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Modifying Spectral Conjugate Gradient Method for Solving Iteration Problems

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Abstract One well-known and simple technique for minimizing the models is to use spectral conjugate gradients. In this work, we construct a unique Spectrum using the gradient approach to expansion a novel search Path, we have proven that the novel spectral approach Possesses the descent property and that the spectrum methodology is globally convergent. The experimental results indicate that for the test problems, the suggested methodology may be computed in conjunction with alternative conjugate gradient techniques.

Keywords Conjugate Gradient Methods, Global convergence, parameter, Iteration problems.

AMS 2010 subject classifications 90C06, 65K05, 90C26

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1. Introduction

We will examine the problem of optimization minimization as delineated below:

$$min_{x \in R^n} f(x) \tag{1}$$

where $f : R^n \longrightarrow R$ is a smooth function, whose gradient is assumed for convenience. For further information, consult [22]. The most effective categories of iterative methodologies are the conjugate gradient (CG) procedures utilized to address iterative challenges, as they do not necessitate the utilization of matrices and are typically characterized by high efficiency. For further insight, refer to [21]. In the context of CG-algorithms, a sequence $\{x_k\}$ is effectively determined by the following iterative process:

$$x_0 \in R^n,\ x_{\tau+1} = x_\tau + \alpha_\tau d_\tau \tag{2}$$

where $\alpha_\tau > 0$ represents the step size and d_τ denotes the direction. The search direction is generated by:

$$d_{\tau+1} = -g_{\tau+1} + \beta_\tau d_\tau \tag{3}$$

where β_τ is designated as the CG-parameter. The step-size α_k adheres to specific line search criteria, such as the conventional Wolfe line search:

$$f(x_\tau + \alpha_\tau d_\tau) \le f(x_\tau) + \delta \alpha_\tau g_\tau^T d_\tau \tag{4}$$

$$g(x_\tau + \alpha_\tau d_\tau)^T d_\tau \ge \sigma g_\tau^T d_\tau \tag{5}$$

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where $0 < \delta < \sigma < 1$, see in [17]. The most recognized expressions for the parameter β_τ in CG-methods is the Hestenes-Stiefel method [13], which is defined as follows:

$$\beta_\tau^{HS} = \frac{g_{\tau+1}^T y_\tau}{d_\tau^T y_\tau} \tag{6}$$

In contrast to conventional CG-methods, in the spectral CG- approach, the search direction $d_{\tau+1}$ is delineated as:

$$d_{\tau+1} = -\theta_\tau g_{\tau+1} + \beta_\tau d_\tau \tag{7}$$

where θ_τ is referred to as a spectral constant. It is readily apparent that (7) simplifies to (3) when $\theta_\tau = 1$. Additional details can be located in [14].

In this influence, enhancements are introduced to the CG-methods, thereby augmenting the effectiveness of the foundational techniques. The ensuing alterations incorporate spectral constants:

$$\theta_\tau^{DY} = \theta_\tau^{HS} = \frac{y_\tau^T d_\tau}{y_\tau^T d_\tau} = 1 \; , \; \theta_\tau^{PR} = \theta_\tau^{FR} = \frac{y_\tau^T d_\tau}{g_\tau^T g_\tau} \; , \; \theta_\tau^{LS} = \beta_\tau^{CD} = \frac{y_\tau^T d_\tau}{|g_\tau^T d_\tau|} \tag{8}$$

further information can be located in [8, 5, 3, 7, 18].

CG-methods have been associated with various global convergence results, which are validated in [25, 19, 23]. The objective of this manuscript is to derive the spectral constant, as well as to investigate its convergence properties, arithmetical results, and subsequent dialogue.

2. A novel CG-method

Basim A. Hassan et al. [8] Suggest a CG parameter based on the Taylor Series (quadratic model), which we represent as follows:

$$\beta_\tau = \frac{\|g_{\tau+1}\|^2}{(f_\tau - f_{\tau+1})/\alpha_\tau + 1/2 d_\tau^T y_\tau} \tag{9}$$

On the basis of a parameter conjugate gradient, we drive a unique spectral conjugate gradient.

We will articulate a conceptual framework for a novel methodology pertaining to the spectral conjugate gradients, which facilitates the generation of a new algorithm. Consequently, the parameters of the conjugate gradient method satisfy the relationship:

$$\beta_\tau \leq \frac{-g_{\tau+1}^T d_{\tau+1}}{-g_\tau^T d_\tau} \tag{10}$$

then it can be regarded as:

$$\beta_\tau g_\tau^T d_\tau = g_{\tau+1}^T d_{\tau+1} = \frac{\|g_{\tau+1}\|^2}{(f_\tau - f_{\tau+1})/\alpha_\tau + 1/2 d_\tau^T y_\tau} g_\tau^T d_\tau \tag{11}$$

From the definition of y_τ, we obtain:

$$y_\tau^T d_\tau = g_{\tau+1}^T d_\tau - g_\tau^T d_\tau \tag{12}$$

It is obvious that:

$$\begin{aligned} g_{\tau+1}^T d_{\tau+1} &= \frac{\|g_{\tau+1}\|^2}{\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} \left[\frac{y_\tau^T d_\tau}{\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} \left(\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau \right) - g_{\tau+1}^T d_\tau \right] \\ &= -\frac{y_\tau^T d_\tau}{\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} \|g_{\tau+1}\|^2 + \frac{\|g_{\tau+1}\|^2}{\frac{f_\tau - f_{\tau+1}}{\alpha_\tau} + \frac{1}{2} d_\tau^T y_\tau} g_{\tau+1}^T d_\tau \end{aligned} \tag{13}$$

$$g_{\tau+1}^T d_{\tau+1} = -\theta_\tau \|g_{\tau+1}\|^2 + \frac{\|g_{\tau+1}\|^2}{\frac{(f_\tau - f_{\tau+1})}{\alpha_\tau} + \frac{1}{2d_\tau^T y_\tau}} g_{\tau+1}^T d_\tau \tag{14}$$

where:

$$\theta_\tau^{NBN} = \frac{y_\tau^T d_\tau}{[(f_\tau - f_{\tau+1})/\alpha_\tau + 1/2d_\tau^T y_\tau]} \tag{15}$$

which we term the NBN method. Based on equation (15), this can provide an innovative search trajectory:

$$d_{\tau+1} = -\theta_\tau^{NBN} g_{\tau+1} + \frac{\|g_{\tau+1}\|^2}{(f(x_\tau) - f(x_{\tau+1}))/\alpha_\tau + 1/2d_\tau^T y_\tau} d_\tau \tag{16}$$

Employing the aforementioned process, an existing method is articulated as follows:

Algorithm 1 NBN Algorithm:

1: **Step 0:** Choose $x_0 \in \mathbb{R}^n$, $0 < \delta_1 < \delta_2 < 1$, set $d_0 = -g_0$.
2: **Step 1:** If $\|g_{\tau+1}\|^2 \leq 10^{-6}$, stop.
3: **Step 2:** Compute β_τ by (9) with θ^{NBN} by (15).
4: **Step 3:** Set $x_{\tau+1} = x_\tau + \alpha_\tau d_\tau$, where α_τ satisfies (4) and (5).
5: **Step 4:** Compute
$$d_{\tau+1} = -\theta_\tau g_{\tau+1} + \beta_\tau d_\tau.$$
6: **Step 5:** Set $\tau = \tau + 1$ and go to Step 1.

The descent property of the proposed process is demonstrated by the following theorem.

Theorem 1
Let Algorithm 1 yield sequence xo, then $g_{\tau+1}^T d_{\tau+1} \leq -C\|g_{\tau+1}\|^2$ for every τ.

Proof
So $d_o = -g_o$ we take $g_o^T d_o = \|g_o\|^2 < 0$. Let $g_\tau^T d_\tau < -C_1\|g_\tau\|^2$. Increasing (16) by $g_{\tau+1}$ and using (12), we have:

$$g_{\tau+1}^T d_{\tau+1} = -\theta^{NBN} g_{\tau+1}^T g_{\tau+1} + \frac{\|g_{\tau+1}\|^2}{(f(x_\tau) - f(x_{\tau+1}))/\alpha_\tau + 1/2d_\tau^T y_\tau} g_{\tau+1}^T d_\tau \tag{17}$$

$$\begin{aligned} g_{\tau+1}^T d_{\tau+1} &= \frac{\|g_{\tau+1}\|^2}{\dfrac{f(x_\tau) - f(x_{\tau+1})}{\alpha_\tau} + \dfrac{1}{2} d_\tau^T y_\tau} \left[-y_\tau^T d_\tau + g_{\tau+1}^T d_\tau\right] \\ &= \frac{\|g_{\tau+1}\|^2}{\dfrac{f(x_\tau) - f(x_{\tau+1})}{\alpha_\tau} + \dfrac{1}{2} d_\tau^T y_\tau} g_\tau^T d_\tau \\ &= \frac{g_\tau^T d_\tau}{\dfrac{f(x_\tau) - f(x_{\tau+1})}{\alpha_\tau} + \dfrac{1}{2} d_\tau^T y_\tau} \|g_{\tau+1}\|^2 \end{aligned} \tag{18}$$

Since $g_\tau^T d_\tau < -C_1\|g_\tau\|^2$ then we have:

$$g_{\tau+1}^T d_{\tau+1} < -C_1 \frac{\|g_\tau\|^2}{(f(x_\tau) - f(x_{\tau+1}))/\alpha_\tau + 1/2d_\tau^T y_\tau} \|g_{\tau+1}\|^2 \tag{19}$$

where $C = C_1 \frac{\|g_\tau\|^2}{(f(x_\tau)-f(x_{\tau+1}))/\alpha_\tau+1/2d_\tau^T y_\tau}$. Since $C_1, f(x_\tau) - f(x_{\tau+1})/\alpha_\tau + 1/2d_\tau^T y_\tau$ and $\|g_\tau\|^2$ are positive, therefore the value of C is similarly positive.

$$\therefore g_{\tau+1}^T d_{\tau+1} \leq -C\|g_{\tau+1}\|^2 \tag{20}$$

□

3. Global convergence

We will examine the global convergence of NBN. In order to support the primary conclusions of this study, we first provide the following modest hypothesis.

Assumption 1

i- Let level set $L = \{x \in R^n \setminus f(x) \leq f(x_0)\}$ is bounded.

ii- In some locality U and L, $f(x)$ is smooth and its gradient is Lipschitz continuous, viz, there exists a constant $\mu_1 > 0$ obtain:

$$\|g_{\tau+1} - g_\tau\| \leq \mu_1 \|x_{\tau+1} - x_\tau\|, \; \forall x_{\tau+1}, x_\tau \in U \tag{21}$$

See [25, 20].

The global convergence of the suggested algorithms is demonstrated using the outcome of the following lemma, commonly known as the Zoutendijk condition. This was first provided by Zoutendijk [26].

Lemma 1

Let assumptions holds. $If\ g_{\tau+1}^T d_{\tau+1} \leq 0$ with α_τ satisfies the (4),(5). Then

$$\sum_{\tau=1}^{\infty} \frac{\left(g_\tau^T d_\tau\right)^2}{\|d_\tau\|^2} < +\infty \tag{22}$$

The following theorem highlights the suggested approaches' global convergence.

Theorem 2

Let assumptions holds and $\{g_{\tau+1}\}$ and $\{d_{k+1}\}$ be generated by Algorithm 1. Then

$$\lim_{\tau\to\infty} inf\ \|g_{\tau+1}\| = 0 \tag{23}$$

Proof

Under the given conditions, Lemma 1 is true. In the following, the inference will be made via implication. Assume, by inconsistency, that near is a positive constant $\in_1 > 0$.

$$\|g_{\tau+1}\| > \in_1 \tag{24}$$

Rewriting (18) as:

$$d_{\tau+1} + \theta_\tau^{NBN} g_{\tau+1} = B_\tau^\beta d_\tau \tag{25}$$

After squaring its two sides, we obtain:

$$\|d_{\tau+1}\|^2 + (\theta_\tau^{NBN})^2 \|g_{\tau+1}\|^2 + 2\theta_\tau^{NBN} d_{\tau+1} g_{\tau+1} = (B_\tau^\beta)^2 \|d_\tau\|^2 \tag{26}$$

From (26), we get

$$\|d_{\tau+1}\|^2 = (B_\tau^\beta)^2 \|d_\tau\|^2 - 2\theta_\tau^{NBN} d_{\tau+1}^T g_{\tau+1} - (\theta_\tau^{NBN})^2 \|g_{\tau+1}\|^2 \tag{27}$$

From the above equation and (10), we have:

$$\|d_{\tau+1}\|^2 \leq \left(\frac{g_{\tau+1}^T d_{\tau+1}}{g_\tau^T d_\tau}\right)^2 \|d_\tau\|^2 - 2\theta_\tau^{NBN} d_{\tau+1}^T g_{\tau+1} - (\theta_\tau^{NBN})^2 \|g_{\tau+1}\|^2 \tag{28}$$

Diving the both side of the inequality by $\left(g_{\tau+1}^T d_{\tau+1}\right)^2$ we have:

$$\begin{aligned}\frac{\|d_{\tau+1}\|^2}{(d_{\tau+1}^T g_{\tau+1})^2} &\leq \frac{\|d_\tau\|^2}{(d_\tau^T g_\tau)^2} - (\theta_\tau^{NBN})^2 \frac{\|d_{\tau+1}\|^2}{(d_{\tau+1}^T g_{\tau+1})^2} - 2\theta_\tau^{NBN} \frac{1}{d_{\tau+1}^T g_{\tau+1}} \\ &\leq \frac{\|d_\tau\|^2}{(d_\tau^T g_\tau)^2} - (\theta_\tau^{NBN})^2 \frac{\|g_{\tau+1}\|^2}{C^2\|g_{\tau+1}\|^4} - 2\theta_\tau^{NBN} \frac{1}{C\|g_{\tau+1}\|^2} \\ &\quad - \frac{1}{\|g_{\tau+1}\|^2} + \frac{1}{\|g_{\tau+1}\|^2} \\ &\leq \frac{\|d_\tau\|^2}{(d_\tau^T g_\tau)^2} - \left(\theta_\tau^{NBN} \frac{\|g_{\tau+1}\|}{C\|g_{\tau+1}\|^2} + \frac{1}{\|g_{\tau+1}\|}\right) + \frac{1}{\|g_{\tau+1}\|^2} \\ \frac{\|d_{\tau+1}\|^2}{(d_{\tau+1}^T g_{\tau+1})^2} &\leq \frac{\|d_\tau\|^2}{(d_\tau^T g_\tau)^2} + \frac{1}{\|g_{\tau+1}\|^2} \end{aligned} \tag{29}$$

Applying (29) to recurrence, see that $\|d_1\|^2 = -g_1^T d_1 = \|g_1\|^2$, we get:

$$\frac{\|d_{\tau+1}\|^2}{\left(d_{\tau+1}^T g_{\tau+1}\right)^2} \leq \sum_{i=1}^{\tau} \frac{1}{\|g_i\|^2} \tag{30}$$

Then we get from (29) and (24) that

$$\frac{\left(g_\tau^T d_\tau\right)^2}{\|d_\tau\|^2} \geq \frac{\varepsilon_1^2}{\tau} \tag{31}$$

which indicates

$$\sum_{\tau=1}^{\infty} \frac{\left(g_\tau^T d_\tau\right)^2}{\|d_\tau\|^2} \geq \sum_{\tau=1}^{\infty} \frac{\varepsilon_1^2}{\tau} = \infty \tag{32}$$

□

4. Arithmetical Results

By calculating the value of the performance measure, each of the following strategies arrived at the best outcome. Depending on the number on the number of function (FUNC) and the number of iterations (ITER) each function has attest.
The results of our numerical tests comparing the NBN in this work with the algorithm assumed by HS in [13] are presented in this section.

Table 1. Arithmetical Results of NBN-Algorithm and HS-Algorithm

Test Function	NBN Algorithm		HS Algorithm	
	ITER	FUNC	ITER	FUNC
1	33	16	42	26
	35	18	44	18
2	13	7	24	11
	12	8	11	7
3	21	14	23	17
	32	13	72	36
4	33	14	77	34

Continued on next page

Table 1 – Continued

Test Function	NBN Algorithm FUNC	NBN Algorithm ITER	HS Algorithm FUNC	HS Algorithm ITER
	26	11	48	16
5	59	11	121	42
	168	32	435	189
6	79	27	85	30
	67	34	59	28
7	73	27	95	35
	239	81	339	93
8	71	62	107	42
	71	51	103	40
9	10	7	14	8
	16	6	16	6
10	84	27	122	66
	84	27	122	66
11	7	5	8	5
	7	4	13	5
12	8	4	14	7
	6	4	7	5
13	41	14	38	11
	63	22	75	22
14	30	13	38	11
	52	23	62	23
15	86	36	123	74
	241	76	251	105
16	120	77	187	144
	87	18	76	22
17	143	7	120	32
	21	11	26	14
18	107	33	191	41
	101	87	120	117
19	48	44	85	77
	71	66	93	88
20	106	63	183	89
	74	55	98	95
21	6	5	33	16
	144	8	167	17
22	92	15	121	20
	86	8	67	14
23	40	22	55	24
	60	49	125	71
24	85	17	91	21
	123	43	88	18
25	103	75	145	66
	32	7	39	16
Total	**2616**	**2204**	**3800**	**3048**

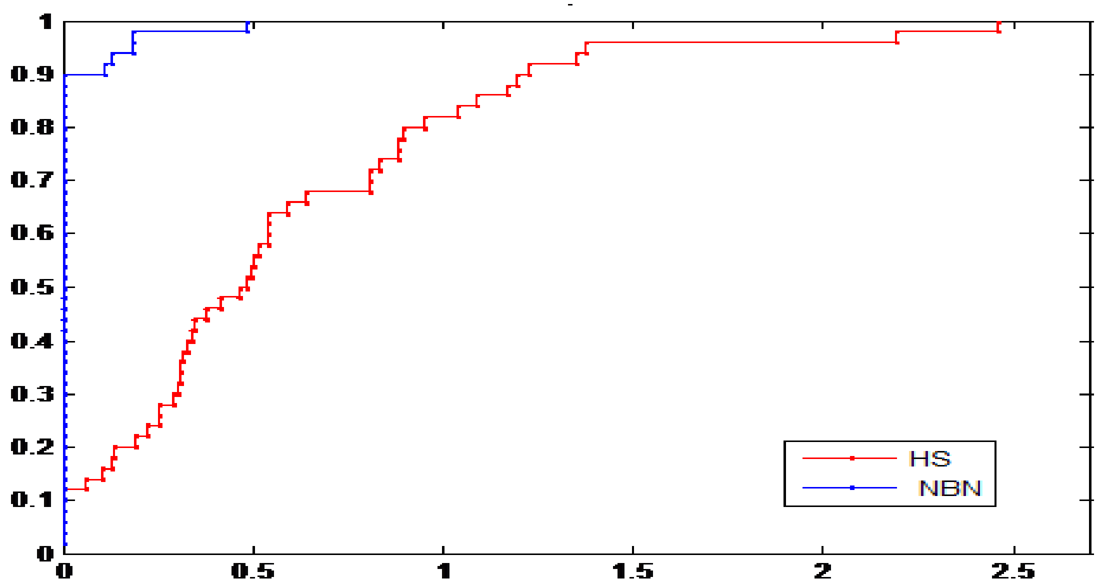
Table 2. Test Functions

NO. Of Function	Test Function	DIM
1	Extended White & Holst	100, 1000
2	Penalty	100, 1000
3	Extended Beale	100, 1000
4	Extended Wood	100, 1000
5	Quadratic Diagonal P.	100, 1000
6	Extended Cliff	100, 1000
7	Perturbed Quadratic	100, 1000
8	Extended Hiebert	100, 1000
9	NONDIA (CUTE)	100, 1000
10	DIXMAAN E (CUTE)	100, 1000
11	DIXMAAN F (CUTE)	100, 1000
12	Extended PSC1	100, 1000
13	TRIDIA (CUTE)	100, 1000
14	Generalized Tridiagonal 2	100, 1000
15	Extended Tridiagonal 1	100, 1000
16	Raydan 1	100, 1000
17	Trigonometric	100, 1000
18	Diagonal 1	100, 1000
19	DIXMAAN K	100, 1000
20	Extended Quadratic QP2	100, 1000
21	Freudenstein & Roth	100, 1000
22	Rosen brock	100, 1000
23	Broyden Tridiagonal	100, 1000
24	ARWHEAD (CUTE)	100, 1000
25	Diagonal 4	100, 1000

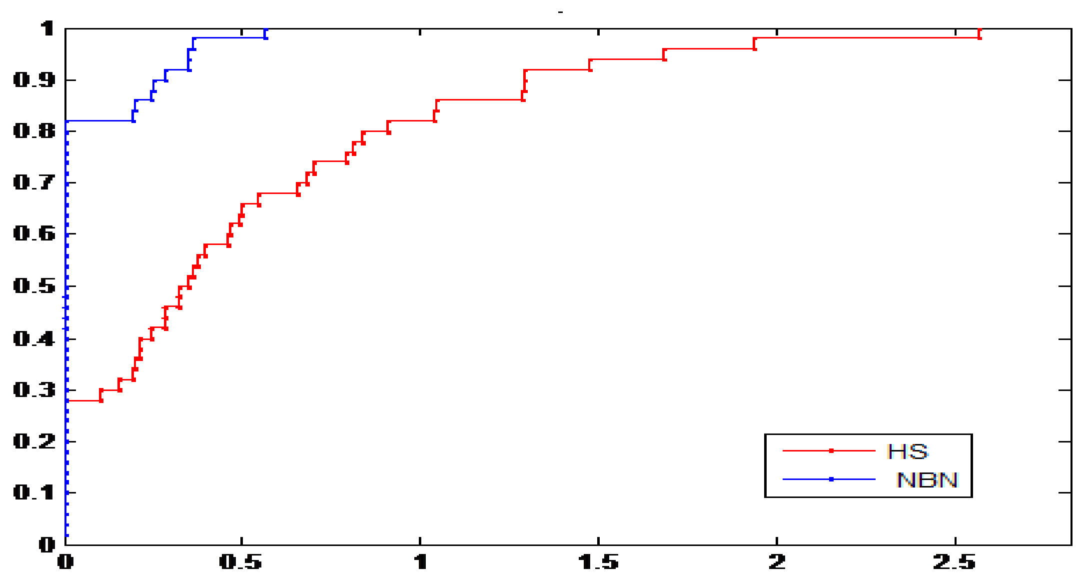
Dolan and More [24] examined how to use a cumulative performance profiling tool (similar to how it was developed) to evaluate our proposed algorithm's performance. We will discuss the application of this tool, which uses a cumulative distribution function (CDF) to describe the ratio of resolution successes for all tested samples to the base reference value. Performance ratios are plotted on the CDF's horizontal axis while each test sample that was resolved is plotted on the CDF's vertical axis. An algorithm's CDF with respect to the reference point has a higher zone than all other algorithms' CDFs and therefore has an advantage because it resolves more samples than any of the other tested algorithms based upon its advantage.

Our first figure presents a comparison of the CDF performance profiles for our modified (NBN) algorithm and the original HS. The cumulative performance profile comparison shows how quickly each algorithm converges to an optimal solution based on the number of iterations needed. The NBN algorithm, despite achieving a higher cumulative performance than the traditional HS algorithm, does so more quickly. As a result, for 25 of the test functions, the proposed NBN algorithm will converge in fewer iterations compared to the traditional HS algorithm.

Conversely, due to converging more slowly than the proposed NBN Algorithm, the traditional HS will require more iterations to converge. Therefore, the traditional HS algorithm is less efficient than the proposed NBN Algorithm.



The data presented in this document indicate that the cost of executing an optimization algorithm is influenced by the number of function evaluations to achieve optimal performance. In addition, the data suggest that the proposed NBN technique has a computational advantage due to a smaller number of function evaluations being needed to reach optimal performance compared to the traditional HS technique. In general, most of the time used to obtain an optimized solution can be attributed to the computational resources consumed to complete function evaluations. As a result, the numerical data supports the recommendation that the NBN technique be employed in actual optimization.



These findings substantiate the endorsement of the proposed NBN method for practical optimization challenges where efficiency is paramount. A wide range of studies have addressed this issue from multiple angles [16, 15, 9, 4, 2, 1], thereby reinforcing the theoretical framework of the current research. Given the recent developments in Quasi-Newton methods reported in [11, 10, 6, 12].

5. Conclusions

One famous and simple method for minimizing the functionality is the conjugate Gradient of Spectrum. We constract an unique globally convergent spectral conjugate gradient to meet the acceptable descent Standards.

To do numerical testing, we employ a range of diffuse test functions. By applying to the calculated results there were 31% less, iterations, and 44% less function evaluation total. We drive anew spectral conjugate gradient based on a CG-parameter. In this part, we will describe an idea of a novel approach to the spectral conjugate of gradients and it is generated anew.

Table 3. The new algorithm's relative efficiency

Tools	**ITER**	**FUNC**
HS-Algorithm	100.%	100.%
NBN-Algorithm	69.23%	57.73%

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