STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, July 2026, pp 790–796.
Published online in International Academic Press (www.IAPress.org)

Multiset dimension of kayak paddles graph and cycles with chord

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Abstract Suppose the set $W = \{s_1, s_2, \ldots, s_k\}$ is a subset of the vertex set $V(G)$. The representation of a vertex v of G with respect to W as follows

$$r_m(v|W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$$

where $d(v, s_i)$ is the distance between the vertex v with the vertices of set W together with their multiplicities. The set W is called the *m-resolving set* of G if every vertices of G have distinct representation with respect to W. If G has an m-resolving set, then an m-resolving set having minimum cardinality is called a multiset basis and its cardinality is called the multiset dimension of G, denoted by $md(G)$. We say that G has an infinite multiset dimension and we write $md(G) = \infty$. In this paper, we determine the multiset dimension of kayak paddles graph and cycles with chord.

Keywords m-resolving set, multiset dimension, kayak paddles graphs, cycle with a chord

AMS 2010 subject classifications 05C12

DOI: 10.19139/soic-2310-5070-3549

1. Introduction

The concept of resolving set have previously introduced by [1, 2] and also Chartrand, *et al* [3] defined the metric dimension of graphs. Slater [1] initiated a metric dimension of a graph as its location number and motivated the study of this invariant by its application to the placement of minimum number of loran/sonar detecting devices in a network so that the position of every vertex in the network can be uniquely represented in terms of its distances to the devices in the set. Application of this concept, which was studied in [5]. A robot moves in a space, which is modeled by a graph. The robot moves from a node to a node, and it can locate itself by the presence of distinctively labeled landmark nodes.

The distance between two vertices $u, v \in V(G)$, denoted by $d(u, v)$ is the length of the shortest path in G. Let $W = \{w_1, w_2, w_3, \ldots, w_k\}$ be a set of the vertices of G. A vertex w resolves a pair of vertices u, v if $d(u, w) \neq d(v, w)$. The reprsentation of each vertex v in G with respect to W is the ordered $k-$tuple

$$r(v|W) = (d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_k))$$

The set W is a resolving set of G if every two vertices in G have distinct representation.The cardinality of a smallest resolving set is called the metric dimension, denoted by $dim(G)$.

Simanjuntak *et al* [8] defined multiset dimension of graph G. Suppose the set $W = \{s_1, s_2, \ldots, s_k\}$ is a subset of the vertex set $V(G)$. The representation of a vertex v of G with respect to W as follows

$$r_m(v|W) = \{d(v, s_1), d(v, s_2), \ldots, d(v, s_k)\}$$

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

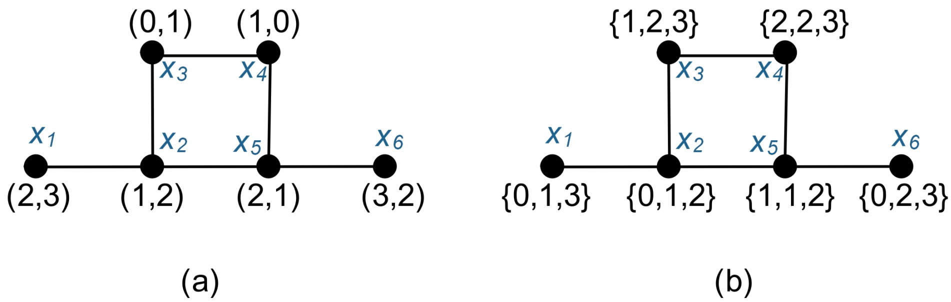

Figure 1. (a) A graph with metric dimension 2; (b) A graph with multiset dimension 3

where $d(v, s_i)$ is the distance between the vertex v with the vertices of the W together with their multiplicities. The set W is called the *m-resolving set* of G if every vertices of G have distinct representation with respect to W. If G has an m-resolving set, then an m-resolving set having minimum cardinality is called a multiset basis and its cardinality is called the multiset dimension of G, denoted by $md(G)$. We say that G has an infinite multiset dimension and we write $md(G) = \infty$.

To find the multiset dimension of some families graphs has been continuous subject for researcher. The multiset dimension of tree, cycle, and cartesian product has been studied in [8, 9] and also Alfarisi, *et al* [12] determined the multiset dimension of some families graphs namely almost hypercube graphs. Khemmani *et al* [14, 15] studiend the multiset dimension of caterpillar graphs and determined the relationship between the elements in multi representations of vertices that belong to the same multi similar equivalence class and Ahmad, *et al* [13] studied metric dimension of Kayak Paddles graph and Cycles with chord.

In Simanjuntak, *et. al.* [8], some bounds are given for the multiset dimension of graphs.

Theorem 1.1
[8] Let G be a graph other than a path. Then $md(G) \geq 3$

We illustrate this concept in Figure 1. In this case, the resolving set is $W = \{x_3, x_4\}$, shown in Figure 1 (a). The muetric dimension is $dim(G) = 2$. The representations of $v \in V(G)$ with respect to W are all distinct.

$$\begin{array}{lll} r(x_1|W) = (2,3), & r(x_2|W) = (1,2), & r(x_3|W) = (0,1) \\ r(x_4|W) = (1,0), & r(x_5|W) = (2,1), & r(x_6|W) = (3,2) \end{array}$$

For the multiset dimension, we only need to make sure the all vertices having distinct representations. Thus, we could have the m-resolving set $W = \{x_1, x_2, x_6\}$, shown in Figure 1 (b). Thus, the multiset dimension is $md(G) = 3$.

$$\begin{array}{lll} r(x_1|W) = \{0,1,3\}, & r(x_2|W) = \{0,1,2\}, & r(x_3|W) = \{1,2,3\} \\ r(x_4|W) = \{2,2,3\}, & r(x_5|W) = \{1,1,2\}, & r(x_6|W) = \{0,2,3\} \end{array}$$

A kayak paddle graphs, denoted by $KP(m, s, n)$ consists of two cycles C_n, C_m which are joined by a path of length $s + 1$ in [10]. Cycle with a chord, denoted by C_n^t is constructed from a cycle C_n by joining two vertices (not adjacent) in C_n whose distance in the cycle is t in [11].

2. Results

In this paper, we determine the multiset dimension of kayak paddles graph and cycles with chord. We proved that both families possess the constant multiset dimension 3.

Theorem 2.1
Let $KP(m, s, n)$ be a kayak paddles graph with $n, m \geq 4$ and $l \geq 1$, then the multiset dimension of $KP(m, s, n)$ is 3.

Proof. We have $V(K_{m,s,n}) = \{x_i, z_l, y_j; 1 \le i \le n, 1 \le l \le s, 1 \le j \le m\}$ and $E(K_{m,s,n}) = \{x_n x_1, x_i x_{i+1}; 1 \le i \le n-1\} \cup \{y_m y_1, y_j y_{j+1}; 1 \le j \le m\} \cup \{z_l z_{l-1}, x_1 z_1, z_s y_1\}$. Based on Theorem 1.1 that

$$md(KP(m,s,n)) \ge 3$$

Furthermore, we prove that $md(KP(m,s,n)) \le 3$, choose the set $W = \{x_{n-1}, x_n, y_m\}$. There are eight different cases, which need to be discussed here.

Case 1. For n, s, m are even
The representation of each vertex in $KP(m,s,n)$ as follows

$$r_m(z_l|W) = \begin{cases} \{l+1, l+2, s-l+2\}, & l \in [1, \frac{s}{2}] \\ \{s-l+2, l+1, l+2\}, & l \in [\frac{s+2}{2}, s] \end{cases}$$

$$r_m(x_i|W) = \begin{cases} \{i, i+1, s+i+1\}, & i \in [1, \frac{n-2}{2}] \\ \{i-1, i, s+i+1\}, & i = \frac{n}{2} \\ \{n-i+1, n-i, n+s-i+3\}, & i \in [\frac{n+2}{2}, n-2] \\ \{0, 1, n+s-i+3\}, & i \in [n-1, n] \end{cases}$$

$$r_m(y_j|W) = \begin{cases} \{j, s+j+1, s+j+2\}, & j \in [1, \frac{m}{2}] \\ \{m-j, s+m-j+3, s+m-j+4\}, & j \in [\frac{m+2}{2}, m] \end{cases}$$

Case 2. For n, s, m are odd
The representation of each vertex in $KP(m,s,n)$ as follows

$$r_m(z_l|W) = \begin{cases} \{l+1, l+2, s-l+2\}, & l \in [1, \lfloor \frac{s}{2} \rfloor] \\ \{s-l+2, s-l+2, s-l+3\}, & l = \lceil \frac{s}{2} \rceil \\ \{s-l+2, l+1, l+2\}, & l \in [\lceil \frac{s}{2} \rceil + 1, s] \end{cases}$$

$$r_m(x_i|W) = \begin{cases} \{i, i+1, s+i+1\}, & i \in [1, \lfloor \frac{n}{2} \rfloor] \\ \{n-i-1, n-i, s+n-i+3\}, & i \in [\lfloor \frac{n}{2} \rfloor + 1, n-2] \\ \{0, 1, s+n-i+3\}, & i \in [n-1, n] \end{cases}$$

$$r_m(y_j|W) = \begin{cases} \{j, s+j+1, s+j+2\}, & j \in [1, \lfloor \frac{m}{2} \rfloor] \\ \{j-1, s+j+1, s+j+2\}, & j = \lceil \frac{m}{2} \rceil \\ \{m-j, s+m-j+3, s+m-j+4\}, & j \in [\lceil \frac{m}{2} \rceil + 1, m] \end{cases}$$

Case 3. For s is odd and n, m are even
The representation of each vertex in $KP(m,s,n)$ as follows

$$r_m(z_l|W) = \begin{cases} \{l+1, l+2, s-l+2\}, & l \in [1, \lfloor \frac{s}{2} \rfloor] \\ \{s-l+2, s-l+2, s-l+3\}, & l = \lceil \frac{s}{2} \rceil \\ \{s-l+2, l+1, l+2\}, & l \in [\lceil \frac{s}{2} \rceil + 1, s] \end{cases}$$

$$r_m(x_i|W) = \begin{cases} \{i, i+1, s+i+1\}, & i \in [1, \frac{n-2}{2}] \\ \{i-1, i, s+i+1\}, & i = \frac{n}{2} \\ \{n-i+1, n-i, n+s-i+3\}, & i \in [\frac{n+2}{2}, n-2] \\ \{0, 1, n+s-i+3\}, & i \in [n-1, n] \end{cases}$$

$$r_m(y_j|W) = \begin{cases} \{j, s+j+1, s+j+2\}, & j \in [1, \frac{m}{2}] \\ \{m-j, s+m-j+3, s+m-j+4\}, & j \in [\frac{m+2}{2}, m] \end{cases}$$

Case 4. For s, n are odd and m is even
The representation of each vertex in $KP(m,s,n)$ as follows

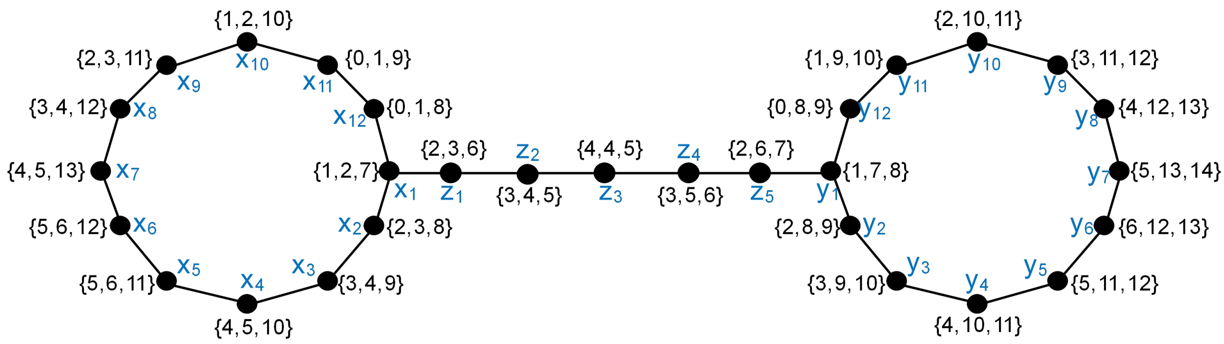

Figure 2. The Multiset Dimension of $KP(12,5,12)$

$$r_m(z_l|W)=\begin{cases}\{l+1,l+2,s-l+2\}, & l\in[1,\lfloor\frac{s}{2}\rfloor]\\ \{s-l+2,s-l+2,s-l+3\}, & l=\lceil\frac{s}{2}\rceil\\ \{s-l+2,l+1,l+2\}, & l\in[\lceil\frac{s}{2}\rceil+1,s]\end{cases}$$

$$r_m(x_i|W)=\begin{cases}\{i,i+1,s+i+1\}, & i\in[1,\lfloor\frac{n}{2}\rfloor]\\ \{n-i-1,n-i,s+n-i+3\}, & i\in[\lfloor\frac{n}{2}\rfloor+1,n-2]\\ \{0,1,s+n-i+3\}, & i\in[n-1,n]\end{cases}$$

$$r_m(y_j|W)=\begin{cases}\{j,s+j+1,s+j+2\}, & j\in[1,\frac{m}{2}]\\ \{m-j,s+m-j+3,s+m-j+4\}, & j\in[\frac{m+2}{2},m]\end{cases}$$

Case 5. For s,m are odd and n is even
The representation of each vertex in $KP(m,s,n)$ as follows

$$r_m(z_l|W)=\begin{cases}\{l+1,l+2,s-l+2\}, & l\in[1,\lfloor\frac{s}{2}\rfloor]\\ \{s-l+2,s-l+2,s-l+3\}, & l=\lceil\frac{s}{2}\rceil\\ \{s-l+2,l+1,l+2\}, & l\in[\lceil\frac{s}{2}\rceil+1,s]\end{cases}$$

$$r_m(x_i|W)=\begin{cases}\{i,i+1,s+i+1\}, & i\in[1,\frac{n-2}{2}]\\ \{i-1,i,s+i+1\}, & i=\frac{n}{2}\\ \{n-i+1,n-i,n+s-i+3\}, & i\in[\frac{n+2}{2},n-2]\\ \{0,1,n+s-i+3\}, & i\in[n-1,n]\end{cases}$$

$$r_m(y_j|W)=\begin{cases}\{j,s+j+1,s+j+2\}, & j\in[1,\lfloor\frac{m}{2}\rfloor]\\ \{j-1,s+j+1,s+j+2\}, & j=\lceil\frac{m}{2}\rceil\\ \{m-j,s+m-j+3,s+m-j+4\}, & j\in[\lceil\frac{m}{2}\rceil+1,m]\end{cases}$$

Case 6. For s is even and n,m are odd
The representation of each vertex in $KP(m,s,n)$ as follows

$$r_m(z_l|W)=\begin{cases}\{l+1,l+2,s-l+2\}, & l\in[1,\frac{s}{2}]\\ \{s-l+2,l+1,l+2\}, & l\in[\frac{s+2}{2},s]\end{cases}$$

$$r_m(x_i|W)=\begin{cases}\{i,i+1,s+i+1\}, & i\in[1,\lfloor\frac{n}{2}\rfloor]\\ \{n-i-1,n-i,s+n-i+3\}, & i\in[\lfloor\frac{n}{2}\rfloor+1,n-2]\\ \{0,1,s+n-i+3\}, & i\in[n-1,n]\end{cases}$$

$$r_m(y_j|W)=\begin{cases}\{j,s+j+1,s+j+2\}, & j\in[1,\lfloor\frac{m}{2}\rfloor]\\ \{j-1,s+j+1,s+j+2\}, & j=\lceil\frac{m}{2}\rceil\\ \{m-j,s+m-j+3,s+m-j+4\}, & j\in[\lceil\frac{m}{2}\rceil+1,m]\end{cases}$$

Case 7. For s, n are even and m are odd
The representation of each vertex in $KP(m, s, n)$ as follows

$$r_m(z_l|W) = \begin{cases} \{l+1, l+2, s-l+2\}, & l \in [1, \frac{s}{2}] \\ \{s-l+2, l+1, l+2\}, & l \in [\frac{s+2}{2}, s] \end{cases}$$

$$r_m(x_i|W) = \begin{cases} \{i, i+1, s+i+1\}, & i \in [1, \frac{n-2}{2}] \\ \{i-1, i, s+i+1\}, & i = \frac{n}{2} \\ \{n-i+1, n-i, n+s-i+3\}, & i \in [\frac{n+2}{2}, n-2] \\ \{0, 1, n+s-i+3\}, & i \in [n-1, n] \end{cases}$$

$$r_m(y_j|W) = \begin{cases} \{j, s+j+1, s+j+2\}, & j \in [1, \lfloor \frac{m}{2} \rfloor] \\ \{j-1, s+j+1, s+j+2\}, & j = \lceil \frac{m}{2} \rceil \\ \{m-j, s+m-j+3, s+m-j+4\}, & j \in [\lceil \frac{m}{2} \rceil + 1, m] \end{cases}$$

Case 8. For s, m are even and n are odd
The representation of each vertex in $KP(m, s, n)$ as follows

$$r_m(z_l|W) = \begin{cases} \{l+1, l+2, s-l+2\}, & l \in [1, \frac{s}{2}] \\ \{s-l+2, l+1, l+2\}, & l \in [\frac{s+2}{2}, s] \end{cases}$$

$$r_m(x_i|W) = \begin{cases} \{i, i+1, s+i+1\}, & i \in [1, \lfloor \frac{n}{2} \rfloor] \\ \{n-i-1, n-i, s+n-i+3\}, & i \in [\lfloor \frac{n}{2} \rfloor + 1, n-2] \\ \{0, 1, s+n-i+3\}, & i \in [n-1, n] \end{cases}$$

$$r_m(y_j|W) = \begin{cases} \{j, s+j+1, s+j+2\}, & j \in [1, \frac{m}{2}] \\ \{m-j, s+m-j+3, s+m-j+4\}, & j \in [\frac{m+2}{2}, m] \end{cases}$$

Based cases $(1)-(8)$ that it can be seen that each vertex has distinct representation with respect to W. Thus, we can conclude that the multiset dimension of $KP(m, s, n)$ is 3.

Theorem 2.2
Let C_n^2 be a cycle with chord graph with $n \geq 7$, then the multiset dimension of C_n^2 is 3.

Proof. We have $V(C_n^2) = \{x_i; 1 \leq i \leq n\}$ and $E(C_n^2) = \{x_1x_3, x_nx_1, x_ix_{i+1}; 1 \leq i \leq n-1\}$. Based on Theorem 1.1 that

$$md(C_n^2) \geq 3$$

Furthermore, we prove that $md(C_n^2) \leq 3$. There are four different cases, which need to be discussed here.
Case 1. For $n = 7$
Consider the set $W = \{x_4, x_6, x_7\}$, then the representation of each vertex of C_7^2 as follows

$$r_m(x_i|W) = \begin{cases} \{1, 2, 2\}, & i = 1 \\ \{2, 2, 3\}, & i = 2 \\ \{1, 2, 3\}, & i = 3 \\ \{0, 2, 3\}, & i = 4 \\ \{1, 1, 2\}, & i = 5 \\ \{0, 1, 2\}, & i = 6 \\ \{0, 1, 3\}, & i = 7 \end{cases}$$

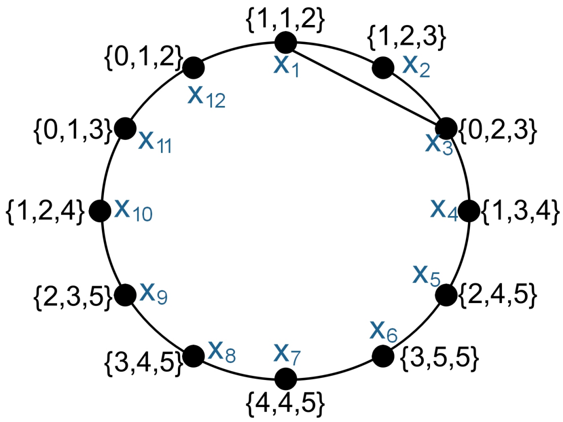

Figure 3. The Multiset Dimension of C_{12}^2

Case 2. For $n = 8$
Consider the set $W = \{x_5, x_7, x_8\}$, then the representation of each vertex of C_8^2 as follows

$$r_m(x_i|W) = \begin{cases} \{1,2,3\}, & i = 1 \\ \{2,3,3\}, & i = 2 \\ \{2,2,3\}, & i = 3 \\ \{1,3,3\}, & i = 4 \\ \{0,2,3\}, & i = 5 \\ \{1,1,2\}, & i = 6 \\ \{0,1,2\}, & i = 7 \\ \{0,1,3\}, & i = 8 \end{cases}$$

Case 3. For n is odd and $n \geq 9$
Consider the set $W = \{x_3, x_{n-1}, x_n\}$, then the representation of each vertex of C_n^2 as follows

$$r_m(x_i|W) = \begin{cases} \{1,1,2\}, & i = 1 \\ \{1,2,3\}, & i = 2 \\ \{0,2,3\}, & i = 3 \\ \{0,1,3\}, & i = n-1 \\ \{0,1,2\}, & i = n \end{cases}$$

$$r_m(x_i|W) = \begin{cases} \{i-3, i-1, i\}, & i \in [4, \lfloor \frac{n}{2} \rfloor] \\ \{i-3, i-2, i-1\}, & i = \lceil \frac{n}{2} \rceil \\ \{i-4, i-3, i-3\}, & i = \lceil \frac{n}{2} \rceil + 1 \\ \{n-i-1, n-i, n-i+2\}, & i \in [\lceil \frac{n}{2} \rceil + 2, n-2] \end{cases}$$

Case 4. For n is even and $n \geq 9$
Consider the set $W = \{x_3, x_{n-1}, x_n\}$, then the representation of each vertex of C_n^2 as follows

$$r_m(x_i|W) = \begin{cases} \{1,1,2\}, & i = 1 \\ \{1,2,3\}, & i = 2 \\ \{0,2,3\}, & i = 3 \\ \{0,1,3\}, & i = n-1 \\ \{0,1,2\}, & i = n \end{cases}$$

$$r_m(x_i|W) = \begin{cases} \{i-3, i-1, i\}, & i \in [4, \frac{n-2}{2}] \\ \{i-3, i-1, i-1\}, & i = \frac{n}{2} \\ \{i-3, i-3, i-2\}, & i = \frac{n+2}{2} \\ \{i-5, i-4, i-3\}, & i = \frac{n+4}{2} \\ \{n-i-1, n-i, n-i+2\}, & i \in [\frac{n+6}{2}, n-2] \end{cases}$$

Based cases $(1)-(4)$ that it can be seen that each vertex has distinct representation with respect to W. Thus, we can conclude that the multiset dimension of C_n^2 is 3.

3. Conclusion

The multiset dimension of Kayak Paddle Graphs and Cycle with Chord is 3.

Acknowledgement

We gratefully acknowledge the financial support provided by Universitas Jember 2026 through Postdoctoral Research Grant with Contract Number: 4307/DST/UN25.C1/LT/2026

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