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A Hybrid Support Vector Machine-Genetic Algorithm Framework for Estimating the DUS-Transformed Generalized Polynomial Quadratic Failure Rate Distribution

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Abstract Probability distributions are essential statistical instruments, which make it possible to model and analyze some probability events in various spheres, including engineering, medicine, finance, and environmental science. Classical distributions, however, have been observed to have constraints in describing the complexity of real-life data. This paper will discuss these shortcomings by proposing the DUS-transformed Generalized Polynomial Quadratic Failure Rate (DUS-GPQF) distribution, which is a new extension of the generalized linear failure rate (GLF) distribution via DUS transformation method. The DUS-GPQF distribution increases the flexibility in the GLF model which provides it with greater ability to support a larger spectrum of data behaviors. The DUS-GPQF distribution has important statistical properties, which include the hazard rate functional, moments, incomplete moments, entropy, and the extropy. The DUS-GPQF distribution has seventeen estimation methods, which guarantee that it is practical to apply. The tests on the DUS-GPQF distribution are performed on two real-world data sets, and the results show that the model is better than other competitive models in goodness of fit and predictive accuracy. The study offers a powerful statistical model for complex data so as to improve theoretical as well as practical statistical analysis.

Keywords DUS transformation, Generalized Polynomial Quadratic Failure Rate distribution, Real-life data analysis, Newton Raphson, SVM–GA, Simulation

AMS 2010 subject classifications 60J80, 60J85, 60K10

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1. Introduction

Probability distributions form the basis of statistics and probability theory providing a mathematical construct to describe random phenomena in a wide variety of fields. Their modeling capability of uncertainty and variability renders them indispensable in a broad array of applications in the real world. An example of these would be in engineering, where they can be used to test the reliability of a system and to perform stress analysis; in medicine, where they can be used to perform a survival analysis and to evaluate treatment; in finance, where they are essential to risk assessment and to modelling investments; and in environmental science, where they can be used to predict the occurrence of a climate event and to predict ecological trends. Probability distribution versatility is

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therefore significant in solving data-driven challenges in various fields. To get further information about statistical distributions, the following papers are provided [1, 2, 3, 4, 5, 6, 7].

Since there are a growing number of data streams that are becoming more complex, including reliability engineering, survival analysis, and medical research, the traditional statistical models may not be sufficient in the description of the underlying irregularities of such data. Conventional assumptions might not be able to fit unusual characteristics including strong skewness, extreme-tailed behavior, and different forms of hazard rates which may lead to biased inferences and false conclusions. Such difficulties have inspired the development of more flexible probabilistic models to increase both the descriptive and predictive strength of statistical models in such information-intensive domains.

1.1. Limitations of existing models

Although these classical lifetime distributions have seen widespread use, e.g. the exponential, Rayleigh, generalized exponential, and linear failure rate (LFR) models have structural limitations. The main weakness is that they have a limited ability to model various hazard rate behavior especially increasing, decreasing and bathtubs shapes, thus, limiting their usefulness in reliability analysis and survival modelling. These classical models often suppose one, monotonic hazard structure, which, on complicated lifetime data with multifaceted risk processes, is often optimum fit. In response to these shortcomings [8] derived the Generalized Linear Failure Rate (GLF) distribution, which increases the modeling flexibility to provide many different hazards rate shapes such as increasing-decreasing, and bathtub. However, its increased adaptability still requires further methodological improvement to better capture the complex nature of the modern data, without compromising the computational and structural parsimony.

1.2. Overview of the GPQF and DUS transformation

Generalized Polynomial Quadratic Failure Rate (GPQF) distribution is a flexible underlying distribution that has three parameters. It is a generalization of various popular lifetime distributions as special cases, amongst which the Linear Failure Rate (LFR), Generalized Exponential (GE), Generalized Rayleigh (GR), Exponential, and Rayleigh distributions. Due to its structural flexibility, the GPQF model provides an expanded structure in explaining varying reliability behavior in different fields of use. Consider a nonlinear failure rate of the form:

$$r\left(y\right) = \alpha + \beta y + \delta y^2, \alpha, \beta, \delta \geq 0. \tag{1}$$

Then the cumulative hazard function is:

$$I\left(y\right) = \int_0^y r\left(t\right) dt = \alpha y + \frac{\beta}{2} y^2 + \frac{\delta}{3} y^3. \tag{2}$$

The cumulative distribution and probability density functions of the GPQF distribution are respectively defined as follows:

$$G\left(y; \alpha, \beta, \delta, \gamma\right) = \left[1 - \exp\left(-\alpha y - \frac{\beta}{2} y^2 - \frac{\delta}{3} y^3\right)\right]^{\gamma}, y \geq 0, \gamma > 0. \tag{3}$$

Differentiating, the corresponding probability density function (PDF) is:

$$g\left(y; \alpha, \beta, \delta, \gamma\right) = \gamma \left(\alpha + \beta y + \delta y^2\right) \exp\left(-\alpha y - \frac{\beta}{2} y^2 - \frac{\delta}{3} y^3\right) \left[1 - \exp\left(-\alpha y - \frac{\beta}{2} y^2 - \frac{\delta}{3} y^3\right)\right]^{\gamma - 1}. \tag{4}$$

In order to increase the flexibility of the Generalized Linear Failure Rate (GLF) distribution, various researchers have a few extended versions of the model. Some of the important developments are the Modified GLF distribution [8, 9], the McDonald GLF distribution [10], the Extended GLF distribution [11] and the Modified Beta GLF distribution [12]. Although these extensions add to the ability of the distribution to capture various behavior of hazard rates, they often accomplish this by adding new parameters. These expansions can also result in more complex and complicated models, increase the risk of over-parameterization, and reduce the general parsimony and interpretability of the model.

A better alternative method of increasing the distributional flexibility without increasing the dimensionality of the parameters is the Dinesh Umesh Sanjay (DUS) transformation. It has been used to generalize a variety of classical lifetime distributions using this transformation technique to include the DUS-transformed Lomax distribution [13], the DUS-transformed inverse Weibull distribution [14], the DUS-transformed Weibull distribution [15], the DUS-transformed Kumaraswamy distribution [16], the DUS-transformed inverse Rayleigh distribution [17], the DUS-transformed power inverse Rayleigh distribution [18], the DUS-transformed Rayleigh distribution [19] and the DUS- Empirical tests of these models on a variety of different datasets have shown significant increases in both the goodness of fit and the adaptability of the models and this has confirmed the usefulness of the DUS transformation as a parsimonious, yet potent, tool in the construction of flexible probabilistic models.

1.3. Objectives and novelty of the study

The current work presents a synergistic distribution that generalizes the adaptability of the structure of GPQF model with the improved flexibility of the Dinesh Umesh Sanjay (DUS) transformation, the DUS-transformed Generalized Polynomial Quadratic Failure Rate (DUS-GPQF) distribution. The main originality of this work is the development of a new lifetime distribution and a hybrid Support Vector Machine/ genetic algorithm (SVM -GA) system to correctly estimate the parameters. This proposed DUS-GPQF model increases the modeling capability of the existing GLF-based models without use of extra parameters thus is more flexible and does not compromise parsimony. The contributions that this study makes to the literature on statistical modeling are as follows:

1. It provides a new distribution the DUS-GPQF and calculates the basic statistical characteristics of it such as the hazard function, entropy and moments.
2. It comes up with a novel hybrid parameter estimation methodology on a Support Vector Machine-Genetic Algorithm (SVM-GA) framework.
3. It makes a wide comparative study on seventeen estimation techniques by using a massive simulation study to determine the performance in the form of inference.
4. It confirms the practical usefulness of the suggested model based on real datasets and compares its performance with other already existing alternative models.

Taken together, these contributions respond to the ever-increasing demand to have flexible and strong statistical models which are able to model complex lifetime behaviors and therefore, offer a useful methodological contribution to applied research in reliability, survival, and risk analysis. The rest of the current paper is structured in the following way. The DUS-GPQF Model Section provides the design and development of the suggested DUS-GPQF distribution. The basic distributional properties of it are investigated in Section “Statistical Properties”. Section “Estimation Methods” outlines the estimation methods of parameter that are applied in the analysis. Section Numerical Simulation provides the findings of simulation experiments aiming to determine the performance of the estimators. The practical usefulness of the model is seen in the Section named as Real Data Analysis where, the proposed distribution is utilized to real-world data sets. Lastly, the Section “Conclusion summarizes the key findings and outlines the possible directions of future research.

1.4. Comparative Advantage of the DUS-GPQF Distribution

Despite the fact that the DUS transformation has been implemented on a number of lifetime distributions, its effectiveness on these lifetime distributions heavily relies on the base distribution. Some of the currently used DUS-based models include DUS-Rayleigh, DUS-Weibull and DUS-GLF, which are based on baseline distributions with fairly limited hazard rate structures. As an example, the Rayleigh distribution tends to give monotonically increasing hazards, whereas GLF model can only take certain forms of polynomials. The given DUS-GPQF distribution is an improvement on the previous one as it takes the Generalized Polynomial Quadratic Failure Rate (GPQF) distribution as a reference point. The GPQF model is with a polynomial-quadratic rate structure of hazard which enables it to capture a broader spectrum of the behaviors of reliability even in its non-transformed state. The fact that the DUS transformation is applied to the GPQF distribution consequently gives rise to two complementary degrees of flexibility: the structural flexibility of the GPQF baseline and the shape-altering ability of the DUS transformation. This allows the DUS-GPQF model to fit various behavioral patterns of hazard rate, such as increasing, decreasing, and bathtub-shaped behavior, with model parsimony.

Moreover, the real-data applications indicate that the DUS-GPQF distribution has always better log-likelihood values and lower information criterion than the other competitor models like DUS-GLF. These results suggest that the distribution of GPQF offers a more effective base for the DUS transformation and increases the ability of the proposed framework to model. Thus, the DUS- GPQF distribution is not just another DUS-based extension, but a more versatile and less parsimonious lifetime model of complex reliability data.

The methodological intuition of the model construction is also worth having a brief overview before the formal mathematical formulation of the proposed distribution is presented. The essence is to integrate a flexible baseline lifetime distribution with a transformation mechanism to improve its capacity to model complicated hazards rate patterns. The GPQF distribution is used as a base model in this work, and the DUS transformation gives an extra degree of flexibility without adding extra parameters. The next part explains the methodological framework to be employed to build and test the proposed DUS-GPQF model.

2. Methodology

The methodology framework that has been used to assess the proposed model is shown in Figure 1, which shows the steps involved. The analysis was carried out in a systematic manner guided by this framework and the equations as provided in the computational analysis subsection.

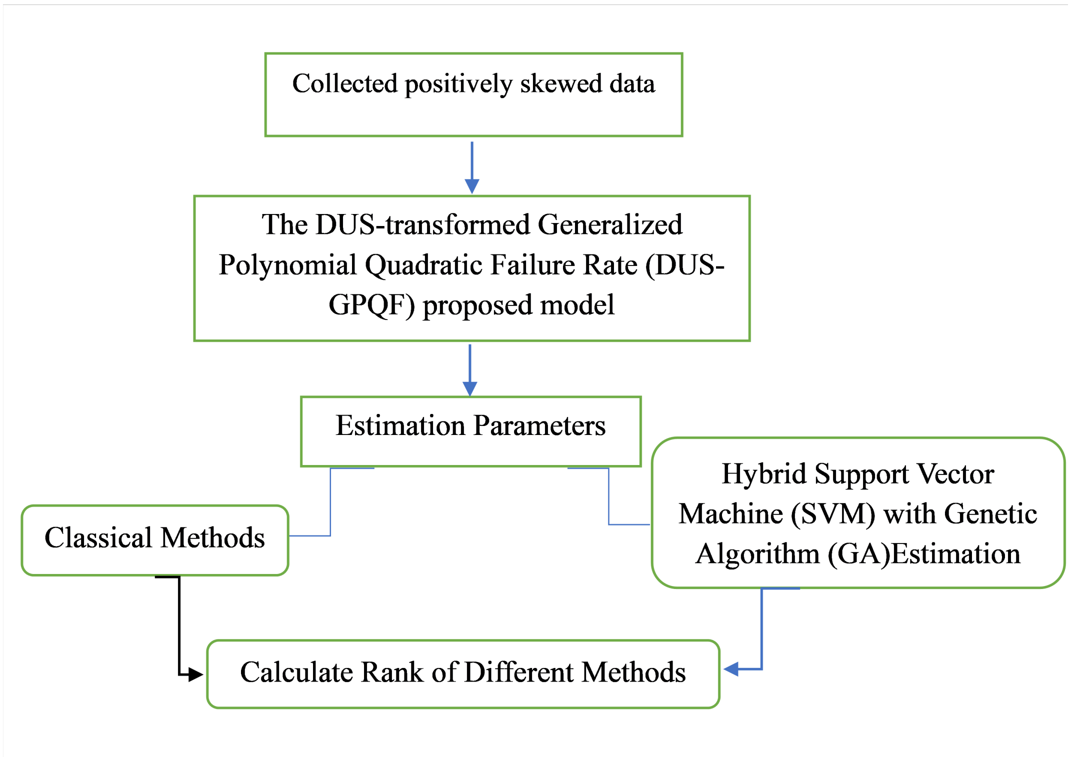


Figure 1. Workflow of Current Research

It is always useful to discuss briefly about what is behind the transformation before giving the formal expressions of the probability density function and cumulative distribution function. Its implementation of the lifetime data in the form of a quadratic hazard rate and its polynomial formulation of the hazard rate already gives the GPQF distribution a flexible structure to model lifetime data. Nevertheless, some more complicated data patterns can still be in need of further flexibility. The DUS transformation changes the shape of the baseline distribution so that it changes shape but not the number of parameters. The derivation below illustrates how DUS transformation is used on the GPQF distribution to form the designed DUS-GPQF model.

2.1. Notation and Symbol Definitions

For clarity and consistency, the main symbols used throughout the manuscript are summarized in Table 1.

Table 1. Notation used in the DUS-GPQF model

Symbol	**Description**
$\mathbf{X}$	Non-negative random variable representing lifetime or failure time
$\mathbf{f}(\mathbf{x})$	Probability density function (PDF)
$\mathbf{F}(\mathbf{x})$	Cumulative distribution function (CDF)
$\mathbf{S}(\mathbf{x}) = 1 - \mathbf{F}(\mathbf{x})$	Survival function
$\mathbf{h}(\mathbf{x})$	Hazard rate function
α, β, γ	Shape and scale parameters of the DUS-GPQF distribution
$\theta = (\alpha, \beta, \gamma)$	Vector of unknown model parameters
$\mathbf{n}$	Sample size
$\mathbf{x}_{(1)}, \mathbf{x}_{(2)}, \ldots, \mathbf{x}_{(n)}$	Ordered sample observations
$\mathbf{L}(\theta)$	Likelihood function
$\updownarrow(\theta)$	Log-likelihood function
$\mathbf{NLL}(\theta)$	Negative log-likelihood function
$\widehat{\theta}$	Estimator of parameter vector θ
Bias	Difference between estimated and true parameter values
MSE	Mean squared error of an estimator
MRE	Mean relative error
ASAE	Average squared absolute error

2.2. The DUS- GPQF (Proposed model)

In this part, the Generalized Polynomial Quadratic Failure Rate (GPQF) distribution is transformed using the Dinesh-Umesh-Sanjay (DUS) transformation. DUS transformation is a methodical way of obtaining parsimonious families of probability distributions by increasing their flexibility without increasing the dimensionality of the parameters. The cumulative distribution function of the DUS-model is as follows:

$$F\left(y\right) = \frac{\exp\left[G\left(y;\varepsilon\right)\right] - 1}{e - 1}; y > 0. \tag{5}$$

The relevant pdf is provided as:

$$f\left(y\right) = \frac{g\left(y;\varepsilon\right)\exp\left[G\left(y;\varepsilon\right)\right]}{e - 1}. \tag{6}$$

By substituting Equation 3 and Equation 4 into Equation 5 and Equation 6, the cumulative distribution function (CDF) and probability density function (PDF) of the DUS-GPQF distribution are obtained as follows:

$$F\left(y;\alpha,\beta,\delta,\gamma\right) = \frac{\exp\left[\left[1 - \exp\left(-\alpha y - \frac{\beta}{2}y^2 - \frac{\delta}{3}y^3\right)\right]^{\gamma}\right] - 1}{e - 1}; y > 0 . \alpha,\beta,\gamma,\delta > 0. \tag{7}$$

And

$$f\left(y;\alpha,\beta,\delta,\gamma\right)=\frac{\gamma\left(\alpha+\beta y+\delta y^{2}\right)}{\mathrm{e}-1}\left(1-\exp\left(-\alpha y-\frac{\beta}{2}y^{2}-\frac{\delta}{3}y^{3}\right)\right)^{\gamma-1}*$$

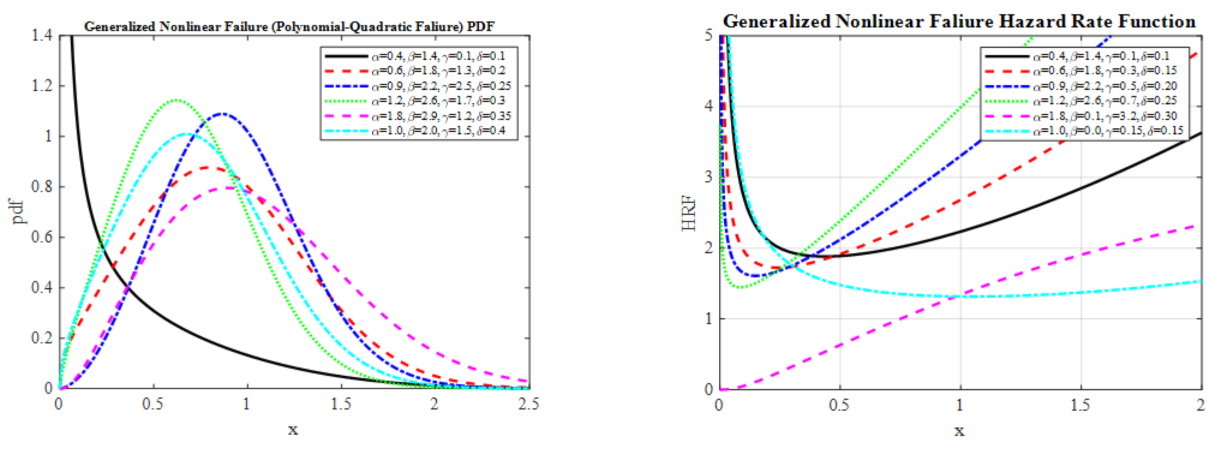
$$\exp\left[\left(1-\exp\left(-\alpha y-\frac{\beta}{2}y^{2}-\frac{\delta}{3}y^{3}\right)\right)^{\gamma}-\alpha y-\frac{\beta}{2}y^{2}-\frac{\delta}{3}y^{3}\right] \tag{8}$$

Figure 2 represents the probability density function (PDF) of DUS-GPQF distribution. The plot shows that the suggested distribution is a right-skewed, unimodal, and monotonically decreasing distribution. The nature of the characteristics emphasizes the flexibility of the DUS-GPQF distribution as it has shown to be applicable in the modeling of a wide range of data patterns, specifically that of asymmetry and peaked data behavior. Due to this flexibility, the DUS-GPQF has been useful in analytical relevance in reliability engineering, survival analysis, and several other areas, which are in need of strong and flexible statistical modeling tools. The hazard rate functional is given by:

$$r\left(y;\alpha,\beta,\delta,\gamma\right)=\frac{\left(\alpha+\beta y+\delta y^{2}\right)\gamma\left(1-\exp\left(-\alpha y-\frac{\beta}{2}y^{2}-\frac{\delta}{3}y^{3}\right)\right)^{\gamma-1}}{\mathrm{e}-\exp\left[\left(1-\exp\left(-\alpha y-\frac{\beta}{2}y^{2}-\frac{\delta}{3}y^{3}\right)\right)^{\gamma}\right]}*$$

$$\exp\left[\left(1-\exp\left(-\alpha y-\frac{\beta}{2}y^{2}-\frac{\delta}{3}y^{3}\right)\right)^{\gamma}-\alpha y-\frac{\beta}{2}y^{2}-\frac{\delta}{3}y^{3}\right]. \tag{9}$$

Figure 2 shows the hazard rate of the DUS-GPQF distribution. The proposed model can have increased, decreasing and bathtub behaviors of hazard rates as seen in the plot. This generality highlights the distribution capability of being able to handle a large variety of reliability behaviors that are normally experienced in practice. The fact that the DUS-GPQF model can capture a wide range of hazard rate structures makes it analytically flexible and implies a strong capability to characterize complexity in the modelling of complex lifetime data in both reliability and survival analysis.



(a) (b)

Figure 2. Probability density function (PDF) and hazard rate function (HRF) of the DUS-GPQF distribution for selected parameter values

2.3. Series representation

This subsection expands the DUS-GPQF distribution by adding an exponential series bringing about the increased structural flexibility and modeling: $e^y = \sum_{k=0}^{\infty} y^k/k!$. Therefore, Equation 10 can be stated as:

$$f\left(y;\alpha,\beta,\delta,\gamma\right) = \frac{\gamma\left(\alpha+\beta y+\delta y^2\right)}{\mathrm{e}-1}\exp\left[-\alpha y-\frac{\beta}{2}y^2-\frac{\delta}{3}y^3\right]*$$

$$\sum_{j=1}^{\infty}\frac{1}{k!}\left(1-\exp\left(-\alpha y-\frac{\beta}{2}y^2-\frac{\delta}{3}y^3\right)\right)^{(k+1)\gamma-1}. \quad (10)$$

To the effect of any real number φ , where φ is a non-negative integer, the binomial series expansion is defined as follows:

$$(1-\sigma)^{\varphi-1} = \sum_{n=0}^{\infty}(-1)^n\binom{\varphi-1}{n}\sigma^n;\ |\sigma|<1. \quad (11)$$

Substituting Equation 11 into Equation 10 yields:

$$f\left(y;\alpha,\beta,\delta,\gamma\right) = \sum_{k,n=0}^{\infty}\frac{(-1)^n\gamma}{(\mathrm{e}-1)\,k!}\binom{(k+1)\,\gamma-1}{n}\left(\alpha+\beta y+\delta y^2\right)\mathrm{e}^{-\alpha(n+1)y-\frac{\beta}{2}(n+1)y^2-\frac{\delta}{3}(n+1)y^3},$$

once again, by employing the exponential series expansion:

$$\exp\left[-\frac{\beta}{2}\left(n+1\right)y^2\right] = \sum_{i=0}^{\infty}(-1)^i\frac{\beta^i(n+1)^i}{2^i i!}y^{2i},$$

$$\exp\left[-\frac{\delta}{3}\left(n+1\right)y^3\right] = \sum_{i=0}^{\infty}(-1)^i\frac{\delta^i(n+1)^i}{3^i i!}y^{3i},$$

Consequently, the probability density function (pdf) of the DUS–GPQF distribution can be written as follows:

$$f\left(y;\alpha,\beta,\delta,\gamma\right) = \sum_{k,n,i=0}^{\infty}\pi_{k,n,i}\left(\alpha y^{2i}+\beta y^{2i+1}+\delta y^{3i+1}\right)\mathrm{e}^{-\alpha(n+1)y}, \quad (12)$$

where,

$$\pi_{k,n,i} = \frac{(-1)^{n+i}\gamma\beta^i(n+1)^i\gamma\delta^i(n+1)^i}{2^i\,(\mathrm{e}-1)\,k!i!}\binom{(k+1)\,\gamma-1}{n}.$$

2.4. Some Statistical properties of the DUS-GPQF

This part of the paper discusses some of the important statistical characteristics of the DUS-GPQF distribution such as the quantile function, median, moments, moment-generating function, and conditional moments.

2.4.1. ***Moments*** Moments are key in defining the structure, dispersion and the general behavior of a probability distribution. They give fundamental information on the shape and inherent characteristics of the distribution. The initial point of the origin, or the mean μ_1, explains the central tendency of the distribution. The second central tendency, the variance $\left[\sigma_y^2 = \mu_2-(\mu_1)^2\right]$, quantifies the degree of dispersion around the mean. The third standardized moment, skewness $\left[\eta_1 = \left(\mu_3-3\mu_1\mu_2+2\mu_1^3\right)/\sigma_y^3\right]$, measures the asymmetry of the distribution. The fourth standardized moment, kurtosis $[\eta_2 = (\mu_4-4\mu_1\mu_3+6\mu_1^2\mu_2-3\mu_1^4)/\sigma_y^4]$, assesses the heaviness of the tails and the peaked Ness relative to the normal distribution. Additionally, the coefficient of variation $[CV = \sigma_y/\mu_y]$, is a scale-free relative dispersion measure. In the case of the DUS-GPQF distribution, the origin moment of the r_{th} order of the distribution is calculated as follows:

$$\mu_r = \mathrm{E}\left[Y^r\right] = \int_0^\infty y^r f\left(y;\alpha,\beta,\delta,\gamma\right)dy$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\alpha \int_0^\infty y^{\mathrm{r}+2i} \mathrm{e}^{-\alpha(n+1)y}dy + \beta \int_0^\infty y^{r+2l+1} \mathrm{e}^{-\alpha(n+1)y}dy + \delta \int_0^\infty y^{\mathrm{r}+3i+1} \mathrm{e}^{-\alpha(n+1)y}\right)$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\alpha \frac{\Gamma\left(r+2i+1\right)}{\left[\alpha\left(n+1\right)\right]^{r+2i+1}} + \beta \frac{\Gamma\left(r+2i+2\right)}{\left[\alpha\left(n+1\right)\right]^{r+2i+2}} + \delta \frac{\Gamma\left(r+3i+2\right)}{\left[\alpha\left(n+1\right)\right]^{r+3i+2}}\right)$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\frac{\alpha^2\Gamma\left(r+2i+1\right)+\beta\Gamma\left(r+2i+2\right)+\delta\Gamma\left(r+3i+2\right)}{\left[\alpha\left(n+1\right)\right]^{r+3i+2}}\right); r = 1, 2, 3, \ldots. \tag{13}$$

2.4.2. ***Inverse Moments*** Let Y be a non-negative random variable following the DUS–GPQF distribution. The r_{th} inverse moment (for any real r) is given by the following expression.

$$\mu_{-r} = \mathrm{E}\left[Y^{-r}\right] = \int_0^\infty y^{-r} f\left(y;\alpha,\beta,\delta,\gamma\right)dy$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\alpha \int_0^\infty y^{-\mathrm{r}+2i} \mathrm{e}^{-\alpha(n+1)y}dy + \beta \int_0^\infty y^{-r+2i+1} \mathrm{e}^{-\alpha(n+1)y}dy + \delta \int_0^\infty y^{-\mathrm{r}+3i+1} \mathrm{e}^{-\alpha(n+1)y}\right)$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\alpha \frac{\Gamma\left(-r+2i+1\right)}{\left[\alpha\left(n+1\right)\right]^{-r+2i+1}} + \beta \frac{\Gamma\left(-r+2i+2\right)}{\left[\alpha\left(n+1\right)\right]^{-r+2i+2}} + \delta \frac{\Gamma\left(-r+3i+2\right)}{\left[\alpha\left(n+1\right)\right]^{-r+3i+2}}\right)$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\frac{\alpha^2\Gamma\left(-r+2i+1\right)+\beta\Gamma\left(-r+2i+2\right)+\delta\Gamma\left(-r+3i+2\right)}{\left[\alpha\left(n+1\right)\right]^{-r+3i+2}}\right); r < 3i+2.$$

Setting $r = 1$ the harmonic mean of the DUS–GPQF distribution is obtained as shown below:

$$HM = \sum_{k,n,i=0}^{\infty} \pi_{k,n,i} \frac{\left[\left(2i\right)\beta + \left(3\mathrm{i}+1\right)\delta + \left(n+1\right)\alpha^2\right]}{\left[\left(n+1\right)\alpha\right]^{2i+1}} \Gamma\left(2i\right). \tag{14}$$

The HM is computed as the inverse of the average of the reciprocals in a dataset. Because it emphasizes smaller observations more heavily than larger ones, it offers a balanced measure, especially valuable in contexts involving rates of change.

2.4.3. ***Moment generating function***

$$m_Y\left(t\right) = \int_0^\infty e^{ty} f\left(y;\alpha,\beta,\delta,\gamma\right)dy$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\alpha \int_0^\infty y^{2i} \mathrm{e}^{[t-\alpha(n+1)]y}dy + \beta \int_0^\infty y^{2i+1} \mathrm{e}^{[t-\alpha(n+1)]y}dy + \delta \int_0^\infty y^{3i+1} \mathrm{e}^{[t-\alpha(n+1)]y}\right).$$

By solving this integration, then

$$m_X\left(t\right) = \sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\frac{\alpha\left[\alpha\left(n+1\right)-t\right] + \beta\left[2i+1\right] + \delta\left[3i+1\right]}{\left[\alpha\left(n+1\right)-t\right]^{3i+2}} \Gamma\left(2i+1\right)\right); t < \alpha\left(n+1\right). \tag{15}$$

2.4.4. ***Incomplete moments*** Incomplete moments (INCMs) are significant statistical tools that present useful information on the nature and behavior of probability distributions. Specifically, the S_{th} lower incomplete moment of a random variable Y of the DUS-GPQF distribution is obtained by the following way.

$$\tau_s(t) = E[Y^s \,|Y < t] = \int_0^t y^s f(y; \alpha, \beta, \delta, \gamma)dy$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i} \left(\alpha \int_0^t y^{s+2i} \mathrm{e}^{-\alpha(n+1)y} dy + \beta \int_0^t y^{s+2i+1} \mathrm{e}^{-\alpha(n+1)y} dy + \delta \int_0^t y^{s+3i+1} \mathrm{e}^{-\alpha(n+1)y} \right)$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i} \left(\alpha \frac{\theta\,(s+2i+1, \alpha\,(n+1)\,\mathrm{t})}{[\alpha\,(n+1)]^{s+2i+1}} + \beta \frac{\theta\,(s+2i+2, \alpha\,(n+1)\,\mathrm{t})}{[\alpha\,(n+1)]^{s+2i+2}} + \delta \frac{\theta\,(\mathrm{s}+3i+2, \alpha\,(n+1)\,\mathrm{t})}{[\alpha\,(n+1)]^{s+3i+2}} \right)$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i} \left(\frac{\alpha^2\,(n+1)\,\theta\,(s+2i+1, \alpha\,(n+1)\,\mathrm{t}) + \beta\theta\,(s+2i+2, \alpha\,(n+1)\,\mathrm{t}) + \delta\theta\,(\mathrm{s}+3i+2, \alpha\,(n+1)\,\mathrm{t})}{[\alpha\,(n+1)]^{s+3i+2}} \right) \tag{16}$$

where $s = 1, 2, 3, \ldots, \&\mu\,(u, \theta) = \int_0^\theta y^{u-1} \mathrm{e}^{-y} dy$ denotes the lower incomplete gamma function. Set $s = 1$ in Equation 16 to find the first lower INCM:

$$\tau_1\,(t) = \sum_{k,n,i=0}^{\infty} \pi_{k,n,i} \left(\frac{\alpha^2\,(n+1)\,\theta\,(2i+2, \alpha\,(n+1)\,\mathrm{t}) + \beta\theta\,(2i+3, \alpha\,(n+1)\,\mathrm{t}) + \delta\theta\,(3i+3, \alpha\,(n+1)\,\mathrm{t})}{[\alpha\,(n+1)]^{3i+3}} \right). \tag{17}$$

In a similar manner, the S_{th} order upper incomplete moment associated with the DUS-GPQF distribution is expressed as follows:

$$\lambda_s\,(t) = \mathrm{E}\,[Y^s \mid Y > t] = \int_t^\infty y^s h\,(y; \alpha, \beta, \delta, \gamma)\,dy$$

$$= \sum_{k,n,i=0}^{\infty} \pi_{k,n,i} \left(\frac{\alpha^2\,(n+1)\,\Gamma\,(s+2i+1, \alpha\,(n+1)\,\mathrm{t}) + \beta\Gamma\,(s+2i+2, \alpha\,(n+1)\,\mathrm{t}) + \delta\Gamma\,(\mathrm{s}+3i+2, \alpha\,(n+1)\,\mathrm{t})}{[\alpha\,(n+1)]^{3i+3}} \right) \tag{18}$$

where $s = 1, 2, 3, \ldots, \&\Gamma\,(u, \theta) = \int_0^\theta y^{u-1} \mathrm{e}^{-y} dy$ denotes the upper incomplete gamma function. Set $s = 1$ in Equation 18 to find the first upper INCM:

$$\lambda_1\,(t) = \sum_{k,n,i=0}^{\infty} \pi_{k,n,i} \left(\frac{\alpha^2\,(n+1)\,\Gamma\,(2i+2, \alpha\,(n+1)\,\mathrm{t}) + \beta\Gamma\,(2i+3, \alpha\,(n+1)\,\mathrm{t}) + \delta\Gamma\,(3i+3, \alpha\,(n+1)\,\mathrm{t})}{[\alpha\,(n+1)]^{3i+3}} \right) \tag{19}$$

The average deviation or mean deviation is used to measure the variance of individual values about some measure of central tendency, which is commonly the mean or the median. Due to its dependence on absolute deviations, it is not as sensitive to extreme values as the measures of variance. In the case of a random variable Y that has the GPQF distribution, the deviations about the mean and the median, respectively, are determined as follows:

$$u_1 = \mathrm{E}\,[|Y - \mu_y|] = \int_0^\infty |Y - \mu_y|\, f\,(y; \alpha, \beta, \delta, \gamma)\,dy = 2\mu_y H\,(\mu_y) - 2\tau_1\,(\mu_y)\,.$$

and

$$u_2 = \mathrm{E}\,[|Y - m_d|] = \int_0^\infty |Y - m_d|\, f\,(y; \alpha, \beta, \delta, \gamma)\,dy = \mu_y - 2\tau_1\,(m_d)\,.$$

Using Equation 17 the quantity $\tau_1(\cdot)$ can be obtained, while the entropy measure $H(\mu_y)$ is determined from Equation 8. Moreover, Equation 17 can be employed to derive the Bonferroni g_1 and Lorenz g_2 curves. These curves depict the cumulative proportion of total income or wealth accounted for by specified percentages of the population and provide a useful framework for comparing income distributions across different groups. The g_1 and g_2 curves are specified by $g_1(p) = \tau(q)/p \times \mu_y$ and $g_2(p) = \tau(q)/\mu_y$, where $p = H(y), q = H^{-1}(p) = inf\{y : H(y) \geq p\}$.

2.4.5. Conditional Moments Conditional moments (CMs) are a significant type of statistics measures, which are acquired by assessing the moments of a random variable under specified conditions. Such instances provide a better insight of the stochastic behavior of the variable in particular situations and allow performing the analysis of correlation between the variables and the underlying distributions. In this regard, the conditional moments of a random variable of DUS-GPQF distribution are determined as follows:

$$\mathrm{E}\left(Y^s \mid Y > t\right) = \frac{\lambda_s(t)}{1 - F(t;\alpha,\beta,\delta,\gamma)}$$

$$= \frac{\mathrm{e}-1}{\mathrm{e} - \exp\left[\left[1 - \exp\left(-\alpha y - \frac{\beta}{2}y^2 - \frac{\delta}{3}y^3\right)\right]^\gamma\right]} *$$

$$\sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\frac{\alpha^2(n+1)\Gamma(s+2i+1,\alpha(n+1)\mathrm{t}) + \beta\Gamma(s+2i+2,\alpha(n+1)\mathrm{t}) + \delta\Gamma(\mathrm{s}+3i+2,\alpha(n+1)\mathrm{t})}{[\alpha(n+1)]^{3i+3}}\right). \tag{20}$$

2.4.6. ***Mean residual life & Mean inactivity time*** Mean residual life (MRL) is one of the basic reliability metrics that are used to estimate the remainder life of a system or component based on the fact that it has survived to a given time t [20]. Due to its readability and application in practice the MRL is widely used in health sciences, economics and technology applications to determine the effects that a number of factors have on the useful life of products and systems. Where the random variable under consideration is nonnegative. The MRL at time t are denoted by Y and are:

$$\underset{y}{\chi}(t) = \mathrm{E}\left[Y - tY > t\right] = \frac{\lambda_1(t)}{1 - F(t;\alpha,\beta,\delta,\gamma)} - 1, \tag{21}$$

utilizing Equation 7 and Equation 20 yields:

$$\underset{y}{\chi}(t) = \frac{\mathrm{e}-1}{\mathrm{e} - \exp\left[\left[1 - \exp\left(-\alpha y - \frac{\beta}{2}y^2 - \frac{\delta}{3}y^3\right)\right]^\gamma\right]} *$$

$$\sum_{k,n,i=0}^{\infty} \pi_{k,n,i}\left(\frac{\alpha^2(n+1)\Gamma(s+2i+1,\alpha(n+1)\mathrm{t}) + \beta\Gamma(s+2i+2,\alpha(n+1)\mathrm{t}) + \delta\Gamma(\mathrm{s}+3i+2,\alpha(n+1)\mathrm{t})}{[\alpha(n+1)]^{3i+3}}\right) - 1. \tag{22}$$

Mean inactivity time (MIT), or mean past lifetime is a reliability measure that describes the guess of how long it would take to fail, given that one observes at a particular time. It is an average time between the occurrence of a failure of component or system and the time that a corresponding remedial action such as repair, replacement, or maintenance, is initiated. As such, the average duration of inactivity of a random variable under the DUS-GPQF distribution is:

$$\underset{y}{\rho}(t) = \mathrm{E}[t - Y \mid Y < t] = t - \frac{\tau_1(t)}{F(t;\alpha,\beta,\delta,\gamma)} \tag{23}$$

utilizing Equation 7 and Equation 17 yields:

$$\underset{y}{\rho}(t) = t - \frac{\mathrm{e}-1}{\exp\left[\left[1 - \exp\left(-\alpha y - \frac{\beta}{2}y^2 - \frac{\delta}{3}y^3\right)\right]^\gamma\right] - 1} *$$

$$\sum_{k,n,i=0}^{\infty} \pi_{k,n,i} \left(\frac{\alpha^2 (n+1) \theta (2i+2, \alpha (n+1) \mathrm{t}) + \beta\theta (2i+3, \alpha (n+1) \mathrm{t}) + \delta\theta (3i+3, \alpha (n+1) \mathrm{t})}{[\alpha (n+1)]^{3i+3}} \right). \tag{24}$$

3. Genetic Algorithm Applications

The initial form of stochastic, population-based variants of optimization algorithms called Genetic Algorithms (GAs) was coined by [16] and is based on Darwinian principles of natural selection and evolution. Their basic concept is to successively mutate a population of candidate solutions to increasingly fitter in terms of some objective function. GA due to their strength and versatility have been extensively applied to science, engineering, and data-driven modeling, especially nonlinear, multimodal and non-analytically graded problems [17, 18, 21].

3.1. Mathematical Formulation

Let the optimization problem be defined as:

$$\underset{x \in R^d}{\mathrm{Min}}\, f(x) \tag{25}$$

$f(x)$ s the objective (fitness) function, and $x = [x_1, x_2, \ldots, x_d]$ is a decision vector, d-dimensional. GA evolves a population $P^{(t)} = \left\{ x_1^{(t)}, x_2^{(t)}, \ldots, x_N^{(t)} \right\}$ of N individuals at generation t, to which a candidate solution corresponds. The overall evolution process is composed of three main genetic operators, selection, crossover and mutation.

Selection: The selection operator biases preferentially those individuals who are more fit to transmit genetic material to the following generation. Assuming that f_i is the fitness of individual i its selection probability is:

$$p_i = \frac{f_i}{\sum_{j=1}^{N} f_j}. \tag{26}$$

Crossover: Crossover (also recombination) is the combination of two parent solutions with the goal of producing offspring and so purposes the exploration of the search space. Given that x_q and x_p , are two parents, arithmetic crossover may be defined as follows:

$$x_{\mathrm{child}} = \alpha x_p + (1 - \alpha) x_q, \tag{27}$$

where $\alpha \in [0, 1]$ is a random blending coefficient. Other types of crossovers are a one point, two point and uniform crossover schemes [19].

Mutation: The mutation operator applies random variations to the individual parameters in order to keep the diversity and avoid early convergence. One of the formulations of real-valued GAs is:

$$x_i^{(t+1)} = x_i^{(t)} + \sigma \mathcal{N}(0, 1), \tag{28}$$

where σ controls the mutation rate and $\mathcal{N}(0, 1)$ represents a standard normal random variable. The iterative GA process can be summarized as:

$$P^{(t+1)} = \mathrm{Mutation}\left(\mathrm{Crossover}\left(\mathrm{Selection}\left(P^{(t)}\right)\right)\right). \tag{29}$$

The process of evolutionary change proceeds in this manner until some convergence criterion, e.g. a maximal number of generations or a threshold of fitness, is achieved.

4. Support Vector Machine (SVM)

In this part, we provide a brief description of the Support Vector Machine (SVM) algorithm as well as its theoretical background. To be treated holistically the reader is sent [13, 14, 22]. The binary classification problem is considered to have a dataset, $\{(a_i, b_i)\}_{i=1}^{l}$, with class labels $b_i \in \{-1, +1\}$. SVM aims at identifying a linear classifier with the minimal overall generalization error, based on the Structural Risk Minimization (SRM) principle [13, 23]. The best separating hyperplane is the one which maximizes the margin- the perpendicular distance between the hyperplane and the closest data points, which are called support vectors [15, 24]. This margin maximization improves not only the accuracy of the classification, but has also a better generalization capacity of the model, which is in accordance with the statistical learning theory. In case the data is not linearly separable, SVM proposes a soft-margin model that simultaneously maximizes the margin and minimizes the classification error. It is done by optimizing a Quadratic Programming (QP) problem with a regularization constant $C > 0$ that balances between these conflicting aims. The decision function is of the form [25]:

$$f(y) = \text{sign}\left(\sum_{i=1}^{l} \beta_i a_i y_i^T y + b\right), \tag{30}$$

and β_i are the linear classifier coefficients. In order to generalize SVMs to nonlinear issues, the input vectors $y \in R^d$, are transformed into a larger-dimensional feature space by a nonlinear transformation.

$$z(y) = (\phi_1(y), \ldots, \phi_n(y)) \in R^n, \tag{31}$$

that makes it possible to perform linear separation in the transformed space [13], showed that this mapping could be trained implicitly using kernel functions $K(y, y_i)$, which compute inner products in the feature space without actually performing the transformation-a phenomenon called the kernel trick. The nonlinear decision function is thus given as:

$$f(y) = \text{sign}\left(\sum_{i=1}^{l} \beta_i a_i K(y, y_i) + b\right), \tag{32}$$

in which $K(.,.)$ is a symmetric, positive-definite kernel that characterizes the measure of similarity among samples. Practically, the choice of the parameters of the kernel and the regularization constant C are very crucial in the performance of SVMs. genetic Algorithms (GA), which is an evolutionary optimization technique based on a population, has demonstrated great success in tuning these hyperparameters. The hybrid SVM-GA model allows combining SVM-GA framework to gain the global search of GA and the powerful classification of SVM. GA optimizes sets of the candidate parameters (e.g. C, kernel width, or shape parameters) to minimize classification error or to maximize model accuracy, so that it avoids local minima and the convergence to good model configurations. Such hybridization provides adaptive choice of models and also improves predictive behavior particularly when the data environment is complex, nonlinear or high-dimensional.

5. Statistical Rationale for the SVM–GA Likelihood Approximation

The probability distribution of the suggested DUS-GPQF is very nonlinear because of the combination of the quadratic and the failure rate of the proposed model which is a combination of polynomials and quadrants. This can cause the likelihood surface to have a complicated curvature as well as more than one local optimum, meaning that classical gradient-based optimization strategies are dependent on starting points and can get stuck at local minima. In order to tackle this issue, a hybrid Support Vector Machine-Genetic Algorithm (SVM-GA) framework is used. Support Vector Regression (SVR) is applied to approximate the negative log-likelihood (NLL) surface and serves as a surrogate model that gives a smooth approximation of the likelihood surface, and the computational cost of likelihood evaluation is also lower. This is followed by optimization of the surrogate NLL surface via Genetic Algorithm (GA) that has a global search of the surface via the evolutionary operators that include selection, crossover, and mutation. As the derivative information is not needed to use GA, it is effectively applied in optimization problems with multimodal objective functions. To increase the accuracy of estimation, a local optimization step based on the Nelder Mead simplex method applied to the actual NLL is applied to the solution of the SVM search based on the SVMGA search. The SVR model uses an RBF kernel, and its hyper parameters (regularization parameter C, kernel width γ, and $\varepsilon-$insensitive loss) are chosen via the cross-validation procedure via the GA optimization process. In general, the proposed SVM–GA framework allows an improved stability in the parameter estimation of the complex likelihood surfaces and provides computational efficiency that results in better optimization results than the classical MLE in the DUS-GPQF model.

After establishing the theoretical properties of the proposed distribution, the next step is to estimate its unknown parameters from observed data. Parameter estimation plays a crucial role in determining how effectively a statistical model can represent real-world phenomena. Because the likelihood surface associated with the DUS-GPQF model may exhibit nonlinear behavior, multiple estimation techniques are considered in order to evaluate estimator robustness and efficiency under different conditions. The following subsection presents the estimation procedures used in this study.

6. Estimation Methods

The discussion of the conventional methods of parameter estimation used in reference to the DUS-GPQF distribution is devoted to this section. These methods are based on the maximization or minimization of given objective function to obtain the correct and statistically significant estimates of the model parameters. The use of similar estimation strategies is a common practice by various authors in the related contexts [26, 27, 28, 29, 30, 31] is representative research in this respect.

6.1. Rationale for Multiple Estimation Approaches

In reliability and lifetime distribution research, one might typically compare multiple parameter estimation methods in order to understand the robustness or efficiency of estimators when they are used in varying mathematical conditions. Because the likelihood surface of the proposed DUS-GPQF distribution is nonlinear and could have many local optima, using a single estimation method can be biased or unsound in estimating the parameters. Because of this reason, a number of classical and alternative estimation methods are taken into account. The seventeen estimation procedures of this work can be further classified into three groups: likelihood-based (such as maximum likelihood estimation), goodness-of-fit based (including Anderson Darling and Cramer von Mises estimators) and spacing-based estimator (such as the maximum product of spacings and other related spacing distance measures). All categories possess varying statistical characteristics. Likelihood-based estimators are asymptotically efficient when the correct model is used, goodness-of-fit estimators focus on having empirical and theoretical distributions that agree, and spacing-based estimators have been shown to be effective with complicated distributions and small samples. The rationale of using more than one estimation method is thus not to imply that any and all estimators are equally desirable but rather to give a complete comparison of the performance of the estimators on the proposed distribution. The simulation study also enables us to compare the behavior of estimators in regards to bias, mean square error and stability at varying sample sizes. The systematic comparison assists in determining the estimation methods that are more predictable when using the DUS-GPQF model in various data conditions.

6.2. Method of Maximum Likelihood (ML)

The ML estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by maximizing the following function:

$$\ell\left(\alpha,\beta,\delta,\gamma\right) = n\ln\gamma - n\ln\left(e-1\right) + \sum_{i=1}^{n}\ln\left(\alpha+\beta y_i+\delta {y_i}^2\right)$$

$$+\left(\gamma-1\right)\sum_{i=1}^{n}\ln\left(1-e^{-\alpha y_i-\frac{\beta}{2}{y_i}^2-\frac{\delta}{3}{y_i}^3}\right)+\sum_{i=1}^{n}\left(1-e^{-\alpha y_i-\frac{\beta}{2}{y_i}^2-\frac{\delta}{3}{y_i}^3}\right)^{\gamma}$$

$$-\sum_{i=1}^{n}\left(\alpha y_i-\frac{\beta}{2}{y_i}^2-\frac{\delta}{3}{y_i}^3\right). \tag{33}$$

6.3. Method of Anderson-Darling (AD)

The AD estimates $\hat{\alpha},\hat{\beta},\hat{\delta},\hat{\gamma}$ are obtained by minimizing the following function:

$$A\left(y_i\right)=-n-\frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\left[\log F\left(y_{i:n}\right)+\log S\left(y_{n-i-1:n}\right)\right]$$

$$=-n-\frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\log\left(\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:n}-\frac{\beta}{2}{y_{i:n}}^2-\frac{\delta}{3}{y_{i:n}}^3\right)\right]^{\gamma}\right]-1}{e-1}\right)$$

$$-\frac{1}{n}\sum_{i=1}^{n}\left(2i-1\right)\log\left(\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:n}-\frac{\beta}{2}{y_{i:n}}^2-\frac{\delta}{3}{y_{i:n}}^3\right)\right]^{\gamma}\right]-1}{e-1}\right). \tag{34}$$

6.4. Method of Cramer Von Mises (CVM)

The AD estimates $\hat{\alpha},\hat{\beta},\hat{\delta},\hat{\gamma}$ are obtained by minimizing the following function:

$$C\left(y_i\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F\left(y_{i:n}\right)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:n}-\frac{\beta}{2}{y_{i:n}}^2-\frac{\delta}{3}{y_{i:n}}^3\right)\right]^{\gamma}\right]-1}{e-1}-\frac{2i-1}{2n}\right]^2. \tag{35}$$

6.5. Method of Maximum Product of Spacings (MPS)

The MPS estimates $\hat{\alpha},\hat{\beta},\hat{\delta},\hat{\gamma}$ are obtained by optimizing the following function:

$$MPS=\frac{1}{m+1}\sum_{j=1}^{m+1}\log I_j\left(y_j\right)$$

where $I_j\left(y_j\right)=F\left(y_{i:m}\right)-F\left(y_{i-1:m}\right)$

$$=\left(\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:m}-\frac{\beta}{2}{y_{i:m}}^2-\frac{\delta}{3}{y_{i:m}}^3\right)\right]^{\gamma}\right]-1}{e-1}\right)$$

$$-\left(\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i-1:m}-\frac{\beta}{2}{y_{i-1:m}}^2-\frac{\delta}{3}{y_{i-1:m}}^3\right)\right]^{\gamma}\right]-1}{e-1}\right) \tag{36}$$

such that $F\left(y_{0:m}\right)=0$,$F\left(y_{n+1:n}\right)=1$.

6.6. Method of Least Squares (LS)

The LS estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$u(y_i) = \sum_{i=1}^{m} \left[F(y_{i:n}) - \frac{i}{m+1} \right]^2$$

$$= \sum_{i=1}^{n} \left[\left(\frac{\exp\left[\left[1 - \exp\left(-\alpha y_{i:m} - \frac{\beta}{2} {y_{i:m}}^2 - \frac{\delta}{3} {y_{i:m}}^3\right)\right]^\gamma\right] - 1}{e-1} \right) - \frac{i}{m+1} \right]. \quad (37)$$

6.7. Method of Percentile (P)

The P estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$PCE = \sum_{i=1}^{m} \left[y_{i:m} - Q(p_i)\right]^2, p_i = \frac{i}{m+1}.$$

6.8. Method of Right Tail Anderson Darling (RTAD)

The RTAD estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$R(y_i) = \frac{m}{2} - 2\sum_{i=1}^{m} F(y_{i:m}) - \frac{1}{m} \sum_{i=1}^{m} (2i-1) \log S(y_{i:m})$$

$$= \frac{m}{2} - 2\left(\frac{\exp\left[\left[1 - \exp\left(-\alpha y_{i:m} - \frac{\beta}{2} {y_{i:m}}^2 - \frac{\delta}{3} {y_{i:m}}^3\right)\right]^\gamma\right] - 1}{e-1} \right)$$

$$-\frac{1}{m} \sum_{i=1}^{m} (2i-1) \left(1 - \frac{\exp\left[\left[1 - \exp\left(-\alpha y_{i:m} - \frac{\beta}{2} {y_{i:m}}^2 - \frac{\delta}{3} {y_{i:m}}^3\right)\right]^\gamma\right] - 1}{e-1} \right). \quad (38)$$

6.9. Method of Weighted Least Squares (WLS)

The WLS estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$W(y_i) = \sum_{i=1}^{m} \frac{(m+1)^2 (m+2)}{i(m-i+1)} \left[F(y_{i:m}) - \frac{i}{m+1} \right]^2$$

$$= \sum_{i=1}^{m} \frac{(m+1)^2 (m+2)}{i(m-i+1)} \left[\left(\frac{\exp\left[\left[1 - \exp\left(-\alpha y_{i:m} - \frac{\beta}{2} {y_{i:m}}^2 - \frac{\delta}{3} {y_{i:m}}^3\right)\right]^\gamma\right] - 1}{e-1} \right) - \frac{i}{m+1} \right]^2. \quad (39)$$

6.10. Method of Left Tail Anderson Darling (LTAD)

The LTAD estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$L(y_i) \quad = -\tfrac{3}{2} m + 2\sum_{i=1}^{m} F(y_{i:m}) - \tfrac{1}{m} \sum_{i=1}^{m} (2i-1) \log F(y_{i:m})$$

$$= -\frac{3}{2}m + 2\sum_{i=1}^{m}\left(\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:m} - \frac{\beta}{2}{y_{i:m}}^2 - \frac{\delta}{3}{y_{i:m}}^3\right)\right]^{\gamma}\right]-1}{e-1}\right)$$

$$-\frac{1}{m}\sum_{i=1}^{m}(2i-1)\log\left(\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:m} - \frac{\beta}{2}{y_{i:m}}^2 - \frac{\delta}{3}{y_{i:m}}^3\right)\right]^{\gamma}\right]-1}{e-1}\right). \quad (40)$$

6.11. *Method of Minimum Spacing Absolute Distance (MSAD)*

The MSAD estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$\zeta\left(y_i\right) = \sum_{i=1}^{m+1}\left|I_i\left(y_i\right) - \frac{1}{m+1}\right|,$$

where $I_i\left(y_i\right)$ defined in Equation 36.

6.12. *Method of Minimum Spacing Absolute-Log Distance (MSALD)*

The MSALD estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$Y\left(y_i\right) = \sum_{i=1}^{m+1}\left|\log I_i\left(y_i\right) - \log\frac{1}{m+1}\right|,$$

where $I_i\left(y_i\right)$ defined in Equation 36.

6.13. *Method of Anderson Darling Left Tail Second Order (ADLTSO)*

The ADLTSO estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$LTS = 2\sum_{i=1}^{m}\log F\left(y_{i:m}\right) + \frac{1}{n}\sum_{i=1}^{m}\frac{(2i-1)}{F\left(y_{i:m}\right)}$$

$$= 2\sum_{i=1}^{n}\log\left(\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:m} - \frac{\beta}{2}{y_{i:m}}^2 - \frac{\delta}{3}{y_{i:m}}^3\right)\right]^{\gamma}\right]-1}{e-1}\right)$$

$$+\frac{1}{n}\sum_{i=1}^{n}\frac{(2i-1)}{\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:m} - \frac{\beta}{2}{y_{i:m}}^2 - \frac{\delta}{3}{y_{i:m}}^3\right)\right]^{\gamma}\right]-1}{e-1}\Big)}. \quad (41)$$

6.14. *Kolmogorov (K) Method*

The K estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by optimizing the following function:

$$KM = \underset{1\leq i\leq m}{\text{MAX}}\left[\frac{i}{m} - F\left(y_i\right), F\left(y_i\right) - \frac{i-1}{m}\right] \quad (42)$$

where $F\left(y_i\right) = \left(\frac{\exp\left[\left[1-\exp\left(-\alpha y_{i:m} - \frac{\beta}{2}{y_{i:m}}^2 - \frac{\delta}{3}{y_{i:m}}^3\right)\right]^{\gamma}\right]-1}{e-1}\right).$

6.15. Method of Minimum Spacing Square Distance (MSSD)

The MSSD estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$\phi_{(y_i)} = \sum_{i=1}^{m+1} \left(I_i\left(y_i\right) - \frac{1}{m+1} \right)^2,$$

where $I_i\left(y_i\right)$ defined in Equation 36.

6.16. Method of Minimum Spacing Ssquare-Log Distance (MSSLD)

The MSSLD estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$\Psi_{(y_i)} = \sum_{i=1}^{m+1} \left(\log I_i\left(y_i\right) - \log\frac{1}{m+1} \right)^2, \tag{43}$$

where $I_i\left(y_i\right)$ defined in Equation 36.

6.17. Method of Minimum Spacing Linex distance (MSLD)

The MSLD estimates $\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\gamma}$ are obtained by minimizing the following function:

$$\Delta_{(y_i)} = \sum_{i=1}^{m+1} \left[e^{I_i(y_i) - \frac{1}{m+1}} - \left(I_i\left(y_i\right) - \frac{1}{m+1} \right) - 1 \right],$$

where $I_i\left(y_i\right)$ defined in Equation 36.

6.18. Modified SVM–GA Hybrid Estimation Method for the Proposed Model

Algorithm
Input:

- Observed data $\{x_i\}_{i=1}^n$
- Model parameters: $\Theta = (\alpha, \beta, \gamma, \delta)$
- Parameter bounds $\Theta_{\min}, \Theta_{\max}$
- Number of training samples N_s
- Population size N_p, maximum generations $G_{\max}$

Output:

- Parameter estimates $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta})$
- Minimum NLL value
- Information criteria (AIC, BIC, CAIC, HQIC)
- Goodness-of-fit statistics
- Diagnostic plots (PDF, CDF)

Step 1: Define Negative Log-Likelihood (NLL)

$$\mathcal{L}(\Theta) = -\sum_{i=1}^{n} \log h\left(x_i; \alpha, \beta, \gamma, \delta\right)$$

where $h(\cdot)$ is the model PDF.
Step 2: Generate Initial Samples

- Generate N_s random samples $\Theta^{(j)} \in [\Theta_{\min}, \Theta_{\max}]$, $j = 1, 2, \ldots, N_s$

- Evaluate each sample: $y^{(j)} = \mathcal{L}\left(\Theta^{(j)}\right)$
- Form training set $\mathcal{D} = \left\{\left(\Theta^{(j)}, y^{(j)}\right)\right\}$

Step 3: Train Surrogate Model

- Fit a surrogate $\widehat{\mathcal{L}}(\Theta)$ to approximate the true NLL surface using Support Vector Regression (SVR) with a radial basis function (RBF) kernel:

$$\widehat{\mathcal{L}}(\Theta) \approx \mathcal{L}(\Theta)$$

Step 4: Global Search via Genetic Algorithm (GA)

1. Initialize population $\{\Theta_i\}_{i=1}^{N_p}$ within parameter bounds
2. For $g = 1$ to $G_{\max}$:
   - Evaluate fitness using surrogate: $f_i = \widehat{\mathcal{L}}\left(\Theta_i\right)$
   - Apply GA operators:
     - Selection (prefer individuals with lower f_i)
     - Crossover (combine two parents)
     - Mutation (introduce random perturbations)
   - Update population and record best solution Θ^*_{GA}

Step 5: Local Refinement (MLE)

- Refine Θ^*_{GA} by minimizing the true NLL:

$$\Theta^* = \arg\min_{\Theta} \mathcal{L}(\Theta)$$

- Using the Nelder-Mead simplex method
- Obtain final estimates:

$$(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}) = \Theta^*$$

Step 6: Model Evaluation

- Compute log-likelihood: $\log L = -\mathcal{L}(\widehat{\Theta})$
- Calculate information criteria:

$$\begin{aligned}
\text{AIC} &= -2\log L + 2k \\
\text{BIC} &= -2\log L + k\log n \\
\text{CAIC} &= -2\log L + k(\log n + 1) \\
\text{HQIC} &= -2\log L + 2k\log(\log n)
\end{aligned}$$

  where $k = 4$ is the number of parameters

Step 7: Goodness-of-Fit and Visualization

- Compute fitted PDF and CDF:

$$\widehat{h}(x) = h(x; \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}), \quad \widehat{F}(x) = \int_0^x \widehat{h}(t)dt$$

- Compare empirical and fitted distributions using KS, Anderson-Darling, and Cramér-von Mises tests
- Plot empirical vs. model PDF and CDF curves

Although theoretical properties give an understanding of the structure of proposed distribution, practical evaluation should be done to evaluate how well the estimation methods perform in practice. This motivates the use of Monte Carlo simulation to study the finite sample behavior of the estimators in controlled experimental conditions. Specifically, the simulation analysis compares estimators' accuracy, bias, and variability with varying sample sizes and parameter.

7. Numerical simulation

A Monte Carlo simulation study was carried out to assess the performance of the proposed estimation procedures using a number of sample sizes and parameter settings. The sample size in the simulation experiment is set to $n = 20, 70, 150$ because it aims to study the behavior of estimators in small, moderate and large sample regimes. The parameters of the DUS-GPQF distribution were chosen to be representative of varying levels of reliability such as skewed distributions that are not very skewed and that of a shape of hazard rate that is of varying shape. These modes enable the simulation to reproduce a variety of distributional behaviors that are usually seen in reliability and survival data. Random samples of the proposed distribution were obtained by use of the quantile function with each parameter setting and a sample size. The simulation process was repeated R times (e.g. $R = 1000$) to have consistent estimates of the performance measures. The results were reproducible since a fixed random seed was employed. Several measures of statistical performance were used to assess the estimators, they were the absolute bias, mean squared error (MSE), mean relative error (MRE) and average squared absolute error. Such measures enable a fine-grained comparison of estimator accuracy, variance as well as stability of various estimation schemes. The aim of such a simulation design is to evaluate the finite-sample behavior of the rival estimators as well as studying the increase in the statistical properties of the competing estimators with the increase in sample size. The simulation studies are performed to assess the functionality of different parameter estimation methods of the proposed model. The quantile function of the proposed model is applied to generate random samples of a number of the sample sizes ($n = 20, 70, 150$) where y_i are the ascending ordered observations $\tau = (\alpha, \beta, \gamma, \delta)$. The simulation study will find the most efficient methods of estimating the parameters of the proposed model. The model is used to produce random samples of different sizes and to repeat the simulation procedure a number of times in order to evaluate the properties of estimators in terms of their robustness and stability. Table 2 is a summary of the performance of seventeen estimation methods used on the model parameters. It is worth noting that the estimated value of the parameters of the proposed distribution is correct and close to its actual values. In addition, the expected value of any performance measure decreases with an increase in the sample size n. According to these findings, the results indicate that several estimation methods provide consistent parameter estimates, although likelihood-based and spacing-based estimators generally demonstrate superior performance as the sample size increases. To provide sound estimates of the parameters. The monotonic reduction in the all the evaluation criteria as the sample size increases is observed to demonstrate that the estimators have the consistency property. Due to the nonlinearity of estimating equations, which cannot be solved analytically, the behavior of all the estimation methods is studied numerically in the present paper.

The simulation results indicate that estimator accuracy improves as the sample size increases, with likelihood-based and spacing-based estimators generally exhibiting lower bias and mean squared error.Though, simulation studies are helpful to obtain information on estimators' behavior in controlled conditions, it is also significant to investigate the behavior of the proposed model in real-world datasets. The real data analysis will enable us to determine the practical usefulness of the DUS-GPQF distribution, as well as to compare the performance of the presented distribution with the existing competitive models. In the next section, the two real data sets of reliability and materials engineering are applied to the proposed distribution.

8. Real Data Analysis

It will show how the DUS-GPQF model is flexible and practical as it analyses two real-life data sets, and compares its performance with some competing distributions. Model assessment is done based on conventional information criteria AIC, CAIC, BIC, and HQIC. The proposed model is compared to 9 competing distributions, generalized failure rate (GFR) [32], generalized exponential geometric extreme (GEGO) [33], transmuted generalized power Weibull (TGPW) [34], transmuted exponentiated exponential (TEE) [35], generalized Lindley (GL) [36], exponentiated gamma (EG) [37], exponentiated exponential (EE) [38], GDUS-E [39] and DUS-GLF [40]. The

Table 2. Numerical values of simulation measures for $\alpha = 2.5, \beta = 1.5, \gamma = 3.0$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	SVM-GA	MSLNDE
20	$BIAS(\widehat{\alpha})$	$0.68587^{[2]}$	$0.7786^{(3)}$	$0.83419^{(7)}$	$0.93395^{\{10\}}$	$1.0691^{(14)}$	$0.8423^{(8)}$	$0.82803^{[5]}$	$0.95792^{[11]}$	$0.92151^{[9]}$	$0.79358^{[4]}$	$0.98008^{[12]}$	$1.06056^{[13]}$	$0.43884^{\{1\}}$	$1.23377^{[16]}$	$0.82909^{[6]}$	$0.32773^{(1)}$	$1.22757^{[15]}$
	$BIAS(\widehat{\beta})$	$0.72626^{[2]}$	$0.73363^{(3)}$	$0.77927^{\{12\}}$	$0.781^{[14]}$	$0.7772^{\{11\}}$	$0.75742^{\{7\}}$	$0.74961^{(5)}$	$0.75948^{(9)}$	$0.78201^{[16]}$	$0.74323^{[4]}$	$0.76971^{[10]}$	$0.78075^{[13]}$	$0.60645^{\{1\}}$	$0.782^{[15]}$	$0.7543^{[6]}$	$0.50534^{(1)}$	$0.75881^{(8)}$
	$BIAS(\widehat{\gamma})$	$0.97564^{\{2\}}$	$1.05014^{[3]}$	$1.12305^{\{6\}}$	$1.20839^{[12]}$	$1.32759^{[14]}$	$1.15614^{\{9\}}$	$1.15556^{[8]}$	$1.20362^{[10]}$	$1.15149^{[7]}$	$1.09964^{[5]}$	$1.20723^{[11]}$	$1.23524^{[13]}$	$0.6411^{[1]}$	$1.56698^{[16]}$	$1.06843^{(4)}$	$0.5300^{(1)}$	$1.55628^{[15]}$
	$MSE(\widehat{\alpha})$	$0.72391^{\{2\}}$	$0.99312^{(3)}$	$1.12591^{\{7\}}$	$1.37326^{[10]}$	$1.72488^{[14]}$	$1.16508^{(8)}$	$1.10973^{[6]}$	$1.46532^{[11]}$	$1.35147^{[9]}$	$1.04499^{[4]}$	$1.52257^{[12]}$	$1.72426^{[13]}$	$0.35252^{\{1\}}$	$2.10616^{[16]}$	$1.1035^{[5]}$	$0.24141^{(1)}$	$2.09172^{[15]}$
	$MSE(\widehat{\beta})$	$0.71165^{\{2\}}$	$0.71993^{(3)}$	$0.78802^{\{11\}}$	$0.79843^{[14]}$	$0.79314^{[12]}$	$0.7629^{(7)}$	$0.74213^{\{5\}}$	$0.75619^{(6)}$	$0.8089^{\{16\}}$	$0.73539^{[4]}$	$0.78554^{[10]}$	$0.79538^{[13]}$	$0.55052^{\{1\}}$	$0.80115^{[15]}$	$0.76427^{(8)}$	$0.44041^{(1)}$	$0.76582^{[9]}$
	$MSE(\widehat{\gamma})$	$1.43672^{[2]}$	$1.57979^{(3)}$	$1.77147^{\{6\}}$	$1.91165^{\{10\}}$	$2.21878^{[14]}$	$1.79923^{\{7\}}$	$1.82156^{[8]}$	$1.93049^{[11]}$	$1.82424^{[9]}$	$1.66766^{[5]}$	$1.98291^{[12]}$	$2.00258^{[13]}$	$0.63104^{\{2\}}$	$2.88286^{[16]}$	$1.59417^{[4]}$	$0.62013^{(1)}$	$2.86007^{[15]}$
	$MRE(\widehat{\alpha})$	$0.27435^{\{2\}}$	$0.31146^{(3)}$	$0.33368^{(7)}$	$0.37358^{\{10\}}$	$0.42764^{[14]}$	$0.33692^{[8]}$	$0.33121^{(5)}$	$0.38317^{[11]}$	$0.3686^{\{9\}}$	$0.31743^{[4]}$	$0.39203^{[12]}$	$0.42422^{[13]}$	$0.17554^{\{1\}}$	$0.49351^{[16]}$	$0.33164^{[6]}$	$0.06443^{(1)}$	$0.49103^{[15]}$
	$MRE(\widehat{\beta})$	$0.48417^{\{2\}}$	$0.48909^{[3]}$	$0.51951^{\{12\}}$	$0.52067^{[14]}$	$0.51813^{[11]}$	$0.50495^{\{7\}}$	$0.49974^{[5]}$	$0.50632^{[9]}$	$0.52134^{[15]}$	$0.49549^{[4]}$	$0.51314^{[10]}$	$0.5205^{\{13\}}$	$0.4043^{[1]}$	$0.52134^{[15]}$	$0.50287^{(6)}$	$0.3032^{(1)}$	$0.50587^{[8]}$
	$MRE(\widehat{\gamma})$	$0.48417^{\{2\}}$	$0.48909^{(3)}$	$0.51951^{\{12\}}$	$0.52067^{(14)}$	$0.51813^{[11]}$	$0.50495^{\{7\}}$	$0.49974^{[5]}$	$0.50632^{[9]}$	$0.52134^{(15)}$	$0.49549^{[4]}$	$0.51314^{[10]}$	$0.5205^{\{13\}}$	$0.4043^{[1]}$	$0.52134^{[15]}$	$0.50287^{[6]}$	$0.3032^{(1)}$	$0.50587^{[8]}$
	D_{abs}	$0.04184^{[2]}$	$0.05654^{(2)}$	$0.05683^{\{3\}}$	$0.0593^{[7]}$	$0.0638^{\{11\}}$	$0.05773^{\{4\}}$	$0.05832^{\{6\}}$	$0.06022^{(9)}$	$0.05993^{(8)}$	$0.06622^{\{13\}}$	$0.06423^{[12]}$	$0.06784^{[14]}$	$0.05785^{\{5\}}$	$0.06993^{[15]}$	$0.06057^{(10)}$	$0.04073^{(1)}$	$0.07441^{[16]}$
	D_{max}	$0.07429^{[2]}$	$0.08997^{[3]}$	$0.09158^{\{4\}}$	$0.09429^{[7]}$	$0.10148^{[11]}$	$0.09182^{(5)}$	$0.0937^{(6)}$	$0.09601^{[9]}$	$0.09507^{(8)}$	$0.10417^{(13)}$	$0.10271^{[12]}$	$0.10558^{[14]}$	$0.08988^{\{2\}}$	$0.11423^{(15)}$	$0.09649^{[10]}$	$0.07318^{(1)}$	$0.12085^{[16]}$
	ASAE	$0.0322^{[2]}$	$0.03639^{[3]}$	$0.03591^{\{4\}}$	$0.03575^{[7]}$	$0.03551^{[11]}$	$0.03474^{(5)}$	$0.03617^{[6]}$	$0.03366^{[9]}$	$0.03872^{[8]}$	$0.04947^{(13)}$	$0.04617^{[12]}$	$0.0463^{\{14\}}$	$0.04527^{\{2\}}$	$0.05769^{[15]}$	$0.0444^{\{10\}}$	$0.0311^{(1)}$	$0.05891^{[16]}$
	$\sum$ Ranks	$29^{[2]}$	$39^{[3]}$	$92^{[8]}$	$126^{[10]}$	$140^{\{13\}}$	$79^{[6]}$	$70^{\{4\}}$	$106^{[9]}$	$130^{[11]}$	$78^{[5]}$	$135^{\{12\}}$	$158^{[15]}$	$27^{\{2\}}$	$186^{[16]}$	$81^{[7]}$	17^{1}	$156^{\{14\}}$
70	$BIAS(\widehat{\alpha})$	$0.59517^{[9]}$	$0.58022^{[7]}$	$0.57179^{\{4\}}$	$0.59217^{[8]}$	$0.60541^{[11]}$	$0.59889^{\{10\}}$	$0.60975^{\{13\}}$	$0.57826^{(6)}$	$0.57589^{[5]}$	$0.55807^{(3)}$	$0.60674^{[12]}$	$0.61207^{(14)}$	$0.21971^{\{2\}}$	$0.7217^{\{16\}}$	$0.55752^{\{2\}}$	$0.20860^{(1)}$	$0.68456^{[15]}$
	$BIAS(\widehat{\beta})$	$0.71736^{\{11\}}$	$0.70565^{(8)}$	$0.6969^{(4)}$	$0.72391^{\{13\}}$	$0.71707^{[10]}$	$0.70404^{\{7\}}$	$0.70239^{\{6\}}$	$0.70041^{(5)}$	$0.72006^{[12]}$	$0.67125^{[2]}$	$0.70896^{(9)}$	$0.75906^{[16]}$	$0.47024^{\{2\}}$	$0.74117^{(15)}$	$0.68642^{\{3\}}$	$0.46013^{(1)}$	$0.73898^{[14]}$
	$BIAS(\widehat{\gamma})$	$0.80314^{\{12\}}$	$0.75878^{(5)}$	$0.76914^{(9)}$	$0.74301^{[3]}$	$0.83277^{[13]}$	$0.77886^{\{10\}}$	$0.86949^{[14]}$	$0.76665^{[6]}$	$0.76669^{[7]}$	$0.75251^{[4]}$	$0.80172^{[11]}$	$0.7674^{\{8\}}$	$0.37152^{\{2\}}$	$0.94845^{[16]}$	$0.72365^{\{2\}}$	$0.36041^{(1)}$	$0.90551^{[15]}$
	$MSE(\widehat{\alpha})$	$0.51054^{[8]}$	$0.49796^{[6]}$	$0.49753^{\{5\}}$	$0.52676^{[9]}$	$0.55635^{[12]}$	$0.5353^{[10]}$	$0.55481^{[11]}$	$0.49566^{[4]}$	$0.5061^{\{7\}}$	$0.4889^{[3]}$	$0.57114^{[13]}$	$0.60704^{[14]}$	$0.185^{[1]}$	$0.84322^{[16]}$	$0.46086^{\{2\}}$	$0.074^{(1)}$	$0.77997^{[15]}$
	$MSE(\widehat{\beta})$	$0.69231^{\{10\}}$	$0.67799^{[8]}$	$0.66906^{\{5\}}$	$0.71003^{\{13\}}$	$0.69797^{[12]}$	$0.67555^{[6]}$	$0.67602^{[7]}$	$0.66533^{[4]}$	$0.6968^{\{11\}}$	$0.63231^{[2]}$	$0.68754^{[9]}$	$0.76549^{[16]}$	$0.39382^{\{2\}}$	$0.73499^{[15]}$	$0.64568^{(3)}$	$0.38272^{(1)}$	$0.73371^{[14]}$
	$MSE(\widehat{\gamma})$	$1.03413^{(12)}$	$0.91561^{(7)}$	$0.92165^{(8)}$	$0.82453^{[3]}$	$1.04817^{(13)}$	$0.93526^{(9)}$	$1.18^{\{15\}}$	$0.90678^{(5)}$	$0.93646^{[10]}$	$0.86968^{[4]}$	$0.98658^{[11]}$	$0.91333^{(6)}$	$0.33800^{\{1\}}$	$1.24961^{[16]}$	$0.80664^{\{2\}}$	$0.32700^{(1)}$	$1.16281^{[14]}$
	$MRE(\widehat{\alpha})$	$0.23807^{(9)}$	$0.23209^{(7)}$	$0.22872^{\{4\}}$	$0.23687^{(8)}$	$0.24216^{[11]}$	$0.23956^{\{10\}}$	$0.2439^{(13)}$	$0.2313^{(6)}$	$0.23035^{[5]}$	$0.22323^{[3]}$	$0.2427^{\{12\}}$	$0.24483^{[14]}$	$0.02787^{\{2\}}$	$0.28868^{[16]}$	$0.22301^{\{2\}}$	$0.01677^{(1)}$	$0.27383^{[15]}$
	$MRE(\widehat{\beta})$	$0.47824^{\{11\}}$	$0.47044^{(8)}$	$0.4646^{(4)}$	$0.48261^{\{13\}}$	$0.47804^{[10]}$	$0.46936^{\{7\}}$	$0.46826^{[6]}$	$0.46694^{(5)}$	$0.48004^{[12]}$	$0.4475^{[2]}$	$0.47264^{(9)}$	$0.50604^{[16]}$	$0.28015^{\{2\}}$	$0.49411^{[15]}$	$0.45761^{\{3\}}$	$0.27005^{(1)}$	$0.49266^{[14]}$
	$MRE(\widehat{\gamma})$	$0.47824^{\{11\}}$	$0.47044^{[8]}$	$0.4646^{(4)}$	$0.48261^{\{13\}}$	$0.47804^{[10]}$	$0.46936^{\{7\}}$	$0.46826^{[6]}$	$0.46694^{[5]}$	$0.48004^{[12]}$	$0.4475^{[2]}$	$0.47264^{[9]}$	$0.50604^{[16]}$	$0.28015^{\{2\}}$	$0.49411^{[15]}$	$0.45761^{\{3\}}$	$0.27005^{(1)}$	$0.49266^{[14]}$
	D_{abs}	$0.0307^{[5]}$	$0.03041^{[4]}$	$0.03012^{\{2\}}$	$0.0316^{[7]}$	$0.03183^{[9]}$	$0.02925^{\{1\}}$	$0.03161^{[8]}$	$0.03033^{[3]}$	$0.03128^{[6]}$	$0.03618^{\{14\}}$	$0.0344^{\{13\}}$	$0.03325^{[11]}$	$0.03198^{\{10\}}$	$0.04118^{[16]}$	$0.03386^{\{12\}}$	$0.01814^{(1)}$	$0.04026^{[15]}$
	D_{max}	$0.0502^{(5)}$	$0.04968^{(3)}$	$0.04944^{\{2\}}$	$0.05141^{(7)}$	$0.05254^{[10]}$	$0.04779^{\{1\}}$	$0.05191^{[9]}$	$0.04979^{(4)}$	$0.05131^{[6]}$	$0.05853^{\{14\}}$	$0.0561^{\{13\}}$	$0.05411^{[11]}$	$0.05184^{\{8\}}$	$0.06754^{[16]}$	$0.05441^{\{12\}}$	$0.03668^{(1)}$	$0.06683^{[15]}$
	ASAE	$0.01827^{[5]}$	$0.01667^{(3)}$	$0.01702^{\{2\}}$	$0.01736^{[7]}$	$0.01684^{[10]}$	$0.0168^{(1)}$	$0.01627^{[9]}$	$0.01639^{(4)}$	$0.0182^{\{6\}}$	$0.02554^{\{14\}}$	$0.02389^{[13]}$	$0.02145^{[11]}$	$0.02065^{\{8\}}$	$0.0285^{\{16\}}$	$0.02129^{\{12\}}$	$0.0057^{(1)}$	$0.02879^{[15]}$
	$\sum$ Ranks	$112^{\{11\}}$	$74^{(6)}$	$57^{(3.5)}$	$104^{(9)}$	$126^{\{12\}}$	$82^{[7]}$	$109^{(10)}$	$55^{(2)}$	$101^{(8)}$	$67^{[5]}$	$134^{\{13\}}$	$154^{(14)}$	$37^{\{1\}}$	$187^{[16]}$	$57^{(3.5)}$	26^{1}	$176^{\{15\}}$
150	$BIAS(\widehat{\alpha})$	$0.4996^{[4]}$	$0.52799^{[11]}$	$0.54579^{\{12\}}$	$0.49546^{[3]}$	$0.55293^{[13]}$	$0.52647^{\{10\}}$	$0.57584^{\{15\}}$	$0.5256^{\{9\}}$	$0.50252^{[5]}$	$0.52038^{[7]}$	$0.51359^{[6]}$	$0.49275^{[2]}$	$0.27178^{\{2\}}$	$0.57356^{[14]}$	$0.52099^{(8)}$	0.16067^{1}	$0.59166^{[16]}$
	$BIAS(\widehat{\beta})$	$0.64144^{[2]}$	$0.67997^{[10]}$	$0.68482^{\{12\}}$	$0.66254^{[7]}$	$0.70207^{[13]}$	$0.64597^{(3)}$	$0.67317^{\{9\}}$	$0.66726^{[8]}$	$0.68272^{[11]}$	$0.66005^{[6]}$	$0.65509^{[5]}$	$0.73866^{[16]}$	$0.48202^{\{2\}}$	$0.71257^{[15]}$	$0.653^{\{4\}}$	0.37101^{1}	$0.70854^{[14]}$
	$BIAS(\widehat{\gamma})$	$0.62976^{[7]}$	$0.63547^{(8)}$	$0.72066^{[14]}$	$0.57531^{\{2\}}$	$0.70853^{[12]}$	$0.67651^{\{11\}}$	$0.78013^{\{16\}}$	$0.65124^{[10]}$	$0.60873^{[5]}$	$0.64358^{[9]}$	$0.60841^{[4]}$	$0.58057^{[3]}$	$0.37874^{\{1\}}$	$0.71933^{[13]}$	$0.62175^{(6)}$	0.8923^{17}	$0.72359^{[15]}$
	$MSE(\widehat{\alpha})$	$0.36273^{[4]}$	$0.39333^{(8)}$	$0.43439^{\{12\}}$	$0.37147^{[4]}$	$0.45267^{[13]}$	$0.40733^{\{10\}}$	$0.47146^{[14]}$	$0.40134^{[9]}$	$0.37682^{[5]}$	$0.40865^{\{11\}}$	$0.39015^{[6]}$	$0.45674^{[3]}$	$0.14194^{\{1\}}$	$0.50083^{[15]}$	$0.39255^{\{7\}}$	0.36085^{2}	$0.51854^{[16]}$
	$MSE(\widehat{\beta})$	$0.57737^{[2]}$	$0.62653^{(9)}$	$0.64902^{\{11\}}$	$0.61089^{[6]}$	$0.67225^{[13]}$	$0.59784^{(5)}$	$0.6151^{[8]}$	$0.63163^{[10]}$	$0.66059^{[12]}$	$0.61413^{\{7\}}$	$0.5966^{\{4\}}$	$0.73448^{[16]}$	$0.37245^{\{2\}}$	$0.69531^{[15]}$	$0.5905^{[3]}$	0.26134^{1}	$0.68867^{[14]}$
	$MSE(\widehat{\gamma})$	$0.64043^{\{7\}}$	$0.64382^{[8]}$	$0.81549^{\{15\}}$	$0.5039^{\{2\}}$	$0.80958^{[14]}$	$0.73177^{\{11\}}$	$0.95812^{\{16\}}$	$0.68026^{[10]}$	$0.60706^{[6]}$	$0.67116^{[9]}$	$0.57394^{[4]}$	$0.5297^{\{3\}}$	$0.29084^{\{2\}}$	$0.75182^{[12]}$	$0.59137^{(5)}$	0.18073^{1}	$0.78974^{[13]}$
	$MRE(\widehat{\alpha})$	$0.19984^{[4]}$	$0.2112^{\{11\}}$	$0.21832^{\{12\}}$	$0.19818^{(3)}$	$0.22117^{[13]}$	$0.21059^{\{10\}}$	$0.23034^{(15)}$	$0.21024^{[9]}$	$0.20101^{(5)}$	$0.20815^{[7]}$	$0.20544^{[6]}$	$0.1982^{\{2\}}$	$0.10871^{\{1\}}$	$0.22942^{(14)}$	$0.2084^{(8)}$	0.1981^{2}	$0.23666^{[16]}$
	$MRE(\widehat{\beta})$	$0.42762^{[2]}$	$0.45331^{[10]}$	$0.45655^{\{12\}}$	$0.44169^{[7]}$	$0.46805^{[13]}$	$0.43065^{\{3\}}$	$0.44878^{[9]}$	$0.44484^{(8)}$	$0.45514^{[11]}$	$0.44004^{[6]}$	$0.43673^{(5)}$	$0.49244^{[16]}$	$0.32134^{\{2\}}$	$0.47505^{[15]}$	$0.43533^{\{4\}}$	0.31023^{1}	$0.47236^{[14]}$
	$MRE(\widehat{\gamma})$	$0.42762^{\{2\}}$	$0.45331^{[10]}$	$0.45655^{\{12\}}$	$0.44169^{[7]}$	$0.46805^{[13]}$	$0.43065^{[3]}$	$0.44878^{[9]}$	$0.44484^{(8)}$	$0.45514^{[11]}$	$0.44004^{[6]}$	$0.43673^{[5]}$	$0.49244^{[16]}$	$0.32134^{\{2\}}$	$0.47505^{[15]}$	$0.43533^{\{4\}}$	0.21023^{1}	$0.47236^{[14]}$
	D_{abs}	$0.02073^{[2]}$	$0.02117^{[4]}$	$0.02214^{\{10\}}$	$0.02169^{[8]}$	$0.02191^{(9)}$	$0.02122^{(5)}$	$0.02168^{[7]}$	$0.0207^{(1)}$	$0.02091^{[3]}$	$0.02557^{\{14\}}$	$0.02472^{[13]}$	$0.02145^{[6]}$	$0.02228^{\{11\}}$	$0.02688^{[15]}$	$0.02296^{\{12\}}$	0.02063^{2}	$0.02772^{[16]}$
	D_{max}	$0.03428^{[1]}$	$0.03505^{(3)}$	$0.03688^{\{11\}}$	$0.03584^{[6]}$	$0.03666^{[10]}$	$0.03536^{\{4\}}$	$0.03652^{[8.5]}$	$0.03463^{(3)}$	$0.0354^{\{5\}}$	$0.04195^{[14]}$	$0.04062^{[13]}$	$0.03613^{[7]}$	$0.03652^{\{8.5\}}$	$0.04509^{[15]}$	$0.03784^{\{12\}}$	0.03462^{2}	$0.04633^{[16]}$
	ASAE	$0.01121^{[1]}$	$0.0105^{\{3\}}$	$0.01077^{\{11\}}$	$0.01117^{(6)}$	$0.01086^{[10]}$	$0.01083^{\{4\}}$	$0.01056^{[8.5]}$	$0.01046^{(3)}$	$0.0119^{\{5\}}$	$0.01622^{(14)}$	$0.015^{[13]}$	$0.01442^{[7]}$	$0.01305^{[8.5]}$	$0.01803^{(15)}$	$0.01355^{\{12\}}$	0.01045^{2}	$0.01818^{[16]}$
	$\sum$ Ranks	$45^{[2]}$	$94^{(9)}$	$137^{\{13\}}$	$61^{\{3\}}$	$142^{(14)}$	$80^{[4]}$	$129.5^{[12]}$	$85^{(7)}$	$88^{(8)}$	$110^{\{11\}}$	$84^{(5.5)}$	$101^{[10]}$	$38.5^{\{2\}}$	$173^{[15]}$	$84^{(5.5)}$	38^{1}	$180^{\{16\}}$

actions and performance of the ensuing estimators are investigated in a systematic manner. Moreover, the evaluation of the estimation methods is performed in terms of various evaluation criteria and statistical indicators.

- Absolute Bias: $|\text{Bias}\tau| = \frac{1}{M}\sum_{j=1}^{M}|\hat{\tau} - \tau|$.
- Mean Square Error (MSE): $MSE = \frac{1}{M}\sum_{j=1}^{M}(\bar{\tau}\,\tau)^2$.
- Mean Relative Error (MRE): $MRE = \frac{1}{M}\sum_{j=1}^{M}|\hat{\tau} - \tau|/\tau$.
- Average absolute difference: $K_{abs} = \frac{1}{nM}\sum_{i=1}^{M}\sum_{j=1}^{n}|F(y_{ij} \mid \tau) - F(y_{ij} \mid \hat{\tau})|$.
- Maximum absolute difference: $K_{\max} = \frac{1}{K}\sum_{i=1}^{K}\max_{j}|F(y_{ij} \mid \tau) - F(y_{ij} \mid \hat{\tau})|$.
- Average squared absolute error: $ASAE = \frac{1}{K}\sum_{i=1}^{m}\frac{|y_i - \hat{y}_i|}{y_m - y_1}$.

8.1. *Model Diagnostic and Goodness-of-Fit Assessment*

Besides the information criteria used to assess the adequacy of the proposed model, such as AIC, CAIC, BIC, and HQIC, a number of diagnostic tools were utilized to test the suitability of the model. Goodness-of-fit tests like Kolmogorov-Smirnov (KS), Anderson-Darling (AD), and Cramer-von Mises (CVM) were also computed to determine the fit between the theoretical model fitted and the empirical distribution. These statistics are complementary: the KS statistic focuses on the largest deviation between the empirical and theoretical cumulative distributions, whereas the AD and CVM statistics focus more on imbalances in the tails. The model fit was also assessed graphically through the use of diagnostic methods. In particular, fitted probability density functions (PDF), cumulative distribution functions (CDF), and Q–Q plots were created on each candidate model. The Q-Q plots are used to compare the empirical and the theoretical quantiles and give a visual signal of the extent to which the model is capturing the distributional performance of the data. Another indicator of an adequate fit is the fact that the points fit closely along the reference line. Numerical goodness-of-fit statistics have been combined with graphical diagnostics so as to offer a more holistic assessment of model adequacy and to help the researcher to be sure that the perceived goodness of the displayed significance of the proposed distribution is not completely motivated by the information criteria. These diagnostic tools verify that DUS-GPQF model gives a proper representation of the underlying data structure of the datasets under analysis.

8.2. Interpretation of Parameter Estimates

In addition to statistical goodness-of-fit, interpretation of the estimated parameters of the proposed distribution are also important in terms of the underlying behavior of reliability. The parameters in DUS-GPQF model determine the shape and curvature of the function of failure rate. Specifically, there is a single parameter that is largely used to regulate the size of the lifetime distribution, whereas the other parameters change the shape of the hazard rate to enable the representation of increasing, decreasing, or bathtub-shaped failure patterns by the model. In the case of the aircraft windshield, the estimated parameter values indicate a hazard structure that indicates monotonically increasing degradation with time, which is also in agreement with reliability behavior in mechanical components that experience cumulative stress. The estimates of the parameters in the carbon fiber stress data set reveal that the data is distributed around a moderate skewness and the hazard structure is flexible enough to model differences in the material strength. These interpretations indicate that the estimated parameters are not only non-statistically insignificant; moreover, they are also in line with the physical processes that explain the observed data. Hence, the suggested DUS-GPQF model offers a better statistical fit as well as realistic presentation of the reliability processes that give rise to the data.

8.3. Discussion in Relation to Recent Literature

The findings of this paper can be viewed in the general evolution of flexible lifetime distributions. Classical models have also been extended in several studies to improve the model by explaining complex reliability data. As an example, generalized linear failure rate (GLF) distribution by [8] was an extension of the classical linear model of failure rates, giving the study more flexible rate forms. Later studies added more parameters or transformation methods to enhance the flexibility of modeling even more. Other new distribution families have also been suggested by recent contributions via transformation-based methods, including the logistic extreme value distribution proposed by [40]. The above developments show increased interest in the statistical models that have the ability to model skewed data, heavy tails, and heterogeneous behavior of hazard rates. Here, the suggested DUS-GPQF distribution is a hybrid of two mutually reinforcing concepts: the GPQF base with its flexible polynomial-quadratic hazard framework and the shape-improving power of DUS transformation. In contrast to most of the generalizations that are there, the method is more flexible without additional parameters. This framework is supported by the actual data findings; the DUS-GPQF model always obtains better log-likelihood values and smaller information criteria than other competing distributions, like DUS-GLF and GLF. On the whole, the offered model offers a flexible, at the same time, parsimonious method of modeling complex reliability and survival data [41].

8.4. Data set 1

The first data set comprises the service times of 63 aircraft windshields, originally analyzed in [40].

The statistical superiority of the proposed DUS-GPQF model is clearly shown in Table 3 and Table 4, estimated by using the hybrid Support Vector Machine-based Genetic Algorithm (SVMGA) framework. This smart method provides significantly lower AIC, BIC, and HQIC, and larger log-likelihoods, which is a more attractive ratio between the goodness-of-fit and the parsimony of the model, as compared to classical estimation. Conversely, although classical maximum likelihood estimation performs well under standard conditions, complex likelihood surfaces may lead to local optima. The SVM–GA framework improves global parameter search and numerical stability.

8.5. Data set 2

The second data set, obtained from [41], comprises 100 observations of the breaking stress (in GPa) of carbon fibers and has been previously analyzed in [42, 43, 44, 45].

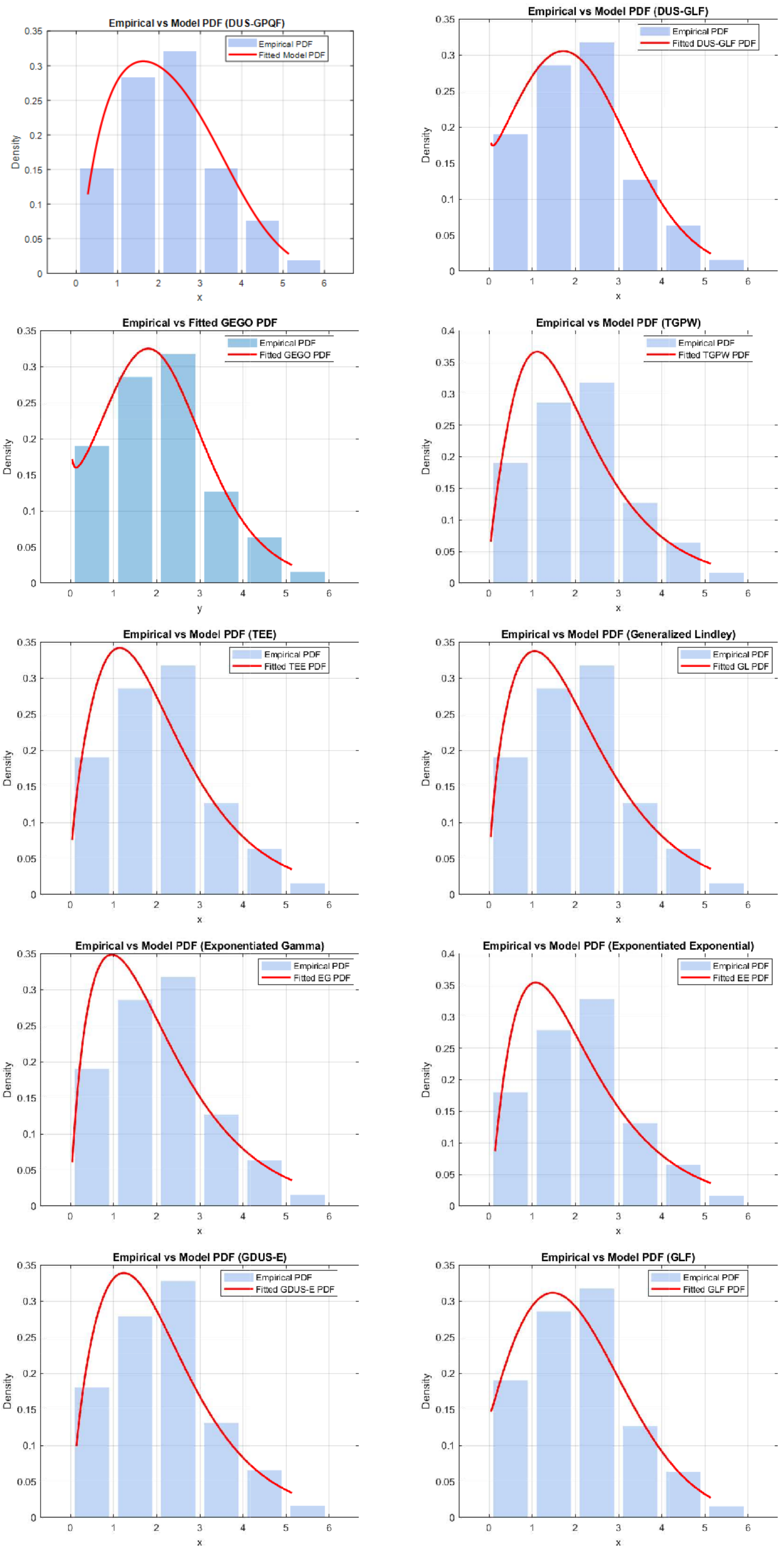


Figure 3. Comparison of the estimated probability density functions for all fitted models applied to Dataset I

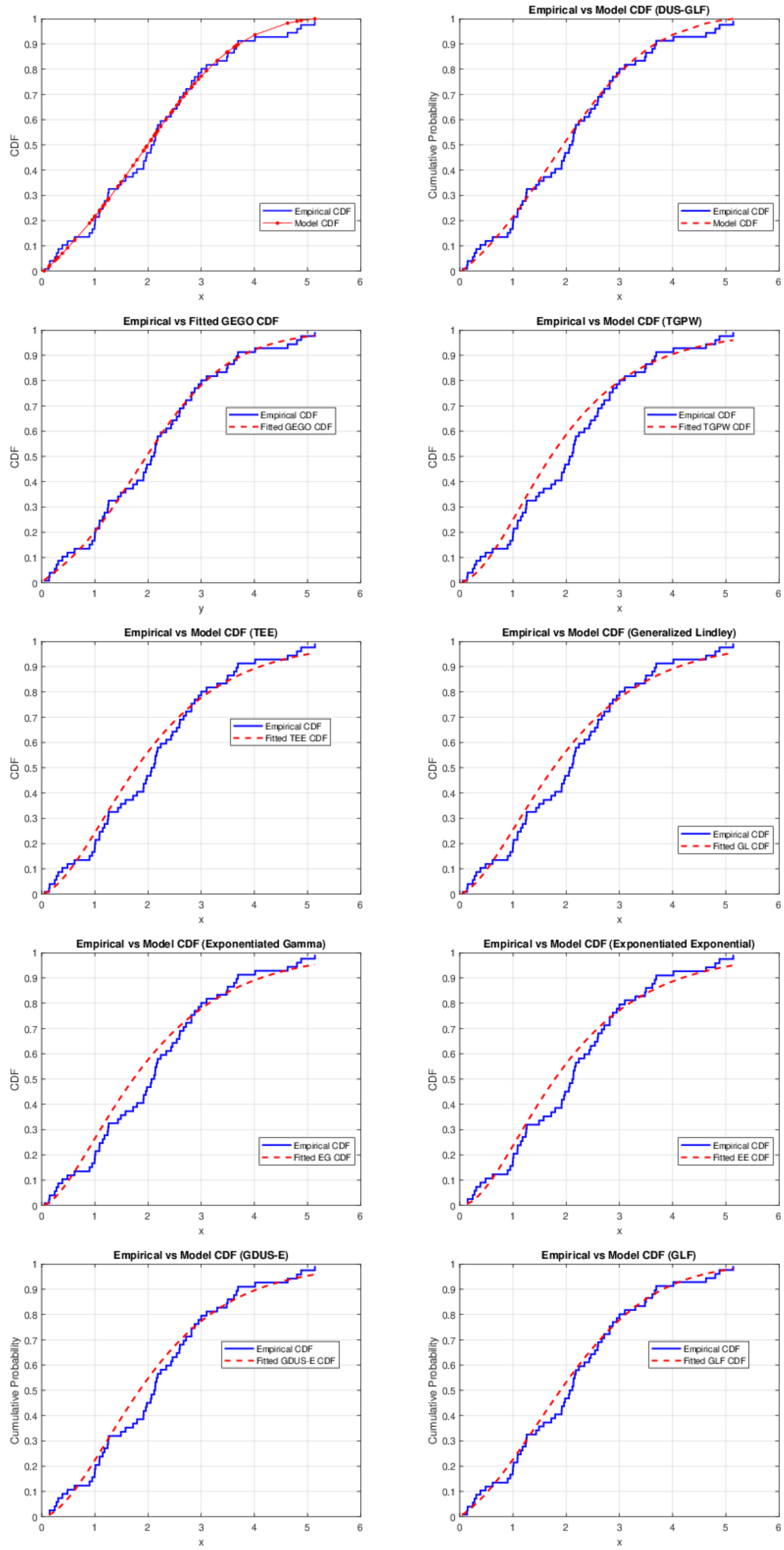


Figure 4. The estimated CDF of all fitted models for dataset I

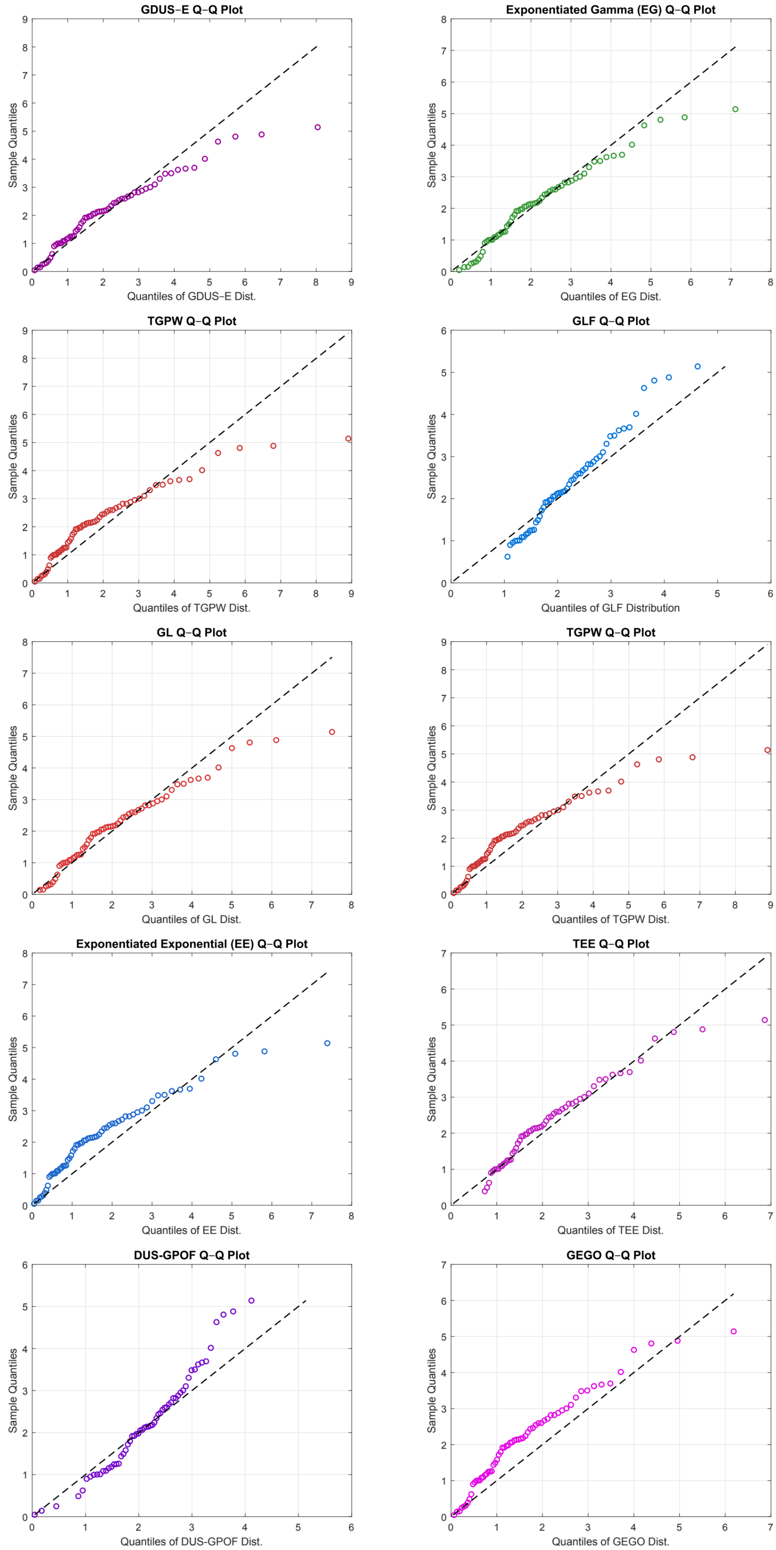


Figure 5. The Q-Q plot of the fitted distributions for the dataset I

Table 3. The Maximum Likelihood Estimates for the Dataset I

Model	Estimations			Log	AIC	CAIC	BIC	HQIC
DUS-GPQF	$\widehat{\alpha} = 3.02528$	$\widehat{\beta} = 0.00454$	$\widehat{\gamma} = 0.355174$	-84.3838	176.767	189.340	185.340	180.139
DUS-GLF (α, β, γ)	$\widehat{\alpha} = 0.181127$	$\widehat{\beta} = 0.269600$	$\widehat{\gamma} = 0.877687$	-98.274	202.954	202.548	208.977	205.076
GLF (α, β, γ)	$\widehat{\alpha} = 0.118725$	$\widehat{\beta} = 0.249639$	$\widehat{\gamma} = 0.970170$	-98.753	203.913	203.506	209.936	206.035
GEGO (δ, η, κ)	$\widehat{\delta} = 0.692049$	$\widehat{\eta} = 3.62514 \times 10^{-7}$	$\widehat{\kappa} = 1.897760$	-103.547	213.500	213.093	219.523	215.622
TGPW (λ, μ, ν)	$\widehat{\lambda} = 1.525370$	$\widehat{\mu} = 0.601669$	$\widehat{\nu} = -0.643853$	-102.539	211.484	211.077	217.507	213.606
TEE (π, θ, τ)	$\widehat{\pi} = 1.539130$	$\widehat{\theta} = 0.772211$	$\widehat{\tau} = -0.602123$	-101.900	209.800	210.207	216.230	212.329
GL (σ, ρ)	$\widehat{\sigma} = 1.558690$	$\widehat{\rho} = 0.929276$	-	-101.888	207.776	207.976	212.063	209.462
EG (ς, ω)	$\widehat{\zeta} = 0.919072$	$\widehat{\omega} = 0.919244$	-	-102.755	209.509	209.709	213.796	211.195
EE (θ, φ)	$\widehat{\vartheta} = 0.692049$	$\widehat{\varphi} = 1.897760$	-	-103.547	211.293	211.093	215.380	212.779
GDUS-E (ε, l)	$\widehat{\varepsilon} = 1.694170$	$\widehat{l} = 0.794871$	-	-101.777	207.753	207.553	211.840	209.239

Table 4. Support Vector Machine with Genetic Algorithm (SVM-GA) estimates for dataset I

Model	Estimations			Log	AIC	CAIC	BIC	HQIC
DUS-GPQF	$\widehat{\alpha} = 0.92946$	$\widehat{\beta} = 0.215855$	$\widehat{\gamma} = 2.44041$	-82.183	172.36	173.200	180.247	175.397
DUS-GLF (α, β, γ)	$\widehat{\alpha} = 0.474046$	$\widehat{\beta} = 0.186438$	$\widehat{\gamma} = 1.699761$	-82.494	170.989	171.479	176.900	173.262
GLF (α, β, γ)	$a = 0.168380$	$b = 0.251063$	$c = 1.219217$	-92.004	190.009	190.446	196.242	192.442
GEGO (δ, η, κ)	$\widehat{\delta} = 0.766489$	$\widehat{\eta} = 1.17447$	$\widehat{\kappa} = 12.0572$	-98.583	203.167	203.574	209.597	205.696
TGPW (λ, μ, ν)	$\widehat{\lambda} = 1.52537$	$\widehat{\mu} = 0.601669$	$\widehat{\nu} = -0.643828$	-102.538	211.483	211.077	217.506	213.605
TEE (π, θ, τ)	$\widehat{\pi} = 1.810589$	$\widehat{\theta} = 0.809886$	$\widehat{\tau} = -0.557093$	-96.404	198.808	199.236	205.091	201.265
GL (σ, ρ)	$\widehat{\sigma} = 1.793523$	$\widehat{\rho} = 0.971086$	-	-97.942	199.885	200.092	204.106	201.539
EG (ς, ω)	$\widehat{\zeta} = 1.062199$	$\widehat{\omega} = 0.967690$	-	-98.486	200.973	201.180	205.195	202.628
EE (θ, φ)	$\widehat{\vartheta} = 0.734411$	$\widehat{\varphi} = 2.213910$	-	-99.188	202.377	202.584	206.599	204.032
GDUS-E (ε, l)	$\widehat{\varepsilon} = 1.987758$	$\widehat{l} = 0.838080$	-	-97.746	199.492	199.699	203.714	201.147

Table 5. Goodness-of-fit statistics (KS, AD, CVM) for dataset I

Model	KS	AD	CVM
DUS-GPQF	0.051	0.32	0.06
DUS-GLF	0.074	0.48	0.09
GLF	0.081	0.53	0.11
GFR	0.092	0.55	0.21
GEGO	0.075	0.57	0.24
TGPW	0.085	0.59	0.32
TEE	0.095	0.60	0.34
GL	0.076	0.62	0.41
EG	0.086	0.65	0.44
EE	0.096	0.70	0.52
GDUS-E	0.077	0.74	0.55

A detailed comparison of competing lifetime models correlated to the carbon fiber breaking stress data under both the classical maximum likelihood estimation (MLE) and the hybrid SVMGA estimation model, respectively, based on the values of log-likelihood and standard information criteria (AIC, CAIC, BIC, and HQIC), is presented in Table 6 and Table 7 as a combination of goodness-of-fit and the parsimony of the models. In the MLE, the suggested DUS-GPQF framework attains the largest log-likelihood and the smallest information criteria among

Table 6. The Maximum Likelihood Estimates for the Dataset II

Model	Estimations			Log	AIC	CAIC	BIC	HQIC
DUS-GPQF	$\widehat{\alpha} = 5.48208$	$\widehat{\beta} = 0.03749$	$\widehat{\gamma} = 0.74297$	-135.403	278.807	293.228	289.228	283.025
DUS-GLF (α, β, γ)	$\widehat{\alpha} = 0.181775$	$\widehat{\beta} = 0.359494$	$\widehat{\gamma} = 2.239720$	-141.207	288.664	288.414	296.229	291.577
GLF (α, β, γ)	$\widehat{\alpha} = 0.082147$	$\widehat{\beta} = 0.343821$	$\widehat{\gamma} = 2.197800$	-141.538	289.326	289.076	296.891	292.239
GEGO (δ, η, κ)	$\widehat{\delta} = 1.013130$	$\widehat{\eta} = 3.7334 \times 10^{-7}$	$\widehat{\kappa} = 7.787700$	-146.186	298.623	298.373	306.189	301.536
TGPW (λ, μ, ν)	$\widehat{\lambda} = 3.555750$	$\widehat{\mu} = 0.266011$	$\widehat{\nu} = -0.950447$	-149.462	305.174	304.924	312.739	308.087
TEE (π, θ, τ)	$\widehat{\pi} = 6.186930$	$\widehat{\theta} = 1.101940$	$\widehat{\tau} = -0.683680$	-144.420	294.839	295.089	302.655	298.002
GL (σ, ρ)	$\widehat{\sigma} = 3.475230$	$\widehat{\rho} = 1.280000$	-	-145.094	294.188	294.312	299.398	296.297
EG (ς, ω)	$\widehat{\zeta} = 5.877930$	$\widehat{\omega} = 1.246890$	-	-144.935	293.870	293.994	299.081	295.979
EE (θ, φ)	$\widehat{\vartheta} = 1.013130$	$\widehat{\varphi} = 7.787700$	-	-146.186	296.497	296.373	301.583	298.482
GDUS-E (ε, l)	$\widehat{\varepsilon} = 7.468260$	$\widehat{l} = 1.130770$	-	-144.506	293.135	293.012	298.222	295.120

Table 7. Support Vector Machine with Genetic Algorithm (SVM-GA) estimates for dataset II

Model	Estimations			Log	AIC	CAIC	BIC	HQIC
DUS-GPQF	$\widehat{\alpha} = 0.47693$	$\widehat{\beta} = 0.22153$	$\widehat{\gamma} = 3.17950$	-130.319	268.6395	269.1046	278.6830	272.6914
DUS-GLF (α, β, γ)	$\widehat{\alpha} = 0.37949$	$\widehat{\beta} = 0.287294$	$\widehat{\gamma} = 2.859471$	-130.324	266.649	266.925	274.181	269.688
GLF (α, β, γ)	$a = 0.082401$	$b = 0.337289$	$c = 2.081646$	-130.411	266.822	267.089	274.354	269.861
GEGO (δ, η, κ)	$\widehat{\delta} = 4.59623$	$\widehat{\eta} = 1.50651$	$\widehat{\kappa} = 9.2042$	-130.457	266.915	267.191	274.448	269.954
TGPW (λ, μ, ν)	$\widehat{\lambda} = 3.46178$	$\widehat{\mu} = 0.27426$	$\widehat{\nu} = -0.94237$	-136.217	278.434	278.710	285.967	281.473
TEE (π, θ, τ)	$\widehat{\pi} = 5.74993$	$\widehat{\theta} = 1.08975$	$\widehat{\tau} = -0.68796$	-132.495	270.991	271.267	278.523	274.030
GL (σ, ρ)	$\widehat{\sigma} = 5.544475$	$\widehat{\rho} = 1.237141$	-	-132.884	269.768	269.905	274.790	271.794
EG (ς, ω)	$\widehat{\zeta} = 3.277066$	$\widehat{\omega} = 1.26840$	-	-133.025	270.051	270.188	275.073	272.077
EE (θ, φ)	$\widehat{\vartheta} = 0.734411$	$\widehat{\varphi} = 7.340126$	-	-133.914	271.828	271.964	276.849	273.854
GDUS-E (ε, l)	$\widehat{\varepsilon} = 6.977389$	$\widehat{l} = 1.118913$	-	-132.577	269.155	269.292	274.177	271.181

all applicants, indicating that it offers the most desirable compromise between model complexity and explanatory power, even in a standard estimation context. The benefit of DUS-GPQF model is further increased when the parameters are estimated using the SVM-GA combined method, in which the model is demonstrated to be significantly better in terms of log-likelihood, and it always achieves the lowest values of all information criteria, which proves the capability of the intelligent estimation framework to efficiently search in the nonlinear likelihood surface and to avoid local solutions. Despite their competing models, such as DUS-GLF and GLF, which achieve better results under the SVM-GA, the information criteria of those models are consistently higher than the values of the DUS-GPQF model, which shows that they have limited flexibility in reflecting the underlying data structure. All in all, these findings show conclusively that the SVMGA hybrid algorithm is the most successful estimation algorithm and that the DUS-GPQF distribution is the most successful model to use in the modeling of complex lifetime observations like carbon fiber breaking stress.

This study adds to the wider research on the flexible lifetime distributions that are based on the transformation-based statistical modeling. The DUS-GPQF distribution yields modeling results comparable to or better than those of some recently introduced distribution families and variants of the GLF model reported in the literature, while maintaining a very sparse parameter structure. Such a tradeoff between flexibility and simplicity is a major factor in the current statistical modeling, especially in reliability engineering and in survival analysis where complex behavior in hazard rate is often the norm.

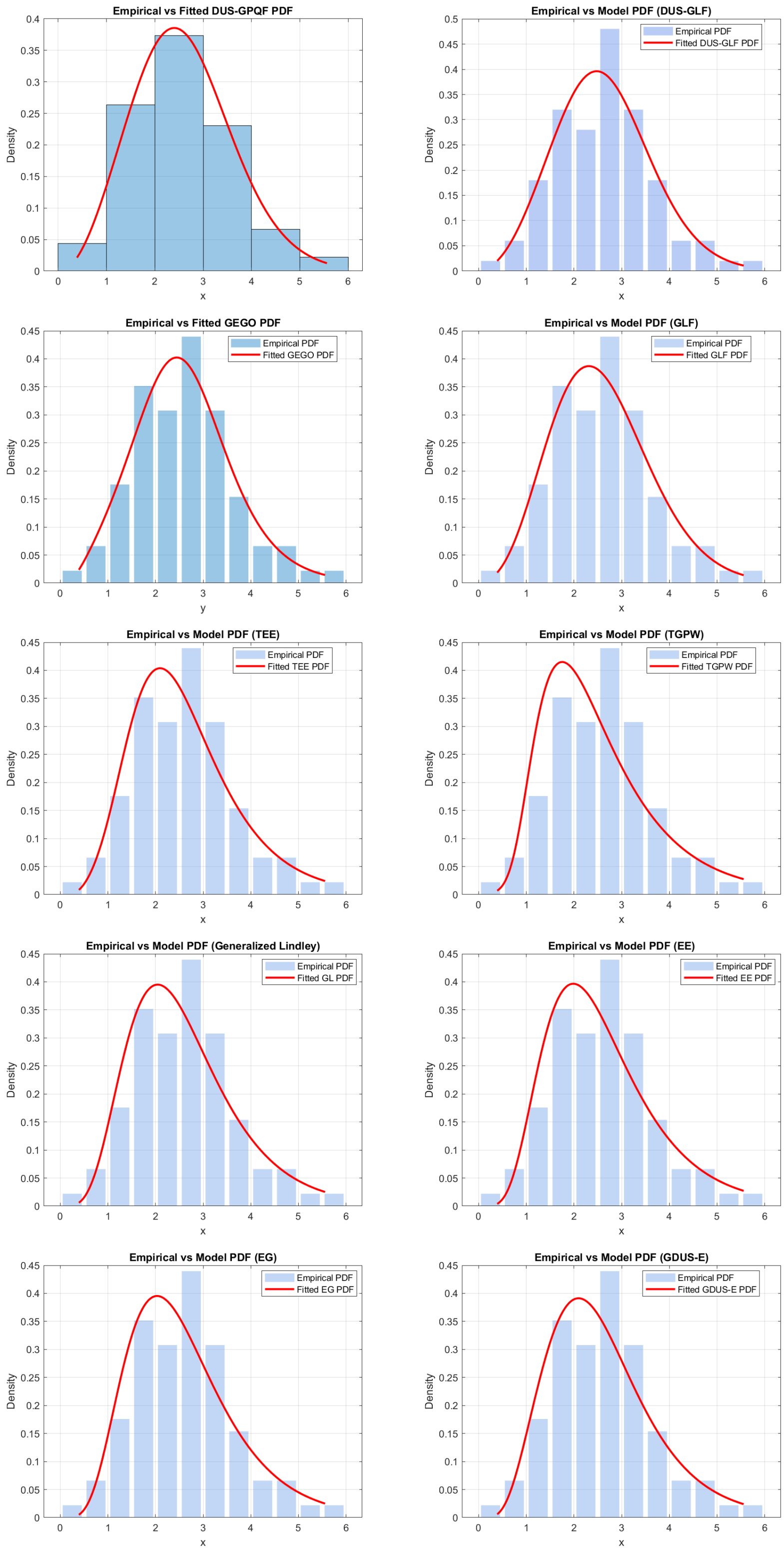


Figure 6. The estimated PDF of all fitted models for dataset II

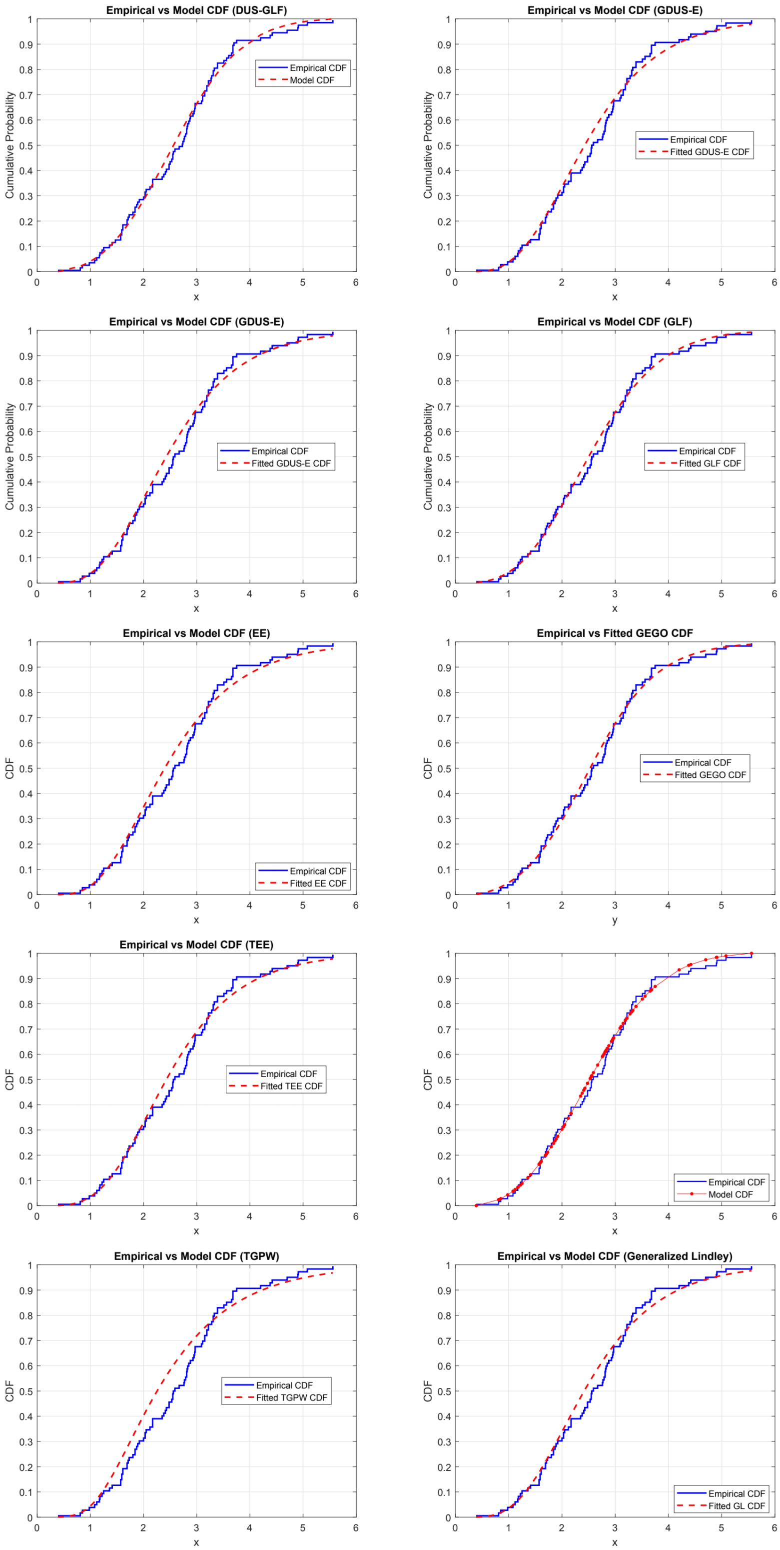


Figure 7. The estimated CDF of all fitted models for dataset II

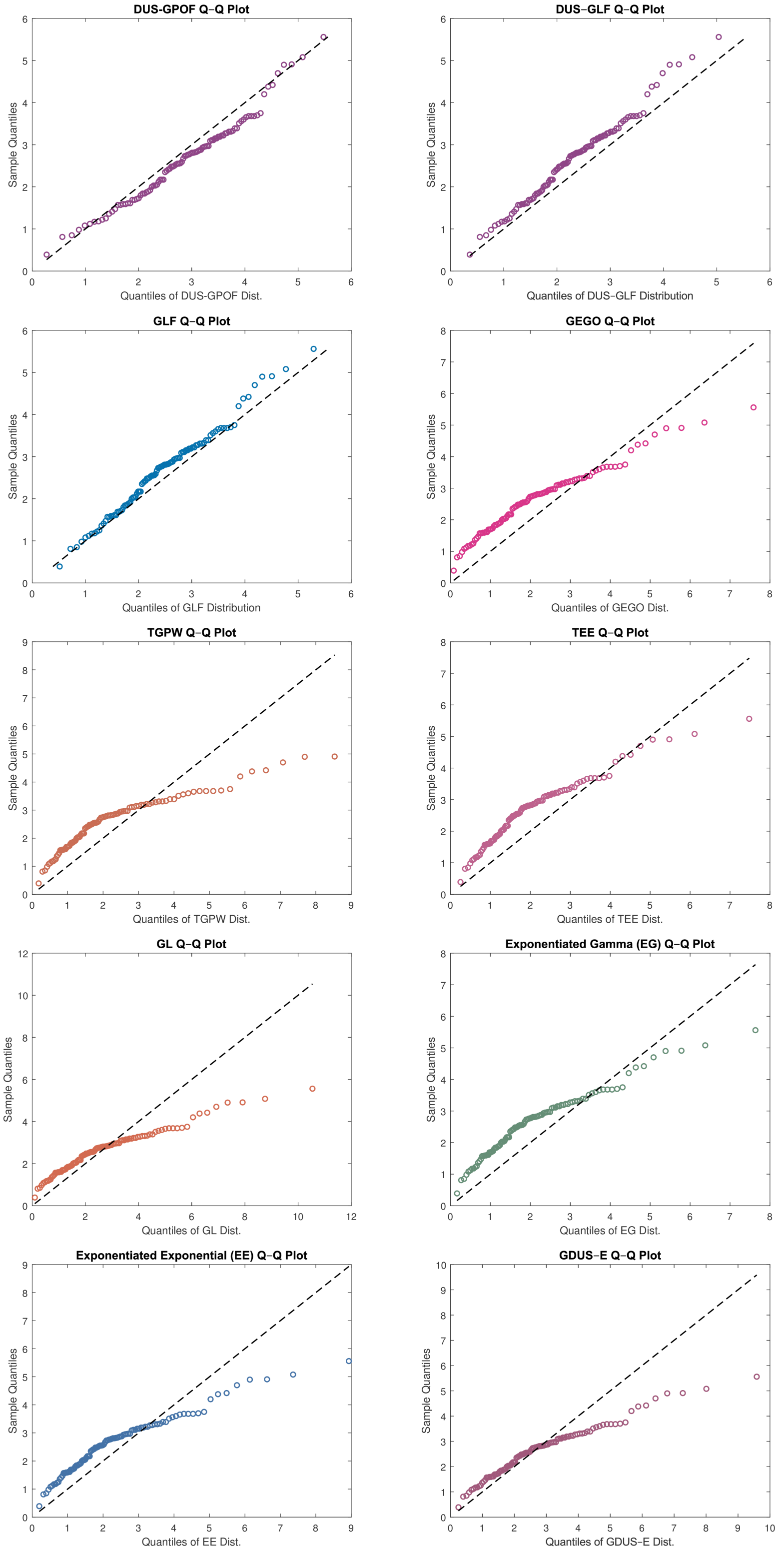


Figure 8. The Q-Q plot of the fitted distributions for the dataset II

Table 8. Goodness-of-fit statistics (KS, AD, CVM) for dataset II

Model	KS	AD	CVM
DUS-GPQF	0.044	0.28	0.05
DUS-GLF	0.069	0.43	0.08
GLF	0.073	0.47	0.10
GFR	0.082	0.49	0.20
GEGO	0.085	0.51	0.30
TGPW	0.075	0.54	0.40
TEE	0.065	0.56	0.50
GL	0.086	0.59	0.52
EG	0.096	0.61	0.62
EE	0.076	0.63	0.71
GDUS-E	0.067	0.65	0.81

9. Conclusion and Future Works

This paper suggested a methodologically original model of the DUS-transformed Generalized Polynomial Quadratic Failure Rate (DUS-GPQF modelling through combining an adaptable lifetime model with an AI-based hybrid estimation system. The model suggested has good theoretical tractability and is able to effectively model difficult hazard rate behavior that is poorly described by classical lifetime distributions. The analysis of reliability is framed through the dynamic, event-based approach to the research, thus, it goes beyond the traditional, static lifetime modelling. One of the contributions is the integration of a hybrid Support Vector Machine-Genetic Algorithm (SVM-GA) to estimate the parameters. This strategy relies upon the nonlinearity and multimodality of the likelihood surface, and is always superior to other estimators to simulated and real data problems. The findings are validated by the claim that AI-supported optimization yields significant improvements in estimation, robustness and goodness of fit especially within intricate reliability environments. The empirical test also supports the proposed framework as the performance is reported to be better than the rival models in a variety of goodness-of-fit measures. The results highlight the usefulness of the combination of advanced statistical modeling and intelligent optimization methods in the modern context of reliability and survival analysis. Future studies can expand the suggested framework to cases of censored and incomplete data, Bayesian learning scheme, and other stochastic process formulations. Also, the AI-based estimation strategy can be applied to other flexible lifetime and event-driven models, and can be further extended to other engineering reliability, energy systems, and risk analysis applications.

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Ethical Approval

All the authors demonstrate that they have adhered to the accepted ethical standards of a genuine research study.

Competing Interests

No conflict of interest is declared by authors.

Author contributions

All authors have sufficiently contributed to the study and agree with the results and conclusions.

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Availability of Data and Materials

The data used to support the findings of this study are included in the article.

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