STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **15**, June 2026, pp 4827–4842.
Published online in International Academic Press (www.IAPress.org)

A New Untied Prakaamy Distribution: Properties and Application

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Abstract This study introduces the Unit Prakaamy Distribution (UPD), a bounded version of the Prakaamy distribution, and examines its statistical and reliability properties. The theoretical properties of the UPD include mean, variance, skewness, kurtosis, non-central moments, Bonferroni and Lorenz curves, as well as the survival and hazard functions. Measures of uncertainty, such as Shannon and Rényi entropy, are also considered. Parameter estimation is performed using Maximum Likelihood Estimation (MLE), Least Squares Estimation (LSE), Weighted Least Squares Estimation (WLS), and the Cramér–von Mises (CvM) method, with Monte Carlo simulations showing that MLE consistently provides the most accurate estimates in terms of bias and mean squared error. The applicability of the UPD is further demonstrated through real-world datasets, where it outperforms competing models, highlighting its flexibility and robustness for modeling data constrained to the unit interval.

Keywords Unit distributions,Prakaamy distribution,Parameter estimation,Simulation,Bounded data.

DOI: 10.19139/soic-2310-5070-3395

1. Introduction

The introduction of novel probability distributions has become a focal point of research in recent years because ofe of their pivotal role in modeling complex real-world phenomena. In various applied fields such as economics, biostatistics, environmental sciences, and psychometrics, bounded random variables frequently arise that encompass proportions, rates, indices, and test scores. However, the current statistical literature is predominantly focused on distributions with unbounded support, revealing a substantial gap in the availability of flexible distributions for bounded data. This gap has stimulated significant research efforts in developing distributions defined on bounded intervals, primarily through the use of transformational techniques on established parent distributions. For further information, the reader is referred to recent studies[31].

Among the probability distributions defined on the (0, 1) interval, the beta distribution remains the most prominent and widely used due to its flexibility in modeling bounded data. Although the beta distribution has proven effective in various applications, alternative distributions such as the Topp–Leone distribution[2],Kumaraswamy distribution, [3] have also been proposed and studied. In recent years, there has been a growing interest among statisticians in constructing nit interval distributions derived from transformations of existing continuous distributions. Notable examples include the Unit-Weibull distribution[4],Unit-Lindley distribution[5], Unit-Birnbaum-Saunders distribution[6], Unit Gamma/Gompertz distribution[7], unit-Half-Normal distribution [8],unit Teissier distribution [9],Half-Logistic unit-Gompertz Type-I distribution[34], unit Exponentiated Fréchet distribution[11], unit-Xgamma distribution[12], unit Half-Logistic Geometric distribution[13], unit-Muth distribution [14],unit-Exponentiated Lomax distribution[15], unit-Chen distribution

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

[16],unit Inverse Weibull distribution [17], unit Gumbel Type-II distribution [18],unit Exponential distribution [19],unit Modified Burr-III distribution [20],unit–Power Burr X distribution [21],unit Exponential Pareto distribution [22],unit Compound Rayleigh disribution [23],unit Extended Exponential distribution [24],Two-Parameter unit Bilal distribution [25],unit Zeghdoudi distribution [26].Consequently, these developments have significantly enriched the current statistical literature. These transformed models offer enhanced flexibility for modeling bounded datasets compared to their original distributions, enabling more precise and adaptable data fitting across diverse fields, including health, environmental studies, engineering, and finance.

Although numerous alternative distributions have been proposed and extensively studied, there remains no definitive consensus on the most suitable mode. As a result, the pursuit of innovative and flexible statistical models continues to be a central focus in contemporary research. Within this scope, the present study contributes by proposing a new unit distribution, constructed based on the Prakaamy distribution (PD).

The Prakaamy distribution proposed by [27], is a one-parameter lifetime model characterized by a single scale parameter θ. This distribution represents a mixture of the exponential distribution (with a scale parameter) and the gamma distribution (with both shape and scale parameters). Shukla's study demonstrated that the PD provides superior fitting performance compared to the Pranav, Ishita, Lindley, and exponential distributions when applied to specific lifetime datasets.The probability density function (PDF) of the PD is given by:

$$f(x) = \frac{\theta^6}{120+\theta^5}(1+x^5)e^{-\theta x}, \quad x>0,\ \theta>0 \tag{1}$$

The cumulative distribution function (CDF) of the PD is given by:

$$F(x) = 1-\left[1+\frac{\theta x\left(\theta^4x^4+5\theta^3x^3+20\theta^2x^2+60\theta x+120\right)}{120+\theta^5}\right]e^{-\theta x}, \quad x>0,\ \theta>0 \tag{2}$$

To improve the flexibility of the PD, various extensions have been proposed, such as the Poisson PD[28],Truncated PD [29]and a new version of PD[30].Although these extensions offer enhanced flexibility, they are not specifically tailored for unit data bounded within the interval [0,1].

The primary objective of this study is to introduce the Unit Prakaamy Distribution (UPD), which represents a transformation of the original Prakaamy Distribution (PD) adapted for proportional data. This work aims to improve the modeling of bounded datasets by presenting a more flexible distribution that preserves the simplicity of the PD while extending its applicability to data constrained to the unit interval.

Motivation for Studying the Distribution

The Unit Prakaamy Distribution (UPD) offers several theoretical and practical advantages for modeling data constrained to the unit interval. Its key properties are available in closed form, facilitating analysis, while classical estimation methods can be applied to fit the model efficiently. Monte Carlo simulations indicate that Maximum Likelihood Estimation (MLE) generally provides the most reliable estimates. Overall, the UPD is a flexible and robust model, capable of fitting three distinct unit datasets.

The structure of this article is organized as follows. Section 2 introduces the Unit Prakaamy Distribution (UPD) and its formulation. In Section 3, the statistical properties of the UPD are examined, including mean, variance, skewness, kurtosis, non-central moments, and Bonferroni and Lorenz curves. Section 4 focuses on reliability measures, specifically the survival and hazard rate functions, while Section 5 addresses uncertainty measures such as Shannon and Rényi entropy. Section 6 presents the parameter estimation methods, including Maximum Likelihood Estimation (MLE), Least Squares Estimation (LSE), Weighted Least Squares (WLS), and the Cramér–von Mises (CvM) method. Section 7 summarizes the Monte Carlo simulation studies conducted to assess the performance and accuracy of these estimators. Section 8 illustrates the application of the UPD to real-world datasets. Finally, Section 9 provides concluding remarks and potential directions for future research.

2. Unit Prakaamy Distribution

Transformation methods have proven to be effective tools for constructing flexible probability distributions defined on bounded supports. According to Condino and Domma [31], numerous recently proposed distributions on the unit interval can be derived by applying straightforward transformations to continuous random variables.

Typical transformations used in this context include $V = \frac{Y}{1+Y}$ and $V = e^{-Y}$ for positive random variables, and $V = \frac{1}{1+e^{-Y}}$ for real-valued variables.

Motivated by this framework, we consider the transformation:

$$y = e^{-\theta x}, \quad \theta > 0.$$

Applying this transformation to the baseline distribution defined in Equation (1), and incorporating the Jacobian of the transformation, the resulting probability density function (PDF) is given by:

$$f_Y(y) = \frac{1}{120 + \theta^5} \left(\theta^5 - (\ln y)^5 \right), \quad 0 < y < 1, \ \theta > 0. \tag{3}$$

The cumulative distribution function (CDF) is defined as:

$$F(y;\theta) = 1 - \left[\frac{1}{\theta} - \frac{\ln y \left((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60 \ln y + 120 \right)}{120\theta + \theta^6} \right], \quad 0 < y < 1, \ \theta > 0. \tag{4}$$

Figures 1 and 2 illustrate the probability density function (PDF) and cumulative distribution function (CDF) of the UPD under varying values of the shape parameter θ. The results highlight the model's flexibility in capturing diverse distributional shapes. As θ increases, the PDF shows sharper peaks near zero, while the CDF reveals faster accumulation at lower values, indicating stronger concentration in the lower tail.

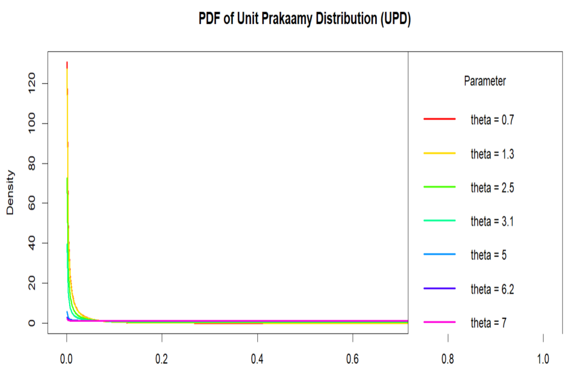

Figure 1. PDF of UPD.

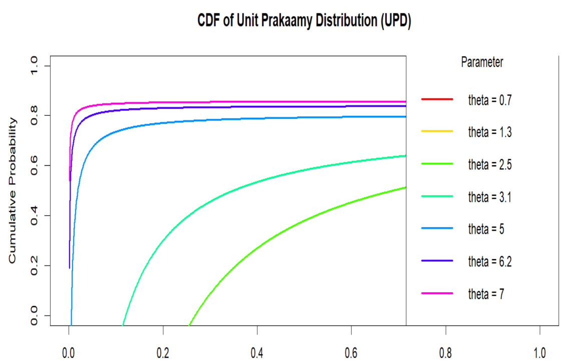

Figure 2. CDF of UPD.

3. Statistical properties

This section explores essential measures that reveal the probabilistic nature of the UPD.

3.1. Mean

Expected Value of the UPD Distribution

The mean of the UPD reflects the average value within its support $[0, 1]$. It is formally defined as the expected value and can be computed using the following integral:

$$\mu = \mathbb{E}(Y) = \int_0^1 y\, f(y;\theta)\, dy$$

Substituting the density function $f(y;\theta)$ into the expression, we obtain:

$$\mathbb{E}(Y) = \frac{1}{120+\theta^5}\int_0^1 y\left(\theta^5 - (\ln y)^5\right)\, dy$$

Derivation

We distribute the integral into two components:

$$\mathbb{E}(Y) = \frac{1}{120+\theta^5}\left[\int_0^1 y\theta^5\, dy - \int_0^1 y(\ln y)^5\, dy\right]$$

Let:

$$K_1 = \int_0^1 y\theta^5\, dy = \theta^5\int_0^1 y\, dy = \frac{\theta^5}{2},$$
$$K_2 = \int_0^1 y(\ln y)^5\, dy$$

To evaluate K_2, we apply integration by parts repeatedly, reducing the power of the logarithmic term at each step. The process results in:

$$K_2 = \frac{-15}{8}$$

Substituting back, we get:

$$\mathbb{E}(Y) = \frac{1}{120+\theta^5}\left(\frac{\theta^5}{2} + \frac{15}{8}\right)$$

Combining terms under a common denominator:

$$\mathbb{E}(Y) = \frac{8\theta^5+30}{16(120+\theta^5)}$$

Thus, the mean of the UPD distribution is given by:

$$\mathbb{E}(Y) = \frac{1}{120+\theta^5}\left(\frac{8\theta^5+30}{16}\right) \tag{5}$$

3.2. variance

The variance of the UPD distribution quantifies the spread of the data around the mean. It is formally defined as:

$$\text{Var}(Y) = \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2,$$

where the second moment is expressed as:

$$\mathbb{E}(Y^2) = \int_0^1 y^2 f(y;\theta)\, dy.$$

By substituting the form of the probability density function $f(y;\theta)$, the second moment becomes:

$$\mathbb{E}(Y^2) = \frac{1}{120+\theta^5}\int_0^1 \left(y^2\theta^5 - y^2(\ln y)^5\right)\, dy = \frac{1}{120+\theta^5}(K_3 - K_4),$$

where

$$K_3 = \int_0^1 y^2\theta^5\, dy = \frac{\theta^5}{3}, \qquad K_4 = \int_0^1 y^2(\ln y)^5\, dy = -\frac{40}{243}.$$

Therefore, the second moment is given by:

$$\mathbb{E}(Y^2) = \frac{81\theta^5 + 40}{243(120+\theta^5)}$$

Recalling the expression for the mean:

$$\mathbb{E}(Y) = \frac{8\theta^5 + 30}{16(120+\theta^5)},$$

the variance of the UPD distribution is obtained as:

$$\text{Var}(Y) = \frac{81\theta^5 + 40}{243(120+\theta^5)} - \left(\frac{8\theta^5 + 30}{16(120+\theta^5)}\right)^2 \tag{6}$$

3.3. Skewness

Skewness quantifies the asymmetry of a distribution around its mean. It is defined by:

$$\gamma_3 = \frac{\mu_3}{\sigma^3},$$

where μ_3 is the third central moment and σ is the standard deviation. For a continuous distribution, the third central moment is given by:

$$\mu_3 = \mu_3' + 2\mu^3 - 3\mu\mu_2',$$

where $\mu = \mathbb{E}(Y)$ is the mean, $\mu_2' = \mathbb{E}(Y^2)$ is the second raw moment, and $\mu_3' = \mathbb{E}(Y^3)$ is the third raw moment. Therefore, skewness can also be expressed as:

$$\gamma_3 = \frac{\mu_3' + 2\mu^3 - 3\mu\mu_2'}{(\mu_2' - \mu^2)^{3/2}}.$$

For the Unit Prakaamy Distribution (UPD), all the necessary moments have been derived analytically and substituted into the above formula, yielding the following closed-form expression:

$$\gamma_3 = \frac{\dfrac{512\theta^5 - 60}{2048(120+\theta^5)} + 2\left(\dfrac{8\theta^5+30}{16(120+\theta^5)}\right)^3 - 3\left(\dfrac{8\theta^5+30}{16(120+\theta^5)}\right)\left(\dfrac{\theta^5-120}{729(120+\theta^5)}\right)}{\left[\dfrac{\theta^5-120}{729(120+\theta^5)} - \left(\dfrac{8\theta^5+30}{16(120+\theta^5)}\right)^2\right]^{3/2}}. \tag{7}$$

3.4. Kurtosis

Kurtosis measures the "tailedness" or peakedness of a probability distribution relative to the normal distribution. It is defined as:

$$\gamma_4 = \frac{\mu_4}{\sigma^4},$$

where μ_4 is the fourth central moment and σ is the standard deviation. For a continuous random variable, the fourth central moment is given by:

$$\mu_4 = \int_0^1 (y-\mu)^4 f(y)\, dy,$$

which can also be expressed in terms of raw moments as:

$$\mu_4 = \mu_4' + 6\mu^2\mu_2' - 4\mu\mu_3' - 3\mu^4,$$

where $\mu = \mathbb{E}(Y)$, $\mu_2' = \mathbb{E}(Y^2)$, $\mu_3' = \mathbb{E}(Y^3)$, and $\mu_4' = \mathbb{E}(Y^4)$.

Substituting into the general formula for kurtosis, we get:

$$\gamma_4 = \frac{\mu_4' + 6\mu^2\mu_2' - 4\mu\mu_3' - 3\mu^4}{(\mu_2' - \mu^2)^2}.$$

For the Unit Prakaamy Distribution (UPD), the moments are computed analytically, yielding the closed-form expression for the kurtosis as follows:

$$\gamma_4 = \frac{\frac{3125\theta^5 - 120}{15625(120+\theta^5)} + 6\left(\frac{\theta^5 - 120}{729(120+\theta^5)}\right)\left(\frac{8\theta^5+30}{16(120+\theta^5)}\right)^2 - 4\left(\frac{8\theta^5+30}{16(120+\theta^5)}\right)\left(\frac{512\theta^5 - 60}{2048(120+\theta^5)}\right) - 3\left(\frac{8\theta^5+30}{16(120+\theta^5)}\right)^4}{\left[\frac{\theta^5-120}{729(120+\theta^5)} - \left(\frac{8\theta^5+30}{16(120+\theta^5)}\right)^2\right]^2}. \tag{8}$$

3.5. Non-Central Moments

The r-th non-central moment of the Unit Prakaamy Distribution (UPD) is defined as:

$$\mu_r' = \mathbb{E}(Y^r) = \int_0^1 y^r f(y;\theta)\, dy,$$

where $f(y;\theta)$ is the probability density function of the UPD with shape parameter $\theta > 0$. After expanding the integrand and applying integration by parts recursively to the logarithmic term, the closed-form expression for the r-th non-central moment is obtained as:

$$\mathbb{E}(Y^r) = \frac{1}{120+\theta^5}\left[\frac{\theta^5}{r+1} + \frac{120}{(r+1)^6}\right]. \tag{9}$$

This expression is valid for all real $r > -1$ and is useful in deriving central moments, skewness, and kurtosis of the distribution.

3.6. Bonferroni and Lorenz Curves

The Lorenz and Bonferroni curves are classical tools for measuring concentration and inequality in distributions. For a non-negative continuous random variable Y with mean μ and distribution function $F(y)$, the **Bonferroni curve** is defined as:

$$B(p) = \frac{1}{p\mu}\int_0^q y f(y)\, dy, \quad \text{where } q = F^{-1}(p),\ 0 < p < 1.$$

The Bonferroni curve for the Unit Prakamy Distribution (UPD) is given by:

$$B(p) = \frac{1}{p\mu}\int_0^q y f(y)\, dy, \quad \text{where } q = F^{-1}(p), \quad 0 < p < 1.$$

Substituting the PDF of the UPD:

$$f(y) = \frac{\theta^5 - (\ln y)^5}{(120 + \theta^5)}, \quad 0 < y < 1,$$

we obtain:

$$B(p) = \frac{64q\theta^5 + 2\gamma(6, -2\ln q)}{128\, p\mu\, (120 + \theta^5)}$$

Splitting the integral:

$$B(p) = \frac{1}{p\mu(120 + \theta^5)} \left[\theta^5 \int_0^q y\, dy - \int_0^q y(\ln y)^5 dy \right].$$

Evaluating the first term:

$$\int_0^q y\, dy = \frac{q^2}{2} \quad \Rightarrow \quad K_1 = \frac{\theta^5 q^2}{2}.$$

The second integral is evaluated via substitution and expressed using the lower incomplete gamma function:

$$\int_0^q y(\ln y)^5 dy = \frac{\gamma(6, -2\ln q)}{64}.$$

Hence, the Bonferroni curve becomes:

$$B(p) = \frac{1}{p\mu(120 + \theta^5)} \left(\frac{\theta^5 q^2}{2} - \frac{\gamma(6, -2\ln q)}{64} \right). \tag{10}$$

Lorenz curveis related to the Bonferroni curve via:

$$L(p) = p \cdot B(p),$$

where $B(p)$ is the Bonferroni curve, μ is the mean, and $q = F^{-1}(p)$.

For the Unit Prakaamy Distribution (UPD), substituting the expression of $B(p)$ yields:

$$L(p) = \frac{1}{\mu(120 + \theta^5)} \left(\frac{\theta^5 q^2}{2} - \frac{\gamma(6, -2\ln q)}{64} \right), \quad 0 < p < 1. \tag{11}$$

3.7. Order Statistics

Let $Y_{(k)}$ denote the k-th order statistic from a random sample of size n drawn from the Unit Prakaamy Distribution (UPD). The probability density function of $Y_{(k)}$ is given by:

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!} \left[F_Y(y)\right]^{k-1} \left[1 - F_Y(y)\right]^{n-k} f_Y(y),$$

where $f_Y(y)$ and $F_Y(y)$ denote the PDF and CDF of the UPD, respectively.

Substituting the specific forms of the UPD:

$$f_Y(y) = \frac{\theta^5 - (\ln y)^5}{120 + \theta^5}, \quad F_Y(y) = 1 - \left[\frac{1}{\theta} - \frac{\ln y \left((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60\ln y + 120 \right)}{120\theta + \theta^6} \right]$$

we obtain:

$$f_{Y_{(k)}}(y) = \frac{n!}{(k-1)!(n-k)!} \left[1 - \left[\frac{1}{\theta} - \frac{\ln y \left((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60\ln y + 120 \right)}{120\theta + \theta^6} \right] \right]^{k-1}$$
$$\cdot \left[\frac{1}{\theta} - \frac{\ln y \left((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60\ln y + 120 \right)}{120\theta + \theta^6} \right]^{n-k} \cdot \frac{\theta^5 - (\ln y)^5}{120 + \theta^5} \tag{12}$$

4. Reliability measures

This section introduces key reliability measures for the UPD, including analytical expressions for the survival function (SF), hazard rate function (HRF). To further explore their behavior, graphical representations of both the SF and HRF are presented. The survival function (SF) describes the probability that a unit or subject will continue to function beyond a given time point, indicating that the event of interest such as failure or death has not yet occurred. As a nonincreasing function, the SF either remains constant or decreases over time. It plays a fundamental role in reliability theory and survival analysis, serving as a primary tool for modeling time-to-event data. The SF corresponding to the UPD is expressed as follows:

$$S(y) = \left[\frac{1}{\theta} - \frac{\ln y\left((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60\ln y + 120\right)}{120\theta + \theta^6}\right] \tag{13}$$

The hazard rate function (HRF)measures the instantaneous risk of failure at a given time, conditional on survival up to that point. It plays a central role in survival analysis and reliability studies by describing how failure likelihood evolves over time. The HRF of the UPD is given by the following expression:

$$h(y) = \frac{\dfrac{\theta^5 - (\ln y)^5}{120 + \theta^5}}{\left[\frac{1}{\theta} - \frac{\ln y((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60\ln y + 120)}{120\theta + \theta^6}\right]} \tag{14}$$

Figures 3 and 4 highlight the flexibility of the UPD model through its survival (SF) and hazard rate (HRF) functions across different values of the shape parameter θ. Higher θ values produce flatter survival curves and nearly constant hazard rates, indicating longer lifetimes and more stable reliability. In contrast, smaller θ values lead to steeper declines in survival and rapidly increasing hazard rates, reflecting higher early-life failure probabilities. These results demonstrate the UPD's capability to capture diverse lifetime behaviors effectively.

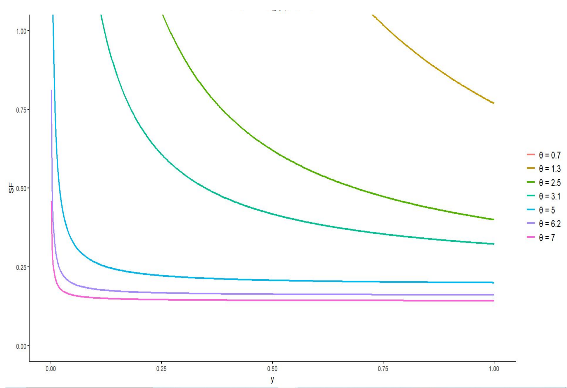

Figure 3. Survival function of the UPD.

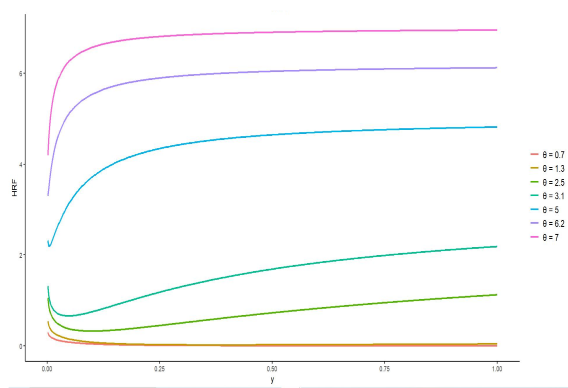

Figure 4. Hazard rate function of the UPD..

The reversed residual life (RRL) function captures the expected time elapsed before failure, given that failure has occurred. It is useful for analyzing early-life failures, left-censored data, and retrospective reliability assessments. RRL complements other reliability measures by offering a backward-looking perspective on failure behavior. The RRL of the UPD is given by the following expression:

$$M(t) = \varphi\left[\frac{\theta^5\, t^{h+1}}{h+1} - \frac{\Gamma\left(6 - \ln t \cdot (h+1)\right)}{(h+1)^6}\right] \tag{15}$$

5. Uncertainty measures

Entropy is a fundamental concept used to quantify the uncertainty or randomness inherent in a probability distribution. In information theory and statistics, entropy provides insight into the information content and dispersion of a random variable. For a continuous random variable Y with a probability density function $f(y)$, several types of entropy can be used to measure this uncertainty.

5.1. Shannon Entropy

Shannon entropy measures the average uncertainty associated with a continuous probability distribution. For a random variable Y with density $f(y)$, it is given by:

$$I(1) = \mathbb{E}[\log f(Y)] = \int_0^1 \log f(y) \cdot f(y)\, dy.$$

For the Unit Prakaamy Distribution (UPD), the density function is:

$$f(y) = \frac{\theta^5 - (\ln y)^5}{(120 + \theta^5)}, \quad 0 < y < 1.$$

Substituting into the entropy definition and simplifying:

$$I(1) = \int_0^1 f(y) \log\left(\frac{\theta^5 - (\ln y)^5}{(120 + \theta^5)}\right) dy,$$

$$= \int_0^1 f(y) \left[\log\theta + \log\left(1 - \left(\frac{\ln y}{\theta}\right)^5\right)\right] dy - \log(120 + \theta^5).$$

Expanding the logarithmic term using the Taylor series:

$$\log\left(1 - \left(\frac{\ln y}{\theta}\right)^5\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{\ln y}{\theta}\right)^{5n},$$

Then the entropy becomes:

$$I(1) = \log\theta + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathbb{E}\left(\frac{\ln y}{\theta}\right)^{5n} - \log(120 + \theta)^5. \quad (16)$$

5.2. Rényi Entropy

Rényi entropy generalizes Shannon entropy by introducing a parameter $\delta > 0, \delta \neq 1$ that controls the sensitivity to the distribution's tail. It is defined for a continuous random variable Y with PDF $f(y)$ as:

$$I(\delta) = \frac{1}{1-\delta} \log \int_0^1 [f(y)]^\delta\, dy.$$

For the Unit Prakamy Distribution (UPD), the PDF is:

$$f(y) = \frac{\theta^5 - (\ln y)^5}{120 + \theta^5}, \quad 0 < y < 1.$$

Raising the PDF to the power δ and applying the binomial expansion:

$$[f(y)]^\delta = \left(\frac{\theta^5}{120 + \theta^5}\right)^\delta \left(1 - \left(\frac{\ln y}{\theta}\right)^5\right)^\delta,$$

$$= \left(\frac{\theta^5}{120+\theta^5}\right)^{\delta} \sum_{r=0}^{\delta} \binom{\delta}{r} \left(\frac{\ln y}{\theta}\right)^{5r}.$$

Substituting into the Rényi entropy formula:

$$I(\delta) = \frac{1}{1-\delta} \log \left[\left(\frac{\theta^5}{120+\theta^5}\right)^{\delta} \sum_{r=0}^{\delta} \binom{\delta}{r} \int_0^1 \left(\frac{\ln y}{\theta}\right)^{5r} dy \right].$$

By change of variables $u = \frac{\ln y}{\theta}$, we get:

$$I(\delta) = \frac{1}{1-\delta} \log \left[\left(\frac{\theta^5}{120+\theta^5}\right)^{\delta} \sum_{r=0}^{\delta} \frac{\binom{\delta}{r}\theta}{(-\theta)^{5r+1}} \Gamma(5r+1) \right]. \tag{17}$$

6. Estimation

This section presents the methodologies utilized for parameter estimation of the UPD. The use of a variety of estimation techniques is essential to achieve robust and reliable parameter estimates, and to assess the applicability and performance of the UPD across different datasets. This approach strengthens the validity of the findings and facilitates a comprehensive evaluation of the model's effectiveness.

6.1. The Maximum Likelihood (ML)

The Maximum Likelihood Estimation (MLE) method estimates unknown parameters by maximizing the likelihood function derived from the observed data. It is a foundational method in statistical inference [32] and has been applied in recent studies for estimating parameters of new probability distributions [33, 34, 35].

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the Unit Prakamy Distribution (UPD) with probability density function:

$$f(y;\theta) = \frac{\theta^5 - (\ln y)^5}{120+\theta^5}, \quad 0 < y < 1,\ \theta > 0.$$

The likelihood function is given by:

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta^5 - (\ln y_i)^5}{120+\theta^5}$$

Taking the natural logarithm, the log-likelihood function becomes:

$$\ell(\theta) = \sum_{i=1}^{n} \log\left(\theta^5 - (\ln y_i)^5\right) - n\log\left(120+\theta^5\right)$$

Differentiating with respect to θ, we obtain the score function:

$$\frac{\partial \ell(\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{5\theta^4}{\theta^5 - (\ln y_i)^5} - \frac{5n\theta^4}{120+\theta^5} = 0 \tag{18}$$

The maximum likelihood estimate $\hat{\theta}$ is obtained by numerically solving Equation (18).

6.2. Method of least squares (LS)

The LSE method estimates parameters by minimizing the squared differences between the empirical cumulative distribution function and the theoretical one. It was introduced in the context of distribution fitting by Swain et al. [36], and has been applied in recent works such as [33, 34]. $\hat{\theta}_{LS}$ of the parameter θ is obtained by minimizing the following objective function:

$$\mathrm{LS}(\theta) = \sum_{i=1}^{n} \left(F(x_i;\theta) - \frac{i}{n+1} \right)^2$$

where $F(x_i;\theta)$ denotes the cumulative distribution function (CDF) of the UPD evaluated at the i-th ordered observation. Differentiating the least squares objective function with respect to θ, the estimating equation is given by:

$$\frac{\partial \mathrm{LS}(\theta)}{\partial \theta} = 2\sum_{i=1}^{n} \left(1 - \left[\frac{1}{\theta} - \frac{\ln y_i \left((\ln y_i)^4 - 5(\ln y_i)^3 + 20(\ln y_i)^2 - 60\ln y_i + 120 \right)}{120\theta + \theta^6} \right] - \frac{i}{n+1} \right) \delta_\theta = 0 \quad (5)$$

where

$$\delta_\theta = \sum_{i=1}^{n} \left[\frac{1}{\theta^2} + \frac{\ln y_i \left((\ln y_i)^4 - 5(\ln y_i)^3 + 20(\ln y_i)^2 - 60\ln y_i + 120 \right)(120 + 6\theta^5)}{(120\theta + \theta^6)^2} - \frac{i}{n+1} \right] \quad (19)$$

6.3. Method of Weighted Least Squares (WLS)

The WLSE method extends OLSE by introducing weights to improve efficiency, particularly in the tails of the distribution. It was also introduced by Swain et al. [36] and has been recently employed in works involving flexible probability models [34, 35]. The WLSE of the parameter θ is obtained by minimizing the following objective function:

$$\mathrm{WLS}(\theta) = \sum_{i=1}^{n} w_i \left(F(x_i;\theta) - \frac{i}{n+1} \right)^2$$

where $F(x_i;\theta)$ is the theoretical cumulative distribution function, and w_i is a weight reflecting the relative importance or variance of the i-th observation.

The Weighted Least Squares Estimator (WLSE) for the parameter θ in the Unit Prakamy Distribution (UPD) is obtained by minimizing the following objective function:

$$\mathrm{WLSE}(\theta) = \sum_{i=1}^{n} \left(\frac{(n+1)^2(n+2)}{(n+1-i)i} \left[F(y_i;\theta) - \frac{i}{n+1} \right] \right)^2$$

where $F(y_i;\theta)$ is the cumulative distribution function (CDF) of the UPD.
The derivative of the WLSE objective function with respect to θ is given by:

$$\frac{\partial \mathrm{WLSE}(\theta)}{\partial \theta} = 2\sum_{i=1}^{n} \left(\frac{(n+1)^2(n+2)}{(n+1-i)i} \left[1 - \left(\frac{1}{\theta} - \frac{\ln y \left((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60\ln y + 120 \right)}{120\theta + \theta^6} \right) - \frac{i}{n+1} \right] \right) \delta_\theta = 0 \quad (20)$$

where $\delta_\theta = \frac{\partial F(y_i;\theta)}{\partial \theta}$ is the derivative of the cumulative distribution function $F(y_i;\theta)$ with respect to θ.

6.4. Method of Cramér–Von Mises (CVM)

The Cramér–von Mises (CVM) estimation method is a goodness-of-fit-based approach that estimates the parameter θ by minimizing the integrated squared difference between the empirical and theoretical cumulative distribution functions. It is known for its sensitivity across the entire support [37], and has been applied in recent literature [33, 35].

$$\mathrm{CVM}(\theta) = \sum_{i=1}^{n} \left(F(x_i;\theta) - \frac{2i-1}{2n} \right)^2$$

The CVM estimator of θ is obtained by minimizing the following objective function:

$$\text{CVM}(\theta) = \sum_{i=1}^{n} \frac{1}{12n} \left(F(y_i;\theta) - \frac{2i-1}{2n} \right)^2$$

where $F(y_i;\theta)$ is the cumulative distribution function (CDF) of the UPD.
The CVM estimator is obtained by solving:

$$\frac{\partial \text{CVM}(\theta)}{\partial \theta} = 2\sum_{i=1}^{n} \frac{1}{12n} \left(1 - \left(\frac{1}{\theta} - \frac{\ln y \left((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60\ln y + 120 \right)}{120\theta + \theta^6} \right) - \frac{2i-1}{2n} \right) \delta_\theta = 0 \tag{21}$$

where $\delta_\theta = \frac{\partial F(y_i;\theta)}{\partial \theta}$ is given by:

$$\frac{\partial F(y,\theta)}{\partial \theta} = -\frac{1}{\theta^2} + \frac{\ln y \left((\ln y)^4 - 5(\ln y)^3 + 20(\ln y)^2 - 60\ln y + 120 \right)(6\theta^5)}{(120\theta + \theta^6)^2}$$

7. Simulation Study

A Monte Carlo simulation was performed to assess the finite-sample performance of the MLE, OLS, WLS, and CvM estimators for the Unit Prakaamy Distribution, using sample sizes $n = 20, 50, 100, 200, 500$ and 1000 replications for $\theta \in \{0.5, 1, 2\}$.

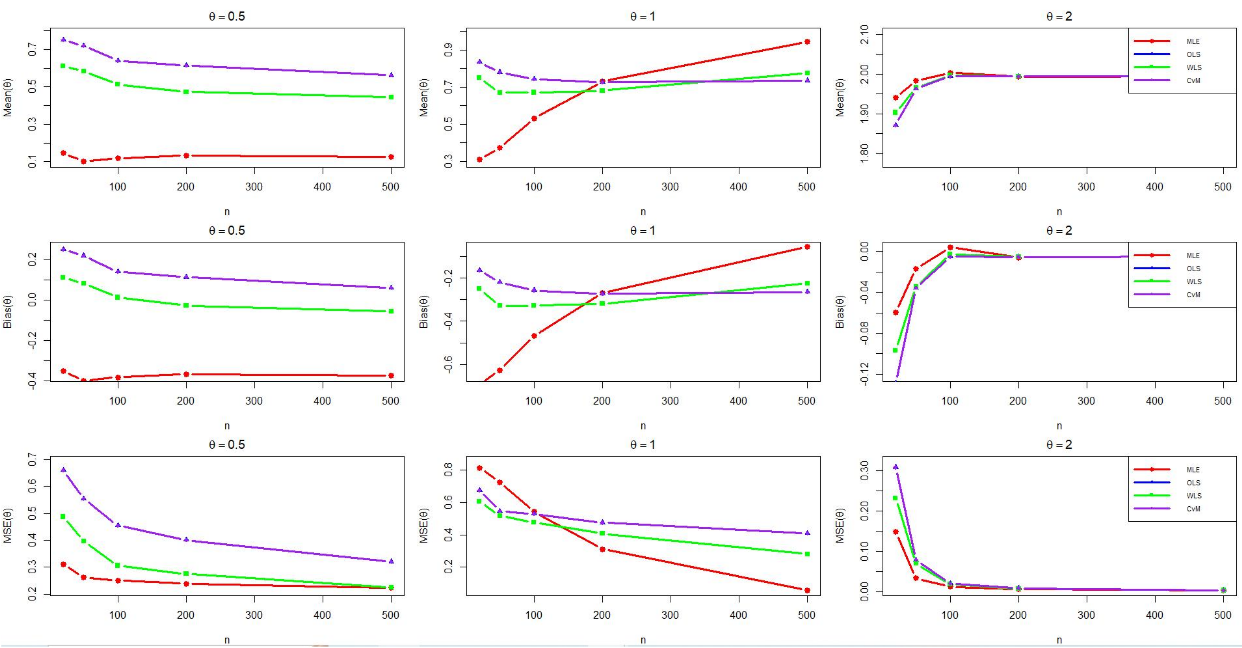

Figure 5. Graphical representation of Mean, Bias, and MSE values presented in Table 1.

As shown in Table 1 and Figure 5, the MLE generally achieves the lowest MSE among all estimators, particularly for $\theta = 0.5$ and $\theta = 2$. For $\theta = 1$, the MLE exhibits slightly higher MSE at small sample sizes, but its performance rapidly improves as n increases, illustrating its asymptotic efficiency and robustness. The WLS estimator consistently ranks second in terms of MSE, providing a reasonable alternative especially for moderate sample sizes. In contrast, the OLS and CvM estimators yield substantially higher MSE values across most scenarios, indicating lower efficiency for the UPD. Furthermore, examination of the bias and mean estimates (Figure 5) confirms that MLE estimates are generally centered around the true parameter values, whereas OLS and CvM show greater deviation in small samples. Overall, the simulation results suggest that MLE is the most reliable estimator for moderate and large samples, while WLS can serve as a viable option when sample sizes are limited.

Table 1. Comparison of Estimators (Mean, Bias, MSE) for $\theta = 0.5, 1, 2$

Sample Size (n)	Estimator	$\theta = 0.5$			$\theta = 1$			$\theta = 2$		
		Mean	Bias	MSE	Mean	Bias	MSE	Mean	Bias	MSE
20	MLE	0.1466	-0.3534	0.3110	0.3093	-0.6907	0.8150	1.9405	-0.0595	0.1478
	OLS	0.7518	0.2518	0.6610	0.8345	-0.1655	0.6748	1.8713	-0.1287	0.3077
	WLS	0.6127	0.1127	0.4871	0.7492	-0.2508	0.6051	1.9032	-0.0968	0.2318
	CvM	0.7518	0.2518	0.6610	0.8345	-0.1655	0.6748	1.8713	-0.1287	0.3077
50	MLE	0.1005	-0.3995	0.2623	0.3728	-0.6272	0.7260	1.9832	-0.0168	0.0330
	OLS	0.7210	0.2210	0.5538	0.7806	-0.2194	0.5464	1.9643	-0.0357	0.0785
	WLS	0.5834	0.0834	0.3962	0.6711	-0.3289	0.5176	1.9657	-0.0343	0.0698
	CvM	0.7210	0.2210	0.5538	0.7806	-0.2194	0.5464	1.9643	-0.0357	0.0785
100	MLE	0.1171	-0.3829	0.2503	0.5316	-0.4684	0.5418	2.0040	0.0040	0.0121
	OLS	0.6409	0.1409	0.4543	0.7427	-0.2573	0.5282	1.9948	-0.0052	0.0204
	WLS	0.5135	0.0135	0.3054	0.6714	-0.3286	0.4775	1.9971	-0.0029	0.0178
	CvM	0.6409	0.1409	0.4543	0.7427	-0.2573	0.5282	1.9948	-0.0052	0.0204
200	MLE	0.1333	-0.3667	0.2382	0.7305	-0.2695	0.3109	1.9940	-0.0060	0.0061
	OLS	0.6143	0.1143	0.4008	0.7261	-0.2739	0.4746	1.9944	-0.0056	0.0082
	WLS	0.4735	-0.0265	0.2745	0.6810	-0.3190	0.4070	1.9945	-0.0055	0.0074
	CvM	0.6143	0.1143	0.4008	0.7261	-0.2739	0.4746	1.9944	-0.0056	0.0082
500	MLE	0.1255	-0.3745	0.2235	0.9457	-0.0543	0.0561	1.9947	-0.0053	0.0023
	OLS	0.5613	0.0613	0.3200	0.7351	-0.2649	0.4095	1.9951	-0.0049	0.0032
	WLS	0.4435	-0.0565	0.2246	0.7748	-0.2252	0.2803	1.9953	-0.0047	0.0029
	CvM	0.5613	0.0613	0.3200	0.7351	-0.2649	0.4095	1.9951	-0.0049	0.0032

8. Application to Real Data

The applicability and flexibility of the *Untied Prakaamy Distribution* (UPD) are demonstrated through the analysis of three real-life datasets. The first dataset consists of leukemia patients' survival times [38], the second arises from endurance tests on ball bearings [39], and the third contains tensile strength measurements of polyester fibers [40].

From Table 2, it is evident that the first and third datasets exhibit moderate skewness values in opposite directions. Specifically, Dataset I displays a moderate negative skewness, indicating a distribution slightly skewed to the left, whereas Dataset III shows a moderate positive skewness, suggesting a right–tailed pattern. In contrast, Dataset II presents a positive skewness of comparable magnitude to Dataset III, reflecting a moderately right–skewed distribution rather than a near-symmetric shape. Regarding kurtosis, all datasets fall within the platykurtic range, as their kurtosis values are lower than that of the normal distribution, indicating flatter shapes with lighter tails.

These descriptive statistics provide an essential foundation for assessing the goodness of fit of the Unit Prakaamy distribution compared with the Unit Zeghdoudi distribution and Pareto Type II [41]models. The comparison highlights the inherent distributional differences among the datasets, which play a crucial role in evaluating the suitability and relative performance of the competing models.

Table 2. Some descriptive analysis of all data sets.

Dataset	**n**	**Mean**	**Median**	**Variance**	**SD**	**Skewness**	**Kurtosis**	**Range**	**Min**	**Max**
Dataset I	39	0.5826	0.6355	0.0808	0.2842	-0.3828	2.0551	0.9740	0.024	0.9980
Dataset II	23	0.3397	0.3210	0.0429	0.2071	0.5049	1.9177	0.6483	0.064	0.7123
Dataset III	30	0.3659	0.3360	0.0721	0.2685	0.5193	2.0832	0.9030	0.023	0.9260

Based on the analysis of the first dataset, which consists of leukemia patients' survival times, the Maximum Likelihood Estimates (MLEs), goodness-of-fit statistics, and Kolmogorov–Smirnov (KS) test results for the Unit Prakaamy, Unit Zeghdoudi, and Pareto Type II distributions are summarized in Table 3.

Table 3. Parameter estimates, log-likelihood, information criteria, and KS test results for Dataset I.

Model	Parameters	LogLik	AIC	BIC	HQIC	KS	P-value
Unit Prakaamy	$\theta = 145.477$	0.000247	1.999	3.663	2.596	0.173	0.193
Unit Zeghdoudi	$\omega = 0.943$	-128.807	259.615	186.351	185.398	0.268	0.007
Pareto Type II	$\alpha = 1.071$	-34.335	70.669	103.579	102.231	0.532	2.34×10^{-17}

As evidenced by the results in Table 3, the Unit Prakaamy distribution clearly outperforms the Unit Zeghdoudi and Pareto Type II models, demonstrating superior agreement with the observed survival data. The competing models show markedly poorer fit statistics, reinforcing the dominance of the Unit Prakaamy distribution. Hence, it represents the most statistically reliable model for describing the leukemia survival times dataset.

The applicability of the Unit Prakaamy Distribution (UPD) is illustrated using the deep groove ball bearings dataset.

Table 4. Parameter estimates, log-likelihood, information criteria, and KS test results for Dataset II.

Model	Parameters	LogLik	AIC	BIC	HQIC	KS	P-value
Unit Prakaamy	$\theta = 4.7139$	0.000247	1.9995	3.6631	2.5964	0.1730	0.1934
Unit Zeghdoudi	$\omega = 3.6215$	-48.2611	98.5223	99.6578	98.8079	0.2310	0.1461
Pareto Type II	$\alpha = 0.5926$	-8.2010	18.4021	19.5376	18.6877	0.5074	2.34×10^{-17}

As observed from Table 4, the Unit Prakaamy distribution provides the best overall fit among the three competing models for this dataset. It exhibits a moderate KS statistic with an acceptable p-value, indicating a reasonable agreement with the observed data. The Unit Zeghdoudi distribution also performs reasonably well, whereas the Unit-Pareto Type II distribution shows a poor fit based on its high KS statistic and very small p-value. Consequently, the Unit Prakaamy distribution is considered the most suitable model.

Table 5. Parameter estimates, log-likelihood, information criteria, and KS test results for Dataset III .

Based on the analysis of the third dataset, which consists of tensile strength measurements of polyester fibers, the Maximum Likelihood Estimates (MLEs), goodness-of-fit statistics, and Kolmogorov–Smirnov (KS) test results for the Unit Prakaamy, Unit Zeghdoudi, and Pareto Type II distributions are summarized.

Model	Parameters	LogLik	AIC	BIC	HQIC	KS	P-value
Unit Prakaamy	$\theta = 232.6806$	-2.4277×10^{-9}	2.0000	3.4012	2.4483	0.237667	0.056438
Unit Zeghdoudi	$\omega = 1.7851$	-32.79958	67.599167	69.000365	68.047423	0.387017	0.000148
Unit-Pareto Type II	$\alpha = 0.863673$	-34.32696	-2.124968	-0.989474	-1.839395	0.557453	2.94×10^{-7}

As evidenced by the results in Table 5, the Unit Prakaamy distribution clearly outperforms the Unit Zeghdoudi and Pareto Type II models, showing the smallest KS statistic and the largest p-value. The competing models exhibit higher KS statistics and very small p-values, indicating poorer agreement with the observed data. Therefore, the Unit Prakaamy distribution is the most suitable and statistically reliable model for describing the tensile strength dataset.

9. Conclusion

In this work, we introduced and studied a bounded version of the Prakaamy distribution, termed the Unit Prakaamy Distribution (UPD). Several theoretical properties of the distribution were derived in closed form, including its moments, entropy, hazard and survival functions, as well as the Bonferroni and Lorenz curves. To estimate the model parameter, four classical estimation methods were applied: Maximum Likelihood Estimation (MLE), Ordinary Least Squares (OLS), Weighted Least Squares Estimation (WLS), and the Cramér–von Mises (CvM) method. A comprehensive Monte Carlo simulation was conducted to evaluate and compare their performance across various sample sizes and parameter values.

The simulation results revealed that MLE generally outperforms the other methods, achieving the lowest mean squared error (MSE) and providing estimates closely centered around the true parameter values, particularly for moderate and large sample sizes. WLS consistently ranks second, offering a reasonable alternative for small samples, while OLS and CvM estimators exhibited higher MSE and greater bias, especially in small samples.

Furthermore, the applicability of the UPD was demonstrated using three real-life datasets: leukemia patients' survival times, deep groove ball bearings, and tensile strength measurements of polyester fibers. Across all datasets, the UPD outperformed competing models, specifically the Unit Zeghdoudi and Pareto Type II distributions, as indicated by lower KS statistics and higher associated p-values.

Overall, the findings suggest that the Unit Prakaamy Distribution is a robust and flexible model for data constrained to the unit interval, and that MLE is the most reliable method for parameter estimation in practical applications. Possible future research directions include applying the Unit Prakaamy Distribution (UPD) in regression studies, including the development of quantile regression models based on this distribution, as well as extending it to bivariate or discrete cases.

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