements. Consequently, the subsequent conditions must be met by the Wolfe conditions, which are commonly utilized in the execution of the Conjugate Gradient technique:

$$f(\mu_k + \alpha_k \text{\d{d}}_k) \le f(\mu_k) + \delta \alpha_k \dot{\mathrm{g}}_k^T \text{\d{d}}_k. \tag{5}$$

$$\text{\d{d}}_k^T \dot{\mathrm{g}}(\mu_k + \alpha_k \text{\d{d}}_k) \ge \sigma\, \text{\d{d}}_k^T \dot{\mathrm{g}}_k. \tag{6}$$

where $0 < \delta < \sigma < 1$. The CG methodology [17, 24] has been employed to tackle the issue delineated in Eq (1) alongside its constrained counterpart. Generally, the search direction associated with the CG method is articulated as:

$$\text{\d{d}}_{k+1} = -\dot{\mathrm{g}}_{k+1} + \beta_k \text{\d{d}}_k. \tag{7}$$

where β_k is a scalar. A variety of iterative frameworks for addressing (2) with dual parameters have been established in the academic literature. For instance, enhancements have been introduced to the well-known quasi-Newton update by Fletcher–Reeves (FR) method [20] and the Dai–Yuan (DY) method [18] defined by:

$$\beta_k^{FR} = \frac{\left\|\dot{\mathrm{g}}_{k+1}\right\|^2}{\left\|\dot{\mathrm{g}}_k\right\|^2} \;,\quad \beta_{k+1}^{DY} = \frac{\left\|\dot{\mathrm{g}}_{k+1}\right\|^2}{\text{\d{d}}_k^T y_k}. \tag{8}$$

where $y_k = \dot{\mathrm{g}}_{k+1} - \dot{\mathrm{g}}_k$. A group of academics has recently released work in this area that is based on modified secant equations, as we can see in [31]. It has taken a lot of effort to find improved versions of the previously described CG equations. These versions satisfy certain conjugacy requirements in addition to furthermore generate suitable descent paths that facilitate effective computation efficiency.

In a recent development, Hideaki and Yasushi [26] and Basim [1, 11, 3, 6, 5] proposed a two-term adaptation of the CG-method for solving the constrained form of (2). The authors proved global convergence of the scheme under some mild assumptions. The parameters derived from their investigations are presented as follows:

$$\beta_k^{HY} = \frac{\left\|\dot{\mathrm{g}}_{k+1}\right\|^2}{2/\alpha_k (f_k - f_{k+1})} \;,\; \beta_k^{B} = \frac{\left\|\dot{\mathrm{g}}_{k+1}\right\|^2}{(f_k - f_{k+1})/\alpha_k - \dot{\mathrm{g}}_k^T \text{\d{d}}_k/2}. \tag{9}$$

By using these methods, a significant level of computational efficiency may be attained while maintaining all the intended features and advantages of the original approaches. A quadratic model is presented as an improved adaptation to improve efficiency in unconstrained optimisation circumstances. This modification aims to maximise the benefits provided by traditional conjugate gradient (CG) algorithms and is influenced by the approaches described in [29] and [24].

We thought it wise to investigate further changes that we believed might improve numerical performance, as in [13, 2, 8, 7, 14, 15]. These changes were predicated on the identification of a new conjugate gradient parameter and the examination of the quadratic model's ability to assess the convergence of that parameter's value. Finally, the numerical experiments are presented, demonstrating the effectiveness of the suggested approach in comparison to the FR-CG algorithm.

2. Proposed Parameter

Deriving novel formulations for the conjugate gradient approach is the main goal of this study. In order to achieve this goal. The function f is expanded using the second-order Taylor series The function f is expanded using the second-order Taylor series, which is expressed as follows:

$$f(\mu) \;=\; f(\mu_{k+1}) - \;\dot{g}_{k+1}^T s_k + \frac{1}{2} s_k^T Q(\mu_{k+1}) s_k. \tag{10}$$

where $s_k = \mu_{k+1} - \mu_k$The gradient obtained from the second-order Taylor series expansion of the function f is given by:

$$\dot{g}_{k+1} \;=\; \dot{g}_k + Q(\mu_{k+1}) s_k. \tag{11}$$

By inserting equation (11) into equation (10), the following relation is derived:

$$s_k^T Q(\mu_k) s_k \;= 1/2\, y_k^T s_k + (\, f_{k+1} - f_k) - \dot{g}_k^T s_k. \tag{12}$$

Through straightforward algebraic manipulation, we can deduce:

$$s_k^T Q(\mu_k) s_k \;= (\dot{g}_k^T s_k)^2 \,/\, (1/2 y_k^T s_k + (\, f_{k+1} - f_k) - \dot{g}_k^T s_k. \tag{13}$$

By applying the **conjugacy condition**, the new parameter can be determined. Recall that the conjugacy condition is defined as follows:

$$\underset{\cdot}{d}_{k+1}^T Q(\mu_k) s_k \;=\; 0. \tag{14}$$

Using equations (7) and (13) in equation (14), we arrive at:

$$\beta_k = \frac{\left[\frac{(\dot{g}_k^T s_k)^2}{s_k^T y_k (1/2 y_k^T s_k + (\, f_{k+1} - f_k) - \dot{g}_k^T s_k)}\right] \dot{g}_{k+1}^T y_k}{\underset{\cdot}{d}_k^T y_k}. \tag{15}$$

When an **exact line search** is applied to the above equation, it results in:

$$\beta_k = \frac{\left[\frac{(\dot{g}_k^T s_k)^2}{s_k^T y_k (1/2 y_k^T s_k + (\, f_{k+1} - f_k) - \dot{g}_k^T s_k)}\right] \left\|\dot{g}_{k+1}\right\|^2}{\underset{\cdot}{d}_k^T y_k}. \tag{16}$$

This expression is designated as **BBP**, and the associated **BBP conjugate gradient method** can be outlined as follows:

3. Global convergence

In this section, we analyze the global convergence properties of the proposed method. To begin, we establish the following:

1. On the specified set $\Omega = \{\mu : \mu \in R^n, f(\mu) \le f(\mu_1)\}$, the function f is **bounded from below**.
2. The function f satisfies the following inequality, as its gradient $\nabla f(\mu)$is **Lipschitz continuous**:

$$\|\dot{g}(\tau) - \dot{g}(\upsilon)\| \le L \,\|\tau - \upsilon\| \;\;, \;\; \forall \tau, \;\; \upsilon \in R^n. \tag{17}$$

See, [21, 28].

Theorem 1
The **BBP algorithm** generates the search directions, which satisfy the following condition:

$$\underset{\cdot}{d}_{k+1}^T \dot{g}_{k+1} \;<\; 0 \text{ and } \underset{\cdot}{d}_{k+1}^T \dot{g}_{k+1} = \beta_k \underset{\cdot}{d}_k^T \dot{g}_k. \tag{18}$$

Proof
Clearly, $\underset{.}{d}_k = -\,\dot{g}_k$ is needed to derive $\underset{.}{d}_1^T \dot{g}_1 < 0$. For any k, the term $\underset{.}{d}_k^T \dot{g}_k < 0$ must be taken into account. It can be directly obtained from equations (7) and (16) as follows:

$$\begin{aligned} \underset{.}{d}_{k+1}^T \dot{g}_{k+1} &= -\dot{g}_{k+1}^T \dot{g}_{k+1} + \beta_k \underset{.}{d}_k^T \dot{g}_{k+1} \\ &= -\,\beta_k \frac{s_k^T y_k}{\left[\frac{(\dot{g}_k^T s_k)^2}{s_k^T y_k (1/2 y_k^T s_k + (f_{k+1} - f_k) - \dot{g}_k^T s_k)}\right]} + \ \beta_k \underset{.}{d}_k^T \dot{g}_{k+1} \ \cdot \end{aligned} \tag{19}$$

thereby guaranteeing that:

$$\underset{.}{d}_{k+1}^T \dot{g}_{k+1} \ = \beta_k \left[\underset{.}{d}_k^T \dot{g}_{k+1} - \frac{s_k^T y_k}{\left[\frac{(\dot{g}_k^T s_k)^2}{s_k^T y_k (1/2 y_k^T s_k + (f_{k+1} - f_k) - \dot{g}_k^T s_k)}\right]} \right]. \tag{20}$$

Equations (11) and (20) may be used to deduce the result:

$$\underset{.}{d}_{k+1}^T \dot{g}_{k+1} = \beta_k \underset{.}{d}_k^T \dot{g}_k. \tag{21}$$

This leads to the following.

$$\underset{.}{d}_{k+1}^T \dot{g}_{k+1} < \ 0. \tag{22}$$

Thus, the proof is finished. □

Analysing the entire convergence properties of the conjugate gradient technique requires a deep understanding of the Zotendijk condition [31].

Lemma 2
Assuming that both conditions (1) and (2) hold, that α_k satisfies the **Wolfe conditions**, and that $\underset{.}{d}_k$ is a **descent direction**, it follows that:

$$\sum_{k=1}^{\infty} \frac{(\dot{g}_k^T \underset{.}{d}_k)^2}{\left\|\underset{.}{d}_k\right\|^2} \ < \ \infty. \tag{23}$$

Theorem 3
Under the assumption that the premises and **Lemma 2** are valid, and denoting a new sequence as $\{u_k\}$, it follows that:

$$\lim_{k \to \infty} (\inf \left\|\dot{g}_k\right\|) = 0. \tag{24}$$

Proof
By **contradiction**, equation (24) is shown to be false. Consequently, we can determine a positive value r>0 such that, for every k:

$$\left\|\dot{g}_{k+1}\right\| > r. \tag{25}$$

By squaring the search step, the result can be expressed as: $\underset{.}{d}_{k+1} + \dot{g}_{k+1} = \beta_k \underset{.}{d}_k$ on both sides:

$$\left\|\underset{.}{d}_{k+1}\right\|^2 + \left\|\dot{g}_{k+1}\right\|^2 + 2\underset{.}{d}_{k+1}^T \dot{g}_{k+1} = (\beta_k)^2 \left\|\underset{.}{d}_k\right\|^2. \tag{26}$$

By applying equation (21) to (26), the following results are obtained:

$$\left\|\underset{.}{d}_{k+1}\right\|^2 = \frac{(\underset{.}{d}_{k+1}^T \dot{g}_{k+1})^2}{(\underset{.}{d}_k^T \dot{g}_k)^2} \left\|\underset{.}{d}_k\right\|^2 - 2\underset{.}{d}_{k+1}^T \dot{g}_{k+1} - \left\|\dot{g}_{k+1}\right\|^2. \tag{27}$$

By dividing equation (27) by $(\underset{\cdot}{\mathrm{d}}_{k+1}^{T}\dot{\mathrm{g}}_{k+1})^2$, we obtain the following result:

$$\begin{aligned}\frac{\left\|\underset{\cdot}{\mathrm{d}}_{k+1}\right\|^2}{(\underset{\cdot}{\mathrm{d}}_{k+1}^{T}\dot{\mathrm{g}}_{k+1})^2} &= \frac{\left\|\underset{\cdot}{\mathrm{d}}_{k}\right\|^2}{(\underset{\cdot}{\mathrm{d}}_{k}^{T}\dot{\mathrm{g}}_{k})^2} - \frac{\left\|\dot{\mathrm{g}}_{k+1}\right\|^2}{(\underset{\cdot}{\mathrm{d}}_{k+1}^{T}\dot{\mathrm{g}}_{k+1})^2} - \frac{2}{\underset{\cdot}{\mathrm{d}}_{k+1}^{T}\dot{\mathrm{g}}_{k+1}} \\ &\leq \frac{\left\|\underset{\cdot}{\mathrm{d}}_{k}\right\|^2}{(\underset{\cdot}{\mathrm{d}}_{k}^{T}\dot{\mathrm{g}}_{k})^2} - \left(\frac{\left\|\dot{\mathrm{g}}_{k+1}\right\|}{(\underset{\cdot}{\mathrm{d}}_{k+1}^{T}\dot{\mathrm{g}}_{k+1})} + \frac{1}{\left\|\dot{\mathrm{g}}_{k+1}\right\|^2}\right) + \frac{1}{\left\|\dot{\mathrm{g}}_{k+1}\right\|^2} \quad . \\ &\leq \frac{\left\|\underset{\cdot}{\mathrm{d}}_{k}\right\|^2}{(\underset{\cdot}{\mathrm{d}}_{k}^{T}\dot{\mathrm{g}}_{k})^2} + \frac{1}{\left\|\dot{\mathrm{g}}_{k+1}\right\|^2}\end{aligned} \tag{28}$$

This provides insight into:

$$\frac{\left\|\underset{\cdot}{\mathrm{d}}_{k+1}\right\|^2}{(\underset{\cdot}{\mathrm{d}}_{k+1}^{T}\dot{\mathrm{g}}_{k+1})^2} \leq \sum_{i=1}^{k+1} \frac{1}{\left\|\dot{\mathrm{g}}_{i}\right\|^2}. \tag{29}$$

Assume that $c_1 > 0$ includes $\left\|\dot{\mathrm{g}}_k\right\| \geq c_1$ for every $k \in n$. Then, it follows that:

$$\frac{\left\|\underset{\cdot}{\mathrm{d}}_{k+1}\right\|^2}{(\underset{\cdot}{\mathrm{d}}_{k+1}^{T}\dot{\mathrm{g}}_{k+1})^2} < \frac{k+1}{c_1^2}. \tag{30}$$

Ultimately, this yields:

$$\sum_{k=1}^{\infty} \frac{(\dot{\mathrm{g}}_k^{T}\underset{\cdot}{\mathrm{d}}_{k})^2}{\left\|\underset{\cdot}{\mathrm{d}}_{k}\right\|^2} = \infty. \tag{31}$$

In a similar vein, the claim about $\lim_{k\to\infty} \inf\left\|\dot{\mathrm{g}}_k\right\| = 0$ is supported, as shown in Lemma 1, however different formulations could provide different results. □

4. Numerical Results

In this article, we clarify how well the BBP algorithm reduces salt-and-pepper impulse noise (2). Current developments in the field of scientific publishing [31, 32, 33]. Table 1 presents a graphical overview of the research procedure. Table 1 contains the original test pictures. Every simulation was carried out using MATLAB 2015a on a personal PC. Evaluations of the BBP methodology's performance were conducted alongside the FR approach. It is crucial to emphasise that the primary goal of this study is to reduce image noising as quickly as possible (2). The quality of the reconstructed pictures is evaluated using the Signal-to-Noise Ratio (PSNR):

$$PSNR = 10 \ \log_{10} \frac{255^2}{\frac{1}{MN}\sum_{\hat{\imath},\hat{\jmath}}(\mu^r_{\hat{\imath},\hat{\jmath}} - \mu^*_{\hat{\imath},\hat{\jmath}})^2}. \tag{32}$$

where $u^r_{\hat{\imath},\hat{\jmath}}$ and $u^*_{\hat{\imath},\hat{\jmath}}$ denote the pixel values in the original image and the reconstructed image, respectively, refer to [16]. The subsequent criteria are established for termination in both methodologies:

$$\frac{|f(\mu_k) - f(\mu_{k-1})|}{|f(\mu_k)|} \leq 10^{-4} \text{ and } \|f(\mu_k)\| \leq 10^{-4}(1 + |f(\mu_k)|). \tag{33}$$

The results of the analysis are summarized in Table 1 below. This table displays the peak signal-to-noise ratio (PSNR), function evaluations (NF), and number of iterations (NI) for each situation.

Table 1. . Performance Results of the FR and BBP Algorithms.

Image	**Noise**	**FR-Method**			**BBP-Method**		
		NI	**NF**	**PSNR (dB)**	**NI**	**NF**	**PSNR (dB)**
Lena	50	82	153	30.5529	64.0	69.0	30.5291
	70	81	155	27.4824	62.0	65.0	27.2348
	90	108	211	22.8583	65.0	67.0	22.9134
House	50	52	53	30.6845	40.0	43.0	34.6144
	70	63	116	31.2564	46.0	47.0	31.1269
	90	111	214	25.2870	51.0	53.0	25.4285
Elaine	50	35	36	33.9129	32.0	33.0	33.9049
	70	38	39	31.8640	36.0	37.0	31.8607
	90	65	114	28.2019	42.0	43.0	28.1791
Camera man	50	59	87	35.5359	32.0	65.0	35.3941
	70	78	142	30.6259	42.0	51.0	30.6653
	90	121	236	24.3962	51.0	57.0	24.7996

In terms of PSNR, number of iterations (NI), and function evaluations (NF), the suggested algorithms perform better than the FR technique, as the table illustrates.

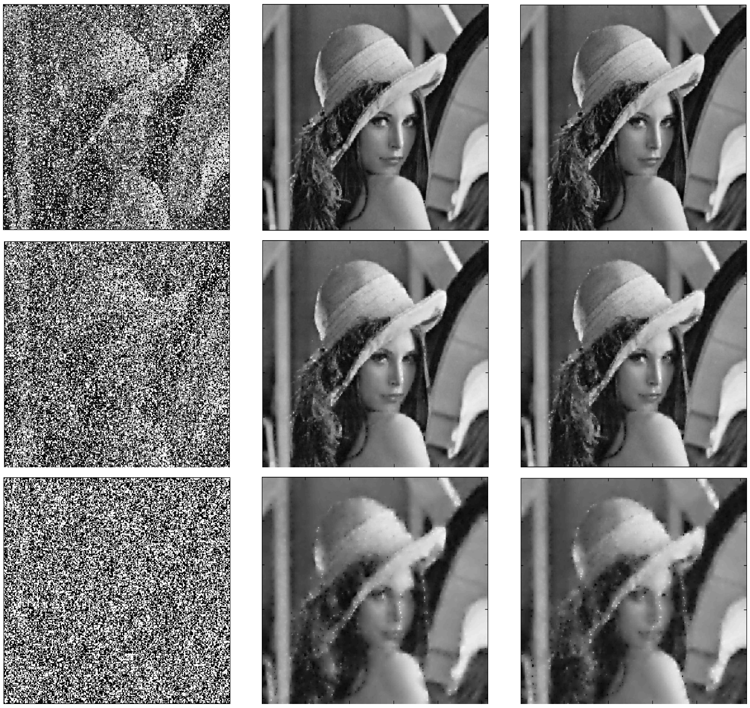

Figure 1. Visual comparison of the Lena image (256 × 256) restored using the FR and BBP algorithms.

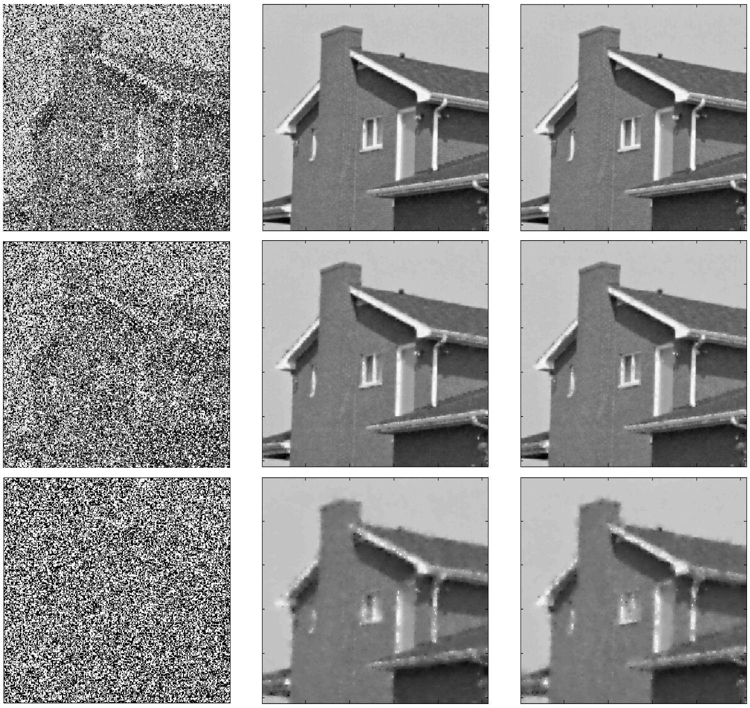

Figure 2. Visual comparison of the House image (256 × 256) restored using the FR and BBP algorithms.

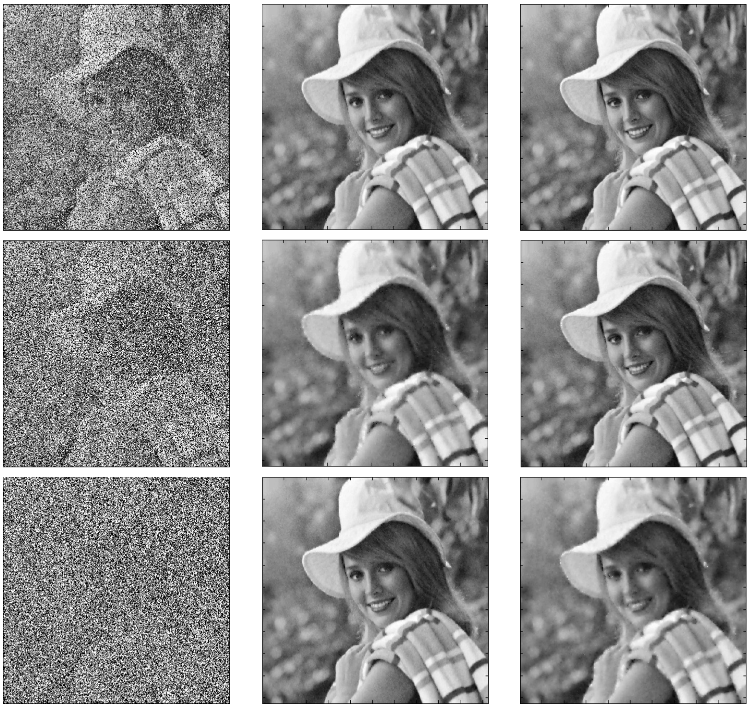

Figure 3. Visual comparison of the Elaine image (256 × 256) restored using the FR and BBP algorithms.

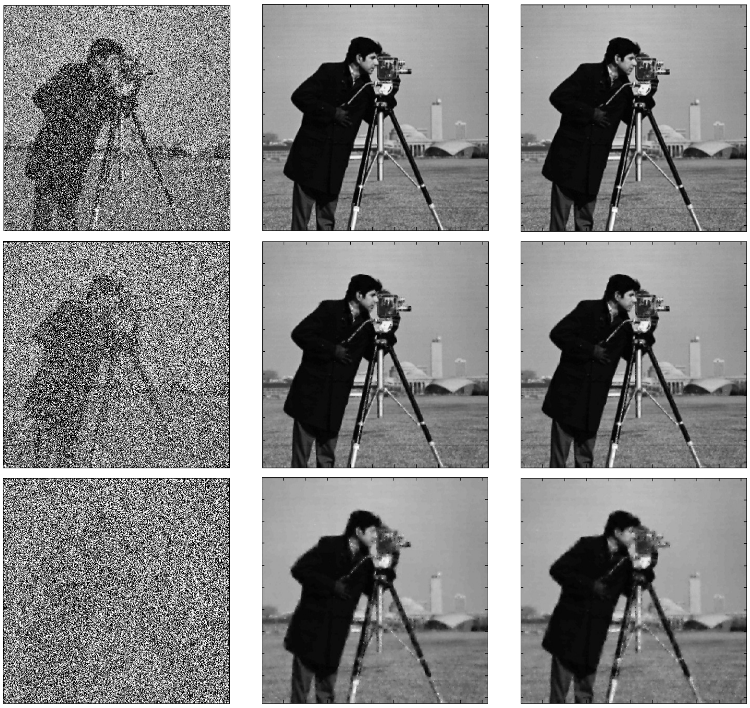

Figure 4. Visual comparison of the Cameraman image (256 × 256) restored using the FR and BBP algorithms.

LenaIt might be misleading to rely solely on raw numerical data for direct comparisons when dealing with a range of problems. Consequently, the data was evaluated using Dolan and Moré's [19] performance profiles, which are now considered a standard in large-scale optimisation. The NOI and NOF performance profiles were subsequently developed as a result of this. According to previous studies in this field [4, 9, 12, 22, 23, 25].

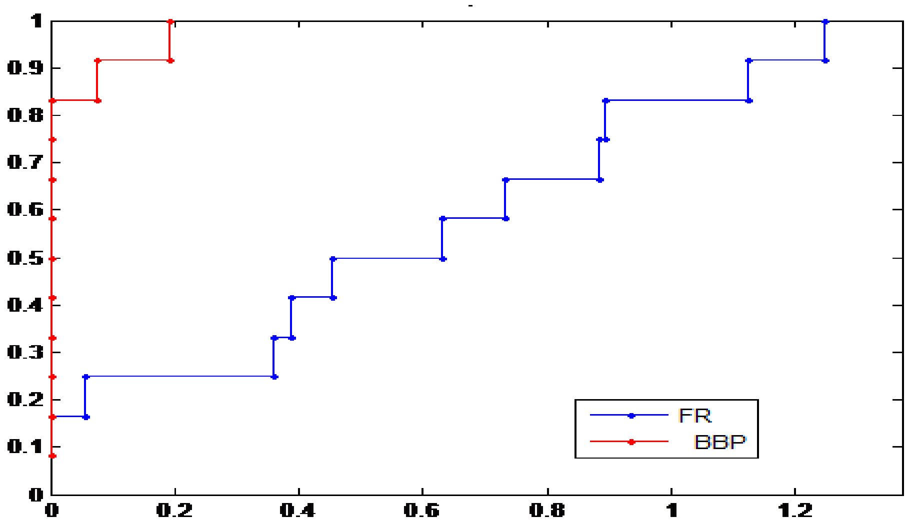

Figure 5. Performance profile for the NI.

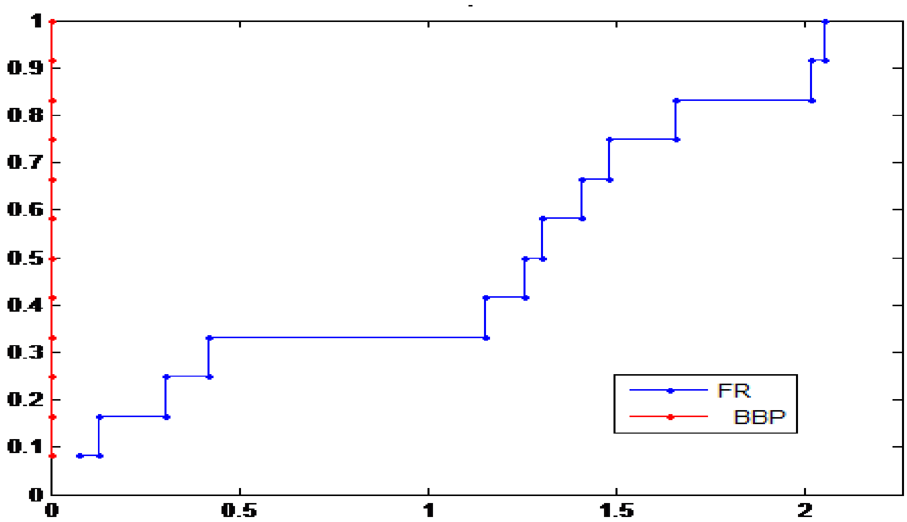

Figure 6. Performance profile for the NF.

5. Conclusions

Finally, we introduced the BBP conjugate gradient technique and suggested a new formula for altering the conjugate gradient approach. We also looked at the consequences of these developments. We used the Wolfe line search technique to determine its global convergence. Simulation-based research has shown that BBP has the ability to drastically lower the number of iterations and function evaluations while preserving comparable image quality standards.

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