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Improved Inference for the XLindley Distribution: A Comparative Analysis of Confidence Intervals and Applications

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Abstract The XLindley distribution is a one-parameter lifetime distribution obtained as a mixture of the exponential and Lindley distributions and is suitable for modeling positively skewed reliability and survival data. Although the XLindley distribution has received increasing attention, there has been no comprehensive comparative study of confidence interval (CI) estimation methods for its parameter. This paper proposes four CI estimation methods for the XLindley distribution parameter: the likelihood-based CI, Wald-type CI, bootstrap-t CI, and bias-corrected and accelerated (BCa) bootstrap CI. Their performance is examined through Monte Carlo simulation studies under various sample sizes and parameter values. Empirical coverage probability (ECP) and average width (AW) are used as evaluation criteria. The results indicate that the likelihood-based and Wald-type CIs provide stable and reliable coverage across most scenarios, whereas the bootstrap-t and BCa bootstrap methods yield narrower intervals but may exhibit under-coverage in small samples. The proposed methods are further illustrated using two real datasets on bladder cancer remission times and light bulb failure times. Model comparison results based on information criteria, goodness-of-fit tests, and graphical analysis confirm the suitability of the XLindley distribution. Overall, the proposed CI methods provide practical tools for inference in lifetime data analysis.

Keywords Lifetime Distribution, Interval Estimation, Likelihood Function, Bootstrap Method, Statistical Inference

AMS 2010 subject classifications 62F25, 62F03, 65C05, 65C60

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1. Introduction

Lifetime and reliability analysis is important in several applied fields, where accurate modeling of time-to-event data is essential for risk assessment and decision-making. It is often necessary to use flexible probability distributions that perform well with positively skewed data from engineering, biomedical, and environmental studies. Although classical lifetime distributions such as the exponential, gamma, and Lindley distributions are analytically tractable, they may lack sufficient flexibility to capture complex hazard rate behaviors observed in real data. Specifically, real-world applications often exhibit increasing or non-monotonic hazard functions, which are difficult to model with simple one-parameter distributions.

To address these limitations, various extensions of the Lindley distribution have been suggested in the literature to improve distribution flexibility. One example is the Power Lindley distribution proposed by Ghitany et al. [13], which includes a power parameter to allow for a broader range of shapes for density and hazard rates. Merovci [18] introduced the Transmuted Lindley distribution, which employs a transmutation mechanism to improve control over skewness and tail behavior. The generalized Lindley distribution proposed by Nadarajah et al. [20] incorporates an additional shape parameter to allow for increasing, decreasing, and bathtub-shaped hazard rates

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and has been shown to outperform classical distributions such as the gamma, lognormal, and Weibull distributions. The exponentiated Lindley distribution [2] and its variants were derived using the exponentiation method, first used for distributions of the exponential type. The exponentiation method introduces an additional shape parameter to these distributions, allowing greater control over skewness and tail behavior. This makes them better at modeling heterogeneous lifetime data.

Although these extensions improve flexibility, adding more parameters may make statistical inference and parameter estimation more difficult, particularly for small or moderate sample sizes. Because of the compromise between inferential simplicity and model flexibility, several researchers had proposed the mixture distributions, which combine two or more baseline distributions while maintaining one-parameter structure. Examples range from distributions like the Shanker [27], Akash [28], Aradhana [29], Rani [33], Gharaibeh [11], Iwueze [10], Juchez [7] and Ola [22] distributions, among others. In recent studies, Chouia and Zeghdoudi [5] introduced the XLindley distribution as a two-component mixture of exponential and Lindley distributions. The exponential distribution has a scale parameter θ, and the Lindley distribution also has a parameter θ. This is a more efficient one-parameter distribution than others. This is indicative of the flexibility of the XLindley distribution, as it was applied to the survival times of individuals in Sierra Leone infected with the Ebola virus.

Furthermore, its hazard rate function is strictly increasing, making it an appropriate alternative for modeling wear-out and aging effects common in reliability and survival data. Also, the XLindley distribution has closed-form formulas for important statistical properties like the moments, skewness, kurtosis, and quantile function.

Although there is increasing interest in the XLindley distribution and its properties, there has been no comprehensive comparative evaluation of confidence interval (CI) estimation methods for its parameter. While standard inferential techniques such as likelihood-based, Wald-type, and bootstrap methods are well established in the literature, their performance has not yet been systematically investigated in the context of the XLindley distribution. To address this gap, this paper applies and compares several well-established CI estimation methods for the XLindley distribution parameter. In particular, we studied likelihood-based CIs and Wald-type CIs based on asymptotic theory. We also introduce bootstrap-based CIs, such as the bootstrap-t and the bias-corrected and accelerated (BCa) bootstrap methods. Through Monte Carlo simulation studies, the performance of the proposed CIs is also evaluated in terms of coverage probability and average width. To the best of our knowledge, this study provides the first comprehensive comparative analysis of these CI methods for the XLindley distribution.

Furthermore, the proposed CI methods are applied on the real datasets to illustrate their practical implementation and to demonstrate the usefulness of the XLindley distribution in real-world applications. The empirical findings demonstrate the efficacy of the proposed CIs for supporting robust statistical inference and uncertainty quantification in lifetime and reliability analysis. Consequently, the findings of this study provide practitioners and researchers with practical tools for parameter inference, facilitating more informed decision-making in applied fields.

The subsequent sections of this paper are structured as follows. Section 2 presents the Lindley and XLindley distributions, discusses point parameter estimation, and presents the four methods for constructing CIs. Section 3 describes the simulation experiments and their results. Section 4 applies the proposed methods to two real datasets. Finally, Section 5 provides the conclusions, and Section 6 covers the recommendations and directions for future research.

2. Methodology

This section reviews the Lindley and XLindley distributions along with their statistical properties. After the point estimation of the XLindley distribution parameter is based on maximum likelihood estimation. The final part of this section discusses the construction of CIs for the XLindley distribution parameter.

2.1. The Lindley and the XLindley Distributions

The Lindley distribution is a continuous probability distribution for non-negative random variables. It is derived as a mixture of the exponential and gamma distributions, with specific mixing probabilities. In this formulation, the exponential distribution is defined with rate parameter θ, and the gamma distribution is defined with shape parameter 2 and rate parameter θ. Let X denotes a random variable that follows the Lindley distribution with parameter θ. The probability density function (PDF) of the Lindley can be derived using a mixed distribution comprising two components, each weighted by their respective mixing probabilities, as follows:

$$f_{\text{Lindley}}(x;\theta) = p \cdot f_{\text{Exp}}(x;\theta) + (1-p) \cdot f_{\text{Gam}}(x;\theta,2), \tag{1}$$

where $f_{\text{Exp}}(x;\theta) = \theta e^{-\theta x}$ is the PDF of the exponential distribution with rate parameter θ and $f_{\text{Gam}}(x;\theta,2)$ is the PDF of the gamma distribution with shape parameter 2 and rate parameter θ and the mixing proportion is $p = \theta/(1+\theta)$. The PDF of the Lindley distribution is expressed as follows:

$$f_{\text{Lindley}}(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \quad x \geq 0,\ \theta > 0. \tag{2}$$

Figure 1 illustrates the PDF plot of the Lindley distribution for various parameter values. In Figure 1, as the parameter θ increases, the density becomes more concentrated near the origin and exhibits a faster decay, indicating shorter lifetimes. Conversely, smaller values of θ produce heavier right tails, reflecting greater variability in the data.

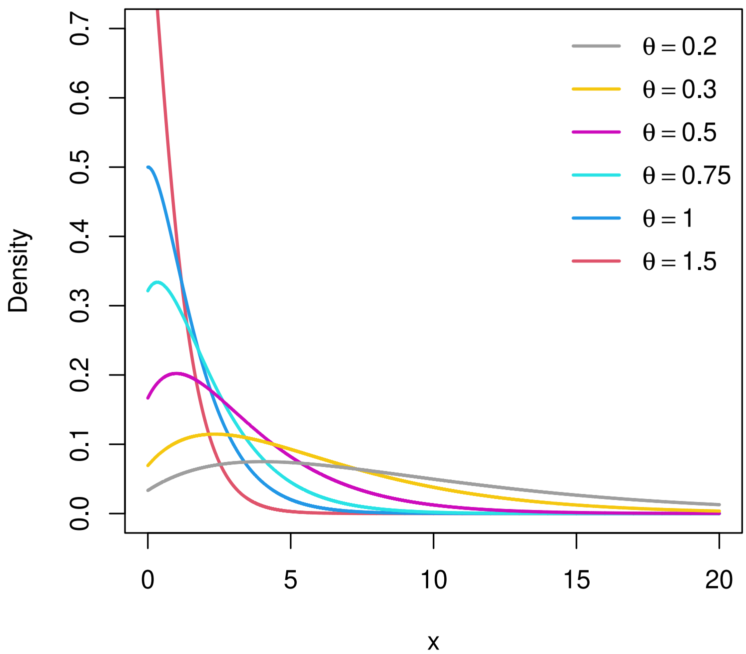

Figure 1. The PDF plot of the Lindley distribution for different parameter values

Chouia and Zeghdoudi [5] proposed the XLindley distribution, a one-parameter continuous distribution that was derived by combining two distributions. The XLindley distribution was developed as a more flexible extension of the Lindley distribution, each weighted by their respective mixing probabilities $p = 1/(1+\theta)$, given in (3):

$$f_{\text{XLindley}}(x;\theta) = p \cdot f_{\text{Exp}}(x;\theta) + (1-p) \cdot f_{\text{Lindley}}(x;\theta), \tag{3}$$

where $f_{\text{Exp}}(x;\theta)$ is the PDF of the exponential distribution with rate parameter θ and $f_{\text{Lindley}}(x;\theta)$ is given in (2). Therefore, the PDF of the XLindley distribution is

$$f_{\text{XLindley}}(x;\theta) = \frac{\theta^2}{(1+\theta)^2}\,(2+\theta+x)\,e^{-\theta x}; \quad x>0, \theta>0. \tag{4}$$

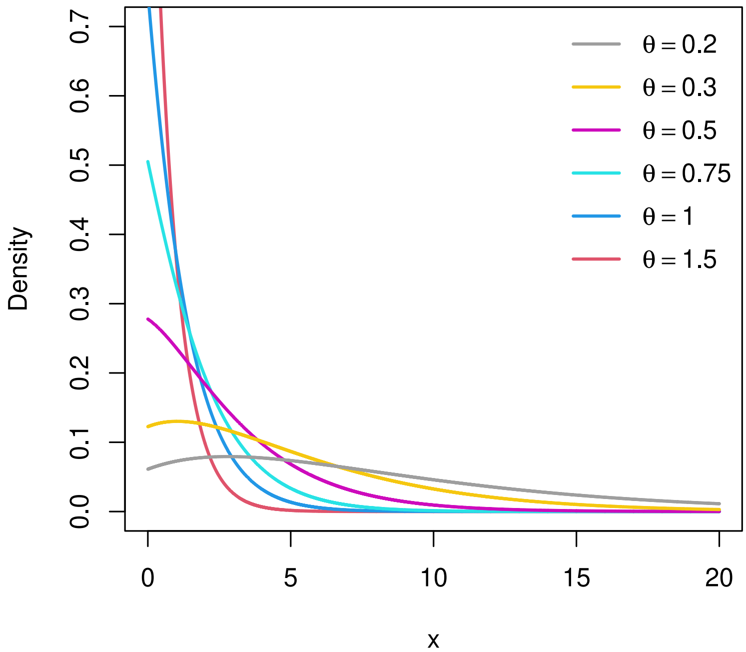

Figure 2. The PDF plot of the XLindley distribution for different parameter values

Figure 2 demonstrates the PDF plot of the XLindley distribution. In comparison to the Lindley distribution, the XLindley distribution can allow greater skewness and heavier tails, dependent on the parameter values. Variations in the parameter substantially influence the peak and tail characteristics of the density function, providing the XLindley distribution appropriate for modeling complicated lifetime data where classical distributions may be inadequate.

Some important statistical properties of the XLindley distribution are outlined below, including the r^{th} moment, mean, variance, survival function, and hazard function. The r^{th} moment of a random variable X following the XLindley distribution is given by

$$\mu_r' = \mathbb{E}(X^r) = \frac{(\theta^2+2\theta+r+1)\,r!}{(1+\theta)^2\,\theta^r}, \qquad r=1,2,\ldots$$

The mean (μ) and variance (σ^2) of the XLindley distribution are given as follows:

$$\mathbb{E}(X) = \mu = \frac{(1+\theta)^2+1}{(1+\theta)^2\,\theta},$$

and

$$\text{Var}(X) = \sigma^2 = \frac{(1+\theta)^4+4\theta^2+6\theta+1}{(1+\theta)^4\,\theta^2}.$$

The skewness coefficient, kurtosis coefficient, and coefficient of variation of the XLindley distribution are given as follows:

$$\text{Skewness} = \sqrt{\beta_1} = \frac{\mathbb{E}(X^3)}{\{\text{Var}(X)\}^{3/2}} = \frac{6\left(\theta^2+2\theta+4\right)(1+\theta)^4}{\left[(1+\theta)^4+4\theta^2+6\theta+1\right]^{3/2}},$$

$$\text{Kurtosis} = \beta_2 = \frac{\mathbb{E}(X^4)}{\{\text{Var}(X)\}^2} = \frac{24\left(\theta^2+2\theta+5\right)(1+\theta)^6}{\left[(1+\theta)^4+4\theta^2+6\theta+1\right]^2},$$

and

$$\text{C.V.} = \gamma = \frac{\sqrt{\text{Var}(X)}}{\mathbb{E}(X)} = \frac{\sqrt{(1+\theta)^4+4\theta^2+6\theta+1}}{(1+\theta)^2+1}.$$

The survival and hazard rate functions are respectively

$$S(x) = \left(1+\frac{\theta x}{(1+\theta)^2}\right)e^{-\theta x}, \quad \text{and} \quad hrf(x) = \frac{\theta^2(x+\theta+2)}{(1+\theta)^2+\theta x}.$$

2.2. *Point Parameter Estimation*

Using the maximum likelihood (ML) method, we can derive the point estimate for the parameter of the XLindley distribution. This method involves constructing the likelihood function based on a given sample and then determining the parameter value that maximizes this function, as explained in the subsequent steps.

Step 1: Find the likelihood function

The likelihood function $L(\theta \mid \mathbf{x})$ denotes the joint probability of obtaining a random sample from the XLindley distribution. It is defined as:

$$L(\theta \mid \mathbf{x}) = \prod_{i=1}^{n} f(x_i;\theta) = \left(\frac{\theta^2}{(1+\theta)^2}\right)^n \cdot \prod_{i=1}^{n}(2+\theta+x_i)\cdot\exp\left(-\theta\sum_{i=1}^{n}x_i\right).$$

Step 2: Find the log-likelihood function

Because directly differentiating the likelihood function is mathematically complicated, the log-likelihood function is used to assist with the differentiation process easier. The log-likelihood function is given by

$$\log L(\theta \mid \mathbf{x}) = 2n\log(\theta) - 2n\log(1+\theta) + \sum_{i=1}^{n}\log(2+\theta+x_i) - \theta\sum_{i=1}^{n}x_i.$$

Step 3: Differentiate the log-likelihood function

The derivative of $\log L(\theta \mid \mathbf{x})$ with respect to θ is computed to find the value of θ that maximizes the log-likelihood function. This yields the score function, $S(\theta \mid \mathbf{x})$, which is expressed as follows:

$$S(\theta \mid \mathbf{x}) = \frac{2n}{\theta} - \frac{2n}{(1+\theta)} + \sum_{i=1}^{n}\frac{1}{(2+\theta+x_i)} - \sum_{i=1}^{n}x_i.$$

Step 4: Set the derivative equal to zero and solve for the ML estimator

The equation $S(\theta \mid \mathbf{x}) \overset{\text{set}}{=} 0$ is solved to determine the value of θ, which involves setting the score function to zero:

$$\frac{2n}{\theta} - \frac{2n}{(1+\theta)} + \sum_{i=1}^{n}\frac{1}{(2+\theta+x_i)} - \sum_{i=1}^{n}x_i \overset{\text{set}}{=} 0.$$

Due to the lack of a closed-form solution for the ML estimator of parameter θ, numerical iteration methods are used to solve the non-linear equation [21]. In this study, the maxLik package [14] was applied to perform ML estimation using the Newton-Raphson technique in the RStudio program [25].

2.3. Proposed Confidence Intervals

The likelihood-based CI, Wald-type CI, bootstrap-t CI, and bias-corrected and accelerated (BCa) bootstrap CI are the four methods for estimating CIs for the XLindley distribution parameter that are proposed in this subsection.

2.3.1. Likelihood-based Confidence Interval One of the parameter interval estimations is the likelihood-based CI, which is based on the likelihood function. In this study, the likelihood function is a function of an unknown parameter that measures how plausible the observed data are under the XLindley distribution. The main concept of the likelihood-based CI is to identify a range of parameter values that maximize the likelihood function while satisfying a nominal confidence level. This is achieved by maximizing the log-likelihood function with respect to the parameter. This interval often has excellent coverage properties and does not rely on normal approximations. When the parameter is estimated using the ML method, a likelihood-based CI is constructed around the ML estimate. This method uses the likelihood ratio, given by

$$\lambda(\theta) = \frac{L(\theta \mid \mathbf{x})}{L(\hat{\theta} \mid \mathbf{x})}.$$

Under regularity conditions, Wilks' theorem states that $-2\log\lambda(\theta)$ asymptotically follows a chi-square distribution with degrees of freedom equal to the number of parameters estimated (typically 1 in this paper) [26]. Therefore, the likelihood-based CI for θ at a $(1-\alpha)100\%$ confidence level is given by

$$\left\{\theta \,\middle|\, -2\log\frac{L(\theta \mid \mathbf{x})}{L(\hat{\theta} \mid \mathbf{x})} \leq \chi^2_{1-\alpha,1}\right\}$$

$$= \left\{\theta \,\middle|\, -2\log\left[\frac{\theta^{2n}(1+\hat{\theta})^{2n}}{\hat{\theta}^{2n}(1+\theta)^{2n}} \cdot \frac{\prod_{i=1}^{n}(2+\theta+x_i)}{\prod_{i=1}^{n}(2+\hat{\theta}+x_i)} \cdot \exp\left(-\theta\sum_{i=1}^{n}x_i + \hat{\theta}\sum_{i=1}^{n}x_i\right)\right] \leq \chi^2_{1-\alpha,1}\right\},$$

where $\chi^2_{1-\alpha,1}$ is the $(1-\alpha)$ quantile of the chi-square distribution with one degree of freedom.

Figure 2 shows the plot $-2\log\lambda(\theta)$ versus θ (solid blue line), $\chi^2_{0.95,1}$ (dashed red line), and 95% likelihood confidence interval (solid green line) when a random sample of size 100 sampled from the XLindley distribution with $\theta = 1$.

Because the cut-point for constructing a likelihood-based CI often involves an asymptotic distribution, such as the chi-square distribution, this approach is grounded in Wilks' theorem. Consequently, the effectiveness of the likelihood-based CI in approximating the true parameter values depends on the assumption that the sample size is sufficiently large for the asymptotic approximation to be valid. However, likelihood-based CI does not always rely on large sample sizes. If the likelihood function behaves well, they can produce precise interval estimates even in situations with smaller sample sizes.

2.3.2. Wald-type Confidence Interval A Wald-type CI is developed using the asymptotic normality of a parameter estimator, generally derived from the maximum likelihood estimator. It defines the interval around the point estimate and uses the estimated standard error and the normal distribution's quantile to determine its width. This method is simple to calculate and is often used in parametric inference. But its accuracy can decrease if the sample size is small or if the estimator's sampling distribution deviates from normality [4]. The foundation of the Wald-type CI lies in the quadratic approximation of the log-likelihood function, $L(\theta \mid \mathbf{x})$, which can be expanded using a Taylor series around $\hat{\theta}$. Expanded to the second-order term surrounding the ML estimate, the Wald statistic approximates the log-likelihood ratio, with the first-order term equal to zero at the ML estimate:

$$\log L(\theta \mid \mathbf{x}) \approx \log L(\hat{\theta} \mid \mathbf{x}) + \left(\theta-\hat{\theta}\right)\frac{\partial}{\partial\theta}\log L(\theta \mid \mathbf{x})\bigg|_{\theta=\hat{\theta}} + \frac{1}{2}\left(\theta-\hat{\theta}\right)^2\frac{\partial^2}{\partial\theta^2}\log L(\theta \mid \mathbf{x})\bigg|_{\theta=\hat{\theta}}$$

$$\log\frac{L(\theta \mid \mathbf{x})}{L(\hat{\theta} \mid \mathbf{x})} \approx \frac{1}{2}\left(\theta-\hat{\theta}\right)^2\frac{\partial^2}{\partial\theta^2}\log L(\theta \mid \mathbf{x})\bigg|_{\theta=\hat{\theta}}$$

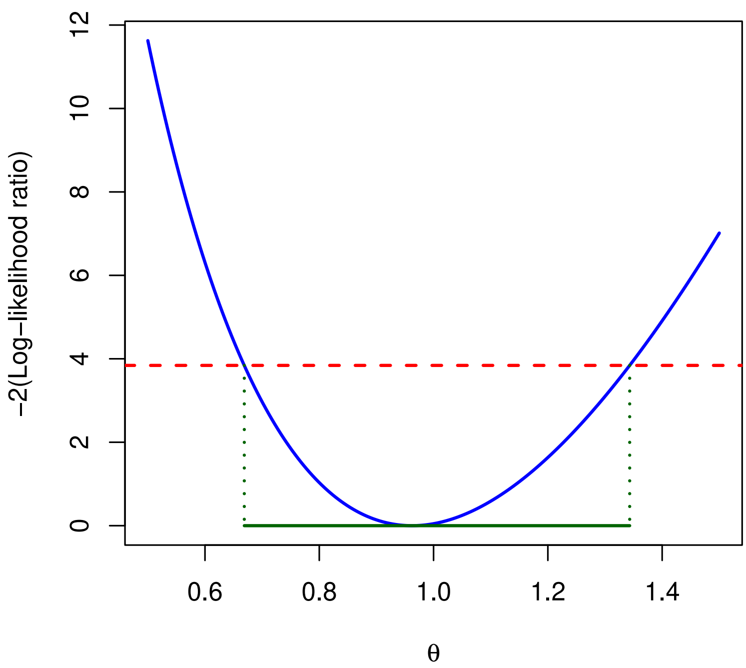

Figure 3. Plot of $-2\log\lambda(\theta)$ as a function of θ.

$$-2\log\frac{L(\theta\mid\mathbf{x})}{L(\hat{\theta}\mid\mathbf{x})}\approx(\theta-\hat{\theta})^2 I(\hat{\theta}),$$

where $I(\hat{\theta})$ is the estimated observed Fisher information. The Wald statistic can be used as an approximation for the likelihood ratio test (LRT) statistic, especially when the sample size is large enough for the asymptotic properties to hold. This gives a quadratic approximation of the log-likelihood ratio [24].

For the XLindley distribution, the observed Fisher information is as follows:

$$I(\hat{\theta})=-\frac{\partial^2}{\partial\theta^2}\log L(\theta\mid\mathbf{x})|\hat{\theta}=\frac{2n}{\hat{\theta}^2}-\frac{2n}{(1+\hat{\theta})^2}+\sum_{i=1}^{n}\frac{1}{(2+\hat{\theta}+x_i)^2},$$

and the Wald-type CI for θ at $(1-\alpha)100\%$ confidence level is given by

$$\left(\hat{\theta}-z_{1-(\alpha/2)}\sqrt{I^{-1}(\hat{\theta})},\ \hat{\theta}+z_{1-(\alpha/2)}\sqrt{I^{-1}(\hat{\theta})}\right),$$

where $z_{1-(\alpha/2)}$ denotes the $(1-(\alpha/2))^{\text{th}}$ quantile of the standard normal distribution.

2.3.3. Bootstrap-t Confidence Interval A bootstrap-t CI can be obtained by standardizing the estimator with its bootstrap-estimated standard error and approximating the sampling distribution of the student's t-statistic through resampling. The interval is obtained by inverting the bootstrap distribution of this statistic rather than relying on a normal approximation. The bootstrap-t CI typically provides more accurate results than the Wald-type CI, especially when the estimator is biased or its distribution is skewed [8]. However, it requires a more reliable estimation of the standard error for each bootstrap sample and is computationally more intensive. The procedure of the bootstrap-t CI can be explained in the following steps:

1) Compute the point estimate $\hat{\theta}$ of the parameter from the original sample, $X_1, \ldots, X_n$.

2) Generate B bootstrap samples, $X_1^*, \ldots, X_n^*$, by random sampling with replacement from the original sample.

3) For each bootstrap sample, calculate the bootstrap replicate of the estimator, represented as $\hat{\theta}^*$, along with its corresponding standard error SE$(\hat{\theta}^*)$.

4) Construct the bootstrap-t statistic for each replicate by standardizing the bootstrap estimates using their estimated standard errors:

$$t^*(\mathbf{x}, \hat{\theta}, \hat{\theta}^*) = \frac{\hat{\theta}^* - \hat{\theta}}{\sqrt{\text{SE}(\hat{\theta}^*)}} = \frac{\hat{\theta}^* - \hat{\theta}}{\sqrt{I^{-1}(\hat{\theta}^*)}}.$$

5) Repeating this process B times yields an empirical distribution of the estimator; from which we can estimate the distribution of the pivotal quantity.

6) Generate the empirical distribution of the bootstrap-t statistics derived from the set of B replicates.

7) Determine the critical values, $t^*_{(\alpha/2)}$ and $t^*_{(1-(\alpha/2))}$, which represent the quantiles of the empirical bootstrap-t distribution, $\alpha/2$ and $1 - (\alpha/2)$:

$$\frac{\#\left(t^*(\mathbf{x}, \hat{\theta}, \hat{\theta}^*) \le t^*_{(\alpha/2)}\right)}{B} = \alpha/2$$

and

$$\frac{\#\left(t^*(\mathbf{x}, \hat{\theta}, \hat{\theta}^*) \le t^*_{(1-(\alpha/2))}\right)}{B} = 1 - (\alpha/2),$$

where the symbol $\#(\cdot)$ denotes the number of bootstrap replications for which the stated inequality holds and B is the number of bootstrap samples.

8) The bootstrap-t CI is then computed as:

$$\left(\hat{\theta} + t^*_{(\alpha/2)}\sqrt{I^{-1}(\hat{\theta})},\ \ \hat{\theta} + t^*_{(1-(\alpha/2))}\sqrt{I^{-1}(\hat{\theta})}\right).$$

2.3.4. Bias-corrected and accelerated bootstrap confidence interval The bias-corrected and accelerated (BCa) bootstrap CI is an advanced bootstrap method that takes into account both bias and skewness in the sampling distribution of an estimator. It has a bias-correction factor and an acceleration parameter that make the interval endpoints more accurate. The BCa bootstrap CI is different from basic or percentile bootstrap CIs in that it applies to non-normal and asymmetric distributions. As a result, it usually achieves higher coverage accuracy, especially for small to medium sample sizes. However, the BCa bootstrap interval requires careful acceleration parameter estimation and is more computationally demanding. The procedure of the BCa bootstrap CI can be described in the subsequent steps:

1) Compute the point estimate $\hat{\theta}$ of the parameter from the original sample, $X_1, \ldots, X_n$.

2) Generate B bootstrap samples, $X_1^*, \ldots, X_n^*$, by random sampling with replacement from the original sample.

3) For each bootstrap sample, calculate the bootstrap replicate of the estimator, represented as $\hat{\theta}^*$.

4) Determine the proportion of bootstrap estimates that are less than the original estimate $\hat{\theta}$, denoted p. The bias-correction factor z_0 is the quantile of the standard normal distribution corresponding to p. Hence, the bias-correction factor z_0 is defined as

$$z_0 = \Phi^{-1}(p),$$

where

$$p = \frac{1}{B}\sum_{b=1}^{B} I\left(\hat{\theta}^{*(b)} < \hat{\theta}\right),$$

$\hat{\theta}^{*(b)}$ denotes the parameter estimate from the b^{th} bootstrap sample, $I(\cdot)$ is the indicator function, and $\Phi^{-1}(\cdot)$ is the quantile function of the standard normal distribution.

5) Compute the acceleration parameter a, which reflects the rate of change of the standard error of the estimator with respect to the data. It is typically obtained using the jackknife method:

$$a = \frac{\sum_{i=1}^{n}\left(\bar{\theta}_{(\cdot)} - \hat{\theta}_{(i)}\right)^3}{6\left[\sum_{i=1}^{n}\left(\bar{\theta}_{(\cdot)} - \hat{\theta}_{(i)}\right)^2\right]^{3/2}},$$

where $\hat{\theta}_{(i)}$ is the leave-one-out estimate, and $\bar{\theta}_{(\cdot)}$ is their average.

6) The corrected quantiles α_1 and α_2 for the CI are computed as:

$$\alpha_1 = \Phi\left(z_0 + \frac{z_0 + z_{\alpha/2}}{1 - a(z_0 + z_{\alpha/2})}\right) \quad \text{and} \quad \alpha_2 = \Phi\left(z_0 + \frac{z_0 + z_{1-(\alpha/2)}}{1 - a(z_0 + z_{1-(\alpha/2)})}\right),$$

where $z_{\alpha/2}$ and $z_{1-(\alpha/2)}$ are the $(\alpha/2)^{\text{th}}$ and $(1-(\alpha/2))^{\text{th}}$ quantiles of the standard normal distribution, respectively.

7) The $(1-\alpha)100\%$ BCa bootstrap CI is calculated using the adjusted percentiles α_1 and α_2 as:

$$\left(\hat{\theta}^*_{(\alpha_1)}, \hat{\theta}^*_{(\alpha_2)}\right),$$

where $\hat{\theta}^*_{(\alpha_1)}$ and $\hat{\theta}^*_{(\alpha_2)}$ are $(\alpha_1)^{\text{th}}$ and $(\alpha_2)^{\text{th}}$ quantiles of of the sorted bootstrap estimates $\hat{\theta}^{*(b)}$.

3. Simulation Studies and Results

A Monte Carlo simulation study was performed to evaluate the efficacy of the proposed CI methods for the parameter of the XLindley distribution. We examined at the empirical coverage probability (ECP) and average width (AW) of four 95% CI methods: the likelihood-based CI, the Wald-type CI, the bootstrap-t CI, and the BCa bootstrap CI. Random samples were generated from the XLindley distribution for a wide range of parameter values and sample sizes. The true parameter values θ were set to 0.2, 0.3, 0.5, 0.75, 1, 1.5, 2, and 2.5, representing small to relatively large parameter ranges. The sample sizes considered were n = 10, 20, 30, 50, 100, 200, and 500, allowing investigation of both small- and large-sample situations. For each combination of (θ, n), the number of simulation replicates was fixed at 2,000. For the bootstrap-based CIs, $B = 2,000$ bootstrap resamples were used in the study. All simulations were implemented in RStudio.

The simulation results for the 95% CIs of the XLindley distribution parameter are summarized in Tables 1 and 2, while graphical presentations of the ECPs and AWs are in Figures 4 and 5, respectively. In general, the ECPs of all CI methods tend to get closer to the nominal confidence level of 0.95 as the sample size increases, regardless of the true parameter value. This indicates that the proposed CIs exhibit consistency as the sample size increases. Specifically, even for small samples, the coverage performance of the likelihood-based CI and the Wald-type CI is comparatively stable across all parameter settings. These two methods often slightly over-cover the nominal confidence level for $n = 10$, particularly for moderate to large values of θ.

The bootstrap-t and BCa bootstrap CIs, on the other hand, exhibit substantial under-coverage when the sample size is small, especially when θ is small. This situation is more noticeable for $n = 10$ and $n = 20$, where the bootstrap-based intervals are usually narrower, which lowers the coverage probabilities. The coverage performance of both bootstrap methods improves significantly as the sample size increases. For samples of moderate to large size ($n \geq 50$), their ECPs become similar to those of the likelihood-based and Wald-type CIs for all parameter values.

With respect to interval precision, the bootstrap-t CI consistently produces the smallest AW among all methods, followed closely by the BCa bootstrap CI. The likelihood-based and Wald-type CIs generally provided wider intervals, particularly for larger values of θ. As expected, the AW consistently decreases with the increase in sample size for all CI methods, indicating improved estimation accuracy in larger samples. This trend is clearly shown in Figure 5.

When comparing the likelihood-based CI and the Wald-type CI, their AWs are almost identical for all parameter values and sample sizes. This is in accordance with the asymptotic relationship between likelihood-based and quadratic-approximation-based inference. However, the likelihood-based CI shows slightly more stable coverage behavior in small to medium samples, especially when θ is small.

These results demonstrate that the performance of CI methods is highly dependent on both sample size and underlying distributional characteristics. In particular, bootstrap-based methods require sufficiently large samples to achieve reliable coverage, whereas the likelihood-based approach remains robust even in small-sample settings. Moreover, a clear balance emerges between coverage accuracy and interval width: bootstrap methods generally produce narrower intervals but may suffer from under-coverage in small samples, while the likelihood-based approach tends to provide more stable coverage at the expense of slightly wider intervals. These findings offer practical guidance for selecting appropriate CI methods, depending on sample size and the desired balance between accuracy and precision.

To explain these findings more rigorously, the under-coverage observed in the bootstrap-t and BCa bootstrap methods for small sample sizes (e.g., $n = 10$ and $n = 20$) can be attributed to instability in estimating the standard error and higher-order correction terms. Specifically, the bootstrap-t method relies on the studentized statistic, which involves the inverse of the Fisher information matrix, $I^{-1}(\hat{\theta}^*)$, evaluated at bootstrap samples. For very small samples, this estimate becomes highly variable, leading to inaccurate standardization and, consequently, overly narrow CIs, resulting in under-coverage.

Similarly, the BCa bootstrap method depends on the accurate estimation of both the bias-correction factor and the acceleration parameter. These quantities require reliable resampling distributions, which may not be well approximated when the sample size is small. As a result, the BCa bootstrap intervals tend to be overly optimistic (too narrow), thereby reducing their ECP.

4. Real Data Applications

This section shows the practical use of the proposed 95% CI methods for the XLindley distribution parameter with two real datasets. We build and compare the likelihood-based, Wald-type, bootstrap-t, and BCa bootstrap CIs to illustrate their performance under real data conditions. These applications demonstrate the performance of the proposed inference procedures in modeling lifetime data. In addition to the XLindley distribution, fifteen competing one-parameter distributions were included for comparative analysis. For brevity, their PDFs are not presented in the main text and are instead provided in Appendix A.

4.1. Remission Times of Bladder Cancer Patients

Uncensored observations of the remission times, expressed in months, for a sample selected at random among 128 bladder cancer patients. Lawless [17] has previously examined this dataset, which has been utilized extensively in survival analysis studies. This dataset has also been analyzed in the study by Abdullahi and Pijitrattana [1]. The data set includes the following observations:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06,7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76,26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69,4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41,7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31,4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69, 5.49.

Table 1. Empirical coverage probability and average width of the 95% CIs for the parameter of the XLindley distribution (θ = 0.20, 0.30, 0.50, and 0.75)

θ	n	Empirical Coverage Probability				Average Width			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
0.2	10	0.9545	0.9590	0.8870	0.8900	0.1895	0.1882	0.1669	0.1791
	20	0.9485	0.9460	0.9155	0.9170	0.1296	0.1291	0.1217	0.1249
	30	0.9475	0.9505	0.9260	0.9255	0.1052	0.1050	0.1002	0.1016
	50	0.9430	0.9470	0.9320	0.9280	0.0806	0.0805	0.0782	0.0790
	100	0.9385	0.9440	0.9325	0.9315	0.0566	0.0566	0.0557	0.0561
	200	0.9525	0.9460	0.9440	0.9405	0.0398	0.0398	0.0393	0.0395
	500	0.9530	0.9515	0.9450	0.9485	0.0252	0.0252	0.0250	0.0251
0.3	10	0.9455	0.9530	0.8855	0.8825	0.2927	0.2904	0.2553	0.2762
	20	0.9505	0.9570	0.9235	0.9250	0.1970	0.1962	0.1843	0.1897
	30	0.9510	0.9505	0.9265	0.9280	0.1594	0.1589	0.1507	0.1532
	50	0.9530	0.9535	0.9300	0.9325	0.1228	0.1226	0.1191	0.1202
	100	0.9500	0.9490	0.9460	0.9435	0.0859	0.0858	0.0843	0.0849
	200	0.9430	0.9460	0.9405	0.9420	0.0606	0.0606	0.0596	0.0600
	500	0.9495	0.9495	0.9435	0.9435	0.0382	0.0382	0.0379	0.0381
0.5	10	0.9470	0.9490	0.9010	0.8900	0.5007	0.4957	0.4366	0.4787
	20	0.9425	0.9485	0.9090	0.9140	0.3425	0.3408	0.3190	0.3293
	30	0.9515	0.9585	0.9345	0.9340	0.2746	0.2737	0.2594	0.2648
	50	0.9540	0.9515	0.9360	0.9390	0.2102	0.2098	0.2025	0.2047
	100	0.9425	0.9430	0.9355	0.9400	0.1474	0.1472	0.1449	0.1459
	200	0.9455	0.9445	0.9435	0.9460	0.1040	0.1039	0.1028	0.1036
	500	0.9465	0.9480	0.9425	0.9445	0.0656	0.0656	0.0648	0.0650
0.75	10	0.9450	0.9565	0.8935	0.8860	0.7950	0.7856	0.6805	0.7559
	20	0.9415	0.9450	0.9100	0.9095	0.5313	0.5281	0.4870	0.5055
	30	0.9560	0.9585	0.9340	0.9325	0.4323	0.4306	0.4094	0.4184
	50	0.9500	0.9550	0.9370	0.9375	0.3287	0.3279	0.3173	0.3220
	100	0.9525	0.9605	0.9455	0.9485	0.2314	0.2312	0.2265	0.2284
	200	0.9440	0.9420	0.9365	0.9410	0.1618	0.1617	0.1599	0.1608
	500	0.9560	0.9555	0.9530	0.9525	0.1019	0.1018	0.1010	0.1015

Descriptive statistics for this dataset are summarized in Table 3, while Figure 6 provides graphical representations, including a histogram, Box and Whisker plot, kernel density plot, and violin plot, effectively highlighting the dataset's positive skewness.

The graphical summary indicates that the remission times exhibit positive skewness, characterized by a long right tail and possible outliers. Such distributional characteristics suggest that symmetric distributions may be inadequate, thereby motivating the use of flexible parametric distributions for estimating the distribution parameter.

A comprehensive evaluation was conducted using several criteria, such as the log-likelihood $(\log(\hat{L}))$, the Akaike information criterion (AIC), and the Bayesian information criterion (BIC), which is also known as the Schwarz information criterion. These criteria provided an objective assessment of model adequacy across various distributions. The AIC and BIC are defined as follows:

$$\text{AIC} = 2k - 2\log\hat{L} \quad \text{and} \quad \text{BIC} = 2k\log(n) - 2\log\hat{L},$$

where k represents the number of estimated parameters in the distribution and $\hat{L}$ denotes the maximized value of the likelihood function for the distribution. The parameter estimates for the dataset under study are given in Table 4, along with the corresponding standard errors (SEs) and goodness-of-fit metrics.

Table 2. Empirical coverage probability and average width of the 95% CIs for the parameter of the XLindley distribution (θ = 1, 1.5, 2, and 2.5)

θ	n	Empirical Coverage Probability				Average Width			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
1	10	0.9475	0.9590	0.8915	0.8905	1.0963	1.0820	0.9320	1.0394
	20	0.9480	0.9535	0.9165	0.9150	0.7402	0.7353	0.6800	0.7087
	30	0.9555	0.9570	0.9390	0.9355	0.5899	0.5873	0.5570	0.5710
	50	0.9470	0.9525	0.9340	0.9345	0.4518	0.4506	0.4369	0.4429
	100	0.9465	0.9505	0.9435	0.9385	0.3167	0.3163	0.3087	0.3118
	200	0.9570	0.9555	0.9530	0.9515	0.2228	0.2226	0.2203	0.2217
	500	0.9495	0.9495	0.9475	0.9465	0.1404	0.1404	0.1395	0.1401
1.5	10	0.9525	0.9565	0.8905	0.8795	1.7429	1.7191	1.4896	1.6768
	20	0.9495	0.9520	0.9245	0.9240	1.1639	1.1558	1.0684	1.1123
	30	0.9495	0.9585	0.9300	0.9325	0.9448	0.9403	0.8955	0.9183
	50	0.9535	0.9525	0.9325	0.9360	0.7157	0.7137	0.6872	0.6976
	100	0.9520	0.9625	0.9485	0.9465	0.4988	0.4981	0.4887	0.4934
	200	0.9425	0.9460	0.9415	0.9410	0.3528	0.3525	0.3487	0.3505
	500	0.9575	0.9560	0.9500	0.9485	0.2213	0.2213	0.2191	0.2203
2	10	0.9460	0.9480	0.8855	0.8745	2.4311	2.3984	2.0514	2.3056
	20	0.9550	0.9580	0.9150	0.9140	1.6306	1.6194	1.4895	1.5517
	30	0.9475	0.9580	0.9330	0.9260	1.3077	1.3016	1.2377	1.2654
	50	0.9585	0.9585	0.9395	0.9375	0.9914	0.9887	0.9564	0.9695
	100	0.9410	0.9435	0.9365	0.9335	0.6937	0.6928	0.6790	0.6842
	200	0.9475	0.9475	0.9460	0.9465	0.4865	0.4862	0.4804	0.4834
	500	0.9435	0.9420	0.9425	0.9435	0.3058	0.3057	0.3039	0.3059
2.5	10	0.9515	0.9465	0.8860	0.8825	3.1164	3.0760	2.6389	2.9740
	20	0.9415	0.9525	0.9160	0.9120	2.0893	2.0756	1.9265	2.0041
	30	0.9490	0.9440	0.9250	0.9260	1.6585	1.6510	1.5587	1.5950
	50	0.9480	0.9515	0.9360	0.9355	1.2773	1.2739	1.2308	1.2486
	100	0.9515	0.9510	0.9440	0.9425	0.8887	0.8875	0.8676	0.8745
	200	0.9480	0.9520	0.9465	0.9460	0.6257	0.6253	0.6188	0.6219
	500	0.9435	0.9435	0.9370	0.9385	0.3937	0.3936	0.3890	0.3904

Table 3. Descriptive statistics for the remission times of bladder cancer patients

Sample Sizes	Minimum	Q1	Median	Mean	Q3	Maximum	St.Dev
128	0.080	3.348	6.395	9.366	11.838	79.050	10.508

Although the exponential distribution yields slightly lower AIC and BIC values in Table 4, it was not selected as the primary distribution in this study. The main objective of this research is not model selection but the development and evaluation of CI methods for the parameter of the XLindley distribution. The real datasets are used primarily to illustrate the practical applicability of the proposed inference methods. The XLindley distribution is of particular interest due to its theoretical structure as a mixture of the exponential and Lindley distributions and its ability to model positively skewed lifetime data with increasing hazard rates. Therefore, the XLindley distribution was retained for the empirical illustration despite the slightly smaller information criteria values obtained for the exponential distribution.

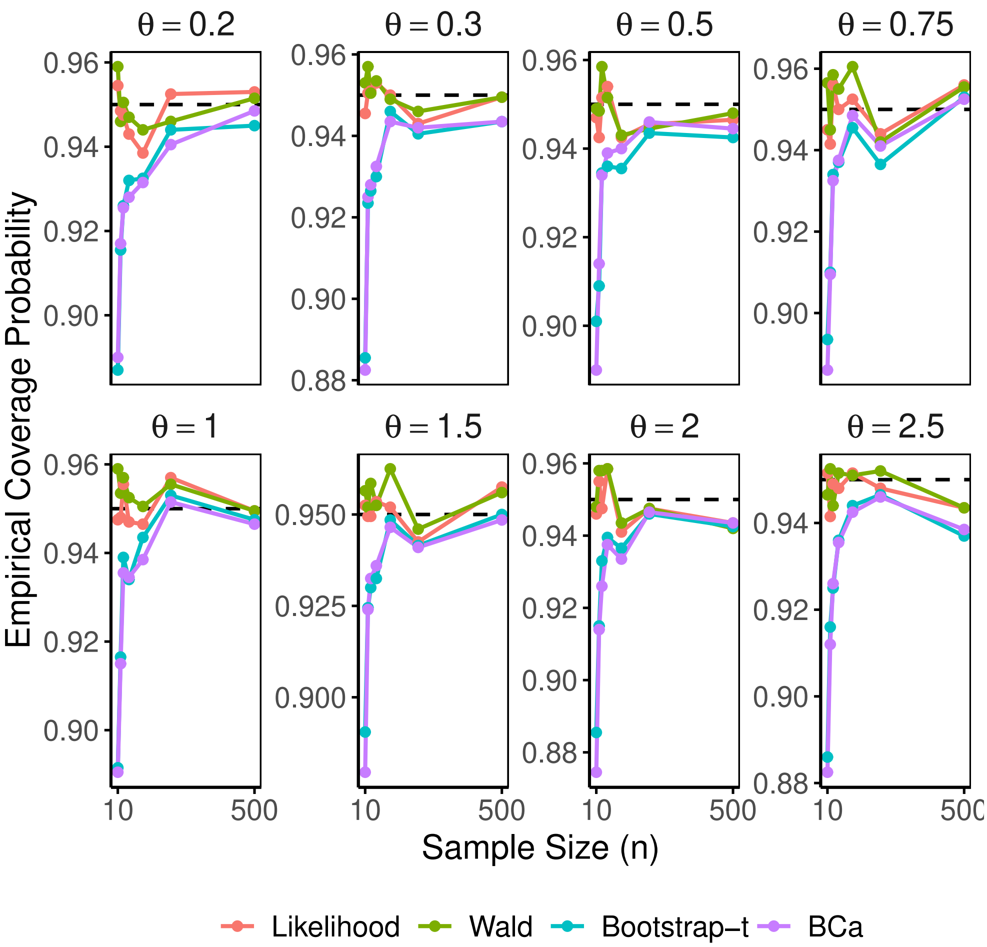

Figure 4. Plots of the ECPs of the CIs for the parameter of the XLindley distribution

We used the Kolmogorov–Smirnov (K–S) statistic [6] to perform a goodness-of-fit test to evaluate whether the bladder cancer remission data follow the XLindley distribution. The K–S statistic is equal to 0.1017 with a p-value of 0.1419. Since the p-value is greater than the 0.05 significance level, there is insufficient evidence to reject the null hypothesis, indicating that the XLindley distribution provides an adequate fit to the data.

The Probability-Probability (P-P) plot shown in Figure 7 compares the empirical cumulative distribution function (CDF) of the observed data with the theoretical CDF of the XLindley distribution, which is based on the estimated parameter. Excellent agreement between the empirical and theoretical distributions is shown by the plotted points' close proximity to the 45-degree reference line. This visual evidence supports the adequacy of the XLindley distribution in modeling the given dataset.

The ML estimate of the XLindley distribution parameter was 0.1832. Table 5 reports the 95% and 99% CIs for the XLindley distribution parameter, constructed using four proposed CI methods. The 95% CI for the likelihood-based method is (0.1614, 0.2069), with an interval width of 0.0455. The Wald-type CI provided a similar interval, ranging from 0.1605 to 0.2059, with the same width of 0.0454. In contrast, the bootstrap-t and BCa bootstrap methods resulted in slightly wider intervals—(0.1546, 0.2163) and (0.1503, 0.2124), respectively.

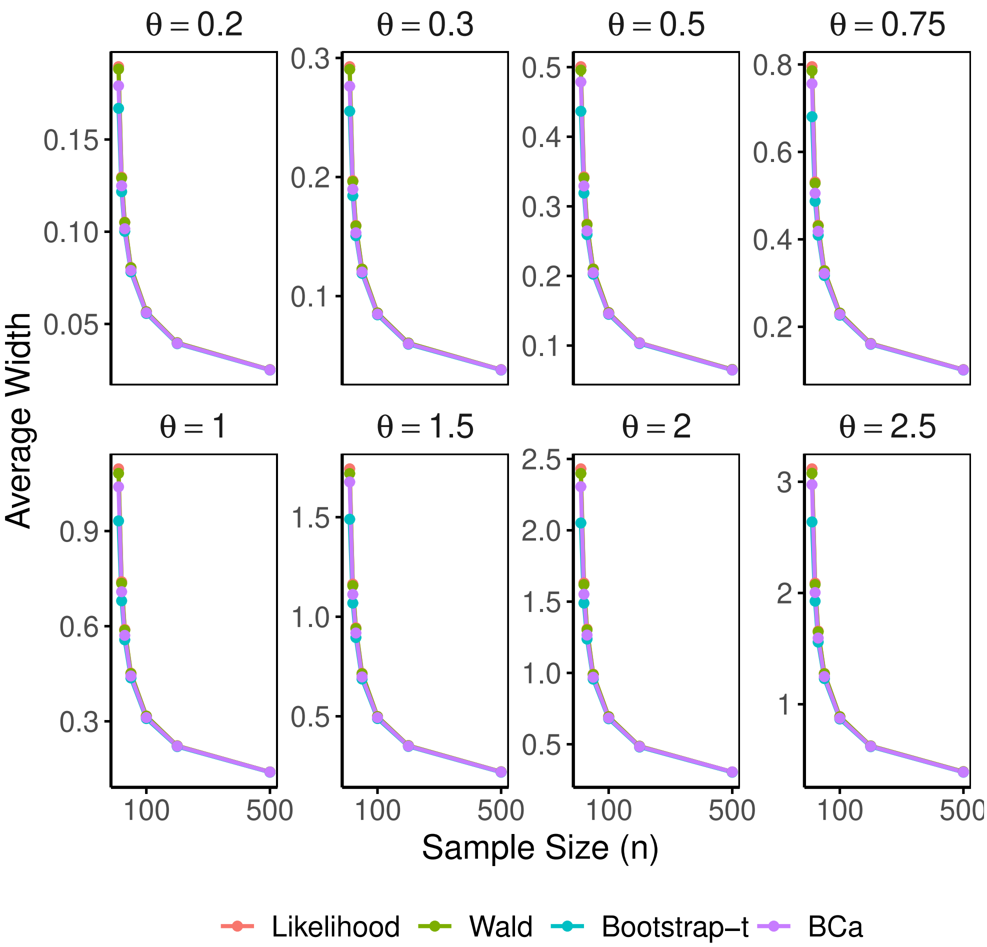

Figure 5. Plots of the AWs of the CIs for the parameter of the XLindley distribution

These results show the efficacy of likelihood-based and Wald-type methods for interval estimation, especially in moderate-large sample settings, as well as the suitability of the XLindley distribution for modeling bladder cancer remission data.

4.2. Failure Times of Electric Bulbs

The following data set, which is positively skewed (right-skewed), shows the failure times of 20 electric bulbs as reported by Murthy et al. [19]. The observations represent the operating lifetimes (in hours) documented until failure and indicate a distinct departure from symmetry, indicating a long right tail. The following are the recorded data:

1.32, 12.37, 6.56, 5.05, 11.58, 10.56, 21.82, 3.60, 1.33, 12.62, 5.36, 7.71, 3.53, 19.61, 36.63, 0.39, 21.35, 7.22, 12.42, 8.92.

Summary statistics of the dataset are presented in Table 6, and the distributional characteristics illustrated in Figure 8 reveal a noticeable right-skewed pattern in the data, along with the presence of one outlier.

Model adequacy was thoroughly examined using several criteria, such as the log-likelihood $(\log(\hat{L}))$, along with AIC and BIC. As shown in Table 7, the estimated parameters, their standard errors (SEs), and relevant model evaluation criteria are presented for the given dataset.

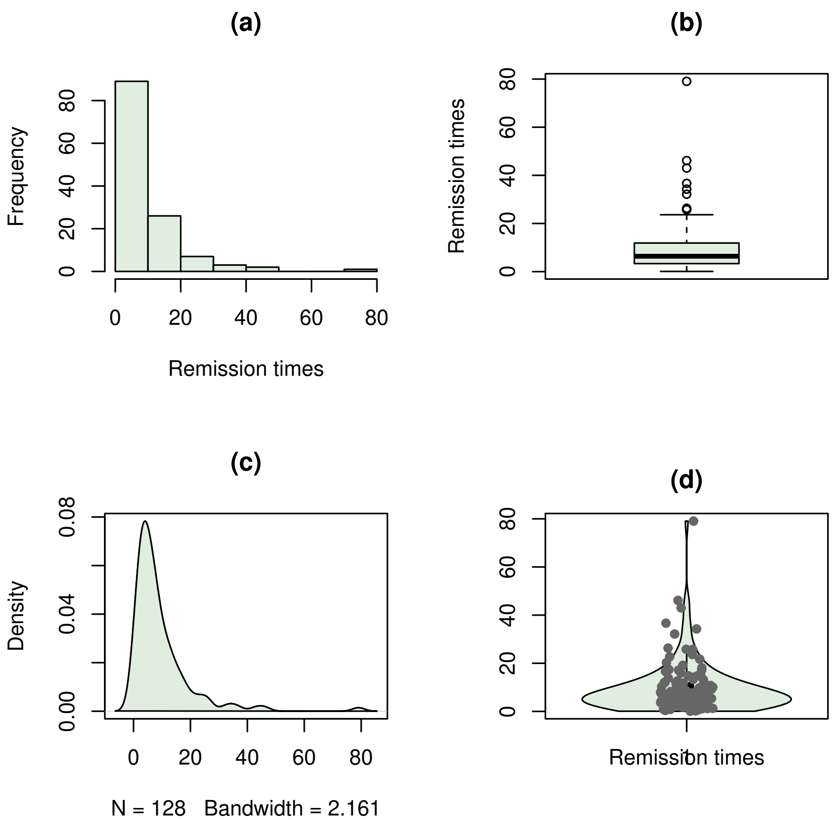

Figure 6. Graphical representations of the remission times of bladder cancer patients: (a) Histogram, (b) Box and Whisker plot, (c) Kernel density plot, and (d) Violin plot

The K–S test was applied to evaluate the suitability of the XLindley distribution for modeling the failure times of electric bulbs. The analysis produced a K–S statistic of 0.1144 with a corresponding p-value of 0.9294. Given that the p-value is greater than the significance level of 0.05, the null hypothesis that the data follow the XLindley distribution cannot be rejected. This result suggests that the XLindley distribution is suitable for the observed data.

Moreover, the plotted points in the P–P plot (Figure 9) exhibit a strong alignment with the diagonal reference line, indicating that the XLindley distribution provides a good fit to the failure times of electric bulbs. The overall pattern suggests that the XLindley distribution effectively captures the underlying structure of the data.

The XLindley distribution demonstrated the best fit among the candidates, evidenced by its lowest AIC and BIC values. Its ML estimate for the distribution parameter was 0.1661. The 95% and 99% CIs for the parameter of the XLindley distribution are presented in Table 8. The 95% CI obtained from the likelihood-based method was (0.1190, 0.2229), corresponding to a width of 0.1039. An interval was constructed by the Wald-type method, spanning from 0.1137 to 0.2172, though with a slightly broader width of 0.1035. The bootstrap-t and BCa bootstrap methods yielded wider intervals–(0.1157, 0.2213) and (0.1144, 0.2219), respectively.

5. Conclusions

This paper proposed and evaluated four methods for estimating confidence intervals (CIs) for the parameter of the XLindley distribution: the likelihood-based, Wald-type, bootstrap-t, and bias-corrected and accelerated

Table 4. Comparison of goodness-of-fit measures for different distributions applied to bladder cancer remission data

Distribution	Estimate (SE)	LogLik	AIC	BIC
XLindley	0.1832 (0.0116)	-417.4519	836.9038	839.7559
Akash	0.3105 (0.0156)	-443.9460	889.8920	892.7440
Akshaya	0.3861 (0.0171)	-462.1868	926.3736	929.2256
Amarendra	0.4067 (0.0178)	-470.4256	942.8512	945.7032
Sujatha	0.2990 (0.0152)	-439.9124	881.8248	884.6768
Shanker	0.2107 (0.0129)	-423.6873	849.3746	852.2266
Adya	0.3163 (0.0156)	-449.2331	900.4661	903.3182
Fav-Jerry	0.1074 (0.0096)	-414.3424	830.6848	833.5368
Garima	0.1555 (0.0120)	-415.4818	832.9637	835.8157
Komal	0.1939 (0.0121)	-419.0231	840.0461	842.8982
Ishita	0.3209 (0.0161)	-450.6836	903.3672	906.2193
Iwok-Nwikpe	0.3061 (0.0157)	-448.5257	899.0513	901.9034
Pratibha	0.3061 (0.0155)	-444.4518	890.9036	893.7557
Chris-Jerry	0.2868 (0.0154)	-434.8029	871.6059	874.4579
Lindley	0.2107 (0.0129)	-420.2047	842.4094	845.2614
Exponential	0.1068 (0.0094)	-414.3419	**830.6838**	**833.5358**

Note. Bold values indicate the best-fitting model according to the minimum AIC and BIC criteria.

Table 5. The 95% and 99% CIs and corresponding widths for the XLindley parameter based on bladder cancer remission data

CI Method	95% CI		99% CI	
	Interval	Width	Interval	Width
Likelihood-based	(0.1614, 0.2069)	0.0455	(0.1550, 0.2147)	0.0597
Wald-type	(0.1605, 0.2059)	0.0454	(0.1534, 0.2131)	0.0597
Bootstrap-t	(0.1546, 0.2163)	0.0617	(0.1443, 0.2270)	0.0827
BCa bootstrap	(0.1503, 0.2124)	0.0621	(0.1397, 0.2240)	0.0843

Table 6. Descriptive statistics for the failure times of electric bulbs

Sample Sizes	Minimum	Q1	Median	Mean	Q3	Maximum	St.Dev
20	0.390	4.688	8.315	10.498	12.470	36.630	8.838

(BCa) bootstrap methods. An explicit analytical equation for the Wald-type CI was derived to enable practical implementation. The Monte Carlo simulation studies were performed to evaluate the performance of these CIs under different sample sizes and parameter values. The main performance criteria for the evaluation were empirical coverage probability (ECP) and average width (AW). The likelihood-based and Wald-type CIs demonstrate generally reliable coverage, with ECPs close to the nominal 0.95 level across most scenarios. However, their performance is more stable compared to the bootstrap-based methods, particularly in small samples, where the bootstrap-t and BCa CIs exhibit noticeable under-coverage (especially for $n \leq 20$). As the sample size increases, the performance of all methods becomes comparable.

The practical utility of the proposed CI methods was further validated using data on bladder cancer remission and failure times of electric bulbs, where the XLindley distribution demonstrated the most suitable fit among fifteen candidate distributions based on AIC and BIC values. Furthermore, we presented probability–probability

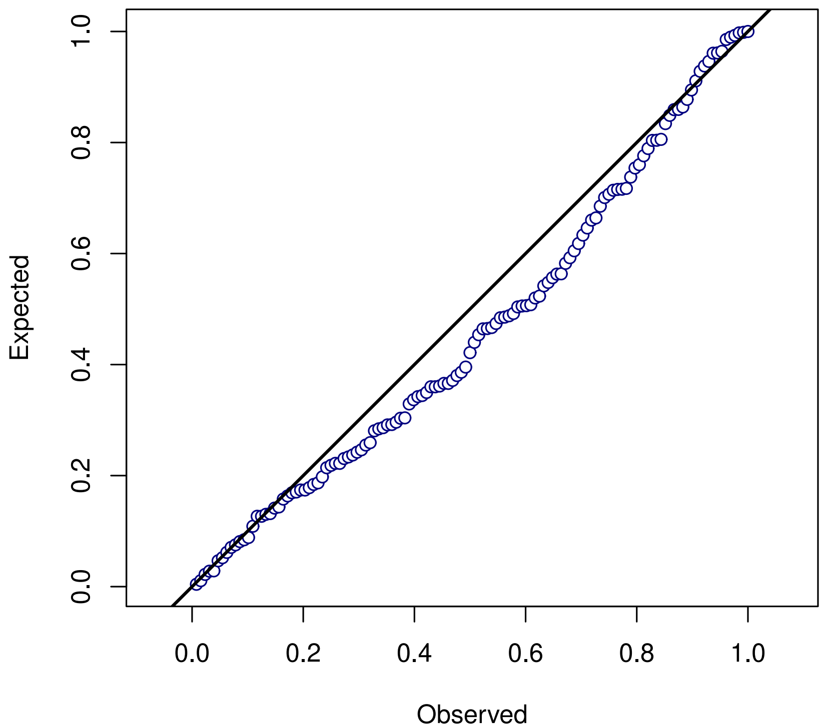

Figure 7. P–P plot of the bladder cancer remission data

Table 7. Comparison of goodness-of-fit measures for different distributions applied to failure times of electric bulbs

Distribution	Estimate (SE)	LogLik	AIC	BIC
XLindley	0.1661 (0.0188)	-131.6284	**265.2569**	**266.9458**
Akash	0.2800 (0.0253)	-135.0512	272.1024	273.7913
Akshaya	0.3496 (0.0277)	-138.7303	279.4606	281.1494
Amarendra	0.3670 (0.0288)	-140.0869	282.1739	283.8628
Sujatha	0.2702 (0.0245)	-134.4905	270.9810	272.6699
Adya	0.2854 (0.0252)	-138.0670	278.1340	279.8229
Fav-Jerry	0.0962 (0.0153)	-133.8476	269.6951	271.3840
Garima	0.1426 (0.0195)	-132.2765	266.5529	268.2418
Komal	0.1753 (0.0195)	-131.8525	265.7050	267.3939
Ishita	0.2894 (0.0260)	-137.8178	277.6356	279.3245
Iwok-Nwikpe	0.2756 (0.0252)	-137.8821	277.7643	279.4532
Pratibha	0.2765 (0.0250)	-136.3477	274.6955	276.3843
Chris-Jerry	0.2599 (0.0248)	-131.8130	265.6261	267.3150
Lindley	0.1899 (0.0208)	-132.1566	266.3132	268.0020
Exponential	0.0957 (0.0151)	-133.8478	269.6956	271.3845

Note. Bold values indicate the best-fitting model according to the minimum AIC and BIC criteria.

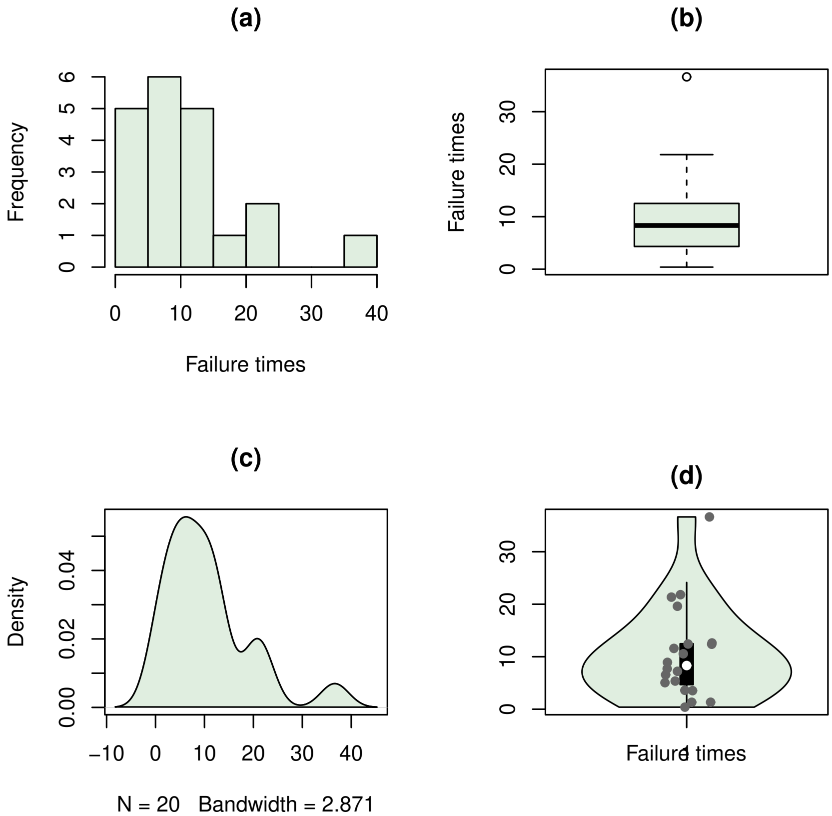

Figure 8. Graphical representations of the failure times of electric bulbs: (a) Histogram, (b) Box and Whisker plot, (c) Kernel density plot, and (d) Violin plot

Table 8. The 95% and 99% CIs and corresponding widths for the XLindley parameter based on failure times of electric bulbs

CI Method	95% CI		99% CI	
	Interval	Width	Interval	Width
Likelihood-based	(0.1190, 0.2229)	0.1039	(0.1065, 0.2434)	0.1369
Wald-type	(0.1137, 0.2172)	0.1035	(0.0974, 0.2335)	0.1361
Bootstrap-t	(0.1157, 0.2213)	0.1056	(0.1059, 0.2444)	0.1385
BCa bootstrap	(0.1144, 0.2219)	0.1075	(0.1054, 0.2392)	0.1338

(P–P) plots for the XLindley distribution to examine whether it fit beyond the AIC and BIC values. The P–P plots demonstrated that the empirical cumulative distribution function (ECDF) of the XLindley distribution closely matched the theoretical cumulative distribution function (CDF), thereby validating its suitability for the data concerning bladder cancer remission and the failure times of electric bulbs. These graphical evaluations improve the information-theoretic criteria and strengthen the evidence for the XLindley distribution's suitableness.

The CIs constructed from these real datasets aligned with the simulation findings. Overall, the likelihood-based and Wald-type methods are recommended for applications requiring accurate coverage and computational stability, while the bootstrap-t and BCa bootstrap methods may be suitable when narrower intervals are desired and sufficient data are available.

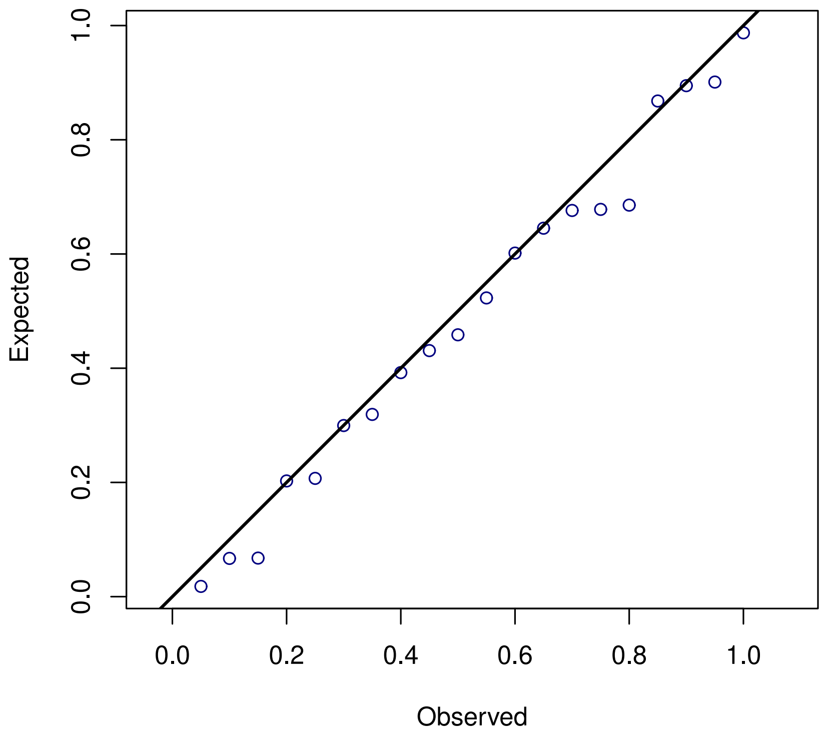

Figure 9. P–P plot of the failure times of electric bulbs

6. Recommendations and Future Work

Based on the simulation results presented in Section 3, more specific and practical guidelines for selecting appropriate confidence interval (CI) methods for the XLindley distribution parameter can be provided.

For small sample sizes ($n \leq 30$), the likelihood-based CI is recommended due to its stable coverage performance across all parameter values. Although this method tends to produce slightly wider intervals, it consistently achieves coverage probabilities close to the nominal level, making it a reliable choice when accuracy is the primary concern.

For moderate sample sizes ($30 < n < 100$), both likelihood-based and Wald-type CIs can be used effectively. The Wald-type CI becomes increasingly reliable as the sample size increases, as the normal approximation improves. However, caution should be exercised when the sample size is still relatively small or when the estimator distribution is skewed.

For large sample sizes ($n \geq 100$), bootstrap-based methods, particularly the BCa bootstrap CI, are recommended. These methods provide narrower confidence intervals and achieve coverage probabilities close to the nominal level, offering an appropriate balance between accuracy and precision. The bootstrap-t method also performs well in this setting, especially when computational resources are sufficient.

Overall, the choice of CI method should be guided by the sample size and the desired balance between coverage accuracy and interval width. Likelihood-based methods are preferable for small samples, while bootstrap methods become increasingly advantageous as the sample size grows.

For ease of application, a summary of recommendations is provided in table below:

Table 9. Guidelines for selecting CI methods based on sample size

Sample Size	Recommended Method	Key Advantage
$n \leq 30$	Likelihood-based CI	Stable coverage
$30 < n < 100$	Likelihood / Wald	Balanced performance
$n \geq 100$	BCa bootstrap CI	High precision

In addition, the bootstrap-t and BCa bootstrap CIs require substantial computational resources, particularly because they involve large-scale resampling. Fortunately, several RStudio packages support the computation of the bootstrap-t and BCa bootstrap confidence intervals, among which the boot package [3] and the bootstrap package [16] are widely used and provide convenient functions for efficiently constructing bootstrap CIs.

Although this study focused on four widely used methods for constructing proposed CIs, other modern approaches such as the Bayesian credible intervals should also be considered. The Bayesian CI, which uses prior information and gives a posterior probability-based interval, could be very helpful in situations where there is already some knowledge about the domain. Although Bayesian credible intervals were excluded from this study due to the additional computational demands and the necessity for supplementary modeling assumptions (e.g., prior selection in Bayesian analysis), future studies should investigate their relevance to the XLindley distribution and evaluate their efficacy regarding ECP and AW. Future research may focus on hypothesis testing procedures and predictive models based on the XLindley distribution and extend CI estimation methods to handle censored lifetime data.

A. Probability Density Functions of Competing Distributions

This appendix presents the probability density functions (PDFs) of the fifteen distributions considered in the real data analysis. Each of these PDFs has one parameter with the value $\theta > 0$ and is defined for $x > 0$:

1. The Akash distribution [28]
$$f(x;\theta) = \frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x}.$$
2. The Akshaya distribution [34]
$$f(x;\theta) = \frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}(1+x)^3e^{-\theta x}.$$
3. The Amarendra distribution [30]
$$f(x;\theta) = \frac{\theta^4}{\theta^3+\theta^2+2\theta+6}(1+x+x^2+x^3)e^{-\theta x}.$$
4. The Sujatha distribution [31]
$$f(x;\theta) = \frac{\theta^3}{\theta^2+\theta+2}\left(1+x+x^2\right)e^{-\theta x}.$$
5. The Shanker distribution [27]
$$f(x;\theta) = \frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}.$$
6. The Adya distribution [36]
$$f(x;\theta) = \frac{\theta^3}{\theta^4+2\theta^2+2}(\theta+x)^2e^{-\theta x}.$$

7. The Fav-Jerry distribution [9]
$$f(x;\theta) = \frac{\theta}{\theta^2+2}\left(2+\theta^3 x\right)e^{-\theta x}.$$
8. The Garima distribution [32]
$$f(x;\theta) = \frac{\theta}{\theta+2}(1+\theta+\theta x)\,e^{-\theta x}.$$
9. The Komal distribution [37]
$$f(x;\theta) = \frac{\theta^2}{\theta^2+\theta+1}(1+\theta+x)\,e^{-\theta x}.$$
10. The Ishita distribution [35]
$$f(x;\theta) = \frac{\theta^3}{\theta^3+2}(\theta+x^2)\,e^{-\theta x}.$$
11. The Iwok-Nwikpe distribution [15]
$$f(x;\theta) = \frac{\theta^3}{\theta+2}(x^2+x)\,e^{-\theta x}.$$
12. The Pratibha distribution [38]
$$f(x;\theta) = \frac{\theta^3}{\theta^3+\theta+2}(\theta+x+x^2)\,e^{-\theta x}.$$
13. The Chris-Jerry distribution [23]
$$f(x;\theta) = \frac{\theta^2}{\theta+2}(1+\theta x^2)\,e^{-\theta x}.$$
14. The Lindley distribution [12]
$$f(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}.$$
15. The exponential distribution
$$f(x;\theta) = \theta\,e^{-\theta x}.$$

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REFERENCES

1. I. Abdullahi, and P. Pijitrattana, *On the new mixture Sushila and Rayleigh distributions: Mathematical properties and application*, Thailand Statistician, vol. 22, no. 3, pp. 565–574, 2024.
2. S. K. Ashour, and M. A. Eltehiwy, *Exponentiated power Lindley distribution*, Journal of Advanced Research, vol. 6, pp. 895–905, 2015.
3. A. Canty, and B. Ripley, *Package boot: Bootstrap functions*, Comprehensive R Archive Network (CRAN), 2024.
4. G. Casella, and R. L. Berger, *Statistical Inference*, Duxbury Thomson Learning, Pacific Grove, CA, 2002.
5. S. Chouia, and H. Zeghdoudi, *The XLindley distribution: Properties and application*, Journal of Statistical Theory and Applications, vol. 20, no. 2, pp. 318–327, 2021.
6. G. W. Corder, and D. I. Foreman, *Nonparametric Statistics: A Step-by-Step Approach*, John Wiley & Sons, Hoboken, NJ, 2014.
7. U. V. Echebiri, and J. I. Mbegbu, *Juchez probability distribution: Properties and applications*, Asian Journal of Probability and Statistics, vol. 20, pp. 56–71, 2022.
8. B. Efron, and R. J. Tibshirani, *An Introduction to the Bootstrap*, Chapman & Hall, New York, NY, 1993.

9. D. F. N. Ekemezie, and O. J. Obulezi, *The Fav-Jerry distribution: Another member in the Lindley class with applications*, Earthline Journal of Mathematical Sciences, vol. 14, no. 4, pp. 793–816, 2024.
10. O. Elechi, E. Okereke, I. Chukwudi, K. Chizoba, and O. Wale, *Iwueze's distribution and its application*, Journal of Applied Mathematics and Physics, vol. 10, pp. 3783–3803, 2022.
11. M. M. Gharaibeh, *Gharaibeh distribution and its applications*, Journal of Statistics Applications & Probability, vol. 10, pp. 441–452, 2021.
12. M. Ghitany, B. Atieh, and S. Nadarajah, *Lindley distribution and its applications*, Mathematics and Computers in Simulation, vol. 78, no. 4, pp. 493–506, 2008.
13. M. E. Ghitany, D. K. Al-Mutairi, N. Balakrishnan, and L. J. Al-Enezi, *Power Lindley distribution and associated inference*, Computational Statistics & Data Analysis, vol. 64, pp. 20–33, 2013.
14. A. Henningsen, and O. Toomet, *MaxLik: A package for maximum likelihood estimation in R*, Computational Statistics, vol. 26, pp. 443–458, 2011.
15. I. A. Iwok, and B. J. Nwikpe, *The Iwok-Nwikpe distribution: Statistical properties and its application*, Asian Journal of Probability and Statistics, vol. 15, no. 1, pp. 35–45, 2021.
16. S. Kostyshak, *Package bootstrap: Functions for the book An Introduction to the Bootstrap*, Comprehensive R Archive Network (CRAN), 2024.
17. J. F. Lawless, *Statistical Models and Methods for Lifetime Data*, John Wiley & Sons, 2011.
18. F. Merovci, *Transmuted Lindley Distribution*, International Journal of Open Problems in Computer Science and Mathematics, vol. 6, pp. 63–72, 2013.
19. D. N. P. Murthy, M. Xie, and R. Jiang, *Weibull Models*, John Wiley & Sons, New York, NY, 2004.
20. S. Nadarajah, H. S. Bakouch, and R. Tahmasbi, *A generalized Lindley distribution*, Sankhya B, vol. 73, pp. 331–359, 2011.
21. A. W. Nwry, H. M. Kareem, R. B. Ibrahim, and S. M. Mohammed, *Comparison between bisection, Newton, and secant methods for determining the root of the non-linear equation using MATLAB*, Turkish Journal of Computer and Mathematics Education, vol. 12, pp. 1115–1122, 2021.
22. A.-T. Ola, and M. G. Mohammed, *Ola distribution: A new one parameter model with applications to engineering and COVID-19 data*, Applied Mathematics & Information Sciences, vol. 17, pp. 242–252, 2023.
23. C. Onyekwere, and O. Obulezi, *Chris-Jerry distribution and its applications*, Asian Journal of Probability and Statistics, vol. 20, pp. 16–30, 2022.
24. Y. Pawitan, *In All Likelihood: Statistical Modelling and Inference Using Likelihood*, Clarendon Press, Oxford, 2001.
25. RStudio Team, *RStudio Desktop IDE*, Posit Software, PBC, 2024.
26. T. A. Severini, *Likelihood Methods in Statistics*, Oxford University Press, Oxford, 2000.
27. R. Shanker, *Shanker distribution and its applications*, International Journal of Statistics and Applications, vol. 5, pp. 338–348, 2015.
28. R. Shanker, *Akash distribution and its applications*, International Journal of Probability and Statistics, vol. 4, pp. 65–75, 2015.
29. R. Shanker, *Aradhana distribution and its applications*, International Journal of Statistics and Applications, vol. 6, pp. 23–34, 2016.
30. R. Shanker, *Amarendra distribution and its applications*, American Journal of Mathematics and Statistics, vol. 6, no. 1, pp. 44–56, 2016.
31. R. Shanker, *Sujatha distribution and its applications*, Statistics in Transition. New Series, vol. 17, pp. 391–410, 2016.
32. R. Shanker, *Garima distribution and its application to model behavioral science data*, Biometrics & Biostatistics International Journal, vol. 4, no. 7, pp. 275–281, 2016.
33. R. Shanker, *Rani distribution and its application*, Biometrics & Biostatistics International Journal, vol. 6, pp. 256–265, 2017.
34. R. Shanker, *Akshaya distribution and its application*, American Journal of Mathematics and Statistics, vol. 7, no. 2, pp. 51–59, 2017.
35. R. Shanker, and K. K. Shukla, *Ishita distribution and its applications*, Biometrics & Biostatistics International Journal, vol. 5, no. 2, pp. 39–46, 2017.
36. R. Shanker, and K. K. Shukla, *Adya distribution with properties and application*, Biometrics & Biostatistics International Journal, vol. 10, pp. 81–88, 2021.
37. R. Shanker, *Komal distribution with properties and application in survival analysis*, Biometrics & Biostatistics International Journal, vol. 12, no. 2, pp. 40–44, 2023.
38. R. Shanker, *Pratibha distribution with properties and application*, Biometrics & Biostatistics International Journal, vol. 13, pp. 136–142, 2023.