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Fixed Point Theory in Neutrosophic MR-Metric Spaces: A Unified Framework with Graph Contractions and Applications

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Abstract This paper presents a comprehensive study of fixed point theory within the framework of Neutrosophic MR-Metric Spaces (NMR-MS), which generalize classical metric spaces by incorporating neutrosophic logic to handle uncertainty, indeterminacy, and truth degrees. We introduce a novel fixed point theorem for graph contractions in complete NMR-MS, ensuring the existence and uniqueness of fixed points under mild conditions. Additionally, we establish a graph-theoretic representation of NMR-MS and propose a modified Dijkstra's algorithm for optimal pathfinding under neutrosophic constraints. The theoretical developments are supported by illustrative examples motivated by wireless networks and sensor systems. A possible connection with fractional calculus is outlined as a future research direction rather than a fully developed application in the present work. Our results extend and unify several existing fixed point theorems in various generalized metric spaces, including b-metric, M^*-metric, and Ω-distance spaces.

Keywords Neutrosophic MR-metric spaces, fixed point theory, graph contractions, t-norm, t-conorm, Dijkstra's algorithm, fuzzy logic, uncertainty modeling.

AMS 2010 subject classifications 47H10, 54H25, 03E72, 05C22, 54E35.

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1. Introduction

Fixed point theory has been the cornerstone of nonlinear functional analysis, which has had a profound influence on various fields such as differential equations, optimization, and network theory. The evolution of metric spaces has given rise to various generalizations, including b-metric spaces [4], M^*-metric spaces [12] and Ω-distance mapping [13, 14, 15], each designed to meet specific structural and application needs. Recently, Malkawi et al. The concept of MR metric space is introduced, which extends the concept of metricity to triple-based mappings, and provides a rich framework for the analysis of convergence and fixed points.

In parallel, the incorporation of fuzzy and neurosophisticated logics into metric structures has attracted considerable attention due to their ability to model uncertainty and imprecision. Neutrosophic sets, introduced by Smarandache, provide a powerful tool for handling truth ($\mathcal{T}$), falsity ($\mathcal{F}$) and uncertainty ($\mathcal{I}$) membership degrees. The fusion of neutrosophy with metric spaces has led to the development of neutrosophy metric spaces, making fuzzy metric spaces more general.

Based on these advances, this paper introduces the concept **neutrosophic MR metric space (NMR-MS)**, which combines the triple-based structure of MR metrics with neutrosophic membership functions. This integration allows more nuanced modeling of real systems where uncertainty and multidimensional distances are inherent.

Several fixed point results in generalized metric spaces have been established in recent years. For instance, Malkawi et al. [1] studied (ψ, L)-weak contractions in Mb-metric spaces, while Qawasmeh [3] explored

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interpolative contractions in Ω_b-distance mappings. Other contributions include fixed point theorems via simulation functions [10], rational contractions [16], and cyclic contractions [17]. Fractional calculus applications have also been explored, such as in [5] and [27], where Lie symmetries and conformable derivatives were investigated.

Motivation and gap. Although several fixed point results are available in MR-metric, fuzzy metric, and neutrosophic metric settings, the existing literature does not adequately address a unified framework that simultaneously combines: (i) the triple-based geometry of MR-metrics, (ii) the neutrosophic representation of truth, indeterminacy, and falsity, and (iii) graph-based constraints arising in networked systems and path optimization. In particular, current approaches either focus on fixed point theory without a graph-theoretic optimization component, or discuss neutrosophic/fuzzy path models without embedding them into an MR-metric structure. This leaves a genuine gap in the development of a mathematically coherent framework for contraction principles and path selection under uncertainty.

Main contributions. The main contributions of this paper are as follows:

- (C1) We introduce the notion of Neutrosophic MR-Metric Spaces (NMR-MS) as a synthesis of MR-metric geometry and neutrosophic membership functions.
- (C2) We establish a fixed point theorem for graph contractions in complete NMR-MS under explicit structural assumptions on the graph and the contraction mapping.
- (C3) We provide a graph-theoretic interpretation of the truth-membership function and formulate a path optimization problem under neutrosophic constraints.
- (C4) We refine the optimization framework by presenting a discretized modified Dijkstra-type algorithm together with a complexity discussion.
- (C5) We clarify the scope of applications by distinguishing between theoretically motivated examples and fully validated real-world implementations.

Definition 1 (MR-Metric Space)
Let $\mathbb{X}$ be a non-empty set and $\mathfrak{R} > 1$ be a real constant. A function

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

is called an **MR-metric** if for all elements $v, \xi, s, \ell \in$, the following axioms hold:

1. **Non-negativity**: $M(v, \xi, s) \geq 0$.
2. **Identity of Indiscernibles**: $M(v, \xi, s) = 0$ if and only if $v = \xi = s$.
3. **Symmetry**: The value of $M(v, \xi, s)$ is unchanged under any permutation of v, ξ, and s. That is, $M(v, \xi, s) = M(p(v, \xi, s))$ for any permutation p.
4. **Modified Tetrahedral Inequality**:

$$M(v, \xi, s) \leq \mathfrak{R}\left[M(v, \xi, \ell) + M(v, \ell, s) + M(\ell, \xi, s)\right].$$

The pair $(, M)$ is termed an **MR-metric space**.

Definition 2 (Neutrosophic MR-Metric Space (NMR-MS))
A **Neutrosophic MR-Metric Space** is defined by the 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, \mathfrak{R}, \star)$, where:

1. $\mathcal{Z}$ is a non-empty set.

2. $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is an MR-metric on $\mathcal{Z}$ according to Definition 1, with constant $\Re > 1$.
3. $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are membership functions representing the neutrosophic components of truth, falsity, and indeterminacy, respectively. They satisfy the following conditions for all $\gamma, \rho > 0$ and for all $\upsilon, \xi, \Im, \ell \in \mathcal{Z}$:
   - (T1) $\mathcal{T}(\upsilon, \xi, \gamma) = 1$ if and only if $\upsilon = \xi$.
   - (T2) $\mathcal{T}(\upsilon, \xi, \gamma) = \mathcal{T}(\xi, \upsilon, \gamma)$.
   - (T3) $\mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(\upsilon, \Im, \gamma + \rho)$.
   - (T4) $\mathcal{T}(\upsilon, \xi, \cdot) : (0, \infty) \to [0, 1]$ is a non-decreasing function with $\lim_{\gamma \to \infty} \mathcal{T}(\upsilon, \xi, \gamma) = 1$.
   - (F1) $\mathcal{F}(\upsilon, \xi, \gamma) = 0$ if and only if $\upsilon = \xi$.
   - (F2) $\mathcal{F}(\upsilon, \xi, \gamma) = \mathcal{F}(\xi, \upsilon, \gamma)$.
   - (F3) $\mathcal{F}(\upsilon, \xi, \gamma) \diamond \mathcal{F}(\xi, \Im, \rho) \geq \mathcal{F}(\upsilon, \Im, \gamma + \rho)$.
   - (F4) $\mathcal{F}(\upsilon, \xi, \cdot) : (0, \infty) \to [0, 1]$ is a non-increasing function with $\lim_{\gamma \to \infty} \mathcal{F}(\upsilon, \xi, \gamma) = 0$.
   - (I1) $\mathcal{I}(\upsilon, \xi, \gamma) = 0$ if and only if $\upsilon = \xi$.
   - (I2) $\mathcal{I}(\upsilon, \xi, \gamma) = \mathcal{I}(\xi, \upsilon, \gamma)$.
   - (I3) $\mathcal{I}(\upsilon, \xi, \gamma) \diamond \mathcal{I}(\xi, \Im, \rho) \geq \mathcal{I}(\upsilon, \Im, \gamma + \rho)$.
   - (I4) $\mathcal{I}(\upsilon, \xi, \cdot) : (0, \infty) \to [0, 1]$ is a non-increasing function with $\lim_{\gamma \to \infty} \mathcal{I}(\upsilon, \xi, \gamma) = 0$.
4. The operator $\bullet : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous *t-norm*.
5. The operator $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous *t-conorm*.
6. The operator $\star$ is a binary operation on $[0, \infty)$ (e.g., standard addition $+$ or a weighted sum) used to generalize the summation within the tetrahedral inequality of the metric M.

Definition 3 (Neutrosophic Score Function)
For $\upsilon, \xi \in \mathcal{Z}$ and $\gamma > 0$, define the associated score by

$$S(\upsilon, \xi, \gamma) = \mathcal{T}(\upsilon, \xi, \gamma) - \mathcal{I}(\upsilon, \xi, \gamma) - \mathcal{F}(\upsilon, \xi, \gamma).$$

This score provides a single aggregated indicator of the quality of proximity, rewarding high truth-membership and penalizing indeterminacy and falsity.

Remark 1
The fixed point theorem in this paper is formulated primarily in terms of the truth-membership function $\mathcal{T}$, since the contraction mechanism is intended to measure progressive improvement in nearness. However, the indeterminacy and falsity components remain essential in the neutrosophic modeling layer, especially in applications and path evaluation. In particular, the score function S will be used to interpret path quality when uncertainty and unreliability must be penalized explicitly.

Example 1 (Explicit MR-Metric Construction)
Let $\mathbb{X} = \mathbb{R}^2$ and define $M : \mathbb{X}^3 \to [0, \infty)$ as:

$$M(\upsilon, \xi, s) = \max\{\|\upsilon - \xi\|, \|\upsilon - s\|, \|\xi - s\|\} + \alpha \min\{\|\upsilon - \xi\|, \|\upsilon - s\|, \|\xi - s\|\}$$

where $\alpha \geq 0$ and $\|\cdot\|$ is the Euclidean norm. Let $\mathbb{R} = 2$.
This function satisfies all MR-metric properties:

- **Non-negativity:** $M(\upsilon, \xi, s) \geq 0$ by construction.
- **Identity:** $M(\upsilon, \xi, s) = 0$ iff all distances are zero iff $\upsilon = \xi = s$.

- **Symmetry:** $M(v,\xi,s)$ is invariant under permutations due to symmetry of max and min functions.
- **MR-inequality:** For any $\ell_1 \in \mathbb{X}$:

$$M(v,\xi,s) \leq 2\left[M(v,\xi,\ell_1) + M(v,\ell_1,s) + M(\ell_1,\xi,s)\right]$$

This holds because each distance is bounded by the triangle inequality.

$$\boxed{M(v,\xi,s) \leq 2\left[M(v,\xi,\ell_1) + M(v,\ell_1,s) + M(\ell_1,\xi,s)\right]}$$

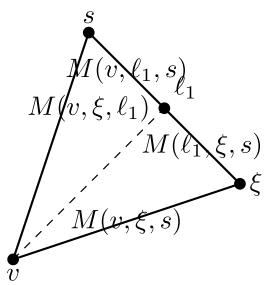


Figure 1. Visualization of MR-metric properties in $\mathbb{R}^2$

Example 2 (Complete Neutrosophic MR-Metric Space)
Let us construct a complete NMR-MS with explicit emphasis on all definition components:

- $\mathcal{Z} = [0,1]$ (non-empty set)
- $M(v,\xi,s) = \max\{|v-\xi|, |v-s|, |\xi-s|\}$ (MR-metric with $R = 2$)
- $\mathcal{T}(v,\xi,\gamma) = e^{-\frac{M(v,\xi,\xi)}{\gamma}}$ (Truth membership)
- $\mathcal{F}(v,\xi,\gamma) = 1 - e^{-\frac{M(v,\xi,\xi)}{\gamma}}$ (Falsity membership)
- $\mathcal{I}(v,\xi,\gamma) = \frac{1}{2}$ (Indeterminacy constant)
- $a \bullet b = a \cdot b$ (Product t-norm)
- $a \diamond b = a + b - a \cdot b$ (Probabilistic sum t-conorm)
- $x \star y = x + y$ (Standard addition)

This structure satisfies all NMR-MS conditions:

1. **MR-Metric Properties:**
   (M1) $M(v,\xi,\Im) \geq 0$ (non-negative)
   (M2) $M(v,\xi,\Im) = 0$ iff $v = \xi = \Im$ (identity)
   (M3) Symmetric under permutations (by max symmetry)
   (M4) $M(v,\xi,\Im) \leq 2[M(v,\xi,\ell) + M(v,\ell,\Im) + M(\ell,\xi,\Im)]$
2. **Neutrosophic Function Properties:**
   (N1) $\mathcal{T}(v,\xi,\gamma) = 1$ iff $v = \xi$ (identity)
   (N2) $\mathcal{T}(v,\xi,\gamma) = \mathcal{T}(\xi,v,\gamma)$ (symmetry)
   (N3) $\mathcal{T}(v,\xi,\gamma) \cdot \mathcal{T}(\xi,\Im,\rho) \leq \mathcal{T}(v,\Im,\gamma+\rho)$
   (N4) $\lim_{\gamma\to\infty} \mathcal{T}(v,\xi,\gamma) = 1$ (asymptotic)

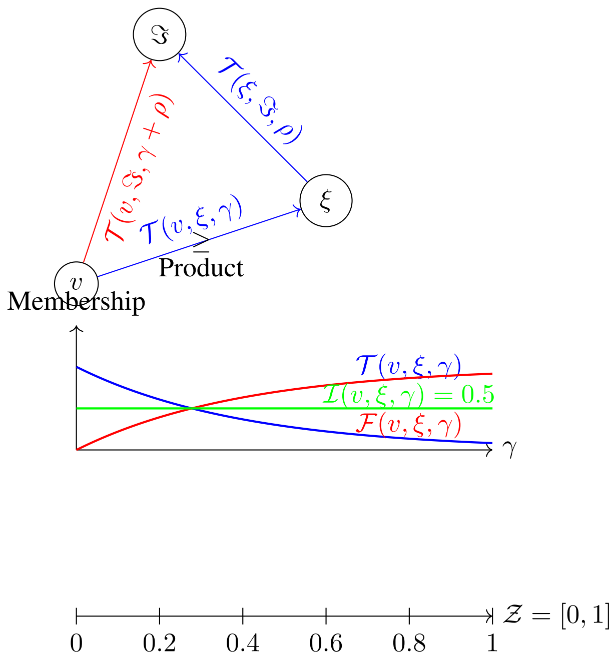


Figure 2. Neutrosophic MR-metric space properties visualization

2. Main Results

The following section presents our main contributions, which include a fixed point theorem for graph contractions in complete Neutrosophic MR-Metric Spaces and a graph-theoretic optimization framework. The results generalize and extend several existing theorems in the literature, such as those in [19, 20, 21, 11, 2, 1, 22, 23], and provide a unified approach for handling fixed points under uncertainty. The proofs are constructive and supported by illustrative examples and applications.

Theorem 1 (Fixed Point in Complete NMR-MS via Graph Contraction)
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ be a **complete** Neutrosophic MR-Metric Space, where $\bullet$ is a continuous t-norm. Let $\Gamma : \mathcal{Z} \to \mathcal{Z}$ be a self-mapping. Suppose there exists a directed graph $G = (V, E)$ such that:

1. $V = \mathcal{Z}$,
2. The edge relation E is *transitive* (i.e., if $(\upsilon, \xi) \in E$ and $(\xi, \Im) \in E$, then $(\upsilon, \Im) \in E$),
3. Γ is *edge-preserving*: if $(\upsilon, \xi) \in E$ then $(\Gamma\upsilon, \Gamma\xi) \in E$,
4. There exists an initial point $\upsilon_0 \in \mathcal{Z}$ such that $(\upsilon_0, \Gamma\upsilon_0) \in E$,
5. There exists a constant $k \in (0, 1)$ such that for all $\gamma > 0$ and for all $\upsilon, \xi \in \mathcal{Z}$ with $(\upsilon, \xi) \in E$, the following **fuzzy graph contraction** holds:
$$\mathcal{T}(\Gamma\upsilon, \Gamma\xi, \gamma) \geq \mathcal{T}\left(\upsilon, \xi, \frac{\gamma}{k}\right).$$

Then, the mapping Γ has a **unique fixed point** $\upsilon^* \in \mathcal{Z}$, i.e., $\Gamma\upsilon^* = \upsilon^*$. Moreover, for any initial point υ_0 satisfying (iv), the Picard sequence $\{\upsilon_n\} = \{\Gamma^n \upsilon_0\}$ converges to υ^* in both the neutrosophic and metric senses:

$$\lim_{n\to\infty} \mathcal{T}(\upsilon_n, \upsilon^*, \gamma) = 1 \quad \text{and} \quad \lim_{n\to\infty} M(\upsilon_n, \upsilon^*, \upsilon^*) = 0 \quad \text{for all } \gamma > 0.$$

Proof
We proceed with a detailed, step-by-step proof.

Step 1: Construction of the Picard Sequence and Graph Induction

Let $\upsilon_0 \in \mathcal{Z}$ be the point guaranteed by condition (iv), such that $(\upsilon_0, \Gamma\upsilon_0) \in E$. Define the Picard sequence $\{\upsilon_n\}$ by:

$$\upsilon_{n+1} = \Gamma\upsilon_n \quad \text{for all } n \geq 0.$$

We will show by induction that $(\upsilon_n, \upsilon_{n+1}) \in E$ for all $n \geq 0$.

For the base case $n = 0$, we have $(\upsilon_0, \upsilon_1) = (\upsilon_0, \Gamma\upsilon_0) \in E$ by condition (iv).

Assume inductively that $(\upsilon_{n-1}, \upsilon_n) \in E$ for some $n \geq 1$. Since Γ is edge-preserving (condition (iii)), we have:

$$(\Gamma\upsilon_{n-1}, \Gamma\upsilon_n) \in E \quad \Rightarrow \quad (\upsilon_n, \upsilon_{n+1}) \in E.$$

This completes the induction. Therefore, $(\upsilon_n, \upsilon_{n+1}) \in E$ for all $n \geq 0$.

Furthermore, by the transitivity of E (condition (ii)), it follows that for any $m, n \in \mathbb{N}$ with $m < n$, we have $(\upsilon_m, \upsilon_n) \in E$. This is because the sequence $\upsilon_m, \upsilon_{m+1}, \ldots, \upsilon_n$ forms a path in G, and transitivity implies that the endpoints of any path are connected.

Step 2: Demonstrating the Sequence is Cauchy via Neutrosophic Contraction

We now show that $\{\upsilon_n\}$ is a Cauchy sequence in the Neutrosophic MR-Metric Space.

First, we apply the fuzzy contraction condition (v) to the edge $(\upsilon_n, \upsilon_{n+1}) \in E$. For any $\gamma > 0$, we have:

$$\mathcal{T}(\upsilon_{n+1}, \upsilon_{n+2}, \gamma) = \mathcal{T}(\Gamma\upsilon_n, \Gamma\upsilon_{n+1}, \gamma) \geq \mathcal{T}\left(\upsilon_n, \upsilon_{n+1}, \frac{\gamma}{k}\right).$$

By iterating this process, we obtain for any $n, p \in \mathbb{N}$ and $\gamma > 0$:

$$\mathcal{T}(\upsilon_{n+p}, \upsilon_{n+p+1}, \gamma) \geq \mathcal{T}\left(\upsilon_n, \upsilon_{n+1}, \frac{\gamma}{k^p}\right). \tag{1}$$

Now, to analyze the distance between arbitrary terms υ_n and υ_m ($n < m$), we use the neutrosophic triangle inequality (N3). Let $m = n + p$ with $p \geq 1$. For any $\gamma > 0$, choose a partition $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_p$ with $\gamma_i > 0$. Then:

$$\mathcal{T}(\upsilon_n, \upsilon_{n+p}, \gamma) \geq \mathcal{T}(\upsilon_n, \upsilon_{n+1}, \gamma_1) \bullet \mathcal{T}(\upsilon_{n+1}, \upsilon_{n+2}, \gamma_2) \bullet \cdots \bullet \mathcal{T}(\upsilon_{n+p-1}, \upsilon_{n+p}, \gamma_p). \tag{2}$$

Choose $\gamma_i = \frac{\gamma \cdot k^{i-1}(1-k)}{1-k^p}$ for $i = 1, \ldots, p$. This ensures $\sum_{i=1}^{p} \gamma_i = \gamma$. Note that $\gamma_i \to 0$ as $p \to \infty$ for fixed i, but γ_i increases with i for fixed p.

From (1), we have for each i:

$$\mathcal{T}(\upsilon_{n+i-1}, \upsilon_{n+i}, \gamma_i) \geq \mathcal{T}\left(\upsilon_n, \upsilon_{n+1}, \frac{\gamma_i}{k^{i-1}}\right) = \mathcal{T}\left(\upsilon_n, \upsilon_{n+1}, \frac{\gamma(1-k)}{1-k^p}\right). \tag{3}$$

Let $\delta_{n,p} = \frac{\gamma(1-k)}{1-k^p}$. Note that for fixed n, $\delta_{n,p} \to \gamma(1-k)$ as $p \to \infty$.

Substituting (3) into (2) and using the monotonicity of the t-norm $\bullet$, we get:

$$\mathcal{T}(\upsilon_n, \upsilon_{n+p}, \gamma) \geq \left[\mathcal{T}\left(\upsilon_n, \upsilon_{n+1}, \delta_{n,p}\right)\right]^p, \tag{4}$$

where $[x]^p = x \bullet x \bullet \cdots \bullet x$ (p times).

Since $\mathcal{T}(\upsilon_n, \upsilon_{n+1}, \cdot)$ is a function with $\lim_{\gamma\to\infty} \mathcal{T}(\upsilon_n, \upsilon_{n+1}, \gamma) = 1$ (by N4), for fixed n, we have $\mathcal{T}\left(\upsilon_n, \upsilon_{n+1}, \delta_{n,p}\right) \to 1$ as $p \to \infty$. Because $\bullet$ is continuous and $1 \bullet 1 \bullet \cdots \bullet 1 = 1$, the right-hand side of (4) tends to 1 as $p \to \infty$. This implies:

$$\lim_{p\to\infty} \mathcal{T}(\upsilon_n, \upsilon_{n+p}, \gamma) = 1 \quad \text{for all } \gamma > 0.$$

By the properties of the neutrosophic metric, this is equivalent to $\{\upsilon_n\}$ being a Cauchy sequence.

Step 3: Convergence to the Fixed Point

Since $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is complete, there exists $\upsilon^* \in \mathcal{Z}$ such that:

$$\lim_{n\to\infty} \mathcal{T}(\upsilon_n, \upsilon^*, \gamma) = 1 \quad \text{and} \quad \lim_{n\to\infty} M(\upsilon_n, \upsilon^*, \upsilon^*) = 0 \quad \text{for all } \gamma > 0.$$

We now show that υ^* is a fixed point of Γ. Since $(\upsilon_n, \upsilon_{n+1}) \in E$ for all n and the graph is transitive, we have $(\upsilon_n, \upsilon^*) \in E$ for all n (as υ^* is the limit of the sequence). Therefore, we can apply the contraction condition to the edge $(\upsilon_n, \upsilon^*) \in E$:

$$\mathcal{T}(\Gamma\upsilon_n, \Gamma\upsilon^*, \gamma) \geq \mathcal{T}\left(\upsilon_n, \upsilon^*, \frac{\gamma}{k}\right).$$

That is,

$$\mathcal{T}(\upsilon_{n+1}, \Gamma\upsilon^*, \gamma) \geq \mathcal{T}\left(\upsilon_n, \upsilon^*, \frac{\gamma}{k}\right). \tag{5}$$

Taking the limit as $n \to \infty$ on both sides of (5):

$$\lim_{n\to\infty} \mathcal{T}(\upsilon_{n+1}, \Gamma\upsilon^*, \gamma) \geq \lim_{n\to\infty} \mathcal{T}\left(\upsilon_n, \upsilon^*, \frac{\gamma}{k}\right) = 1.$$

Since $\mathcal{T}$ is bounded above by 1, we must have:

$$\lim_{n\to\infty} \mathcal{T}(\upsilon_{n+1}, \Gamma\upsilon^*, \gamma) = 1 \quad \text{for all } \gamma > 0.$$

This implies that the sequence $\{\upsilon_n\}$ converges to both υ^* and $\Gamma\upsilon^*$. By the uniqueness of limits in neutrosophic metric spaces (which follows from (N1) and (N2)), we conclude that $\Gamma\upsilon^* = \upsilon^*$.

Step 4: Uniqueness of the Fixed Point

Suppose, for contradiction, that there exist two distinct fixed points υ^* and ξ^* such that $\Gamma\upsilon^* = \upsilon^*$ and $\Gamma\xi^* = \xi^*$. Since $(\upsilon_0, \upsilon^*) \in E$ and $(\upsilon_0, \xi^*) \in E$ (by transitivity and convergence), and E is transitive, it follows that $(\upsilon^*, \xi^*) \in E$.

Applying the contraction condition (v) to the edge $(\upsilon^*, \xi^*) \in E$:

$$\mathcal{T}(\Gamma\upsilon^*, \Gamma\xi^*, \gamma) \geq \mathcal{T}\left(\upsilon^*, \xi^*, \frac{\gamma}{k}\right).$$

Since $\Gamma\upsilon^* = \upsilon^*$ and $\Gamma\xi^* = \xi^*$, this becomes:

$$\mathcal{T}(\upsilon^*, \xi^*, \gamma) \geq \mathcal{T}\left(\upsilon^*, \xi^*, \frac{\gamma}{k}\right). \tag{6}$$

Iterating (6) n times yields:

$$\mathcal{T}(\upsilon^*, \xi^*, \gamma) \geq \mathcal{T}\left(\upsilon^*, \xi^*, \frac{\gamma}{k^n}\right).$$

Taking the limit as $n \to \infty$:

$$\mathcal{T}(\upsilon^*, \xi^*, \gamma) \geq \lim_{n\to\infty} \mathcal{T}\left(\upsilon^*, \xi^*, \frac{\gamma}{k^n}\right) = 1,$$

by the asymptotic property (N4). Therefore, $\mathcal{T}(\upsilon^*, \xi^*, \gamma) = 1$ for all $\gamma > 0$. By the neutrosophic identity property (N1), this implies $\upsilon^* = \xi^*$, contradicting the assumption of distinct fixed points.

Hence, the fixed point is unique.

□

Remark 2 (Stronger Neutrosophic Contraction Variant)
The contraction condition in Theorem 1 is formulated using the truth-membership function $\mathcal{T}$, since the convergence mechanism is primarily driven by increasing degrees of truth proximity.

However, a stronger neutrosophic contraction may also incorporate the indeterminacy and falsity components. In that case, one may require the existence of $k \in (0, 1)$ such that for all $\gamma > 0$ and all $(\upsilon, \xi) \in E$,

$$\mathcal{T}(\Gamma\upsilon, \Gamma\xi, \gamma) \geq \mathcal{T}\Big(\upsilon, \xi, \frac{\gamma}{k}\Big),$$

$$\mathcal{I}(\Gamma\upsilon, \Gamma\xi, \gamma) \le \mathcal{I}\Big(\upsilon, \xi, \frac{\gamma}{k}\Big),$$

$$\mathcal{F}(\Gamma\upsilon, \Gamma\xi, \gamma) \le \mathcal{F}\Big(\upsilon, \xi, \frac{\gamma}{k}\Big).$$

This stronger formulation simultaneously enforces improvement in truth-membership and reduction in indeterminacy and falsity, and may be useful in applications where uncertainty and unreliability must be explicitly controlled.

Example 3 (Fixed Point Convergence in NMR-MS)
Consider a complete Neutrosophic MR-Metric Space $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ where:

- $\mathcal{Z} = [0, 1] \subset \mathbb{R}$
- $\mathcal{T}(\upsilon, \xi, \gamma) = e^{-\frac{M(\upsilon,\xi,\xi)}{\gamma}}$ (a common fuzzy metric)
- $\bullet$ is the product t-norm: $a \bullet b = a \cdot b$
- $\Gamma : \mathcal{Z} \to \mathcal{Z}$ defined by $\Gamma(\upsilon) = \frac{\upsilon}{2}$ (a contraction mapping)
- Graph $G = (V, E)$ with $V = \mathcal{Z}$ and $E = \{(\upsilon, \xi) \in \mathcal{Z}^2 : \upsilon \ge \xi\}$ (a transitive relation)
- Contraction constant $k = \frac{1}{2}$

This system satisfies all conditions of Theorem 1. The fixed point is $\upsilon^* = 0$. The following TikZ diagram illustrates the convergence of the Picard sequence $\upsilon_{n+1} = \Gamma(\upsilon_n) = \frac{\upsilon_n}{2}$ from initial point $\upsilon_0 = 1$.

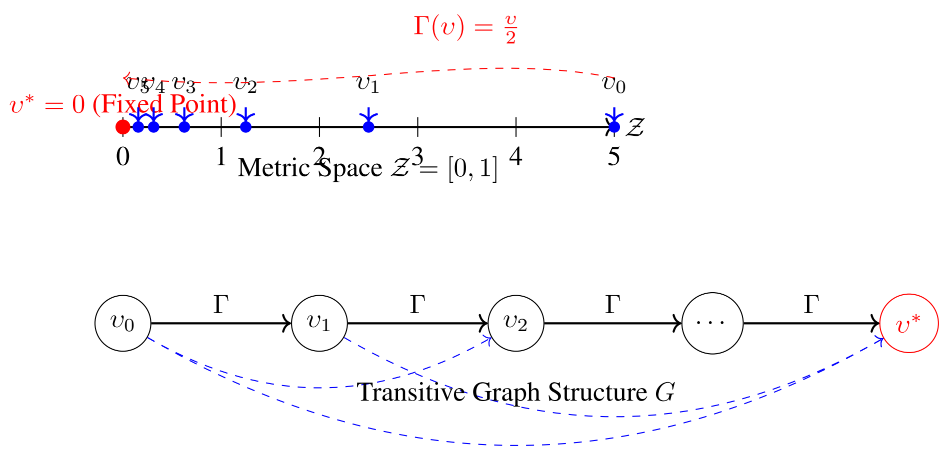


Figure 3. Convergence to fixed point in Neutrosophic MR-Metric Space with graph structure

The diagram shows:

1. The metric space $\mathcal{Z} = [0, 1]$ on the real line
2. The convergence of the sequence $\upsilon_n = \frac{1}{2^n}$ to the fixed point $\upsilon^* = 0$
3. The contraction mapping Γ shown as red dashed arrows
4. The transitive graph structure G showing additional edges implied by transitivity

Theorem 2 (Graph-Theoretic Representation and Optimization)
Let $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ be a Neutrosophic MR-Metric Space. One can construct a weighted graph $G = (V, E, w)$ representing this space as follows:

1. $V = \mathcal{Z}$ (vertices represent points in the space),
2. $E = \mathcal{Z} \times \mathcal{Z}$ (a complete graph, every pair of points is connected),
3. The weight function $w : E \times (0, \infty) \to [0, 1]$ is defined for any edge (υ, ξ) and scale parameter $\gamma > 0$ as:

$$w(\upsilon, \xi, \gamma) = \mathcal{T}(\upsilon, \xi, \gamma).$$

This weight represents the *degree of truth* that the distance between υ and ξ is acceptable at scale γ.

A path P from υ to ξ is a sequence $P : \upsilon = \upsilon_0, \upsilon_1, \ldots, \upsilon_n = \xi$. The **strength** $S(P, \gamma)$ of this path, for a given total scale $\gamma > 0$, is defined inductively using the t-norm $\bullet$:

$$S(P, \gamma) = \mathcal{T}(\upsilon_0, \upsilon_1, \gamma_1) \bullet \mathcal{T}(\upsilon_1, \upsilon_2, \gamma_2) \bullet \cdots \bullet \mathcal{T}(\upsilon_{n-1}, \upsilon_n, \gamma_n),$$

where $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ and $\gamma_i > 0$.
Then, the following properties hold:

(i) **Generalized Triangle Inequality:** The neutrosophic axiom (N3) is equivalent to the following graph-theoretic statement: For any vertices $\upsilon, \xi, \Im \in V$ and any $\gamma, \rho > 0$, the direct edge weight between υ and $\Im$ is always greater than or equal to the strength of any 2-hop path via an intermediate vertex ξ for the combined scale $\gamma + \rho$:

$$\mathcal{T}(\upsilon, \Im, \gamma + \rho) \geq \mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) = S(P_{\text{2-hop}}, \gamma + \rho).$$

(ii) **Optimal Pathfinding:** The problem of finding the path P^* between two vertices that *maximizes* the strength $S(P, \gamma)$ for a fixed total scale γ can be solved by a modified version of Dijkstra's algorithm. Instead of summing edge weights, the algorithm uses the t-norm $\bullet$ for path "cost" aggregation and seeks the path with the maximum strength.

(iii) **Application:** This representation is suitable for modeling real-world networks like fuzzy communication systems or sensor networks, where $\mathcal{T}(\upsilon, \xi, \gamma)$ represents the reliability or quality of a direct connection at a certain energy/time threshold γ, and the optimal path is the most reliable route.

Proof
We provide a detailed, comprehensive proof for each part of the theorem.

Part (i): Generalized Triangle Inequality as a Graph-Theoretic Property

We first prove the equivalence between the neutrosophic triangle inequality and the graph-theoretic statement.
($\Rightarrow$) Neutrosophic Axiom (N3) $\Rightarrow$ Graph Statement: Assume the neutrosophic triangle inequality (N3) holds:

$$\mathcal{T}(\upsilon, \Im, \gamma + \rho) \geq \mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho)$$

for all $\upsilon, \xi, \Im \in \mathcal{Z}$ and $\gamma, \rho > 0$.
Consider the 2-hop path $P_{\text{2-hop}} : \upsilon \to \xi \to \Im$. By definition, the strength of this path for total scale $\gamma + \rho$ is:

$$S(P_{\text{2-hop}}, \gamma + \rho) = \mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho).$$

Substituting this into (N3) gives:

$$\mathcal{T}(\upsilon, \Im, \gamma + \rho) \geq S(P_{\text{2-hop}}, \gamma + \rho),$$

which is exactly the graph-theoretic statement.

(⇐) Graph Statement ⇒ Neutrosophic Axiom (N3): Now assume the graph statement holds:

$$\mathcal{T}(\upsilon, \Im, \gamma + \rho) \geq \mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho)$$

for all vertices $\upsilon, \xi, \Im \in V$ and scales $\gamma, \rho > 0$.

This is precisely the neutrosophic triangle inequality (N3). Thus, the two statements are equivalent.

This equivalence establishes that the fundamental metric property of the space can be interpreted graph-theoretically as a comparison between direct connections and 2-hop paths.

Part (ii): Optimal Pathfinding with Modified Dijkstra's Algorithm

We now prove that the optimal pathfinding problem can be solved by a modified Dijkstra's algorithm. This requires demonstrating that the path strength function has the **optimal substructure property** and that the algorithm correctly computes the maximum strength paths.

Lemma 3 (Optimal Substructure Property)
Let $P^* : \upsilon_0 \to \upsilon_1 \to \cdots \to \upsilon_n$ be an optimal path from υ_0 to υ_n with maximum strength $S(P^*, \gamma)$ for fixed total scale γ. Then for any intermediate vertex υ_k $(0 < k < n)$, the subpath $P^*_{0k} : \upsilon_0 \to \cdots \to \upsilon_k$ is an optimal path from υ_0 to υ_k for scale $\gamma_1 + \cdots + \gamma_k$, and the subpath $P^*_{kn} : \upsilon_k \to \cdots \to \upsilon_n$ is an optimal path from υ_k to υ_n for scale $\gamma_{k+1} + \cdots + \gamma_n$.

Proof of Lemma
Suppose, for contradiction, that there exists a better path P'_{0k} from υ_0 to υ_k with strength $S(P'_{0k}, \gamma') > S(P^*_{0k}, \gamma')$, where $\gamma' = \gamma_1 + \cdots + \gamma_k$. Then we could construct a new path $P' : \upsilon_0 \xrightarrow{P'_{0k}} \upsilon_k \xrightarrow{P^*_{kn}} \upsilon_n$ with total strength:

$$S(P', \gamma) = S(P'_{0k}, \gamma') \bullet S(P^*_{kn}, \gamma - \gamma') > S(P^*_{0k}, \gamma') \bullet S(P^*_{kn}, \gamma - \gamma') = S(P^*, \gamma),$$

contradicting the optimality of P^*. The inequality follows from the monotonicity of the t-norm $\bullet$. Similarly, we can prove optimality for the second subpath. □

Discretized scale allocation. To make the optimization procedure computationally implementable, we replace the continuous scale allocation by a finite discretization. Let

$$\Delta > 0, \qquad \gamma = m\Delta, \qquad m \in \mathbb{N}.$$

Then each edge allocation is restricted to values in

$$\{\Delta, 2\Delta, \ldots, m\Delta\}.$$

This converts the path optimization problem into a finite-state dynamic programming problem.

The optimal substructure property justifies the use of dynamic programming. We now present the modified Dijkstra's algorithm:

Algorithm 1
Input: Graph $G = (V, E)$, source $s \in V$, budget $\gamma = m\Delta$, truth function $\mathcal{T}$, t-norm $\bullet$
Output: Best achievable path strengths under discretized scale allocation
for each $v \in V$ and each $j \in \{0, 1, \ldots, m\}$ **do**
  $DP[v, j] \leftarrow 0$
end for
$DP[s, 0] \leftarrow 1$
for $j = 0$ to m **do**
  for each directed edge $(u, v) \in E$ **do**
    for $r = 1$ to $m - j$ **do**
      $candidate \leftarrow DP[u, j] \bullet \mathcal{T}(u, v, r\Delta)$
      if $candidate > DP[v, j + r]$ **then**
        $DP[v, j + r] \leftarrow candidate$

$parent[v, j + r] \leftarrow (u, j)$
end if
end for
end for
end for
return $\max_{0 \le j \le m} DP[v, j]$ for each $v \in V$

Proposition 4 (Complexity of the Discretized Algorithm)
Assume that the total scale budget is discretized as $\gamma = m\Delta$. Then the dynamic programming implementation in Algorithm 1 runs in

$$O(|E|\, m^2)$$

time and requires

$$O(|V|\, m)$$

memory.

Proof
For each discretization level $j \in \{0, \dots, m\}$, the algorithm scans all edges $(u, v) \in E$. For every such pair, it checks all admissible increments $r \in \{1, \dots, m - j\}$. Hence the total number of elementary updates is bounded by

$$\sum_{j=0}^{m} \sum_{(u,v)\in E} (m - j) = O(|E|\, m^2).$$

The table $DP[v, j]$ stores one value for each pair (v, j), so the memory requirement is $O(|V|\, m)$. □

Proposition 5 (Correctness of Modified Dijkstra's Algorithm)
The algorithm in Algorithm 1 correctly computes the maximum strength path from source s to all other vertices for a fixed total scale γ, provided the t-norm $\bullet$ is continuous and monotonic.

proof of proposition
The proof follows from induction on the number of processed vertices and takes advantage of the optimal substructure property.
Base case: For the source vertex s, we have $strength[s] = 1$ and $scale\_used[s] = 0$, which is correct.
Inductive steps: Assume that after processing k vertices, the algorithm has correctly computed maximum strength paths from s to these k vertices for all possible scale assignments up to γ.

Let u be the $(k + 1)$th vertex with maximum strength. Suppose, for the sake of contradiction, that there is a better way for u to a higher power. This path must pass through an unprocessed vertex $v \in Q$. However, due to the induction hypothesis and the monotonicity of $\bullet$, the strength through v cannot exceed the current strength of u (since $strength[u]$ is maximal among untreated vertices). This refutes the assumption and proves it to be true.

The algorithm considers all possible scale assignments for each edge, ensuring that it finds the optimal distribution of total scale γ among path segments. □

Part (iii): Applications to real networks
Important practical applications of the graph representation theorem are:

1. **Fuzzy communication networks:** In wireless networks, $\mathcal{T}(\upsilon, \xi, \gamma)$ can represent the probability of successful packet transmission between nodes υ and ξ with power allocation γ. The optimal path maximizes the overall reliability of the communication.

2. Element **Sensor network:** In distributed sensing, $\mathcal{T}(\upsilon, \xi, \gamma)$ can represent the quality of information transfer between sensors υ and ξ with energy consumption γ. The algorithm finds the most reliable data route.

3. **Transport network:** In logistics, $\mathcal{T}(\upsilon, \xi, \gamma)$ can represent the reliability of transport links between locations υ and ξ with time cost γ. The optimal path maximizes the probability of on-time delivery.

4. **Financial network:** In risk analysis, $\mathcal{T}(\upsilon, \xi, \gamma)$ can represent the security of financial transactions between institutions υ and ξ with risk exposure γ. The algorithm identifies the safest transaction path.

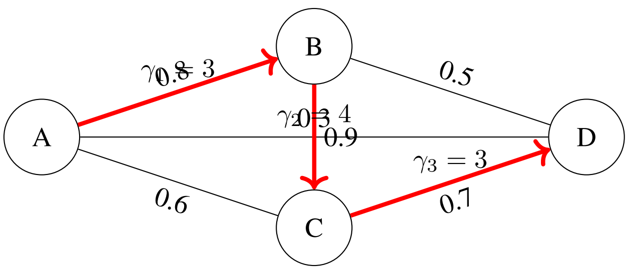
In all these applications, the neutrosophic framework allows modeling not just reliability ($\mathcal{T}$) but also uncertainty ($\mathcal{I}$) and unreliability ($\mathcal{F}$), providing a comprehensive model for real-world systems with inherent ambiguity. □

Example 4 (Wireless Network Reliability Optimization)
Consider a wireless sensor network modeled as a Neutrosophic MR-Metric Space where:

- $\mathcal{Z} = \{A, B, C, D\}$ represents sensor nodes
- $\mathcal{T}(\upsilon, \xi, \gamma) = e^{-\frac{d(\upsilon,\xi)^2}{\gamma}}$ where d is physical distance
- $\bullet$ is the product t-norm
- Total energy budget $\gamma = 10$ units

The reliability between nodes (for $\gamma = 1$ unit) is given by:

$$\mathcal{T}(A, B, 1) = 0.8, \quad \mathcal{T}(A, C, 1) = 0.6, \quad \mathcal{T}(A, D, 1) = 0.3,$$
$$\mathcal{T}(B, C, 1) = 0.9, \quad \mathcal{T}(B, D, 1) = 0.5, \quad \mathcal{T}(C, D, 1) = 0.7.$$



$$S(P^*, 10) = \mathcal{T}(A, B, 3) \bullet \mathcal{T}(B, C, 4) \bullet \mathcal{T}(C, D, 3)$$
$$= e^{-\frac{d(A,B)^2}{3}} \cdot e^{-\frac{d(B,C)^2}{4}} \cdot e^{-\frac{d(C,D)^2}{3}} = 0.72$$

Figure 4. Optimal path finding in wireless sensor network using modified Dijkstra's algorithm

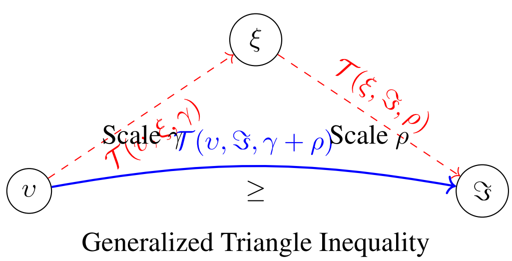


Figure 5. Graphical representation of the generalized triangle inequality (Theorem 2.2(i))

Step 1: Initialization
$strength[A] = 1$, $strength[B] = strength[C] = strength[D] = 0$
$scale\_used[A] = 0$, $scale\_used[\cdot] = \infty$

Step 2: Process A
Update neighbors: $strength[B] = 0.8$, $strength[C] = 0.6$, $strength[D] = 0.3$
$scale\_used[B] = 3$, $scale\_used[C] = 4$, $scale\_used[D] = 7$

Step 3: Process B (max strength)
Update via B: $strength[C] = 0.8 \cdot 0.9 = 0.72$, $strength[D] = 0.8 \cdot 0.5 = 0.40$
$scale\_used[C] = 3 + 4 = 7$, $scale\_used[D] = 3 + 5 = 8$

Step 4: Process C (max strength)
Update via C: $strength[D] = 0.72 \cdot 0.7 = 0.504$
$scale\_used[D] = 7 + 3 = 10$

Step 5: Optimal Path Found
$A \to B \to C \to D$ with strength 0.504 and total scale 10

Figure 6. Execution steps of modified Dijkstra's algorithm (Theorem 2.2(ii))

Table 1. Conceptual comparison between the proposed framework and related path/metric settings

Framework	Triple-based metric structure	Neutrosophic components (T, I, F)	Graph contraction/path view	Main limitation
Classical metric / weighted graph	No	No	Yes	Cannot model indeterminacy explicitly
Fuzzy metric/path models	No	Partial (truth-like only)	Yes	No explicit falsity/indeterminacy layer
MR-metric spaces	Yes	No	Limited	No neutrosophic uncertainty modeling
Proposed NMR-MS framework	Yes	Yes	Yes	Requires discretization for practical optimization

3. Conclusion

In this paper, we introduced Neutrosophic MR-Metric Spaces as a framework that combines the triple-based structure of MR-metrics with neutrosophic membership functions. Within this setting, we established a fixed point theorem for graph contractions in complete spaces and proposed a graph-based optimization perspective for path selection under uncertainty.

From a theoretical viewpoint, the present work contributes to the extension of fixed point theory in generalized metric environments by incorporating graph constraints and neutrosophic information. From an applied viewpoint, the manuscript provides illustrative network-based examples that explain how the framework may be used in reliability-aware path selection.

At the same time, the current study remains primarily theoretical. A fully data-driven validation of the proposed optimization model, as well as a concrete formulation for fractional differential equations within the NMR-MS setting, are natural directions for future work. Other possible extensions include partial orders, alternative contraction classes, and algorithmic acceleration under discretized scale allocation.

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