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A Highly Accurate Upper Bound for the Cumulative Standard Normal Distribution Function and Related Functions

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Abstract Although there is extensive literature concerning upper bounds for the cumulative standard normal distribution function $\Phi(x)$, many existing bounds are not uniformly accurate over the entire domain $x \geq 0$. This paper proposes a new explicit upper bound for $\Phi(x)$ that is uniformly accurate over $x \geq 0$ and improves the classical Polya upper bound and several related bounds available in the literature. The proposed bound is constructed by modifying the classical Polya structure through the introduction of a polynomial correction term inside the exponential function, leading to a flexible parametric form. The coefficients of this correction polynomial are determined using a constrained minimax-type optimization framework combined with local accuracy conditions obtained through Taylor-series matching near the origin, ensuring both global tightness and local stability. It is shown that the maximum absolute error between the proposed bound and the exact function $\Phi(x)$ does not exceed 5.785×10^{-5} for all $x \geq 0$. Numerical comparisons and graphical illustrations demonstrate that the proposed bound is consistently tighter than several well-known alternative bounds, particularly in the small and moderate range of x. The structure of the bound also preserves analytical tractability, making it suitable for theoretical analysis and computational applications. Finally, due to the functional relationships between $\Phi(x)$, the Q-function and the error function, the proposed construction can be directly extended to derive corresponding bounds for these related special functions.

Keywords Bounds for the standard normal CDF; Polya bound; Error function; Q-function; Maximum absolute error.

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1. Introduction

The cumulative standard normal (Gaussian) distribution function $\Phi(x)$ and its related special functions appear frequently in applied mathematics, probability theory, statistics, and communication theory (Simon, 2006). The cumulative distribution function (CDF) of a standard normal random variable X is defined as,

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt \tag{1}$$

where $\phi(t)$ is the standard normal probability density function, which is given by,

$$\phi(t) = \frac{\exp(-t^2/2)}{\sqrt{2\pi}}, \quad -\infty < t < \infty \tag{2}$$

The function $\Phi(x)$ plays a central role in statistical inference, reliability analysis, stochastic modeling, signal processing, and communication systems. However, despite its importance, the integral defining $\Phi(x)$ does not

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admit a closed-form antiderivative in terms of elementary functions. Consequently, numerous approximations, inequalities, and upper/lower bounds have been developed in the literature.

The probability that a standard normal random variable X exceeds a positive value x is known as the Q-function and is defined by,

$$Q(x) = \int_x^\infty \phi(t)dt, \tag{3}$$

which is related to $\Phi(x)$ by,

$$Q(x) = 1 - \Phi(x). \tag{4}$$

Two additional special functions closely related to $\Phi(x)$ are the error function $\mathrm{erf}(x)$ and the complementary error function $\mathrm{erfc}(x)$ defined respectively by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)dt \tag{5}$$

and

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x). \tag{6}$$

These functions can also be expressed in terms of $\Phi(x)$ as follows:

$$\mathrm{erf}(x) = 2\Phi(\sqrt{2}x) - 1 \tag{7}$$

and

$$\mathrm{erfc}(x) = 2 - 2\Phi(\sqrt{2}x). \tag{8}$$

The functions $\Phi(x)$, $Q(x)$, $\mathrm{erf}(x)$ and $\mathrm{erfc}(x)$ arise naturally in many practical applications, including bit error rate (BER) analysis in wireless communication systems, diffusion theory, reliability engineering, statistical quality control, and mathematical finance (Trigui et al., 2021). Because of the importance of these functions, many researchers have proposed approximations and bounds with varying degrees of simplicity and accuracy (see Edous & Eidous, 2018; Bercu, 2020; Hanandeh & Eidous, 2021; Eidous & Abu-Hawwas, 2021; Hanandeh & Eidous, 2022; Lipoth et al., 2022; Eidous & Ananbeh, 2022; Ananbeh & Eidous, 2025; Obeidat et al., 2025; Eidous et al., 2026). A comprehensive review of 45 approximations for $\Phi(x)$ was provided by Eidous and Abu-Shareefa (2020), while more recent works such as Eidous and Al-Rawwash (2025) proposed highly accurate analytic approximations.

Several studies have also focused on deriving analytically tractable bounds for $\Phi(x)$ that preserve both simplicity and accuracy over wide intervals (see Eidous, 2023, 2025). However, despite these developments, most existing upper bounds exhibit non-uniform accuracy: some are tight near the origin, while others are accurate only in the tail region. This motivates the construction of a uniformly sharp upper bound over $x \geq 0$, which maintains controlled approximation error across the entire domain.

In this paper, we propose a new upper bound for $\Phi(x)$ obtained by modifying the classical Polya structure using a polynomial correction term inside the exponential function. The coefficients are chosen using a constrained numerical optimization procedure (minimax-style fitting) that minimizes the maximum absolute error while ensuring the resulting function remains an upper bound for $\Phi(x)$ on $x \geq 0$. Additional emphasis is placed on maintaining accuracy near $x = 0$. Accordingly, the main contributions of this paper can be summarized as follows:

- A new analytically tractable upper bound for the cumulative standard normal distribution function is proposed.
- The bound is constructed using a Polya-type structure with a polynomial correction term in the exponent, where the coefficients are obtained via a constrained minimax optimization procedure ensuring both sharpness and validity as an upper bound.
- The maximum absolute error of the proposed bound $\Phi_{EI}(x)$ satisfies $\max_{x>0} |\Phi_{EI}(x) - \Phi(x)| < 5.785 \times 10^{-5}$.
- Extensions of the proposed result to the Q-function, error function, and complementary error function are established.

- Extensive numerical comparisons demonstrate the superior accuracy of the proposed bound relative to several well-known upper bounds available in the literature.

The remainder of this paper is organized as follows. Section 2 reviews several existing upper bounds for $\Phi(x)$. Section 3 presents the proposed upper bound together with its theoretical properties and approximation accuracy. Numerical comparisons are reported in Section 4, while discussion and concluding remarks are provided in Section 5.

2. Existing Upper Bounds for $\Phi(x)$

The integral in $\Phi(x)$, as well as those defining $Q(x)$, $\mathrm{erf}(x)$ and $\mathrm{erfc}(x)$, does not admit a closed-form expression in terms of elementary functions. Consequently, both numerical evaluation and analytic approximation techniques are required in practical applications and theoretical analysis. Therefore, a variety of bounds and approximations have been proposed in the literature (see Eidous, 2023). In particular, many works focus on constructing explicit upper bounds that are both analytically simple and computationally efficient. For $x \geq 0$, some of the most relevant upper bounds are summarized below.

- Polya (1949) proposed the classical upper bound for $\Phi(x)$,

$$\Phi(x) \leq \Phi_{PO}(x), \tag{9}$$

where

$$\Phi_{PO}(x) = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \exp\left(-\frac{2x^2}{\pi}\right)}. \tag{10}$$

This bound is simple and widely used due to its closed-form structure. However, its accuracy varies across different ranges of x, as noted in Iacono (2021).

- Kouba (2006) derived the bound,

$$\Phi(x) \leq \Phi_{KO}(x), \tag{11}$$

where

$$\Phi_{KO}(x) = 1 - \frac{\phi(x)}{t} \tag{12}$$

and $t = \sqrt{1 + (x/2)^2} + x/2$.

- Alzer (2010) proposed the following bound,

$$\Phi(x) \leq \Phi_{AL}(x) \tag{13}$$

where,

$$\Phi_{AL}(x) = 0.5 + \frac{1.0407}{2}\tanh(\sqrt{2/\pi}x). \tag{14}$$

- Abreu (2012) gave the following upper bound,

$$\Phi(x) \leq \Phi_{AB}(x) \tag{15}$$

where

$$\Phi_{AB}(x) = 1 - \frac{\exp(-x^2)}{12} - \frac{\phi(x)}{1+x}. \tag{16}$$

- Neuman (2013) gave the following upper bound for $\Phi(x)$,

$$\Phi(x) \leq \Phi_{NE}(x), \tag{17}$$

where

$$\Phi_{NE}(x) = \frac{1}{2} + \frac{x}{3}\frac{2 + \exp(-x^2/2)}{\sqrt{2\pi}}. \tag{18}$$

- Yang et al. (2018) improved the Neuman bound $\Phi_{NE}(x)$ by proposing the following bound,

$$\Phi(x) \le \Phi_{YA}(x), \tag{19}$$

where

$$\Phi_{YA}(x) = \frac{1}{2} + \frac{x}{9}\frac{4 + 5\exp(-3x^2/10)}{\sqrt{2\pi}}. \tag{20}$$

- Let $y = x/\sqrt{2}$. Then, for $0 \le y \le 4.418$, Bercu (2020) gave the following upper bound,

$$\Phi(x) \le \Phi_{BE}(x), \tag{21}$$

where

$$\Phi_{BE}(x) = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\frac{113400y}{29y^8 - 660y^6 + 1260y^4 + 37800y^2 + 113400}. \tag{22}$$

A common limitation of the above bounds is that none of them provides uniformly tight accuracy over the entire range $x \ge 0$. Instead, the performance of most of them is typically range-dependent. Some bounds are more accurate for small values of x (e.g., Yang et al., 2018), while others perform better for moderate or large values of x (e.g., Kouba, 2006). This non-uniform behavior across the domain of x (see Table 1) motivates the need for a single unified upper bound with controlled and balanced accuracy across the entire interval $x \ge 0$.

3. Proposed Upper Bound of $\Phi(x)$

The main result of this paper is presented as follows. Let

$$\Phi_{EI}(x) = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \exp\left(-\frac{2x^2}{\pi}\left(1 + \frac{(-\pi+3)x^2}{3\pi} + \left(\frac{7}{90} + \frac{40001}{30000\pi^2} - \frac{2}{3\pi}\right)x^4\right)\right)}. \tag{23}$$

Then, numerical evidence strongly suggests that

$$\Phi(x) \le \Phi_{EI}(x), \quad \forall x \ge 0. \tag{24}$$

Numerical verification and computational analysis: Let $h(x) = \Phi_{EI}(x) - \Phi(x)$. To show that $\Phi_{EI}(x)$ is an upper bound of $\Phi(x)$ for all $x \ge 0$, it is sufficient to verify that $h(x) \ge 0$ on $[0, \infty)$.

At the origin, both functions coincide. Indeed, since the exponential term in $\Phi_{EI}(x)$ vanishes at $x = 0$, we obtain $\Phi_{EI}(0) = \frac{1}{2} = \Phi(0)$, and hence $h(0) = 0$.

For large values of x, observe that the exponent in the definition of $\Phi_{EI}(x)$ contains a polynomial whose leading term is proportional to x^6 with positive coefficient

$$\frac{7}{90} + \frac{40001}{30000\pi^2} - \frac{2}{3\pi} \approx 0.000669 > 0. \tag{25}$$

Therefore, the exponential term decays to zero as $x \to \infty$, which implies $\Phi_{EI}(x) \to 1$. Since also $\Phi(x) \to 1$, it follows that

$$\lim_{x\to\infty} h(x) = 0. \tag{26}$$

The function $h(x)$ is continuously differentiable on $(0, \infty)$, and its derivative $h'(x)$ consists of a difference between the standard normal density and a smooth exponential-polynomial expression. Although the explicit algebraic form of $h'(x)$ is tedious, it is well-defined and continuous over the whole domain. Figure 1 shows the curve plot of $h'(x)$ over $[0, 5]$.

A detailed symbolic inspection of the structure of $h'(x)$, combined with high-precision numerical evaluation, shows that $h'(x)$ admits exactly one zero on $(0, \infty)$. The uniqueness of the stationary point was verified numerically

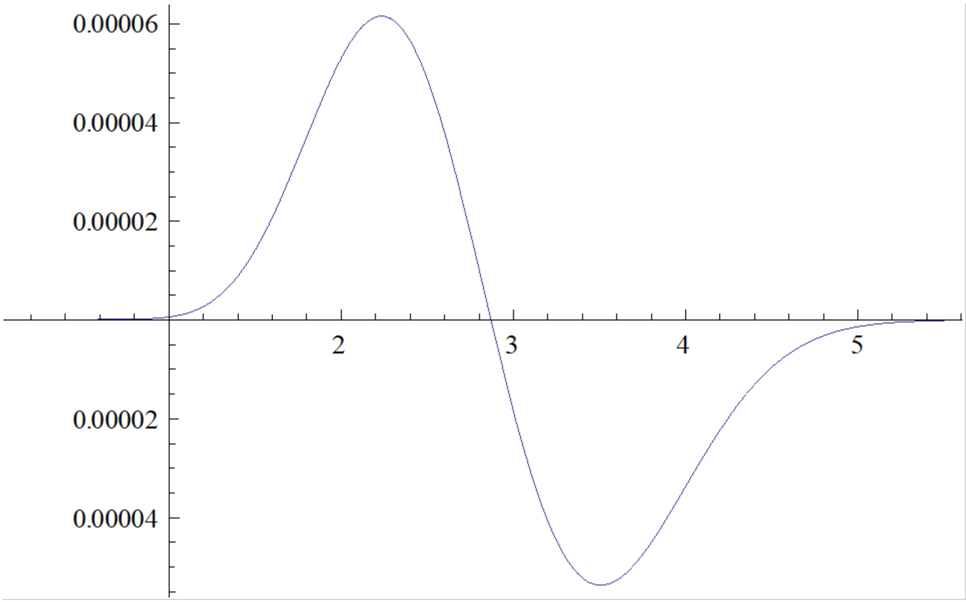

Figure 1. Plot of $h'(x)$.

by using Mathematica 13 with 50-digit precision and adaptive interval subdivision together with sign analysis of $h'(x)$ over a dense computational grid. Moreover, numerical root refinement (using high-precision arithmetic) confirms that this point is located at $x^* \approx 2.86991$. Accordingly, the validity of the proposed upper bound should be interpreted as numerically verified rather than analytically proven.

Further numerical evaluation of $h'(x)$ on both sides of x^* indicates a sign change from positive to negative, suggesting that $h(x)$ is strictly increasing on $(0, x^*)$ and strictly decreasing on (x^*, ∞). Consequently, the numerical evidence indicates that x^* corresponds to the unique global maximizer of $h(x)$ (see also Figure 2). Evaluating $h(x)$ at this point yields

$$h(x^*) \approx 5.784 \times 10^{-5}. \tag{27}$$

Since the numerical computations indicate that $h(x)$ increases from $h(0) = 0$ up to $h(x^*) > 0$ and then decreases back to 0 as $x \to \infty$, the numerical evidence strongly suggests that $h(x) \geq 0$ for all $x \geq 0$ (see Figure 2). Therefore,

$$\Phi(x) \leq \Phi_{EI}(x), \quad \forall x \geq 0. \tag{28}$$

This completes the numerical verification and computational investigation.

Based on the numerical verification above, we obtain the corresponding numerical bound

$$Q(x) = 1 - \Phi(x) \geq 1 - \Phi_{EI}(x) \tag{29}$$

and

$$\text{erf}(x) \leq 2\Phi_{EI}(\sqrt{2}x) - 1. \tag{30}$$

Because of its high accuracy, the proposed upper bound $\Phi_{EI}(x)$ can be used as an approximation for $\Phi(x)$, i.e.

$$\Phi(x) \approx \Phi_{EI}(x), \quad x \geq 0 \tag{31}$$

with maximum absolute error

$$\max_{x \geq 0} |\Phi_{EI}(x) - \Phi(x)| \approx 5.784 \times 10^{-5}. \tag{32}$$

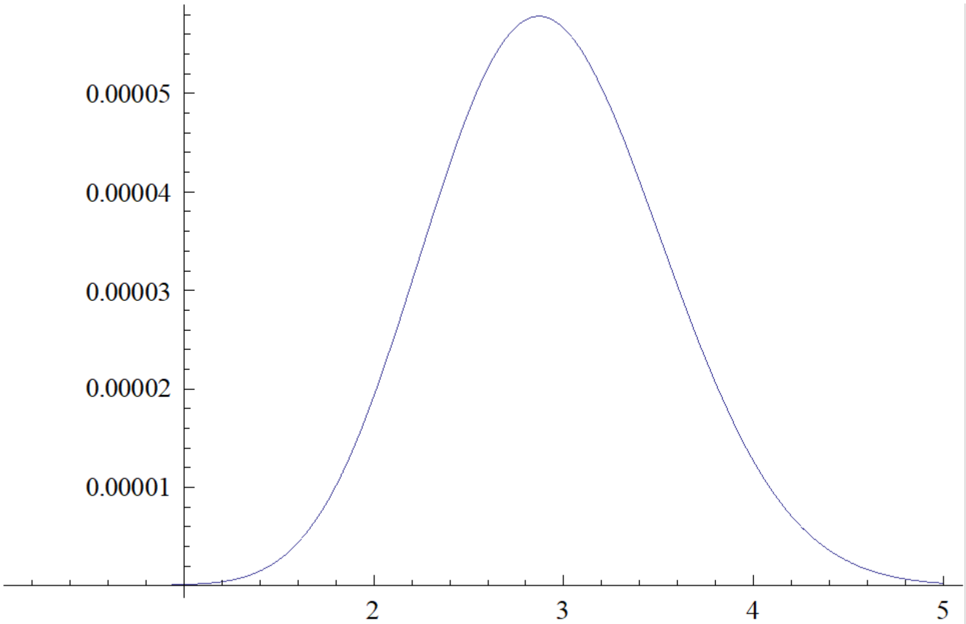

Figure 2. Plot of $h(x) = \Phi_{EI}(x) - \Phi(x)$.

To further improve accuracy, a refined approximation is obtained by modifying the coefficients in the polynomial correction term, leading to,

$$\Phi^*_{EI}(x) = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \exp\left(-\frac{2x^2}{\pi}\left(1 - 0.01506x^2 + 0.00063x^4\right)\right)}. \tag{33}$$

For this refined approximation, numerical evaluation shows that

$$\max_{x \geq 0} |\Phi^*_{EI}(x) - \Phi(x)| = 3.1677 \times 10^{-5} \tag{34}$$

which is smaller than the error of $\Phi_{EI}(x)$. Moreover, the two errors satisfy the approximate proportionality relation

$$\max_{x \geq 0} |\Phi_{EI}(x) - \Phi(x)| \approx 1.826 \max_{x \geq 0} |\Phi^*_{EI}(x) - \Phi(x)|. \tag{35}$$

It is important to emphasize that $\Phi^*_{EI}(x)$ is intended purely as an approximation and is neither an upper nor lower bound for $\Phi(x)$.

Lemma 1: For $0 \leq x \leq 4.74915$, the following inequality holds:

$$\Phi_{EI}(x) \leq \Phi_{PO}(x). \tag{36}$$

The interval $0 \leq x \leq 4.74915$ characterizes the region where the proposed bound lies below the classical Polya bound, indicating improved tightness relative to $\Phi_{PO}(x)$ in that range. For $x > 4.74915$ it is found that the maximum absolute error of $\Phi_{EI}(x)$ is less than 9.16×10^{-7}, while the corresponding error for the bound $\Phi_{PO}(x)$ is less than 9.15×10^{-7} (see the results of Table 1).

Proof of Lemma 1: It is straightforward to show that (for $x \geq 0$),

$$\Phi_{EI}(x) \leq \Phi_{PO}(x), \tag{37}$$

If and only if,

$$\frac{1}{2}\sqrt{1-\exp\left(-\frac{2x^2}{\pi}\left(1+\frac{(-\pi+3)x^2}{3\pi}+\left(\frac{7}{90}+\frac{40001}{30000\pi^2}-\frac{2}{3\pi}\right)x^4\right)\right)} \leq \frac{1}{2}\sqrt{1-\exp\left(-\frac{2x^2}{\pi}\right)} \tag{38}$$

If and only if,

$$\exp\left(-\frac{2x^2}{\pi}\left(1+\frac{(-\pi+3)x^2}{3\pi}+\left(\frac{7}{90}+\frac{40001}{30000\pi^2}-\frac{2}{3\pi}\right)x^4\right)\right) \geq \exp\left(-\frac{2x^2}{\pi}\right), \tag{39}$$

If and only if,

$$1+\frac{(-\pi+3)x^2}{3\pi}+\left(\frac{7}{90}+\frac{40001}{30000\pi^2}-\frac{2}{3\pi}\right)x^4 \leq 1 \tag{40}$$

If and only if,

$$\left(\frac{7}{90}+\frac{40001}{30000\pi^2}-\frac{2}{3\pi}\right)x^2 \leq \frac{\pi-3}{3\pi}, \tag{41}$$

If and only if,

$$x \leq \sqrt{\frac{\pi-3}{3\pi\left(\frac{7}{90}+\frac{40001}{30000\pi^2}-\frac{2}{3\pi}\right)}} \cong 4.74915. \tag{42}$$

This completes the proof.

4. Numerical Comparisons

To evaluate the accuracy of the proposed upper bound, consider the error function $h_U(x) = \Phi_U(x) - \Phi(x)$ where $\Phi_U(x)$ denotes a given upper bound for $\Phi(x)$. In particular, the subscripts $U = KO, AL, AB, NE, YA, BE$ and PO correspond respectively to the bounds of Kouba (2006), Alzer (2010), Abreu (2012), Neuman (2013), Yang et al. (2018), Bercu (2020), and Polya (1949). For the proposed upper bound, the corresponding error function is defined by

$$h_{EI}(x) = \Phi_{EI}(x) - \Phi(x). \tag{43}$$

Table 1 reports the numerical values of these error functions for selected values of x ranging from 0.1 to 8.5. Although most of the considered bounds are valid for all $x \geq 0$, the comparison is focused primarily on small and moderate values of x, since both the proposed bound and the classical Polya bound become extremely accurate for large argument values. The numerical results presented in Table 1 lead to several observations.

1. The proposed upper bound is very accurate for small values of x (say, $x < 1.9$) and for large values of x (say, $x > 3.9$). It remains accurate for other values of x when compared with the other upper bounds considered in this study.

2. One can observe that the proposed bound, as well as $\Phi_{YA}(x)$ and $\Phi_{BE}(x)$ is highly accurate for small argument values. However, the bounds $\Phi_{YA}(x)$ and $\Phi_{BE}(x)$ are not as tight as the proposed bound for moderate to large argument values. It is worthwhile to note that the bound $\Phi_{BE}(x)$ is developed for $0 \leq y = x/\sqrt{2} \leq 4.418$ (i.e., $0 \leq x \leq 6.248$) as stated in Section 2. The numerical study confirms this behavior, which is clear from the negative values of $h_{BE}(x)$ for $x \geq 6.5$. The disadvantage of using the bound $\Phi_{BE}(x)$ for only a sub-interval of $[0, \infty)$ limits its applicability in various fields of analysis.

3. Similar to the proposed bound $\Phi_{EI}(x)$, the bounds $\Phi_{KO}(x)$ and $\Phi_{AB}(x)$ are very close to the exact $\Phi(x)$ as x increases beyond 3. Moreover, one can conclude that each bound becomes even more accurate as $x \to \infty$. However, the numerical values show clearly the significant improvement provided by the proposed bound over $\Phi_{KO}(x)$ and $\Phi_{AB}(x)$ for $0 < x < 2.5$.

4. By taking into account all the considered values of the argument x in this numerical study, one can conclude that the tightness of the two bounds $\Phi_{AL}(x)$ and $\Phi_{NE}(x)$ is not as good as the other bounds.

5. A direct comparison between the proposed bound $\Phi_{EI}(x)$ and the classical Polya bound $\Phi_{PO}(x)$ shows that the proposed bound is consistently sharper for approximately $0 \le x < 5$. For larger values of x, both bounds become extremely close to the exact distribution function, with errors below approximately 3.0×10^{-7}.

Overall, the numerical study demonstrates that the proposed upper bound achieves a balanced and uniformly high level of accuracy over the entire interval $x > 0$, thereby improving upon several classical upper bounds while preserving a simple closed-form structure.

Table 1. The values of error function $h_U(x) = \Phi_U(x) - \Phi(x)$ for some values of x.

x	$h_{KG}(x)$	$h_{AU}(x)$	$h_{AR}(x)$	$h_{WE}(x)$	$h_{YA}(x)$	$h_{BE}(x)$	$h_{EO}(x)$	$h_{RG}(x)$
0.1	1.11×10^{-2}	1.60×10^{-3}	1.68×10^{-3}	6.63×10^{-4}	1.90×10^{-1}	0.00	2.98×10^{-4}	4.63×10^{-11}
0.3	9.13×10^{-3}	4.32×10^{-3}	1.26×10^{-2}	1.58×10^{-5}	4.08×10^{-2}	3.11×10^{-12}	7.72×10^{-5}	1.48×10^{-10}
0.5	6.77×10^{-3}	5.77×10^{-3}	8.93×10^{-3}	1.96×10^{-4}	1.41×10^{-6}	8.05×10^{-10}	3.29×10^{-4}	1.29×10^{-10}
0.7	4.66×10^{-3}	5.72×10^{-3}	7.23×10^{-3}	9.96×10^{-4}	1.41×10^{-5}	2.96×10^{-10}	7.96×10^{-4}	4.21×10^{-8}
0.9	3.03×10^{-3}	4.46×10^{-3}	6.94×10^{-3}	3.25×10^{-2}	7.66×10^{-7}	4.16×10^{-7}	1.43×10^{-3}	8.92×10^{-8}
1.1	1.88×10^{-3}	2.64×10^{-3}	7.08×10^{-3}	8.10×10^{-3}	2.87×10^{-8}	3.25×10^{-10}	2.11×10^{-3}	2.29×10^{-7}
1.3	1.12×10^{-3}	1.01×10^{-3}	6.92×10^{-3}	1.68×10^{-2}	8.38×10^{-4}	1.71×10^{-5}	2.70×10^{-3}	8.14×10^{-7}
1.5	6.40×10^{-4}	9.57×10^{-4}	6.22×10^{-3}	3.05×10^{-2}	2.04×10^{-3}	6.80×10^{-5}	3.06×10^{-3}	2.65×10^{-6}
1.7	3.53×10^{-4}	1.63×10^{-4}	5.10×10^{-3}	5.00×10^{-2}	4.31×10^{-3}	2.17×10^{-4}	3.14×10^{-3}	6.85×10^{-6}
1.9	1.88×10^{-4}	1.19×10^{-4}	3.84×10^{-3}	7.56×10^{-2}	8.18×10^{-3}	5.86×10^{-4}	2.94×10^{-3}	1.43×10^{-5}
2.1	9.67×10^{-5}	2.98×10^{-2}	2.66×10^{-3}	1.07×10^{-1}	1.42×10^{-2}	1.38×10^{-3}	2.54×10^{-3}	2.48×10^{-5}
2.3	4.82×10^{-5}	5.23×10^{-1}	1.72×10^{-3}	1.44×10^{-1}	2.28×10^{-2}	2.92×10^{-3}	2.03×10^{-3}	3.70×10^{-5}
2.5	2.32×10^{-5}	7.65×10^{-2}	1.04×10^{-3}	1.86×10^{-1}	3.45×10^{-2}	5.65×10^{-3}	1.51×10^{-3}	4.82×10^{-5}
2.7	1.08×10^{-5}	1.00×10^{-2}	5.94×10^{-4}	2.31×10^{-1}	4.94×10^{-2}	1.01×10^{-2}	1.05×10^{-3}	5.57×10^{-5}
2.9	4.88×10^{-6}	2.13×10^{-2}	3.21×10^{-4}	2.79×10^{-1}	6.76×10^{-2}	1.71×10^{-2}	6.82×10^{-4}	5.78×10^{-5}
3.1	2.13×10^{-6}	1.41×10^{-2}	1.65×10^{-4}	3.29×10^{-1}	8.91×10^{-2}	2.75×10^{-2}	4.17×10^{-4}	5.41×10^{-5}
3.3	9.00×10^{-7}	1.55×10^{-2}	8.13×10^{-5}	3.80×10^{-1}	1.13×10^{-1}	4.23×10^{-2}	2.40×10^{-4}	4.61×10^{-5}
3.5	3.68×10^{-7}	1.67×10^{-2}	3.83×10^{-5}	4.32×10^{-1}	1.40×10^{-1}	6.30×10^{-2}	1.30×10^{-4}	3.58×10^{-5}
3.7	1.45×10^{-7}	1.76×10^{-2}	1.73×10^{-5}	4.85×10^{-1}	1.70×10^{-1}	9.12×10^{-2}	6.68×10^{-5}	2.53×10^{-5}
3.9	5.56×10^{-8}	1.83×10^{-2}	7.53×10^{-6}	5.38×10^{-1}	2.01×10^{-1}	1.29×10^{-1}	3.25×10^{-5}	1.63×10^{-5}
4.1	2.05×10^{-8}	1.89×10^{-2}	3.15×10^{-6}	5.91×10^{-1}	2.33×10^{-1}	1.80×10^{-1}	1.50×10^{-5}	9.57×10^{-6}
4.4	4.33×10^{-9}	1.94×10^{-2}	7.93×10^{-7}	6.70×10^{-1}	2.83×10^{-1}	2.86×10^{-1}	4.30×10^{-6}	3.60×10^{-6}
4.7	8.43×10^{-10}	1.98×10^{-2}	1.83×10^{-7}	7.50×10^{-1}	3.35×10^{-1}	4.37×10^{-1}	1.11×10^{-6}	1.09×10^{-6}
5.0	1.52×10^{-10}	2.00×10^{-2}	3.89×10^{-8}	8.30×10^{-1}	3.87×10^{-1}	6.21×10^{-1}	2.56×10^{-7}	2.71×10^{-7}
5.5	7.33×10^{-12}	2.02×10^{-2}	2.42×10^{-9}	9.63×10^{-1}	4.75×10^{-1}	7.00×10^{-1}	1.79×10^{-8}	1.89×10^{-8}
6.0	2.83×10^{-12}	2.03×10^{-2}	1.19×10^{-10}	1.10	5.64×10^{-1}	1.83×10^{-1}	9.59×10^{-10}	9.87×10^{-10}
6.5	8.66×10^{-15}	2.03×10^{-2}	4.57×10^{-12}	1.23	6.53×10^{-1}	-2.14×10^{-1}	3.96×10^{-11}	4.02×10^{-11}
7.0	2.22×10^{-16}	2.03×10^{-2}	1.38×10^{-13}	1.36	7.41×10^{-1}	-3.75×10^{-1}	1.27×10^{-12}	1.28×10^{-12}
7.5	1.11×10^{-16}	2.03×10^{-2}	3.33×10^{-15}	1.49	8.30×10^{-1}	-4.40×10^{-1}	3.19×10^{-14}	3.19×10^{-14}
8.0	0.00	2.03×10^{-2}	1.11×10^{-16}	1.63	9.18×10^{-1}	-4.68×10^{-1}	6.66×10^{-16}	6.66×10^{-16}
8.5	0.00	2.04×10^{-2}	0.00	1.76	1.01	-4.82×10^{-1}	0.00	0.00

5. Conclusions

This paper introduced a new sharp upper bound for the standard normal cumulative distribution function $\Phi(x)$. The proposed bound was constructed by introducing a polynomial correction term into the exponent of a Polya-type expression, thereby preserving analytical simplicity while significantly improving approximation accuracy.

Numerical evidence indicates that the obtained bound provides uniformly high accuracy over the interval $x \geq 0$, with maximum absolute error bounded by 5.785×10^{-5}. Consequently, the proposed expression can also serve as a highly accurate analytic approximation for $\Phi(x)$.

Numerical comparisons demonstrated that the proposed bound improves the classical Polya bound and several other well-known upper bounds, particularly for small and moderate values of the argument. At the same time, the proposed formula retains excellent asymptotic behavior for large values of x.

Because of the well-known relationships among $\Phi(x)$, the Q-function, the error function, and the complementary error function, the proposed result immediately yields corresponding bounds and approximations for these related special functions.

An additional advantage of the proposed bound is its explicit closed-form structure, which makes it suitable for theoretical analysis, symbolic manipulation, and applications requiring analytically tractable Gaussian approximations. Such applications arise naturally in statistics, communication theory, reliability analysis, stochastic modeling, and computational mathematics.

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