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Ma-Minda Type of Bi-Starlike Functions Associated with a Shell-Shaped Region in q-Calculus

Abdullah Alsoboh [1], Ala Amourah [2,3], Jamal Salah [1,*], Abed Al-Rahman M. Malkawi [3], Omar Alnajar[4], Tala Sasa [5]

[1] *College of Applied and Health Sciences, A'Sharqiyah University, P.O. Box 42, Post Code 400, Ibra, Sultanate of Oman. abdullah.alsoboh@asu.edu.om*
[2] *Mathematics Education Program, Faculty of Education and Arts, Sohar University,Sohar 311, Oman.*
[3] *Department of Mathematics, Faculty of Arts and Science, Amman Arab University, Amman 11953, Jordan. a.malkawi@aau.edu.jo*
[4] *Department of Mathematics, Faculty of Science and Technology, Irbid National University, P.O. Box: 2600 Irbid 21110, Jordan. o.alnjar@inu.edu.jo*
[5] *Department of Mathematics, Faculty of Science, Applied Science Private University, Amman, Jordan. t_sasa@asu.edu.jo*

Abstract This article introduces a bi-starlike class linked with a shell-shaped region by using the principle of subordination and q-calculus, where $q \in (0,1)$. We then apply the well-known method of Ma and Minda to investigate a novel subclass of bi-univalent functions. We prove that the class is nonempty by using a geometric approach and derive explicit upper estimates for the Taylor–Maclaurin coefficients $|\eta_2|$ and $|\eta_3|$. Additionally, we study the Fekete–Szegö functional. The study examines shell-shaped image domains using the subordination principle under quantum q- calculus, supported by illustrative geometric plots.

Keywords Analytic functions; q−calculus; Shell-Shaped Region; Subordination.

AMS 2010 subject classifications 30C45, 30C50.

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1. Introduction

Consider the family $\mathscr{A}$ consisting of functions $\mathtt{f}$ of the form

$$\mathtt{f}(\zeta) = \zeta + \sum_{\kappa=2}^{\infty} \eta_\kappa \zeta^\kappa, \qquad \zeta \in \mho = \{\zeta \in \mathbb{C} : |\zeta| < 1\}, \tag{1.1}$$

and $\mathtt{f}$ is analytic in $\mho$. Additionally, $\mathtt{f}$ must satisfy the normalization condition

$$\mathtt{f}'(0) - 1 = 0 = \mathtt{f}(0). \tag{1.2}$$

Further, the subclass $\mathcal{P}$ of analytic functions in $\mho$ that have real parts and have the form

$$\varphi(\zeta) = 1 + \sum_{\kappa=1}^{\infty} \varphi_\kappa \zeta^\kappa, \qquad (\zeta \in \mho), \tag{1.3}$$

*Correspondence to: Jamal Salah (Email: drjamals@asu.edu.om)

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where, by Carathéodory's lemma (see, for example, [5]), the coefficients φ_σ satisfy the inequality

$$|\varphi_\sigma| \leq 2, \qquad \text{for all } \sigma \geq 1. \tag{1.4}$$

The subset of $\mathscr{A}$ of all univalent (one to one or injective) function on $\mho$ of the form (1.1) is denoted by $\mathtt{f}^{-1}$ and defined by

$$\zeta = \mathtt{f}^{-1}(\mathtt{f}(\zeta)), \quad \varpi = \mathtt{f}(\mathtt{f}^{-1}(\varpi)) \qquad \left(r_0(\mathtt{f}) \geq \frac{1}{4};\ |\varpi| < r_0(\mathtt{f}); \zeta \in \mho\right),$$

where

$$\mathtt{f}^{-1}(\varpi) = \hbar(\varpi) = \varpi\left(1 - \alpha_2\varpi + \varpi^2(-\eta_3 + 2\eta_2^2) - \varpi^3(\eta_4 + 5\alpha_2^3 - 5\eta_3\eta_2) + \cdots\right). \tag{1.5}$$

The collection of all univalent functions $\mathtt{f}$ with their inverse are denoted by $\sum$ and known by the bi-univalent class.

A new trend on GFT, the gemetric function theory, was created by Ma and Minda [31], they developed the method of generating a starlike class and denoted by $K(\Theta)$ by using the principal of subordination by:

$$K(\Theta) = \left\{\mathtt{f} \in \mathscr{A} : \quad \frac{\zeta\mathtt{f}'(\zeta)}{\mathtt{f}(\zeta)} \prec \Theta(\zeta), \qquad (\zeta \in \mho)\right\}.$$

In this formulation, $\Theta(\zeta)$ denotes an analytic function with positive real part, normalized according to the condition (1.2). Various subclasses of analytic functions have been introduced in the literature by prescribing specific choices of the function Θ in the corresponding subordination conditions. For instance, Sokol and Stankiewicz [1] considered $\Theta(\zeta) = \sqrt{1+\zeta}$, while Piejko and Sokól [2] employed $\Theta(\zeta) = \zeta/(1-\zeta)$. Later, Singh and Kaur [4] introduced the function $\Theta(\zeta) = \zeta + \sqrt[3]{1+\zeta^3}$. Each of these choices generates a distinct subclass exhibiting specific geometric characteristics. More recently, Priya and Sharma [3] investigated the choice $\Theta(\zeta) = \zeta + \sqrt{1+\zeta^2}$.

The mapping

$$\Theta(\zeta) = \zeta + \sqrt{1+\zeta^2}$$

transforms the open unit disk conformally onto a shell-shaped domain lying entirely in the right half-plane. In addition, Θ admits analyticity and univalence throughout $\mathbb{C} \setminus \{-i, i\}$, exhibits symmetry with respect to the real axis, and belongs to the Carathéodory class by virtue of its positive real part. In addition, it satisfies the normalization conditions

$$\Theta(0) = 1 \quad \text{and} \quad \Theta'(0) = 1.$$

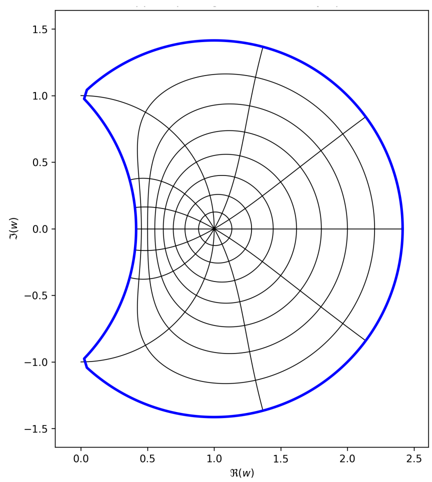
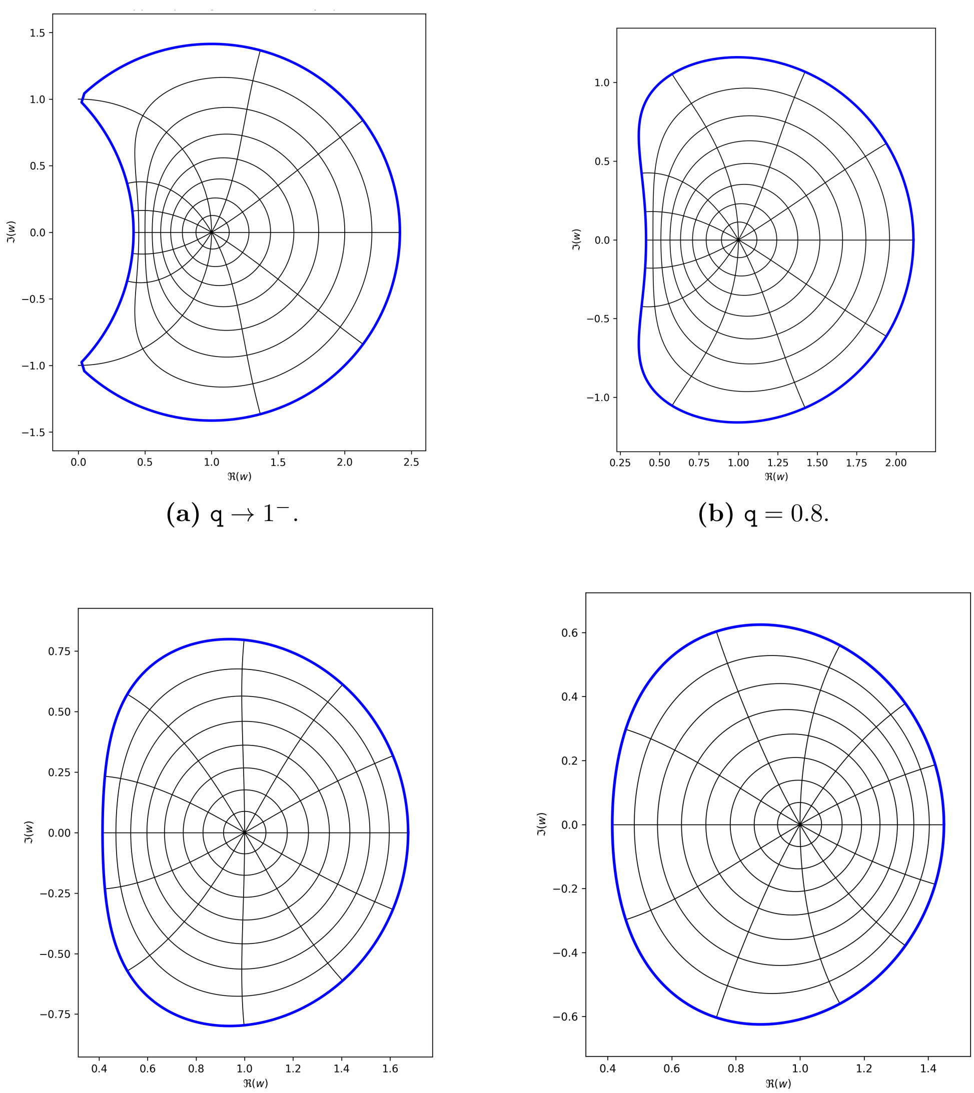


Figure 1. $\Theta(\mho)$ where $\Theta(\zeta) = \zeta + \sqrt{1+\zeta^2}$.

This article provides a concise overview of q-calculus, originally introduced by Jackson and subsequently developed by many researchers [7]–[10]. Instead of reiterating standard material, we recall only the essential concepts required for the present investigation. All definitions, notation, and fundamental properties of q-calculus, together with comprehensive treatments of the q-difference operator and its applications, may be found in the cited literature, particularly in the monograph of Gasper and Rahman [6]. Throughout this work, the parameter $\mathtt{q}$ is assumed to satisfy $0 < \mathtt{q} < 1$, and all subsequent results are based on these well-established foundations.

Motivated by the work of Priya and Sharma [3] and the Ma–Minda subordination framework [31], we introduce a fundamental class of analytic functions generated via q-calculus. This construction serves as the core setting of the paper and provides the basis for the investigation of new subclasses with rich geometric properties.

Figure 2. The image of $\Theta(e^{i\theta};\mathtt{q})$ (blue curve) for several values of $\mathtt{q}$.



(a) $\mathtt{q} \to 1^-$.

(b) $\mathtt{q} = 0.8$.

(c) $\mathtt{q} = 0.4$.

(d) $\mathtt{q} = 0.1$.

Definition 1.1. Let $\mathtt{f}$ be an analytic function of the form (1.1). We say that $\mathtt{f}$ belongs to the family $\mathcal{S}^*_{\mathtt{q}}(\Theta)$ whenever its $\mathtt{q}$-derivative is subordinate to the generating function $\Theta(\zeta; q)$, that is,

$$\frac{\zeta\,\partial_q \mathtt{f}(\zeta)}{\mathtt{f}(\zeta)} \prec \frac{\zeta\,\lfloor 2\rceil_{\mathtt{q}}}{\zeta\,(1-\mathtt{q})+2} + \sqrt{1+\left(\frac{\lfloor 2\rceil_{\mathtt{q}}\,\zeta}{\zeta\,(1-\mathtt{q})+2}\right)^2} =: \Theta(\zeta;\mathtt{q}). \tag{1.6}$$

Here, $\lfloor \kappa\rceil_{\mathtt{q}}$ denotes the associated $\mathtt{q}$-number (see [7] for further details). The function $\Theta(\cdot;\mathtt{q})$ is normalized by $\Theta(0;\mathtt{q}) = 1$, and the square root is understood throughout in its principal branch.

We emphasize that the contribution of the present paper is not a new general coefficient-estimation method: the Ma–Minda subordination scheme and the Carathéodory coefficient technique are standard. The specific contribution is the analysis of the $\mathtt{q}$-dependent shell mapping $\Theta(\zeta;\mathtt{q})$ simultaneously for a function and its analytic inverse. Unlike merely replacing the derivative by ∂_q while retaining a fixed dominant, the dominant itself varies with $\mathtt{q}$. Consequently, changing $\mathtt{q}$ deforms the boundary of $\Theta(\mho;\mathtt{q})$, as illustrated in Figure 2, and changes the admissible coefficient region through the explicit factors $\lfloor 2\rceil_{\mathtt{q}}$, $\mathtt{q}^{-1}$, and $\mathtt{q}^{-2}$ appearing in Theorems 3.1 and 4.1. The two subordination conditions are coupled through the identity $\ell_1 = -\jmath_1$, so the resulting estimates describe the interaction between this domain deformation and inverse-function constraints. The classical shell-like class is recovered continuously as $\mathtt{q} \to 1^-$, which serves as a consistency test rather than the sole justification for the construction.

Accordingly, the novelty claimed here is deliberately limited to (i) the definition and geometric verification of this particular $\mathtt{q}$-deformed shell-like bi-starlike class, (ii) proof that it is nonempty, and (iii) explicit $\mathtt{q}$-dependent initial-coefficient and Fekete–Szegö estimates together with their classical limits. We do not claim that introducing an additional parameter, by itself, constitutes a fundamentally new method or resolves the general coefficient problem for bi-univalent functions.

For visualization purposes, Figure 2 illustrates the image of the open unit disk under the generating function $\Theta(\zeta;\mathtt{q})$ for selected values of the parameter $\mathtt{q}$.

Remark 1.2 (Qualitative geometric effect of $\mathtt{q}$)**.** Write the dominant function as

$$\Theta(\zeta;\mathtt{q}) = \Phi\big(T_{\mathtt{q}}(\zeta)\big)\,, \qquad \Phi(\xi) = \xi + \sqrt{1+\xi^2}, \qquad T_{\mathtt{q}}(\zeta) = \frac{\lfloor 2\rceil_{\mathtt{q}}\zeta}{2+(1-\mathtt{q})\zeta}.$$

For the standard $\mathtt{q}$-number $\lfloor 2\rceil_{\mathtt{q}} = 1+\mathtt{q}$, the map $T_{\mathtt{q}}$ is a genuine Möbius deformation of the disk and not a uniform rescaling. Indeed,

$$T_{\mathtt{q}}(-1) = -1, \qquad T_{\mathtt{q}}(1) = \frac{1+\mathtt{q}}{3-\mathtt{q}}, \qquad T'_{\mathtt{q}}(0) = \frac{1+\mathtt{q}}{2}.$$

Thus the negative real boundary point is fixed for every $\mathtt{q}$, whereas the positive real side is compressed whenever $0 < \mathtt{q} < 1$; the compression becomes stronger as $\mathtt{q}$ decreases. After composition with the classical shell map Φ, this produces an asymmetric change in the width and curvature of the shell-shaped target, rather than a simple change of scale. The plots in Figure 2 visualize this directional deformation.

This has a qualitative consequence for the function class. Membership requires the logarithmic $\mathtt{q}$-derivatives of both $\mathtt{f}$ and $\hbar$ to remain in a target region whose positive-real reach and boundary curvature vary with $\mathtt{q}$. Hence functions admitted for one value of $\mathtt{q}$ need not satisfy the same two subordinations for another value; $\mathtt{q}$ indexes a family of geometrically different admissibility conditions, not merely a numerical constant in the final bounds. At $\mathtt{q} = 1$, $T_{\mathtt{q}}$ becomes the identity and the asymmetric deformation disappears, recovering the classical shell-shaped region.

Bounds for the initial Taylor–Maclaurin coefficients have been obtained. However, the problem of determining sharp estimates for the higher-order coefficients $|\eta_\kappa|$ for $\kappa \geq 3$, $\kappa \in \mathbb{N}$, remains open and has been highlighted in several studies (see, for example, [12]–[32]).

The present work does not resolve this general higher-order coefficient problem. The main obstruction is that, from the third order onward, the coefficient comparison simultaneously involves several coefficients of the two Schwarz functions associated with $\mathtt{f}$ and $\hbar$, together with nonlinear mixed products of the preceding Taylor coefficients. The elementary Carathéodory estimates $|\ell_k| \leq 2$ and $|\jmath_k| \leq 2$, used below, do not retain the dependence among these coefficients and therefore become too coarse to identify optimal bounds. Moreover, the inverse coefficients are increasingly complicated polynomials in $\eta_2, \ldots, \eta_\kappa$, so the two subordination conditions yield a coupled nonlinear extremal problem rather than independent coefficient inequalities. Progress toward $|\eta_\kappa|$, $\kappa \geq 4$, will consequently require sharper coefficient-body parametrizations for the Carathéodory class, or an appropriate variational or Loewner-type method that preserves these dependencies.

2. The bi-univalent Class $\Sigma^*(\Theta; \mathtt{q})$

This section presents a basic class of bi-univalent functions linked with Shell-Shaped Region using the principal of subordination and using $\mathtt{q}$-calculus where $\mathtt{q} \in (0, 1)$.

Definition 2.1. A bi-univalent function $\mathtt{f}$ of the form (1.1) is said to be in the class $\Sigma^*(\Theta; \mathtt{q})$ if the following subordinations are satisfied:

$$\psi_1(\mathtt{f}(\zeta)) = \frac{\zeta\, \partial_q \mathtt{f}(\zeta)}{\mathtt{f}(\zeta)} \prec \frac{\zeta \langle 2; \mathtt{q} \rangle}{\zeta\,(1-\mathtt{q})+2} + \sqrt{1 + \left(\frac{\langle 2; \mathtt{q} \rangle \zeta}{\zeta\,(1-\mathtt{q})+2} \right)^2} = \Theta(\zeta; \mathtt{q}), \tag{2.1}$$

and

$$\psi_2(\hbar(\varpi)) = \frac{\varpi \partial_q \hbar(\varpi)}{\hbar(\varpi)} \prec \frac{\varpi \langle 2; \mathtt{q} \rangle}{\varpi\,(1-\mathtt{q})+2} + \sqrt{1 + \left(\frac{\langle 2; \mathtt{q} \rangle \varpi}{\varpi\,(1-\mathtt{q})+2} \right)^2} = \Theta(\varpi; \mathtt{q}), \tag{2.2}$$

with $\Theta(0) = 1$.

This class is nonempty. due to the function

$$\mathtt{f}_*(\zeta) = \frac{\zeta}{1 - \alpha \zeta}, \qquad |\alpha| < 2.$$

That has the inverse inverse and given by

$$\hbar_*(\varpi) = \mathtt{f}_*^{-1}(\varpi) = \frac{\varpi}{1 + \alpha \varpi}.$$

Appling the derivative operator on the function with their inverse, one can find easily that

$$\begin{aligned} \psi_1(\mathtt{f}_*(\zeta)) &= \frac{\zeta\, \partial_q \mathtt{f}_*(\zeta)}{\mathtt{f}_*(\zeta)} = \frac{1}{1 - \alpha\, \mathtt{q}\, \zeta}, \\ \psi_2(\hbar_*(\varpi)) &= \frac{\varpi\, \partial_q \hbar_*(\varpi)}{\hbar_*(\varpi)} = \frac{1}{1 + \alpha\, \mathtt{q}\, \varpi}. \end{aligned} \tag{2.3}$$

Moreover, for each $\zeta \in \mho$, we have the identity

$$\psi_1(\mathtt{f}_*(-\zeta)) = \psi_2\big(\mathtt{f}_*^{-1}(\zeta)\big)\,,$$

and consequently,

$$\psi_1(\mathtt{f}_*(\mho)) = \psi_2\big(\mathtt{f}_*^{-1}(\mho)\big)\,.$$

This implies $\psi(\zeta) \in \Sigma_{\mathtt{q}}(\Theta)$, and there exist suitable values of the parameter χ such that

$$\Sigma_{\mathtt{q}}(\Theta) \setminus \{\psi_1(\mathtt{f}_*)\} \neq \varnothing.$$

For illustrative purposes, the obtained results are visualized using the GeoGebra computational software. Although the corresponding GeoGebra scripts are not included, the software is employed to plot the boundary $\partial\mho$ through the mappings induced by the functions Θ, ψ_1, and ψ_2, as depicted in Figure 3. These graphical illustrations remain valid for admissible choices of the parameters satisfying $|\alpha| < 0.27$ and $\mathtt{q} \in (0,1)$.

It is important to emphasize that the function Θ is univalent in the unit disk $\mho$. As a consequence, the subordination relations

$$\psi_1(\mathtt{f}_*(\zeta)) \prec \Theta(\zeta) \quad \text{and} \quad \psi_2\big(\mathtt{f}_*^{-1}(\omega)\big) \prec \Theta(\omega)$$

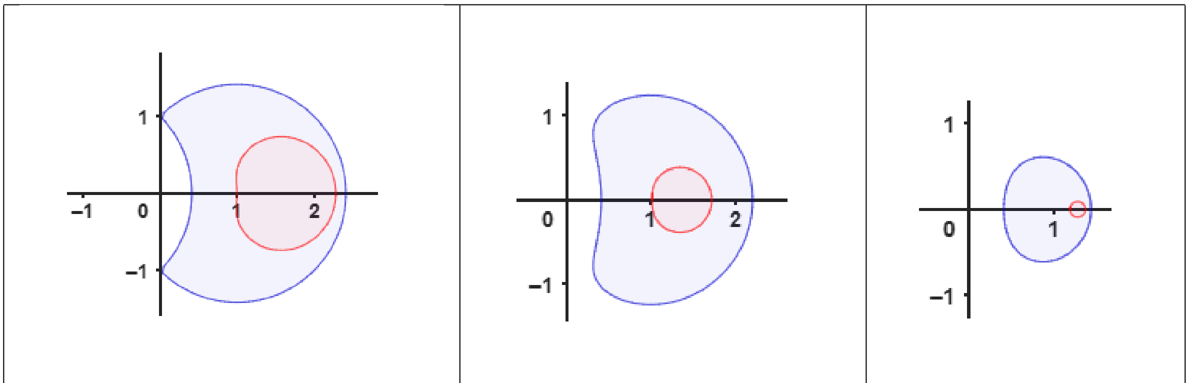
are satisfied. These follow directly from the normalization conditions

$$\psi_1(\mathtt{f}_*(0)) = \psi_2\big(\mathtt{f}_*^{-1}(0)\big) = \Theta(0) = 1,$$

together with the inclusion relations

$$\psi_1(\mathtt{f}_*(\mho)) \subset \Theta(\mho), \qquad \psi_2(\hbar_*(\mho)) \subset \Theta(\mho).$$

Figure 3. Blue curves show $\Theta(e^{i\theta};\mathtt{q})$ for various values of $\mathtt{q}$, while the red curve represents $\psi_1(\mathtt{f}_*(\mho))$.



(a) $\alpha = 0.58,\ \mathtt{q} \to 1^-$	(b) $\alpha = 0.21,\ \mathtt{q} = 0.87$	(c) $\alpha = 0.27,\ \mathtt{q} = 0.38$

Example 2.2
Consider the limiting case as $\mathtt{q} \to 1^-$, for which the class $\Sigma^*(\Theta;\mathtt{q})$ naturally reduces to the classical bi-starlike class $\Sigma^*(\Theta)$. An analytic function $\mathtt{f}$ is said to belong to the class $\Sigma^*(\Theta)$ provided that its logarithmic derivative and that of its inverse function satisfy the subordination conditions

$$\frac{\zeta \mathtt{f}'(\zeta)}{\mathtt{f}(\zeta)} \prec \zeta + \sqrt{1+\zeta^2}, \qquad \frac{\varpi\, \hbar'(\varpi)}{\hbar(\varpi)} \prec \varpi + \sqrt{1+\varpi^2},$$

for all $\zeta, \omega \in \mho$.

3. Coefficient Estimates for the Initial Terms of $\Sigma^*(\Theta;\mathtt{q})$

In the first theorem, we derive upper bounds for the initial coefficients in the Taylor–Maclaurin expansion of bi-univalent functions belonging to the class $\Sigma^*(\Theta;\mathtt{q})$, as introduced in Definition 2.1.

Theorem 3.1
Let $\mathtt{f} \in \Sigma$, given by (1.1), belong to the class $\Sigma^*(\Theta;\mathtt{q})$. Then

$$|\eta_2| \leq \frac{\lfloor 2 \rceil_{\mathtt{q}}}{\mathtt{q}\sqrt{2\lfloor 2 \rceil_{\mathtt{q}}^2 + 3 - \mathtt{q}}},$$

and

$$|\eta_3| \leq \frac{\lfloor 2 \rceil_{\mathtt{q}}^2}{\mathtt{q}^2\left(2\lfloor 2 \rceil_{\mathtt{q}}^2 + 3 - \mathtt{q}\right)} + \frac{1}{2\,\mathtt{q}\,\lfloor 2 \rceil_{\mathtt{q}}}.$$

Proof
Let $\mathtt{f} \in \Sigma^*(\Theta;\mathtt{q})$. By Definition 2.1, there exist two analytic functions Υ_1 and Υ_2 such that

$$\Upsilon_1(0) = \Upsilon_2(0) = 0, \qquad |\Upsilon_1(\zeta)| < 1, \quad |\Upsilon_2(\varpi)| < 1, \qquad \zeta, \varpi \in \mho.$$

Define the functions

$$P_1(\zeta) = \frac{1+\Upsilon_1(\zeta)}{1-\Upsilon_1(\zeta)} = 1 + \ell_1\zeta + \ell_2\zeta^2 + \ell_3\zeta^3 + \cdots, \quad (\zeta \in \mho),$$

and

$$P_2(\varpi) = \frac{1+\Upsilon_2(\varpi)}{1-\Upsilon_2(\varpi)} = 1 + \jmath_1\varpi + \jmath_2\varpi^2 + \jmath_3\varpi^3 + \cdots, \quad (\varpi \in \mho).$$

Clearly, $P_1, P_2 \in \mathcal{P}$, and therefore $|\ell_k| \leq 2$ and $|\jmath_k| \leq 2$ for $k \geq 1$.
From these definitions, we obtain

$$\Upsilon_1(\zeta) = \frac{\ell_1}{2}\zeta + \left(\frac{\ell_2}{2} - \frac{\ell_1^2}{4}\right)\zeta^2 + O(\zeta^3),$$

which yields

$$\Theta(\Upsilon_1(\zeta);\mathtt{q}) = 1 + \frac{\lfloor 2 \rceil_{\mathtt{q}}\,\ell_1}{4}\,\zeta + \frac{\lfloor 2 \rceil_{\mathtt{q}}\left(8\ell_2 + (\mathtt{q}-3)\ell_1^2\right)}{32}\,\zeta^2 + O(\zeta^3). \tag{3.1}$$

Similarly,

$$\Theta(\Upsilon_2(\varpi);\mathtt{q}) = 1 + \frac{\lfloor 2 \rceil_{\mathtt{q}}\,\jmath_1}{4}\,\varpi + \frac{\lfloor 2 \rceil_{\mathtt{q}}\left(8\jmath_2 + (\mathtt{q}-3)\jmath_1^2\right)}{32}\,\varpi^2 + O(\varpi^3). \tag{3.2}$$

On the other hand, the $\mathtt{q}$-derivatives of $\mathtt{f}$ and its inverse $\hbar$ satisfy

$$\psi_1(\mathtt{f}(\zeta)) = \frac{\zeta\,\partial_q\mathtt{f}(\zeta)}{\mathtt{f}(\zeta)} = 1 + \mathtt{q}\,\eta_2\zeta + \mathtt{q}\Big(\lfloor 2 \rceil_{\mathtt{q}}\,\eta_3 - \eta_2^2\Big)\zeta^2 + \cdots, \tag{3.3}$$

and

$$\psi_2(\hbar(\varpi)) = \frac{\varpi\partial_q\hbar(\varpi)}{\hbar(\varpi)} = 1 - \mathtt{q}\,\eta_2\varpi + \mathtt{q}\Big((1+2\mathtt{q})\eta_2^2 - \lfloor 2 \rceil_{\mathtt{q}}\,\eta_3\Big)\,\varpi^2 + \cdots. \tag{3.4}$$

Comparing coefficients in (3.1)–(3.4), we obtain

$$\mathtt{q}\,\eta_2 = \frac{\lfloor 2 \rceil_{\mathtt{q}}}{4}\ell_1, \tag{3.5}$$

$$-\mathtt{q}\,\eta_2 = \frac{\lfloor 2 \rceil_{\mathtt{q}}}{4}\jmath_1, \tag{3.6}$$

$$\mathtt{q}\Big(\lfloor 2 \rceil_{\mathtt{q}}\,\eta_3 - \eta_2^2\Big) = \frac{8\ell_2 + (\mathtt{q}-3)\ell_1^2}{32}, \tag{3.7}$$

and

$$\mathsf{q}\Big((1+2\mathsf{q})\eta_2^2 - \lfloor 2\rceil_{\mathsf{q}}\,\eta_3\Big) = \frac{8\jmath_2 + (\mathsf{q}-3)\jmath_1^2}{32}. \tag{3.8}$$

From (3.5) and (3.6), we have $\ell_1 = -\jmath_1$ and hence

$$\eta_2^2 = \frac{\lfloor 2\rceil_{\mathsf{q}}^2}{32\mathsf{q}^2}\left(\ell_1^2 + \jmath_1^2\right) \quad \Longleftrightarrow \quad \frac{32\,\mathsf{q}^2}{\lfloor 2\rceil_{\mathsf{q}}^2}\eta_2^2 = \left(\ell_1^2 + \jmath_1^2\right). \tag{3.9}$$

Adding (3.7) and (3.8), and using (3.9), we obtain

$$\eta_2^2 = \frac{\lfloor 2\rceil_{\mathsf{q}}^2\,(\ell_2 + \jmath_2)}{4\,\mathsf{q}^2\left(2\lfloor 2\rceil_{\mathsf{q}}^2 + 3 - \mathsf{q}\right)}. \tag{3.10}$$

Since $|\ell_2| < 2$ and $|\jmath_2| < 2$, it follows that $|\ell_2 + \jmath_2| < 4$, and therefore

$$|\eta_2| \leq \frac{\lfloor 2\rceil_{\mathsf{q}}}{\mathsf{q}\sqrt{2\lfloor 2\rceil_{\mathsf{q}}^2 + 3 - \mathsf{q}}}.$$

Next, subtracting (3.8) from (3.7) and using $\ell_1^2 = \jmath_1^2$, we obtain

$$\eta_3 = \eta_2^2 + \frac{\ell_2 - \jmath_2}{8\,\mathsf{q}\,\lfloor 2\rceil_{\mathsf{q}}}. \tag{3.11}$$

Using (3.10), yields

$$\eta_3 = \frac{\lfloor 2\rceil_{\mathsf{q}}^2\,(\ell_2 + \jmath_2)}{4\,\mathsf{q}^2\left(2\lfloor 2\rceil_{\mathsf{q}}^2 + 3 - \mathsf{q}\right)} + \frac{\ell_2 - \jmath_2}{8\,\mathsf{q}\,\lfloor 2\rceil_{\mathsf{q}}}$$

Applying the bounds $|\ell_2| < 2$ and $|\jmath_2| < 2$ completes the proof:

$$|\eta_3| \leq \frac{\lfloor 2\rceil_{\mathsf{q}}^2}{\mathsf{q}^2\left(2\lfloor 2\rceil_{\mathsf{q}}^2 + 3 - \mathsf{q}\right)} + \frac{1}{2\,\mathsf{q}\,\lfloor 2\rceil_{\mathsf{q}}}.$$

□

Corollary 3.2
Let $\mathtt{f} \in \Sigma$, given by (1.1), belong to the class $\Sigma^*(\Theta)$. Then

$$|\eta_2| \leq \frac{2}{\sqrt{10}}, \qquad \text{and} \quad |\eta_3| \leq \frac{13}{20}.$$

Remark 3.3. Since $\lfloor 2\rceil_{\mathsf{q}} \to 2$ as $\mathsf{q} \to 1^-$, the two estimates in Theorem 3.1 reduce exactly to

$$|\eta_2| \leq \frac{2}{\sqrt{10}} \quad \text{and} \quad |\eta_3| \leq \frac{13}{20},$$

respectively. Thus, the quantum estimates consistently recover the corresponding classical estimates obtained from the same coefficient-comparison argument for the shell-like bi-starlike class $\Sigma^*(\Theta)$. These classical estimates, like their q-analogues, are not asserted to be sharp: the separate use of the Carathéodory bounds for the two auxiliary functions does not prove that equality can occur simultaneously while both bi-univalence and inverse subordination are preserved. In particular, the limit $\mathsf{q} \to 1^-$ reduces the formulas to known classical upper estimates, but not to known sharp results. No extremal function establishing equality in either $2/\sqrt{10}$ or $13/20$ is presently available for this shell-like bi-starlike class. Consequently, passage to the classical limit preserves the validity of the estimates but does not improve their sharpness status; determining the best possible classical and quantum bounds remains an open extremal problem.

Remark 3.4. The bounds in Theorem 3.1 are upper estimates and their sharpness has not been established. In particular, no pair of admissible Schwarz functions is presently known for which all applications of the triangle inequality and the Carathéodory coefficient bounds in the proof become equalities while the resulting function and its inverse both satisfy the defining subordinations. This simultaneous compatibility is the principal obstacle to constructing an extremal function. Possible improvements may be obtained by replacing the coefficient-wise estimates $|\ell_2|, |\jmath_2| \leq 2$ with a joint Carathéodory coefficient parametrization and then optimizing under the relation $\ell_1 = -\jmath_1$, or by formulating the problem through Schur parameters. We have therefore removed the terms “best” and “sharp” wherever they referred to Theorem 3.1.

4. The Fekete–Szegö Functional

In their seminal 1933 work, Fekete and Szegö [33] obtained an exact bound for the functional $\mu\eta_2^2 - \eta_3$ for real parameters $\mu \in [0, 1]$, a result that has since become known as the classical Fekete–Szegö inequality. Extending such sharp estimates to more general settings-namely, for functions $\mathtt{f} \in \mathscr{A}$ belonging to compact families and for arbitrary complex values of μ remains a challenging and nontrivial problem.

Motivated by the approach introduced by Zaprawa [34], we investigate the Fekete–Szegö functional for functions in the class $\Sigma^*(\Theta; \mathtt{q})$, deriving corresponding bounds within the framework of $\mathtt{q}$-calculus.

Theorem 4.1
Let $\mathtt{f} \in \Sigma$ be given by (1.1) and suppose that $\mathtt{f} \in \Sigma^*(\Theta; \mathtt{q})$. For $\mu \in \mathbb{R}$, we have

$$\left|\eta_3 - \mu\,\eta_2^2\right| \leq \begin{cases} \dfrac{1}{2\,\mathtt{q}\,\lfloor 2\rceil_{\mathtt{q}}}, & |1-\mu| \leq \dfrac{\mathtt{q}\left|2\lfloor 2\rceil_{\mathtt{q}}^2 + 3 - \mathtt{q}\right|}{2\lfloor 2\rceil_{\mathtt{q}}^3}, \\[2ex] \dfrac{\lfloor 2\rceil_{\mathtt{q}}^2\,|1-\mu|}{\mathtt{q}^2\left|2\lfloor 2\rceil_{\mathtt{q}}^2 + 3 - \mathtt{q}\right|}, & |1-\mu| \geq \dfrac{\mathtt{q}\left|2\lfloor 2\rceil_{\mathtt{q}}^2 + 3 - \mathtt{q}\right|}{2\lfloor 2\rceil_{\mathtt{q}}^3}. \end{cases}$$

Proof
Since $\mathtt{f} \in \Sigma^*(\Theta; \mathtt{q})$, we obtain from (3.10)–(3.11) that

$$\begin{aligned} \eta_3 - \mu\eta_2^2 &= \frac{\lfloor 2\rceil_{\mathtt{q}}^2\,(1-\mu)\,(\ell_2 + \jmath_2)}{4\,\mathtt{q}^2\left(2\lfloor 2\rceil_{\mathtt{q}}^2 + 3 - \mathtt{q}\right)} + \frac{\ell_2 - \jmath_2}{8\,\mathtt{q}\,\lfloor 2\rceil_{\mathtt{q}}} \\ &= \frac{1}{4\,\mathtt{q}}\left[\left(\mathscr{D}(\mu) + \frac{1}{2\,\lfloor 2\rceil_{\mathtt{q}}}\right)\ell_2 + \left(\mathscr{D}(\mu) - \frac{1}{2\,\lfloor 2\rceil_{\mathtt{q}}}\right)\jmath_2\right], \end{aligned}$$

where

$$\mathscr{D}(\mu) := \frac{\lfloor 2\rceil_{\mathtt{q}}^2\,(1-\mu)}{\mathtt{q}\left(2\lfloor 2\rceil_{\mathtt{q}}^2 + 3 - \mathtt{q}\right)}.$$

Since $P_1, P_2 \in \mathcal{P}$, we have $|\ell_2| \leq 2$ and $|\jmath_2| \leq 2$. Therefore, by the triangle inequality,

$$\begin{aligned} \left|\eta_3 - \mu\eta_2^2\right| &\leq \frac{1}{4\,\mathtt{q}}\left(\left|\mathscr{D}(\mu) + \frac{1}{2\lfloor 2\rceil_{\mathtt{q}}}\right||\ell_2| + \left|\mathscr{D}(\mu) - \frac{1}{2\lfloor 2\rceil_{\mathtt{q}}}\right||\jmath_2|\right) \\ &\leq \frac{1}{2\,\mathtt{q}}\left(\left|\mathscr{D}(\mu) + \frac{1}{2\lfloor 2\rceil_{\mathtt{q}}}\right| + \left|\mathscr{D}(\mu) - \frac{1}{2\lfloor 2\rceil_{\mathtt{q}}}\right|\right). \end{aligned}$$

Using the elementary estimate

$$|x+a|+|x-a|=2\max\{|x|,|a|\}\qquad (x\in\mathbb{R},\ a\geq 0),$$

with $x=\mathscr{D}(\mu)$ and $a=\dfrac{1}{2\lfloor 2\rceil_{\mathtt{q}}}$, we obtain

$$|\eta_3-\mu\eta_2^2|\leq \frac{1}{\mathtt{q}}\max\left\{|\mathscr{D}(\mu)|,\frac{1}{2\lfloor 2\rceil_{\mathtt{q}}}\right\}.$$

Equivalently,

$$|\eta_3-\mu\,\eta_2^2|\leq \begin{cases}\dfrac{1}{2\,\mathtt{q}\,\lfloor 2\rceil_{\mathtt{q}}}, & |\mathscr{D}(\mu)|\leq \dfrac{1}{2\lfloor 2\rceil_{\mathtt{q}}},\\[2mm] \dfrac{|\mathscr{D}(\mu)|}{\mathtt{q}}, & |\mathscr{D}(\mu)|\geq \dfrac{1}{2\lfloor 2\rceil_{\mathtt{q}}}.\end{cases}$$

Since

$$|\mathscr{D}(\mu)|=\frac{\lfloor 2\rceil_{\mathtt{q}}^2\,|1-\mu|}{\mathtt{q}\,|2\lfloor 2\rceil_{\mathtt{q}}^2+3-\mathtt{q}|},$$

the above estimate can be rewritten in terms of $|1-\mu|$ as

$$|\eta_3-\mu\,\eta_2^2|\leq \begin{cases}\dfrac{1}{2\,\mathtt{q}\,\lfloor 2\rceil_{\mathtt{q}}}, & |1-\mu|\leq \dfrac{\mathtt{q}\,|2\lfloor 2\rceil_{\mathtt{q}}^2+3-\mathtt{q}|}{2\lfloor 2\rceil_{\mathtt{q}}^3},\\[2mm] \dfrac{\lfloor 2\rceil_{\mathtt{q}}^2\,|1-\mu|}{\mathtt{q}^2\,|2\lfloor 2\rceil_{\mathtt{q}}^2+3-\mathtt{q}|}, & |1-\mu|\geq \dfrac{\mathtt{q}\,|2\lfloor 2\rceil_{\mathtt{q}}^2+3-\mathtt{q}|}{2\lfloor 2\rceil_{\mathtt{q}}^3}.\end{cases}$$

□

Corollary 4.2
Let $\mathtt{f}\in\Sigma$ be given by (1.1) and suppose that $\mathtt{f}\in\Sigma^*(\Theta)$. Then, for $\mu\in\mathbb{R}$, we have

$$\left|\eta_3-\mu\,\eta_2^2\right|\leq \begin{cases}\dfrac{1}{4}, & |1-\mu|\leq \dfrac{5}{8},\\[2mm] \dfrac{2|1-\mu|}{5}, & |1-\mu|\geq \dfrac{5}{8}.\end{cases}$$

5. Conclusion

In this paper, a bi-starlike class of analytic functions associated with a shell-shaped region has been introduced by employing the subordination principle within the framework of quantum ($\mathtt{q}$-) calculus for $\mathtt{q}\in(0,1)$. By applying the well-known Ma–Minda methodology, a novel subclass of bi-univalent functions has been investigated. The non-emptiness of the proposed class has been established through a geometric approach, confirming the validity of the construction.

Furthermore, explicit upper bounds for the Taylor–Maclaurin coefficients $|\eta_2|$ and $|\eta_3|$ have been derived, and the associated Fekete–Szegö functional has been investigated. The sharpness of the coefficient estimates has not been established, because equality in the separate Carathéodory estimates need not be simultaneously compatible with both defining subordinations. The geometric behavior of the introduced functions has been further illustrated through graphical representations of shell-shaped image domains, which support and complement the theoretical findings.

The contribution should therefore be understood as a specific $\mathtt{q}$-deformation within the established Ma–Minda framework, rather than as a new general method. Its added content consists of a dominant shell mapping that varies with $\mathtt{q}$, the corresponding deformation of the image domain, the simultaneous treatment of the function and its inverse, and explicit parameter-dependent estimates. The recovery of the classical shell-like class as $\mathtt{q} \to 1^-$ verifies the consistency of the construction. It does not, by itself, establish sharpness or solve the broader coefficient problem.

The extension to higher-order coefficients remains open. At those orders, the coefficient equations contain coupled mixed terms from the function, its inverse, and two auxiliary Schwarz functions; coefficient-wise Carathéodory bounds discard the dependencies needed for a sharp optimization. Joint Carathéodory parametrizations, Schur parameters, and variational methods are natural directions for improving the present estimates and studying extremal functions.

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