STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **15**, Month 2026, pp 0–10.
Published online in International Academic Press (www.IAPress.org)

On the Inclusive Local Irregularity Coloring of Blooming Graph Families

Arika Indah Kristiana [1,*], Marcha Dwi Prihandini [1], Rafiantika Megahnia Prihandini [1], Susanto [1], Lioni Anka Monalisa [1], Hutkemri Zulnaidi[2], Ridho Alfarisi[3]

[1]*Department of Mathematics Education, Universitas Jember, Indoensia*
[2]*Department of Mathematics and Science Education, Universiti Malaya, Malaysia*
[3]*Department of Mathematics, Universitas Jember, Indoensia*

Abstract A graph G is an ordered pair of sets denoted by $G = (V, E)$ where $V(G)$ is the vertex set and $E(G)$ is the edge set. Graph coloring requires that all vertices be colored using as few colors as possible such that no two adjacent vertices share the same color. One extension of graph coloring is the inclusive local irregular vertex coloring, which is a coloring that also takes into account the label of each vertex itself. The number of distinct colors obtained is called the inclusive local irregular chromatic number, denoted by $\chi^i_{lis}(G)$. This research presents five new theorems related to inclusive local irregular vertex coloring of blooming graph families, namely the octopus graph (O_n), sandat graph (St_n), butterfly graph $(BF_{m,n})$, tunjung graph (Tj_n), and the sunflower graph (Sf_n).

Keywords Inclusive local irregular vertex coloring; chromatic number; blooming graph families

AMS 2010 subject classifications 05C78

DOI: 10.19139/soic-2310-5070-3479

1. Introduction

Graph theory has developed since 1736 and continues to be a relevant topic to study today. Graph theory can be understood as a subject studied to represent certain problems so that they are easier to solve [1]. The emergence of graph theory was motivated by the Konigsberg bridges problem, which was solved by the Swiss mathematician Leonhard Euler. This problem originated from the attempt of local residents to cross the seven bridges over the Pregel River exactly once and return to the starting point. Euler attempted to solve the problem by developing a graph-theoretic proof, which led to the concept of an Eulerian circuit. Since that time, graph theory has evolved into a significant area of research that can represent various types of problems.

A graph G is an ordered pair of sets denoted by $G = (V, E)$ where $V(G)$ is a finite nonempty set of vertices and $E(G)$ is a set of unordered pairs of vertices that are elements of V(G), called edges. Based on its type, a graph G is called a multigraph if there exists a pair of vertices that are connected by two or more edges. A graph is called a pseudograph if it contains a loop, that is, an edge whose endpoints coincide at a single vertex. A graph G that has neither multiple edges nor loops is called a simple graph. A graph G has vertex and edge cardinalities known as the order and the size, respectively, which are denoted by $|V(G)|$ and $|E(G)|$ Moreover, the concept of vertex degree in a graph is denoted by $d(v)$ [2].

Graph coloring is one of the fundamental topics in graph theory. Graph coloring is defined as the assignment of colors to vertices, edges, or regions of a graph with the minimum number of colors, where adjacent vertices

*Correspondence to: Arika Indah Kristiana (Email: arika.fkip@unej.ac.id). Department of Mathematics Education, Universitas Jember, Indonesia.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

must receive distinct colors. Inclusive local irregular vertex coloring is an extension of graph coloring. This type of coloring combines two concepts, distance irregular labeling and vertex coloring [3]. A graph satisfies inclusive local irregular vertex coloring if a labeling with the minimum possible values is assigned such that adjacent vertices have distinct weights, where the weight of each vertex includes the label of the vertex itself. The number of distinct colors obtained is called the inclusive local irregular chromatic number, denoted by $\chi^i_{lis}(G)$. The concept of this coloring was initially proposed in 2020, as presented in study [4], which examines path graphs, cycle graphs, and star graphs. Other similar studies are presented in [3, 5, 6, 7, 8, 9]. Based on the result of studies [4] several definitions, observations, Lemmas, and Propositions concerning inclusive local irregular vertex coloring have been established and will serve as the basis for this study. The relevant definitions are presented as follows:

Definition 1
In inclusive local irregular vertex coloring, let the labeling function be defined as $l : V(G) \to \{1, 2, \ldots, k\}$ and the weight function $w^i : V(G) \to N$. The inclusive weight of a vertex v is given by $w^i(v) = l(v) + \sum_{u \in N(v)} l(u)$ A graph is said to satisfy inclusive local irregular vertex coloring if the following conditions hold:

1. $opt(l) = min\{max\{l_a\}\}$, where l_a denotes an inclusive local irregular vertex labeling, and
2. for every $uv \in E(G)$, $w^i(u) \neq w^i(v)$.

Definition 2
The inclusive local irregular chromatic number is defined as the minimum number of colors required for inclusive local irregular vertex coloring and is denoted by $\chi^i_{lis}(G)$.

The following observations, lemmas, and propositions are provided to simplify the proof of the theorems.

Observation 1
If a connected graph G contains a pair of adjacent vertices with distinct degrees, then $opt(l) = 1$.

Observation 2
If a connected graph G contains a pair of adjacent vertices with equal degrees, then $opt(l) \geq 2$.

Observation 3
A graph G that contains a cycle C_3 does not admit an inclusive local irregular vertex coloring.

Lemma 1
Let G be a simple and connected graph. Then $\chi^i_{lis}(G) \geq \chi_{lis}(G)$.

Proposition 1
The local irregular chromatic number of the sandat graph (St_n) for $n \geq 3$ is $\chi_{lis}(St_n) = 3$

Proposition 2
The local irregular chromatic number of the sunflower graph (Sf_n) for $n \geq 3$ is given by

$$\chi_{lis}(Sf_n) = \begin{cases} 7, & \text{for } n \text{ is odd} \\ 5, & \text{for } n \text{ is even} \end{cases}$$

Based on the types of graphs that have been investigated in relation to this topic, there are still several graphs that have not yet been studied. One example is the family of blooming graphs, which comprises several vertices adjacent to a central vertex, subject to the condition that every vertex in the graph must be assigned a color distinct from that of the central vertex [11]. The types of graphs to be investigated include the octopus graph (O_n), which is formed from the combination of the fan graph (F_n) and the star graph (S_n) [12], the sandat graph (St_n), which is constructed from the star graph (S_n) and the Dutch windmill graph (D^m_n) [10], the butterfly graph $(BF_{m,n})$, which is formed by two wings of sizes m and n sharing a common central vertex t and having two pendant edges [13], the tunjung graph (Tj_n), which is an extension of the helm graph (H_n) with the addition of new pendant vertices [14], and the sunflower graph (Sf_n), which is built from the combination of the flower graph (Fl_n) and the star graph (S_n) [15].

2. Results and Discussion

This article examines inclusive local irregular vertex coloring of the family of blooming graphs, namely the octopus graph (O_n), the sandat graph (St_n), the butterfly graph $(BF_{m,n})$, the tunjung graph (Tj_n), and the sunflower graph (Sf_n). The resulting theorems are presented below.

Observation 4
The local irregular chromatic number of the octopus graph (O_n) for $n \geq 2$ is given by

$$\chi_{lis}(O_n) = \begin{cases} 4, & \text{for } n = 2, 3 \\ 5, & \text{for } n \geq 4 \end{cases}$$

Theorem 1
The inclusive local irregular chromatic number of the octopus graph (O_n) for $n \geq 3$ is given by

$$\chi_{lis}(O_n) = \begin{cases} 4, & \text{for } n = 3 \\ 5, & \text{for } n \geq 4 \end{cases}$$

Proof
Let O_n be the octopus graph with the vertex set $V(O_n) = \{x\} \cup \{y_i; 1 \leq i \leq n\} \cup \{z_i; 1 \leq i \leq n\}$ and the edge set $E(O_n) = \{xy_i; 1 \leq i \leq n\} \cup \{xz_i; 1 \leq i \leq n\} \cup \{y_i y_{i+1} \leq i \leq n-1\}$. The cardinalities of the vertex and edge sets of this graph can be expressed as $|V(O_n)| = 2n + 1$ and $|E(O_n)| = 3n - 1$.
Case 1. For $n = 3$
Given that $y_1 y_2 \in E(O_3)$, the graph (O_3) contains a pair of adjacent vertices with different degrees. Therefore, based on Observation 1, this graph satisfies $opt(l) = 1$. Based on this, the labeling function on the graph can be defined as $l : V(O_3) \to \{1\}$. The lower bound of the inclusive local irregular chromatic number of the octopus graph (O_n) with $n = 3$ is $\chi^i_{lis}(O_n) \geq \chi_{lis}(O_n) = 4$, so that $\chi^i_{lis}(O_n) \geq 4$. The labeling used is as follows.

$$l(x) = 1$$
$$l(y_i) = 1, \text{ for } 1 \leq i \leq n$$
$$l(z_i) = 1, \text{ for } 1 \leq i \leq n$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(x) = 2n + 1$$
$$w^i(y_i) = \begin{cases} 3, & \text{for } i \text{ is odd} \\ 4, & \text{for } i \text{ is even} \end{cases}$$
$$w^i(z_i) = 2, \text{ for } 1 \leq i \leq n$$

From the labeling function, $|w^i(V(O_n))| = 4$ providing the upper bound $\chi^i_{lis}(O_n) \leq |w^i(V(O_n))|$ or equivalently $\chi^i_{lis}(O_n) \leq 4$. Based on the lower bound from Observation 4 and the upper bound obtained through the vertex weight function for $n = 3$, it follows that $4 \leq \chi^i_{lis}(O_n) \leq 4$ or equivalently $\chi^i_{lis}(O_n) = 4$. Therefore, it is proven that the inclusive local irregular chromatic number of the octopus graph with $n = 3$ is $\chi^i_{lis}(O_n) = 4$.
Case 2. For $n \geq 4$
Given that $y_2 y_3 \in E(O_n)$ for $n \geq 4$, the octopus graph (O_n) contains a pair of adjacent vertices with equal degrees, and hence, by Observation 2, it holds that $opt(l) \geq 2$. Based on this, the labeling function on the graph can be defined as $l : V(O_n) \to \{1, 2\}$. The lower bound of the inclusive local irregular chromatic number of the octopus graph (O_n) with $n \geq 4$ is $\chi^i_{lis}(O_n) \geq \chi_{lis}(O_n) = 5$, so that $\chi^i_{lis}(O_n) \geq 5$. The labeling used is as follows.

$$l(x) = 1$$
$$l(y_i) = \begin{cases} 1, & \text{for } i \text{ is odd} \\ 2, & \text{for } i \text{ is even} \end{cases}$$
$$l(z_i) = 1, \text{ for } 1 \leq i \leq n$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(x) = \begin{cases} \frac{5n+1}{2}, & \text{for } n \text{ is odd} \\ \frac{5n+2}{2}, & \text{for } n \text{ is even} \end{cases}$$
$$w^i(y_i) = \begin{cases} 4, & \text{for } i = 1, n \\ 5, & \text{for } i \text{ is even} \\ 6, & \text{for } i \text{ is odd, } i \neq 1 \end{cases}$$
$$w^i(z_i) = 2, \text{ for } 1 \leq i \leq n$$

From the labeling function, $|w^i(V(O_n))| = 5$ providing the upper bound $\chi^i_{lis}(O_n) \leq |w^i(V(O_n))|$ or equivalently $\chi^i_{lis}(O_n) \leq 5$. Based on the lower bound from Observation 4 and the upper bound obtained through the vertex weight function for $n \geq 4$, it follows that $5 \leq \chi^i_{lis}(O_n) \leq 5$ or equivalently $\chi^i_{lis}(O_n) = 5$. Therefore, it is proven that the inclusive local irregular chromatic number of the octopus graph with $n \geq 4$ is $\chi^i_{lis}(O_n) = 5$. □

Theorem 2
The inclusive local irregular chromatic number of the sandat graph (St_n) for $n \geq 3$ is $\chi^i_{lis}(St_n) = 3$.

Proof
Let St_n be the sandat graph with the vertex set $V(St_n) = \{x\} \cup \{y_i; 1 \leq i \leq n\} \cup \{y_{i,1}; 1 \leq i \leq n\} \cup \{y_{i,2}; 1 \leq i \leq n\}$ and the edge set $E(St_n) = \{xy_i; 1 \leq i \leq n\} \cup \{xy_{i,1}; 1 \leq i \leq n\} \cup \{xy_{i,2}; 1 \leq i \leq n\} \cup \{y_iy_{i,1}; 1 \leq i \leq n\} \cup \{y_iy_{i,2}; 1 \leq i \leq n\}$. The cardinalities of the vertex and edge sets of this graph can be expressed as $|V(St_n)| = 3n + 1$ and $|E(St_n)| = 5n$.
Given that $y_2y_{2,1} \in E(St_n)$ for $n \geq 3$, the graph (St_n) contains a pair of adjacent vertices with different degrees. Therefore, based on Observation 1, this graph satisfies $opt(l) = 1$. Based on this, the labeling function on the graph can be defined as $l : V(St_3) \to \{1\}$. The lower bound of the inclusive local irregular chromatic number of the sandat graph (St_n) with $n = 3$ is $\chi^i_{lis}(St_n) \geq \chi_{lis}(St_n) = 3$, so that $\chi^i_{lis}(St_n) \geq 3$. The labeling used is as follows.

$$l(x) = 1$$
$$l(y_i) = 1, \text{ for } 1 \leq i \leq n$$
$$l(y_{i,1}) = 1, \text{ for } 1 \leq i \leq n$$
$$l(y_{i,2}) = 1, \text{ for } 1 \leq i \leq n$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(x) = 3n + 1$$
$$w^i(y_i) = 4, \text{ for } 1 \leq i \leq n$$
$$w^i(y_{i,1}) = 3, \text{ for } 1 \leq i \leq n$$
$$w^i(y_{i,2}) = 3, \text{ for } 1 \leq i \leq n$$

From the labeling function, $|w^i(V(St_n))| = 3$ providing the upper bound $\chi^i_{lis}(St_n) \leq |w^i(V(St_n))|$ or equivalently $\chi^i_{lis}(St_n) \leq 3$. Based on the lower bound from Proposition 1 and the upper bound obtained through the vertex weight function, it follows that $3 \leq \chi^i_{lis}(St_n) \leq 3$ or equivalently $\chi^i_{lis}(St_n) = 3$. Therefore, it is proven that the inclusive local irregular chromatic number of the sandat graph is $\chi^i_{lis}(St_n) = 3$. □

Observation 5
The local irregular chromatic number of the butterfly graph $(BF_{m,n})$ for $n \geq 2$ is given by

$$\chi_{lis}(BF_{m,n}) = \begin{cases} 4, & \text{for } m, n = 2, 3 \\ 5, & \text{for } m, n \geq 4 \end{cases}$$

Theorem 3
The inclusive local irregular chromatic number of the butterfly graph $(BF_{m,n})$ for $n \geq 3$ is given by

$$\chi^i_{lis}(BF_{m,n}) = \begin{cases} 4, & \text{for } m,n=3 \\ 5, & \text{for } m,n \geq 4 \end{cases}$$

Proof
Let $BF_{m,n}$ be the butterfly graph with the vertex set $V(BF_{m,n}) = \{t\} \cup \{u\} \cup \{v\} \cup \{u_i; 1 \leq i \leq m\} \cup \{v_i; 1 \leq i \leq n\}$ and the edge set $E(BF_{m,n}) = \{tu\} \cup \{tv\} \cup \{tu_i; 1 \leq i \leq m\} \cup \{tv_i; 1 \leq i \leq n\} \cup \{u_iu_{i+1}; 1 \leq i \leq m-1\} \cup \{v_iv_{i+1}; 1 \leq i \leq n-1\}$. The cardinalities of the vertex and edge sets of this graph can be expressed as $|V(BF_{m,n})| = m+n+3$ and $|E(BF_{m,n})| = 2(m+n)$.
Case 1. For $m,n=3$
Given that $u_1u_2 \in E(BF_{3,3})$, the graph $(BF_{3,3})$ contains a pair of adjacent vertices with different degrees. Therefore, based on Observation 1, this graph satisfies $opt(l)=1$. Based on this, the labeling function on the graph can be defined as $l: V(BF_{3,3}) \rightarrow \{1\}$. The lower bound of the inclusive local irregular chromatic number of the butterfly graph $(BF_{m,n})$ with $m,n=3$ is $\chi^i_{lis}(BF_{m,n}) \geq \chi_{lis}(BF_{m,n}) = 4$, so that $\chi^i_{lis}(BF_{m,n}) \geq 4$. The labeling used is as follows.

$$l(t) = 1$$
$$l(u) = 1$$
$$l(v) = 1$$
$$l(u_i) = 1, \text{ for } 1 \leq i \leq m$$
$$l(v_i) = 1, \text{ for } 1 \leq i \leq n$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(t) = 3n$$
$$w^i(u) = 2$$
$$w^i(v) = 2$$
$$w^i(u_i) = \begin{cases} 3, & \text{for } i \text{ is odd} \\ 4, & \text{for } i \text{ is even} \end{cases}$$
$$w^i(v_i) = \begin{cases} 3, & \text{for } i \text{ is odd} \\ 4, & \text{for } i \text{ is even} \end{cases}$$

From the labeling function, $|w^i(V(BF_{m,n})| = 4$ providing the upper bound $\chi^i_{lis}(BF_{m,n}) \leq |w^i(V(BF_{m,n})|$ or equivalently $\chi^i_{lis}(BF_{m,n}) \leq 4$. Based on the lower bound from Observation 5 and the upper bound obtained through the vertex weight function for $m,n=3$, it follows that $4 \leq \chi^i_{lis}(BF_{m,n}) \leq 4$ or equivalently $\chi^i_{lis}(BF_{m,n}) = 4$. Therefore, it is proven that the inclusive local irregular chromatic number of the butterfly graph with $m,n=3$ is $\chi^i_{lis}(BF_{m,n}) = 4$.
Case 2. For $m,n \geq 4$
Given that $u_1u_2 \in E(BF_{m,n})$ for $m,n \geq 4$, the butterfly graph $(Bf_{m,n})$ contains a pair of adjacent vertices with equal degrees, and hence, by Observation 2, it holds that $opt(l) \geq 2$. Based on this, the labeling function on the graph can be defined as $l: V(BF_{m,n}) \rightarrow \{1,2\}$. The lower bound of the inclusive local irregular chromatic number of the butterfly graph $(BF_{m,n})$ with $m,n \geq 4$ is $\chi^i_{lis}(BF_{m,n}) \geq \chi_{lis}(BF_{m,n}) = 5$, so that $\chi^i_{lis}(BF_{m,n}) \geq 5$. The labeling used is as follows.

$$l(t) = 1$$
$$l(u) = 1$$
$$l(v) = 1$$
$$l(u_i) = \begin{cases} 1, & \text{for } i \text{ is odd} \\ 2, & \text{for } i \text{ is even} \end{cases}$$
$$l(v_i) = \begin{cases} 1, & \text{for } i \text{ is odd} \\ 2, & \text{for } i \text{ is even} \end{cases}$$

Subcase 1. For $m = n$
The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(t) = \begin{cases} 3n+2, & \text{for } m, n \text{ odd} \\ 3n+3, & \text{for } m, n \text{ even} \end{cases}$$
$$w^i(u) = 2$$
$$w^i(v) = 2$$
$$w^i(u_i) = \begin{cases} 4, & \text{for } i = 1, m \\ 5, & \text{for } i \text{ is even} \\ 6, & \text{for } i \text{ is odd}, i \neq 1 \end{cases}$$
$$w^i(v_i) = \begin{cases} 4, & \text{for } i = 1, n \\ 5, & \text{for } i \text{ is even} \\ 6, & \text{for } i \text{ is odd}, i \neq 1 \end{cases}$$

Subcase 2. For $m \neq n$
The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(t) = \begin{cases} \frac{3(m+n)+4}{2}, & \text{for } m, n \text{ odd} \\ \frac{3(m+n)+5}{2}, & \text{for } m \text{ is odd}, n \text{ is even}; m \text{ is even}, n \text{ is odd} \\ \frac{3(m+n)+6}{2}, & \text{for } m, n \text{ even} \end{cases}$$
$$w^i(u) = 2$$
$$w^i(v) = 2$$
$$w^i(u_i) = \begin{cases} 4, & \text{for } i = 1, m \\ 5, & \text{for } i \text{ is even} \\ 6, & \text{for } i \text{ is odd}, i \neq 1 \end{cases}$$
$$w^i(v_i) = \begin{cases} 4, & \text{for } i = 1, n \\ 5, & \text{for } i \text{ is even} \\ 6, & \text{for } i \text{ is odd}, i \neq 1 \end{cases}$$

From the labeling function, $|w^i(V(BF_{m,n})| = 5$ providing the upper bound $\chi^i_{lis}(BF_{m,n}) \leq |w^i(V(BF_{m,n})|$ or equivalently $\chi^i_{lis}(BF_{m,n}) \leq 5$. Based on the lower bound from Observation 5 and the upper bound obtained through the vertex weight function for $m = n$ and $m \neq n$, it follows that $5 \leq \chi^i_{lis}(BF_{m,n}) \leq 5$ or equivalently $\chi^i_{lis}(BF_{m,n}) = 5$. Therefore, it is proven that the inclusive local irregular chromatic number of the butterfly graph with $n \geq 4$ with $m = n$ and $m \neq n$ is $\chi^i_{lis}(BF_{m,n}) = 5$. □

Observation 6
The local irregular chromatic number of the tunjung graph (Tj_n) for $n \geq 3$ is given by

$$\chi_{lis}(Tj_n) = \begin{cases} 7, & \text{for } n \text{ is odd} \\ 4, & \text{for } n \text{ is even} \end{cases}$$

Theorem 4
The inclusive local irregular chromatic number of the tunjung graph (Tj_n) for $n \geq 3$ is given by

$$\chi^i_{lis}(Tj_n) = \begin{cases} 7, & \text{for } n \text{ is odd} \\ 4, & \text{for } n \text{ is even} \end{cases}$$

Proof
Let Tj_n be the tunjung graph with the vertex set $V(Tj_n) = \{v\} \cup \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\} \cup \{z_i; 1 \leq i \leq n\}$ and the edge set $E(Tj_n) = \{vx_i; 1 \leq i \leq n\} \cup \{x_iy_i; 1 \leq i \leq n\} \cup \{y_iz_i; 1 \leq i \leq n1\} \cup \{vy_i; 1 \leq i \leq$

$n\} \cup \{vz_i; 1 \leq i \leq n\} \cup \{x_i x_{i+1}; 1 \leq i \leq n-1\} \cup \{y_i y_{i+1}; 1 \leq i \leq n-1\} \cup \{x_1 x_n\} \cup \{y_1 y_n\}$. The cardinalities of the vertex and edge sets of this graph can be expressed as $|V(Tj_n)| = 3n+1$ and $|E(Tj_n)| = 7n$.

Case 1. For n is odd

Subcase 1. For $n = 3$

Given that $u_1 u_2 \in E(Tj_3)$, the graph (Tj_3) contains a pair of adjacent vertices with equal degrees, and hence, by Observation 2, it holds that $opt(l) \geq 2$. Based on this, the labeling function on the graph can be defined as $l : V(Tj_3) \to \{1, 2, 3\}$. The lower bound of the inclusive local irregular chromatic number of the tunjung graph (Tj_n) with $n = 3$ is $\chi^i_{lis}(Tj_n) \geq \chi_{lis}(Tj_n) = 7$, so that $\chi^i_{lis}(Tj_n) \geq 7$. The labeling used is as follows.

$$l(v) = 1$$
$$l(x_i) = 1, \text{ for } 1 \leq i \leq n$$
$$l_(y_i) = \begin{cases} 1, & \text{for } i \text{ is odd, } i \neq n \\ 2, & \text{for } i \text{ is even} \\ 3, & \text{for } i = n \end{cases}$$
$$l_(z_i) = \begin{cases} 1, & \text{for } i = n \\ 2, & \text{for } i \text{ is even} \\ 3, & \text{for } i \text{ is odd, } i \neq n \end{cases}$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(v) = 5n + 1$$
$$w^i(x_i) = \begin{cases} 5, & \text{for } i \text{ is odd, } i \neq n \\ 6, & \text{for } i \text{ is even} \\ 7, & \text{for } i = n \end{cases}$$
$$w^i(y_i) = \begin{cases} 9, & \text{for } i = n \\ 10, & \text{for } i \text{ is even} \\ 11, & \text{for } i \text{ is odd, } i \neq 1 \end{cases}$$
$$w^i(z_i) = 5, \text{ for } 1 \leq i \leq n$$

Subcase 2. For $n \geq 5$

Given that $x_1 x_2 \in E(Tj_n)$ for $n \geq 5$, the tunjung graph Tj_n contains a pair of adjacent vertices with equal degrees, and hence, by Observation 2, it holds that $opt(l) \geq 2$. Based on this, the labeling function on the graph can be defined as $l : V(Tj_n) \to \{1, 2\}$. The lower bound of the inclusive local irregular chromatic number of the tunjung graph (Tj_n) for n odd is $\chi^i_{lis}(Tj_n) \geq \chi_{lis}(Tj_n) = 7$, so that $\chi^i_{lis}(Tj_n) \geq 7$. The labeling used is as follows.

$$l(v) = 1$$
$$l_(x_i) = \begin{cases} 1, & \text{for } i = n-1 \\ 2, & \text{for } 1 \leq i \leq n, i \neq n-1 \end{cases}$$
$$l_(y_i) = \begin{cases} 1, & \text{for } i \text{ is odd} \\ 2, & \text{for } i \text{ is even} \end{cases}$$
$$l_(z_i) = \begin{cases} 1, & \text{for } i \text{ is even, } i = n \\ 2, & \text{for } i \text{ is odd, } i \neq n \end{cases}$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(v) = 5n - 1$$
$$w^i(x_i) = \begin{cases} 7, & \text{for } i = n-2, n \\ 8, & \text{for } i \text{ is odd, } i \neq n-2, n; i = n-1 \\ 9, & \text{for } i \text{ is even, } i \neq n-1 \end{cases}$$

$$w^i(y_i) = \begin{cases} 7, & \text{for } i = n-1 \\ 8, & \text{for } i \text{ is even, } i \neq n-1; i = n \\ 9, & \text{for } i = 1 \\ 10, & \text{for } i \text{ is odd, } i \neq 1, n \end{cases}$$

$$w^i(z_i) = \begin{cases} 3, & \text{for } i = n \\ 4, & \text{for } 1 \leq i \leq n, i \neq n \end{cases}$$

From the labeling function, $|w^i(V(Tj_n))| = 7$ providing the upper bound $\chi^i_{lis}(Tj_n) \leq |w^i(V(Tj_n))|$ or equivalently $\chi^i_{lis}(Tj_n) \leq 7$. Based on the lower bound from Observation 6 and the upper bound obtained through the vertex weight function, it follows that $7 \leq \chi^i_{lis}(Tj_n) \leq 7$ or equivalently $\chi^i_{lis}(Tj_n) = 7$. Therefore, it is proven that the inclusive local irregular chromatic number of the tunjung graph for n odd is $\chi^i_{lis}(Tj_n) = 7$.

Case 2. For n is even

Given that $y_1y_2 \in E(Tj_n)$ for $n \geq 4$, the tunjung graph contains a pair of adjacent vertices with equal degrees, and hence, by Observation 2, it holds that $opt(l) \geq 2$. Based on this, the labeling function on the graph can be defined as $l : V(Tj_n) \to \{1, 2\}$. The lower bound of the inclusive local irregular chromatic number of the tunjung graph (Tj_n) for n even is $\chi^i_{lis}(Tj_n) \geq \chi_{lis}(Tj_n) = 4$, so that $\chi^i_{lis}(Tj_n) \geq 4$. The labeling used is as follows.

$$l(v) = 1$$

$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ is odd} \\ 2, & \text{for } i \text{ is even} \end{cases}$$

$$l(y_i) = 1, \text{ for } 1 \leq i \leq n$$

$$l(z_i) = 1, \text{ for } 1 \leq i \leq n$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(v) = \frac{7n+2}{2}$$

$$w^i(x_i) = \begin{cases} 6, & \text{for } i \text{ is even} \\ 7, & \text{for } i \text{ is odd} \end{cases}$$

$$w^i(y_i) = \begin{cases} 6, & \text{for } i \text{ is odd} \\ 7, & \text{for } i \text{ is even} \end{cases}$$

$$w^i(z_i) = 3, \text{ for } 1 \leq i \leq n$$

From the labeling function, $|w^i(V(Tj_n))| = 4$ providing the upper bound $\chi^i_{lis}(Tj_n) \leq |w^i(V(Tj_n))|$ or equivalently $\chi^i_{lis}(Tj_n) \leq 4$. Based on the lower bound from Observation 6 and the upper bound obtained through the vertex weight function, it follows that $4 \leq \chi^i_{lis}(Tj_n) \leq 4$ or equivalently $\chi^i_{lis}(Tj_n) = 4$. Therefore, it is proven that the inclusive local irregular chromatic number of the tunjung graph for n even is $\chi^i_{lis}(Tj_n) = 4$. □

Theorem 5

The inclusive local irregular chromatic number of the sunflower graph (Sf_n) for $n \geq 3$ is given by

$$\chi^i_{lis}(Sf_n) = \begin{cases} 7, & \text{for } n \text{ is odd} \\ 5, & \text{for } n \text{ is even} \end{cases}$$

Proof

Let Sf_n be the sunflower graph with the vertex set $V(Sf_n) = \{v\} \cup \{u_i; 1 \leq i \leq n\} \cup \{t_i; 1 \leq i \leq n\} \cup \{x_i; 1 \leq i \leq n\}$ and the edge set $E(Sf_n) = \{vu_i; 1 \leq i \leq n\} \cup \{vt_i; 1 \leq i \leq n\} \cup \{vx_i; 1 \leq i \leq n1\} \cup \{t_ix_i; 1 \leq i \leq n\} \cup \{t_1t_n\} \cup \{t_it_{i+1}; 1 \leq i \leq n-1\}$. The cardinalities of the vertex and edge sets of this graph can be expressed as $|V(Sf_n)| = 3n + 1$ and $|E(Sf_n)| = 5n$.

Case 1. For n is odd

Subcase 1. For $n = 3$

Given that $t_1t_2 \in E(Sf_3)$, the graph (Sf_3) contains a pair of adjacent vertices with equal degrees, and hence, by Observation 2, it holds that $opt(l) \geq 2$. Based on this, the labeling function on the graph can be defined as $l : V(Sf_3) \rightarrow \{1, 2, 3\}$. The lower bound of the inclusive local irregular chromatic number of the sunflower graph (Sf_n) with $n = 3$ is $\chi^i_{lis}(Sf_n) \geq \chi_{lis}(Sf_n) = 7$, so that $\chi^i_{lis}(Sf_n) \geq 7$. The labeling used is as follows.

$$l(v) = 1$$
$$l(u_i) = 2, \text{ for } 1 \leq i \leq n$$
$$l(t_i) = \begin{cases} 1, & \text{for } i \text{ is odd, } i \neq n \\ 2, & \text{for } i \text{ is even, } i = n \end{cases}$$
$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ is odd, } i \neq n \\ 2, & \text{for } i \text{ is even} \\ 3, & \text{for } i = n \end{cases}$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(v) = 4n + 6$$
$$w^i(u_i) = 3, \text{ for } 1 \leq i \leq n$$
$$w^i(t_i) = \begin{cases} 7, & \text{for } i \text{ is odd, } i \neq n \\ 8, & \text{for } i \text{ is even} \\ 9, & \text{for } i = n \end{cases}$$
$$w^i(x_i) = \begin{cases} 3, & \text{for } i \text{ is odd, } i \neq n \\ 5, & \text{for } i \text{ is even} \\ 6, & \text{for } i = n \end{cases}$$

Subcase 2. For $n \geq 5$
Given that $t_1t_2 \in E(Sf_n)$ for $n \geq 5$, the sunflower graph contains a pair of adjacent vertices with equal degrees, and hence, by Observation 2, it holds that $opt(l) \geq 2$. Based on this, the labeling function on the graph can be defined as $l : V(Sf_n) \rightarrow \{1, 2\}$. The lower bound of the inclusive local irregular chromatic number of the sunflower graph (Sf_n) for n odd is $\chi^i_{lis}(Sf_n) \geq \chi_{lis}(Sf_n) = 7$, so that $\chi^i_{lis}(Sf_n) \geq 7$. The labeling used is as follows.

$$l(v) = 1$$
$$l(u_i) = 2, \text{ for } 1 \leq i \leq n$$
$$l_(t_i) = \begin{cases} 1, & \text{for } i \text{ is even} \\ 2, & \text{for } i \text{ is odd} \end{cases}$$
$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ is odd, } i \neq 1, i = 2 \\ 2, & \text{for } i \text{ is even, } i = 1, i \neq 2 \end{cases}$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(v) = 4n + 6$$
$$w^i(u_i) = 3, \text{ for } 1 \leq i \leq n$$
$$w^i(t_i) = \begin{cases} 6, & \text{for } i \text{ is odd, } i \neq 1, n \\ 7, & \text{for odd } i = 2, n \\ 8, & \text{for } i \text{ is even, } i = 1, i \neq 2 \end{cases}$$
$$w^i(x_i) = \begin{cases} 3, & \text{for } i = 2, n \\ 4, & \text{for } 1 \leq i \leq n, i \neq 1, 2 \\ 5, & \text{for } i = 1 \end{cases}$$

From the labeling function, $|w^i(V(Sf_n))| = 7$ providing the upper bound $\chi^i_{lis}(Sf_n) \leq |w^i(V(Sf_n))|$ or equivalently $\chi^i_{lis}(Sf_n) \leq 7$. Based on the lower bound from Proposition 2 and the upper bound obtained through the vertex weight function, it follows that $7 \leq \chi^i_{lis}(Sf_n) \leq 7$ or equivalently $\chi^i_{lis}(Sf_n) = 7$. Therefore, it is proven that the inclusive local irregular chromatic number of the sunflower graph for n odd is $\chi^i_{lis}(Sf_n) = 7$.

Case 2. For n is even

Given that $t_1t_2 \in E(Sf_n)$ for $n \geq 4$, the sunflower graph contains a pair of adjacent vertices with equal degrees, and hence, by Observation 2, it holds that $opt(l) \geq 2$. Based on this, the labeling function on the graph can be defined as $l : V(Sf_n) \to \{1, 2\}$. The lower bound of the inclusive local irregular chromatic number of the sunflower graph (Sf_n) for n even is $\chi^i_{lis}(Sf_n) \geq \chi_{lis}(Sf_n) = 5$, so that $\chi^i_{lis}(Sf_n) \geq 5$. The labeling used is as follows

$$l(v) = 1$$
$$l(u_i) = 1, \text{ for } 1 \leq i \leq n$$
$$l(t_i) = \begin{cases} 1, & \text{for } i \text{ is odd} \\ 2, & \text{for } i \text{ is even} \end{cases}$$
$$l(x_i) = \begin{cases} 1, & \text{for } i \text{ is even} \\ 2, & \text{for } i \text{ is odd} \end{cases}$$

The weight function of the vertices, obtained by summing the labels, is given as follows.

$$w^i(v) = 4n + 1$$
$$w^i(u_i) = 2, \text{ for } 1 \leq i \leq n$$
$$w^i(t_i) = \begin{cases} 6, & \text{for } i \text{ is even} \\ 8, & \text{for } i \text{ is odd} \end{cases}$$
$$w^i(x_i) = 4, \text{ for } 1 \leq i \leq n$$

From the labeling function, $|w^i(V(Sf_n))| = 5$ providing the upper bound $\chi^i_{lis}Sf_n) \leq |w^i(V(Sf_n))|$ or equivalently $\chi^i_{lis}(Sf_n) \leq 5$. Based on the lower bound from Proposition 2 and the upper bound obtained through the vertex weight function, it follows that $5 \leq \chi^i_{lis}(Sf_n) \leq 5$ or equivalently $\chi^i_{lis}(Sf_n) = 5$. Therefore, it is proven that the inclusive local irregular chromatic number of the sunflower graph for n even is $\chi^i_{lis}(Sf_n) = 5$. □

Figure 1 illustrates the octopus graph and the butterfly graph for $n \geq 4$ and $m, n \geq 4$. Based on this figure, both graphs are shown to have the same inclusive local irregular chromatic number, namely 5.

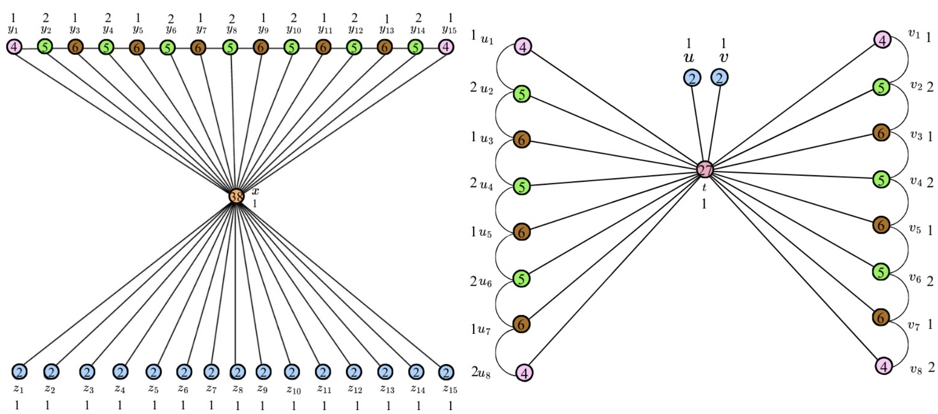

Figure 1. Octopus graph (O_{15}) and Butterfly Graph $(BF_{m,n})$.

3. Concluding Remarks

Based on the foregoing discussion, it can be concluded that the inclusive local irregular chromatic numbers of the octopus graph (O_n) and the butterfly graph $(BF_{m,n})$ for $n \geq 4$ and $m, n \geq 4$ are the same, namely $\chi^i_{lis}(G) = 5$, which is different from those of the sandat graph, the tunjung graph, and the sunflower graph. The inclusive local irregular chromatic numbers of the tunjung graph (Tj_n) and the sunflower (Sf_n) graph are divided into two cases, namely the odd and even cases.

Acknowledgments

The authors would like to express their sincere gratitude to LP2M and the Faculty of Teacher Training and Education, University of Jember, Indonesia, for the support provided in 2026.

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