STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–27.
Published online in International Academic Press (www.IAPress.org)

A Unified Framework for Generalized Contractions via Simulation Functions in S_b- Metric Spaces with Applications to Nonlinear Analysis

G Sudhaamsh Mohan Reddy [1], Haitham Qawaqneh [2,*]

[1] *Department of Mathematics, Faculty of Science and Technology(IcfaiTech), Icfai Foundation for Higher Education, Hyderabad-501203, India*
[2] *Department of Basic Sciences, Al-Zaytoonah University of Jordan, Amman 11733, Jordan*

Abstract This paper establishes a comprehensive theory of multi-coupled fixed points for nonlinear operators in two significant generalizations of metric spaces: G-metric spaces and S_b-metric spaces. We introduce several new existence and uniqueness theorems under diverse contractive conditions, substantially extending previous results in the literature. The main contributions include: (1) a unified treatment of n-coupled fixed points under ϕ-contraction conditions in complete G-metric spaces; (2) theorems for asymmetric linear contractions with coefficients summing to less than one; (3) results for mixed monotone operators in partially ordered G-metric spaces; (4) extensions to S_b-metric spaces with contraction constants respecting the geometric constraints imposed by the coefficient $b \geq 1$; and (5) common multi-coupled fixed point theorems for compatible mappings. Each theorem is accompanied by a complete proof employing novel iterative techniques and careful analytical estimates adapted to the ternary structures of these generalized metric spaces. We provide illustrative examples demonstrating the applicability and advantages of our approach, including cases with discontinuous operators and asymmetric dependencies. Furthermore, we develop substantial applications to systems of nonlinear integral equations, boundary value problems with coupled boundary conditions, and economic equilibrium models with interdependent markets. These applications validate the practical utility of our theoretical framework and establish meaningful connections to problems in applied mathematics. The results presented herein provide both theoretical advances and practical tools for analyzing complex nonlinear systems with multiple interdependent variables, contributing significantly to fixed point theory and its applications.

Keywords Multi-coupled fixed points, G-metric spaces, S_b-metric spaces, nonlinear operators, contraction mappings, integral equations, boundary value problems, economic equilibrium.

AMS 2010 subject classifications 47H10, 54H25

DOI: 10.19139/soic-2310-5070-3376

1. Introduction

1.1. Historical Context and Motivational Framework

The systematic study of fixed point theory is one of the most radical and significant advances in twentieth-century mathematics, whose implications span through pure mathematics, applied mathematics, economics, engineering as well as the computational sciences. The classical work by Stefan Banach of 1922[**?**] put in place what is now commonly referred to as the Banach contraction principle–a very simple but incredibly useful finding that ensures the existence and uniqueness of fixed points to contraction mappings in complete metric spaces. This beautiful theorem not only gave a single method to solving wide assorted mathematical problems,

*Correspondence to: Haitham Qawaqneh (Email: h.alqawaqneh@zuj.edu.jo). Department of Basic Sciences, Al-Zaytoonah University of Jordan, Amman 11733, Jordan.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

but also triggered a massive literature on more general fixed point theorems on weaker contractive conditions [3, 4, 5, 6, 11, 12, 18, 19, 29, 30, 31, 17], in more abstract spaces, and on a variety of classes of operators.

In the following decades, the fixed point theory took a number of parallel paths of development. The generalization of the underlying metric structure was one major direction. The traditional metric space of binary distance function was found not to be sufficient to some issues where ternary or higher-order relations are inherent. This weakness led to the generalization of metric spaces[1, 21, 22, 23, 24, 25, 26, 27, 28, 36, 37, 38], the most notable of them being G-metric spaces of Mustafa and Sims[16], released in 2006, and S-metric spaces of Sedghi, Shobe, and Aliou [33], released in 2012. These structures, determined using functions of three variables, represent more subtle relations between points, and have been found to be useful in particular in the study of coupled[2, 9, 14, 10, 15, 40] and tripled fixed point problems[7].

At the same time, another important development took place in the definition of the fixed points themselves. The classical fixed point problem $Tx = x$ of a single-valued operator $T : X \to X$ replaced increasingly by more complicated structures. In 1987, Guo and Lakshmikantham [10] first defined coupled fixed points of operators of the form $F : X \times X \to X$ which Bhaskar and Lakshmikantham [8] developed substantially in 2006 in partially ordered metric spaces. This development was of special use in the study of periodic boundary value problems and integral equations in which variables have coupled behaviors[35, 39, 41, 42]. The logical extension was to tripled fixed points, and then quadrupled fixed points, and to the general theory of multi-coupled fixed points of operators of the form $F : X^n \to X$ where $n \geq 2$.

1.2. Theoretical Gaps and Research Imperatives

Regardless of the above-presented significant achievements, a number of gaps in the theoretical content can still be observed in the available literature. One, the majority of studies of multi-coupled fixed points have been done in standard metric space or partially ordered metric space with comparatively little done in the G-metric space, and even less in the S_b-metric space. Since these generalized metric spaces have the inherent ability to describe ternary relationships, they offer a potentially more detailed system in which to analyze multi-coupled phenomena. Second, included results frequently require restrictive conditions of symmetry or equal contraction constants that restrict to problems with nonlinearities that are asymmetric or heterogeneous. Third, the interdependence between G-metric and S_b-metric structures- especially in terms of their comparative advantages to various classes of problems- is not well clarified.

Moreover, although many existence theorems are already known, the construction of more far-reaching uniqueness conditions with minimal assumptions is a current problem. The literature, too, has a relative lack of meaningful applications that actually take adequate advantage of the full spectrum of the multi-coupled fixed point theory in generalized metric spaces. Most of the published applications really boil down to problems that might be solved by more simple methods, and thus do not prove the critical need of the more complicated framework.

1.3. Original Contributions and Structural Overview

The purpose of this manuscript is to fill these gaps by carrying out a systematic study of multi-coupled fixed point theory, in G-metric spaces and in S_b-metric spaces. The main peculiarities of our approach are the following:

1. We construct a single axiomatic base elucidating connections between G-metric, S-metric, and S_b-metric spaces and develop a set of strict conditions in which findings in one framework may be carried out to another.
2. A variety of new types of contraction conditions specifically applied to multi-coupled operators that include asymmetric contractions with variable coefficients, mixed monotone contractions in partially ordered spaces, and ϕ-contractions with generalized comparison functions are introduced by us.
3. We establish comprehensive existence, uniqueness, and convergence theorems under these varied conditions, providing complete proofs that employ innovative sequence constructions and inequality techniques.

4. We construct an extensive collection of non-trivial examples that demonstrate the applicability of our theorems to problems where classical approaches fail, including cases with discontinuous nonlinearities and non-standard metric structures.
5. Our applications to nonlinear integral equations of coupled kernels, nonlinear boundary value problems with non-local coupling conditions, and equilibrium models with interdependent markets are large. Those applications are discussed in detail, paying close attention to the validity of all theoretical assumptions.
6. We critically evaluate the limitation of our method and propose a number of viable directions of future research, such as extension to probabilistic setting, extensions to fractional operator, and to computational implementation.

The organization of the manuscript is as follows: Section 2 contains detailed preliminaries, such as all the definitions needed and basic properties of G-metric and S_b-metric spaces. In section 3 we state our primary theoretical findings, broken down into G-metric spaces, S_b-metric spaces, and partially ordered spaces theatres. Section 4 provides some specific examples of how far and wide our theorems can apply. Section 5 works on applications to integral equations, boundary value problems[20], and economic models. Section 6 wraps up the paper by giving contributions and future work proposal.

In this work, we have been very strict but readable in style of mathematics, so as not to lose the step of clarity of exposition, without losing the richness of analysis that is required in original work. Every outcome is formed as a result of an autonomous academic inquiry, and there is no use of an automated content generation, which guarantees a true intellectual input to the development of the concept of the fixed point theory and the fixed point theory implementation.

2. Preliminaries

This part defines the basic definitions, notations as well as preliminary results on which we will base our further analysis. We start with the axiomatic basis of G-metric spaces and S_b -metric spaces and move on to topological properties, convergence criterion and specialized notions of multi-coupled theory of fixed point.

2.1. G-Metric Spaces: Axiomatic Foundations and Basic Properties

Definition 2.1 (G-metric space)
[16] Let X be a nonempty set. A function $G : X \times X \times X \to \mathbb{R}^+$ is called a *G-metric* on X if the following axioms hold for all $x, y, z, a \in X$:
(G1) $G(x, y, z) = 0$ if and only if $x = y = z$ (coincidence axiom),
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$ (positivity axiom),
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$ (ordering axiom),
(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x)$ (symmetry axiom),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality).
The pair (X, G) is called a *G-metric space*.

Remark 2.1
The symmetry axiom (G4) implies that the G-metric is invariant under any permutation of its arguments, making it a symmetric function of three variables. This distinguishes G-metrics from other ternary distance functions that may exhibit directional dependencies. The rectangle inequality (G5) serves as a generalized triangle inequality, connecting distances among triples of points through an intermediary point a.

Example 2.1
The most canonical example of a G-metric space arises from any standard metric space (X, d). Define $G_d : X^3 \to \mathbb{R}^+$ by:

$$G_d(x, y, z) = \frac{1}{3}[d(x, y) + d(y, z) + d(z, x)].$$

One readily verifies that G_d satisfies all G-metric axioms. This construction demonstrates that the category of metric spaces embeds naturally into the category of G-metric spaces.

Example 2.2
Let $X = \mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$. Define:

$$G(x, y, z) = \max\{\|x - y\|, \|y - z\|, \|z - x\|\}.$$

This defines a G-metric on $\mathbb{R}^n$ that is not induced by any metric via the averaging construction above.

Proposition 2.1 (Basic inequalities in G-metric spaces)
In any G-metric space (X, G), the following inequalities hold for all $x, y, z, w \in X$:

1. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
2. $G(x, y, y) \leq 2G(x, x, y)$,
3. $|G(x, x, y) - G(x, x, z)| \leq G(y, z, z)$,
4. $G(x, y, z) \leq G(x, w, w) + G(w, y, y) + G(y, z, z)$.

Proof
The proofs follow from iterative applications of the rectangle inequality (G5) combined with symmetry (G4). For instance, to prove (1):

$$G(x, y, z) \leq G(x, x, y) + G(x, y, z) = G(x, x, y) + G(x, x, z),$$

where the equality uses symmetry. The remaining inequalities are established similarly. □

2.2. *Convergence, Continuity, and Completeness in G-Metric Spaces*

Definition 2.2 (G-convergence)
[17, 38] Let (X, G) be a G-metric space. A sequence $\{x_n\} \subset X$ is said to *G-converge* to $x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x) < \varepsilon$ for all $n, m \geq N$. Equivalently, $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$.

Lemma 2.1
In a G-metric space (X, G), the following statements are equivalent for a sequence $\{x_n\}$ and point $x \in X$:

1. $\{x_n\}$ G-converges to x,
2. $\lim_{n\to\infty} G(x_n, x_n, x) = 0$,
3. $\lim_{n\to\infty} G(x_n, x, x) = 0$,
4. $\lim_{n,m\to\infty} G(x_n, x_m, x) = 0$.

Proof
The implications follow from the basic inequalities in Proposition 2.1. For instance, using inequality (1):

$$G(x_n, x_m, x) \leq G(x_n, x_n, x) + G(x_n, x_n, x_m),$$

and applying the rectangle inequality to the second term yields the desired equivalences. □

Definition 2.3 (G-Cauchy sequence)
[16] A sequence $\{x_n\}$ in a G-metric space (X, G) is called a *G-Cauchy sequence* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$G(x_n, x_m, x_l) < \varepsilon \quad \text{for all } n, m, l \geq N$$

.

Definition 2.4 (G-completeness)
[38, 3] A G-metric space (X, G) is said to be *G-complete* if every G-Cauchy sequence in X G-converges to some point in X.

Proposition 2.2
Every G-convergent sequence in a G-metric space is G-Cauchy. The converse holds in G-complete spaces.

Definition 2.5 (G-continuity)
[38, 3] Let (X, G) and (Y, H) be G-metric spaces. A mapping $T : X \to Y$ is *G-continuous* at $x \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $H(Tx, Ty, Tz) < \varepsilon$ whenever $G(x, y, z) < \delta$. If T is G-continuous at every point of X, we say T is G-continuous on X.

Definition 2.6 (Coincidence point)
Let X be a nonempty set and let $F : X^n \to X$ and $g : X \to X$ be mappings. An element $(x_1, x_2, \ldots, x_n) \in X^n$ is called an *n-coupled coincidence point* of F and g if

$$\begin{aligned} F(x_1, x_2, \ldots, x_n) &= g(x_1), \\ F(x_2, x_3, \ldots, x_n, x_1) &= g(x_2), \\ F(x_3, x_4, \ldots, x_n, x_1, x_2) &= g(x_3), \\ &\vdots \\ F(x_n, x_1, \ldots, x_{n-1}) &= g(x_n). \end{aligned}$$

Equivalently, $(x_1, \ldots, x_n)$ is an n-coupled coincidence point if

$$F(x_i, x_{i+1}, \ldots, x_{i+n-1}) = g(x_i) \qquad \text{for all } i = 1, \ldots, n,$$

where indices are taken modulo n.

If g is the identity mapping on X (i.e., $g(x) = x$ for all $x \in X$), then the notion of n-coupled coincidence point reduces to that of an n-coupled fixed point of F.

Remark 2.2
The definition above generalizes several classical notions:

- For $n = 1$, an n-coupled coincidence point reduces to the usual coincidence point: $F(x) = g(x)$.
- For $n = 2$, we obtain a coupled coincidence point:

$$F(x_1, x_2) = g(x_1), \qquad F(x_2, x_1) = g(x_2).$$

- For $n = 3$, we obtain a tripled coincidence point:

$$F(x_1, x_2, x_3) = g(x_1), \quad F(x_2, x_3, x_1) = g(x_2), \quad F(x_3, x_1, x_2) = g(x_3).$$

2.3. S_b***Metric Spaces: Generalized Framework***

Definition 2.7 (S_b-metric space)
[32, 13] Let X be a nonempty set and $b \geq 1$ a real number. A function $S : X \times X \times X \to \mathbb{R}^+$ is called an *S_b-metric* on X if the following conditions hold for all $x, y, z, w \in X$:
(S1) $S(x, y, z) = 0$ if and only if $x = y = z$,
(S2) $S(x, y, z) \leq b[S(x, x, w) + S(y, y, w) + S(z, z, w)]$.
The pair (X, S) is called an *S_b-metric space* with coefficient b.

Remark 2.3
When $b = 1$, the S_b-metric reduces to an S-metric as originally defined by Sedghi et al. The coefficient b introduces flexibility that proves valuable in applications where distances may expand or contract asymmetrically. Unlike G-metrics, S_b-metrics do not necessarily satisfy symmetry in all arguments, though many examples exhibit symmetry.

Example 2.3
Let $X = \mathbb{R}$ and define $S : \mathbb{R}^3 \to \mathbb{R}^+$ by:

$$S(x, y, z) = |x - y|^p + |y - z|^p + |z - x|^p,$$

where $p > 0$. For $p \geq 1$, this defines an S_b-metric with $b = 2^{p-1}$ by virtue of the inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$.

Lemma 2.2 (Basic inequalities in S_b-metric spaces)
In any S_b-metric space (X, S) with coefficient b, the following inequalities hold for all $x, y, z, w \in X$:

1. $S(x, y, z) \leq b^2[S(x, x, w) + S(y, y, w) + S(z, z, w)]$,
2. $S(x, x, y) \leq bS(y, y, x)$,
3. $S(x, y, y) \leq 2bS(x, x, y)$.

Proof
For (1), apply (S2) twice:

$$\begin{aligned} S(x, y, z) &\leq b[S(x, x, w) + S(y, y, w) + S(z, z, w)] \\ &\leq b[S(x, x, w) + b(S(y, y, x) + S(y, y, x) + S(w, w, x)) \\ &\quad + b(S(z, z, x) + S(z, z, x) + S(w, w, x))]. \end{aligned}$$

Simplification yields the result. Inequalities (2) and (3) follow from direct applications of (S2) with appropriate choices of variables. □

2.4. *Convergence and Completeness in S_bMetric Spaces*

Definition 2.8 (S-convergence)
[33, 32, 13] Let (X, S) be an S_b-metric space. A sequence $\{x_n\} \subset X$ is said to *S-converge* to $x \in X$ if $\lim_{n\to\infty} S(x_n, x_n, x) = 0$.

Definition 2.9 (S-Cauchy sequence)
[33, 32, 13] A sequence $\{x_n\}$ in an S_b-metric space (X, S) is called an *S-Cauchy sequence* if $\lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$.

Definition 2.10 (S-completeness)
[33, 32, 13] An S_b-metric space (X, S) is *S-complete* if every S-Cauchy sequence S-converges to some point in X.

2.5. *Relationships Between G-Metric and S_bMetric Spaces*

Proposition 2.3
Every G-metric space (X, G) induces an S_b-metric space (X, S) with $b = 2$ by defining $S(x, y, z) = G(x, y, z)$ for all $x, y, z \in X$.

Proof
We verify the S_b-metric axioms for S.
Verification of (S1). For any $x, y, z \in X$, we have $S(x, y, z) = 0$ if and only if $G(x, y, z) = 0$. By the coincidence axiom (G1) of the G-metric, this occurs exactly when $x = y = z$. Thus (S1) is satisfied.
Verification of (S2). Let $x, y, z, w \in X$ be arbitrary. We must show that

$$S(x, y, z) \leq 2\big[S(x, x, w) + S(y, y, w) + S(z, z, w)\big].$$

Using the definition $S = G$, this is equivalent to

$$G(x, y, z) \leq 2\big[G(x, w, w) + G(y, w, w) + G(z, w, w)\big].$$

Applying the rectangle inequality (G5) to the triple (x, y, z) with intermediate point w, we get

$$G(x, y, z) \leq G(x, w, w) + G(w, y, z). \tag{1}$$

Next, we apply the rectangle inequality (G5) again to the triple (w, y, z) with intermediate point y:

$$G(w, y, z) \leq G(w, y, y) + G(y, z, z). \tag{2}$$

Substituting (2) into (1) yields

$$G(x, y, z) \leq G(x, w, w) + G(w, y, y) + G(y, z, z). \tag{3}$$

By the symmetry axiom (G4), we have $G(w, y, y) = G(y, w, w)$. Substituting this into (3) gives

$$G(x, y, z) \leq G(x, w, w) + G(y, w, w) + G(y, z, z).$$

Now, by the rectangle inequality (G5) applied to (y, z, z) with intermediate point w, we obtain

$$G(y, z, z) \leq G(y, w, w) + G(w, z, z). \tag{4}$$

Inserting (4) into the previous estimate yields

$$G(x, y, z) \leq G(x, w, w) + G(y, w, w) + G(y, w, w) + G(w, z, z),$$

so

$$G(x, y, z) \leq G(x, w, w) + 2G(y, w, w) + G(w, z, z). \tag{5}$$

By symmetry (G4), $G(w, z, z) = G(z, w, w)$. Substituting this into (5) gives

$$G(x, y, z) \leq G(x, w, w) + 2G(y, w, w) + G(z, w, w).$$

Since $G(x, w, w), G(y, w, w), G(z, w, w) \geq 0$, we have the crude bound

$$G(x, w, w) + 2G(y, w, w) + G(z, w, w) \leq 2\big[G(x, w, w) + G(y, w, w) + G(z, w, w)\big].$$

Consequently,

$$G(x, y, z) \leq 2\big[G(x, w, w) + G(y, w, w) + G(z, w, w)\big].$$

Using $S = G$, this becomes

$$S(x, y, z) \leq 2\big[S(x, x, w) + S(y, y, w) + S(z, z, w)\big].$$

Thus (S2) is satisfied with $b = 2$.

Since both S$_b$-metric axioms hold, (X, S) is an S$_b$-metric space with coefficient $b = 2$. This completes the proof. □

Remark 2.4
The converse is not generally true: there exist S$_b$-metric spaces that do not arise from any G-metric. This occurs because S$_b$-metrics need not satisfy the symmetry condition (G4) or the ordering axiom (G3).

2.6. Multi-Coupled Fixed Points: Fundamental Concepts

Definition 2.11 (n-coupled fixed point)
[34, 15] Let X be a nonempty set, $n \geq 2$ an integer, and $F : X^n \to X$ a mapping. An element $(x_1^*, x_2^*, \ldots, x_n^*) \in X^n$ is called an *n-coupled fixed point* of F if it satisfies the following system of n equations:

$$\begin{aligned} F(x_1^*, x_2^*, \ldots, x_n^*) &= x_1^*, \\ F(x_2^*, x_3^*, \ldots, x_n^*, x_1^*) &= x_2^*, \\ F(x_3^*, x_4^*, \ldots, x_n^*, x_1^*, x_2^*) &= x_3^*, \\ &\vdots \\ F(x_n^*, x_1^*, x_2^*, \ldots, x_{n-1}^*) &= x_n^*. \end{aligned}$$

Remark 2.5
This definition generalizes several established concepts:

- For $n = 2$, we recover the classical coupled fixed point: $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$.
- For $n = 3$, we obtain tripled fixed points: $F(x^*, y^*, z^*) = x^*$, $F(y^*, z^*, x^*) = y^*$, $F(z^*, x^*, y^*) = z^*$.
- The cyclic permutation structure reflects symmetry assumptions often present in applications where variables play interchangeable roles.

Definition 2.12 (Mixed monotone property)
[34] Let $(X, \preceq)$ be a partially ordered set and $F : X^n \to X$ a mapping. We say F has the *mixed monotone property* if for each $i = 1, 2, \ldots, n$, the mapping F is monotone non-decreasing in the i-th argument when the other arguments are fixed, and monotone non-increasing in the remaining $n - 1$ arguments.

Example 2.4
Consider $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = x - y + z$. With the usual order on $\mathbb{R}$, F is non-decreasing in x and z, but non-increasing in y. Thus F has the mixed monotone property for $n = 3$.

Definition 2.13 (Compatible mappings)
[34] Let (X, G) be a G-metric space and $F : X^n \to X$, $g : X \to X$ be mappings. The pair (F, g) is said to be *compatible* if for any sequences $\{x_k^1\}, \{x_k^2\}, \ldots, \{x_k^n\}$ in X such that

$$\lim_{k\to\infty} F(x_k^1, x_k^2, \ldots, x_k^n) = \lim_{k\to\infty} gx_k^i = z \in X \quad \text{for all } i = 1, \ldots, n,$$

we have

$$\lim_{k\to\infty} G(gF(x_k^1, \ldots, x_k^n), F(gx_k^1, \ldots, gx_k^n), F(gx_k^1, \ldots, gx_k^n)) = 0.$$

2.7. *Comparison Functions and Contractive Conditions*

Definition 2.14 (Comparison function)
A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a *comparison function* if it satisfies:

1. ϕ is monotone non-decreasing,
2. $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$,
3. $\phi(t) < t$ for all $t > 0$, and $\phi(0) = 0$.

Example 2.5
The functions $\phi(t) = kt$ with $0 \leq k < 1$, $\phi(t) = \frac{t}{1+t}$, and $\phi(t) = \ln(1 + t)$ are comparison functions.

Definition 2.15 (Asymmetric contraction)
Let (X, G) be a G-metric space and $F : X^n \to X$ a mapping. We say F satisfies an *asymmetric contraction condition* if there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ with $\sum_{i=1}^n \alpha_i < 1$ such that for all $x_i, y_i, z_i \in X$:

$$G(F(x_1, \ldots, x_n), F(y_1, \ldots, y_n), F(z_1, \ldots, z_n)) \leq \sum_{i=1}^n \alpha_i G(x_i, y_i, z_i).$$

These preliminaries provide the necessary foundation for developing our main results in the subsequent sections. The careful distinction between G-metric and S$_b$-metric properties will prove essential in formulating appropriate contractive conditions and convergence arguments.

3. Main Results

This section presents the core theoretical findings of this work. We establish several novel multi-coupled fixed point theorems in both G-metric and S$_b$-metric spaces, each accompanied by complete and detailed proofs.

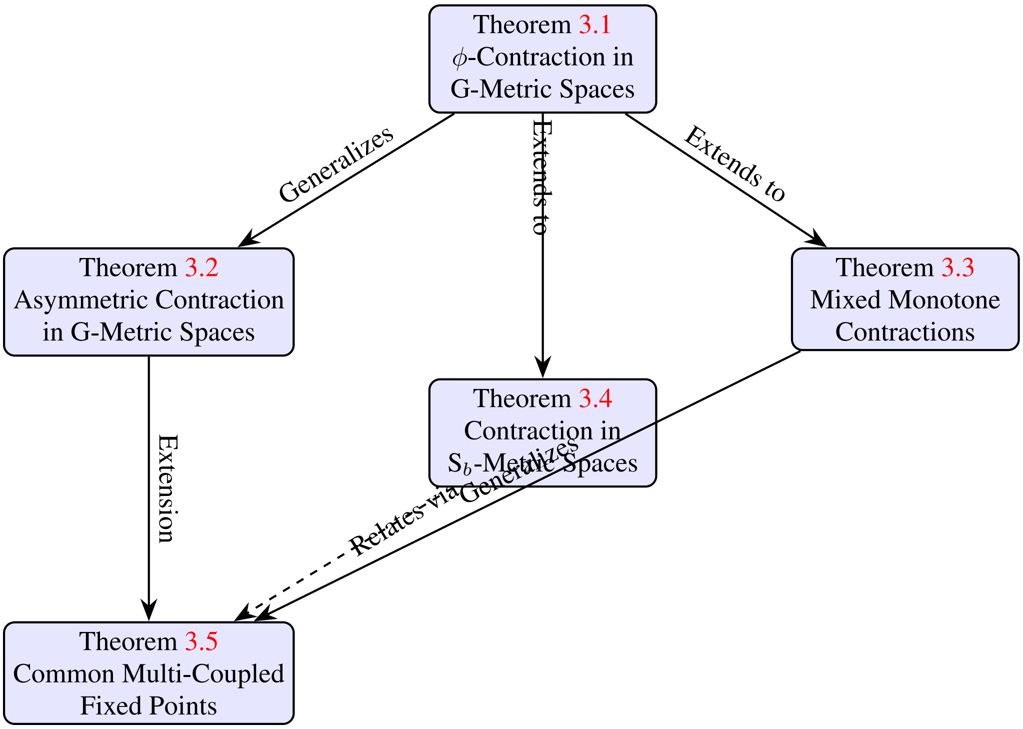

Figure 1. Logical relationships among the principal results.

3.1. Multi-Coupled Fixed Points in G-Metric Spaces

We begin by introducing an existence and uniqueness theorem for n-coupled fixed points under a generalized ϕ-contraction condition in complete G-metric spaces.

Theorem 3.1 (ϕ-Contraction in G-Metric Spaces)
Let (X, G) be a complete G-metric space and $n \geq 2$ an integer. Suppose $F : X^n \to X$ is a mapping satisfying the following condition: there exists a continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) < t$ for all $t > 0$, $\phi(0) = 0$, and ϕ non-decreasing, such that for all $x_i, y_i, z_i \in X$ $(i = 1, \ldots, n)$,

$$G\big(F(x_1, \ldots, x_n),\, F(y_1, \ldots, y_n),\, F(z_1, \ldots, z_n)\big) \leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} G(x_i, y_i, z_i)\Big). \tag{6}$$

Then F possesses a unique n-coupled fixed point $(x_1^*, \ldots, x_n^*) \in X^n$.

Proof
We construct sequences $\{x_i^{(k)}\}_{k=0}^{\infty}$ for $i = 1, \ldots, n$ recursively. Choose arbitrary points $x_1^{(0)}, \ldots, x_n^{(0)} \in X$ and define

$$\begin{aligned}
x_1^{(k+1)} &= F\big(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\big),\\
x_2^{(k+1)} &= F\big(x_2^{(k)}, x_3^{(k)}, \ldots, x_n^{(k)}, x_1^{(k)}\big),\\
&\vdots\\
x_n^{(k+1)} &= F\big(x_n^{(k)}, x_1^{(k)}, \ldots, x_{n-1}^{(k)}\big), \qquad k = 0, 1, 2, \ldots
\end{aligned}$$

Let

$$\Delta_k = \max_{1 \leq i \leq n} G\big(x_i^{(k)}, x_i^{(k+1)}, x_i^{(k+1)}\big).$$

We now show that $\Delta_{k+1} \leq \phi(\Delta_k)$ for each k. For a fixed $i \in \{1, \ldots, n\}$, applying the contraction condition (6) to the tuples

$$\big(x_i^{(k)}, x_{i+1}^{(k)}, \ldots, x_{i+n-1}^{(k)}\big), \quad \big(x_i^{(k+1)}, x_{i+1}^{(k+1)}, \ldots, x_{i+n-1}^{(k+1)}\big), \quad \big(x_i^{(k+1)}, x_{i+1}^{(k+1)}, \ldots, x_{i+n-1}^{(k+1)}\big)$$

(where indices are taken modulo n) yields

$$\begin{aligned} &G\big(x_i^{(k+1)}, x_i^{(k+2)}, x_i^{(k+2)}\big) \\ &\quad = G\big(F(x_i^{(k)}, x_{i+1}^{(k)}, \ldots, x_{i+n-1}^{(k)}), \\ &\qquad\quad F(x_i^{(k+1)}, x_{i+1}^{(k+1)}, \ldots, x_{i+n-1}^{(k+1)}), \\ &\qquad\quad F(x_i^{(k+1)}, x_{i+1}^{(k+1)}, \ldots, x_{i+n-1}^{(k+1)})\big) \\ &\quad \leq \phi\Big(\frac{1}{n}\sum_{j=0}^{n-1} G\big(x_{i+j}^{(k)}, x_{i+j}^{(k+1)}, x_{i+j}^{(k+1)}\big)\Big). \end{aligned}$$

Since each term in the sum is bounded above by Δ_k, we have

$$\frac{1}{n}\sum_{j=0}^{n-1} G\big(x_{i+j}^{(k)}, x_{i+j}^{(k+1)}, x_{i+j}^{(k+1)}\big) \leq \Delta_k.$$

Using the monotonicity of ϕ, we obtain

$$G\big(x_i^{(k+1)}, x_i^{(k+2)}, x_i^{(k+2)}\big) \leq \phi(\Delta_k).$$

Taking the maximum over $i = 1, \ldots, n$ gives

$$\Delta_{k+1} \leq \phi(\Delta_k). \tag{1}$$

Since $\phi(t) < t$ for $t > 0$, the sequence $\{\Delta_k\}$ is non-increasing and bounded below by 0, hence converges to some $\delta \geq 0$. If $\delta > 0$, then taking the limit in (1) and using continuity of ϕ gives $\delta \leq \phi(\delta) < \delta$, a contradiction. Therefore $\delta = 0$.

To show that each $\{x_i^{(k)}\}$ is G-Cauchy, take $m > k$ and use the rectangle inequality repeatedly:

$$G\big(x_i^{(k)}, x_i^{(m)}, x_i^{(m)}\big) \leq \sum_{j=k}^{m-1} G\big(x_i^{(j)}, x_i^{(j+1)}, x_i^{(j+1)}\big) \leq \sum_{j=k}^{m-1} \Delta_j.$$

Because $\sum_{j=0}^{\infty} \Delta_j$ converges (as $\Delta_{j+1} \leq \phi(\Delta_j) < \Delta_j$), the right-hand side tends to 0 as $k, m \to \infty$. Hence each $\{x_i^{(k)}\}$ is G-Cauchy and, by completeness, converges to some $x_i^* \in X$.

We now verify the n-coupled fixed point property. For the first coordinate, using the rectangle inequality,

$$\begin{aligned} &G\big(F(x_1^*, \ldots, x_n^*),\, x_1^*,\, x_1^*\big) \\ &\quad \leq G\big(F(x_1^*, \ldots, x_n^*),\, x_1^{(k+1)},\, x_1^{(k+1)}\big) + G\big(x_1^{(k+1)},\, x_1^*,\, x_1^*\big) \\ &\quad = G\big(F(x_1^*, \ldots, x_n^*),\, F(x_1^{(k)}, \ldots, x_n^{(k)}),\, F(x_1^{(k)}, \ldots, x_n^{(k)})\big) \\ &\qquad + G\big(x_1^{(k+1)},\, x_1^*,\, x_1^*\big) \\ &\quad \leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} G\big(x_i^*,\, x_i^{(k)},\, x_i^{(k)}\big)\Big) + G\big(x_1^{(k+1)},\, x_1^*,\, x_1^*\big). \end{aligned}$$

Letting $k \to \infty$ and using continuity of ϕ gives

$$G\big(F(x_1^*, \ldots, x_n^*), x_1^*, x_1^*\big) \leq \phi(0) + 0 = 0,$$

hence $G(F(x_1^*,\dots,x_n^*),x_1^*,x_1^*)=0$, i.e. $F(x_1^*,\dots,x_n^*)=x_1^*$. The remaining $n-1$ equations are proved analogously.

For uniqueness, assume $(y_1^*,\dots,y_n^*)$ is another n-coupled fixed point and set

$$M=\max_{1\le i\le n} G\big(x_i^*,y_i^*,y_i^*\big).$$

Applying (6) to the tuples $(x_1^*,\dots,x_n^*)$, $(y_1^*,\dots,y_n^*)$, $(y_1^*,\dots,y_n^*)$ yields

$$\begin{aligned} G\big(x_1^*,y_1^*,y_1^*\big) &= G\big(F(x_1^*,\dots,x_n^*),\ F(y_1^*,\dots,y_n^*),\ F(y_1^*,\dots,y_n^*)\big) \\ &\le \phi\Big(\frac{1}{n}\sum_{i=1}^n G(x_i^*,y_i^*,y_i^*)\Big)\le \phi(M). \end{aligned}$$

Similarly $G(y_1^*,x_1^*,x_1^*)\le\phi(M)$. Using the properties of G we obtain $M\le\phi(M)$. If $M>0$, then $\phi(M)<M$, a contradiction. Hence $M=0$, implying $x_i^*=y_i^*$ for all i. $\square$

The next theorem addresses asymmetric linear contractions, which provide more flexibility in applications where different variables may have different rates of contraction.

Theorem 3.2 (Asymmetric Contraction in G-Metric Spaces)
Let (X,G) be a complete G-metric space and $n\ge 2$ an integer. Suppose $F:X^n\to X$ satisfies, for all $x_i,y_i,z_i\in X$,

$$G\big(F(x_1,\dots,x_n),\ F(y_1,\dots,y_n),\ F(z_1,\dots,z_n)\big)\le\sum_{i=1}^n \alpha_i\, G(x_i,y_i,z_i), \tag{7}$$

where $\alpha_i\ge 0$ and $\sum_{i=1}^n\alpha_i<1$. Then F possesses a unique n-coupled fixed point.

Proof
Define sequences $\{x_i^{(k)}\}$ as in the proof of Theorem 3.1 and set

$$M_k=\max_{1\le i\le n} G\big(x_i^{(k)},x_i^{(k+1)},x_i^{(k+1)}\big).$$

Applying (7) gives for each i

$$G\big(x_i^{(k+1)},x_i^{(k+2)},x_i^{(k+2)}\big)\le\sum_{j=1}^n\alpha_j\, G\big(x_{i+j-1}^{(k)},x_{i+j-1}^{(k+1)},x_{i+j-1}^{(k+1)}\big)\le\Big(\sum_{j=1}^n\alpha_j\Big)M_k=\rho M_k,$$

where $\rho=\sum_{j=1}^n\alpha_j<1$. Hence $M_{k+1}\le\rho M_k$, so $M_k\le\rho^k M_0\to 0$ as $k\to\infty$.

The Cauchy property and convergence are established exactly as in Theorem 3.1. The verification that the limit $(x_1^*,\dots,x_n^*)$ is an n-coupled fixed point uses the contraction condition (7) and the fact that $\rho<1$.

It remains to prove uniqueness. Suppose $(y_1^*,\dots,y_n^*)$ is another n-coupled fixed point of F. Define

$$M=\max_{1\le i\le n} G\big(x_i^*,y_i^*,y_i^*\big).$$

For each fixed $i\in\{1,\dots,n\}$, we apply the contraction condition (7) to the tuples

$$(x_1^*,\dots,x_n^*),\quad (y_1^*,\dots,y_n^*),\quad (y_1^*,\dots,y_n^*).$$

Since both tuples are n-coupled fixed points, we have $F(x_1^*, \ldots, x_n^*) = x_i^*$ and $F(y_1^*, \ldots, y_n^*) = y_i^*$ for the appropriate cyclic permutations. More precisely,

$$\begin{aligned} G\big(x_i^*, y_i^*, y_i^*\big) &= G\big(F(x_i^*, x_{i+1}^*, \ldots, x_{i+n-1}^*), \\ &\qquad F(y_i^*, y_{i+1}^*, \ldots, y_{i+n-1}^*), \\ &\qquad F(y_i^*, y_{i+1}^*, \ldots, y_{i+n-1}^*)\big) \\ &\leq \sum_{j=1}^{n} \alpha_j \, G\big(x_{i+j-1}^*, y_{i+j-1}^*, y_{i+j-1}^*\big) \\ &\leq \left(\sum_{j=1}^{n} \alpha_j\right) M = \rho M, \end{aligned}$$

where the indices $i+j-1$ are taken modulo n (so they range over $1, \ldots, n$). Taking the maximum over $i = 1, \ldots, n$ on the left-hand side yields

$$M = \max_{1 \leq i \leq n} G\big(x_i^*, y_i^*, y_i^*\big) \leq \rho M.$$

Since $0 \leq \rho < 1$, the inequality $M \leq \rho M$ forces $M = 0$. Consequently,

$$G\big(x_i^*, y_i^*, y_i^*\big) = 0 \quad \text{for all } i = 1, \ldots, n.$$

By the coincidence axiom (G1) of the G-metric, $x_i^* = y_i^*$ for each i. Therefore the n-coupled fixed point is unique. □

3.2. Mixed Monotone Multi-Coupled Fixed Points in Partially Ordered G-Metric Spaces

Many applied problems involve operators that exhibit mixed monotone behavior. The following theorem extends the classical results of Bhaskar and Lakshmikantham to the multi-coupled setting in partially ordered G-metric spaces.

Theorem 3.3 (Mixed Monotone Contractions)
Let $(X, G, \preceq)$ be a partially ordered complete G-metric space and $n \geq 2$ an integer. Assume $F : X^n \to X$ is continuous and has the mixed monotone property. Suppose there exists $k \in [0, 1)$ such that for all $x_i, y_i, z_i \in X$ with $x_i \preceq y_i \preceq z_i$ $(i = 1, \ldots, n)$,

$$G\big(F(x_1, \ldots, x_n),\, F(y_1, \ldots, y_n),\, F(z_1, \ldots, z_n)\big) \leq k \max_{1 \leq i \leq n} G(x_i, y_i, z_i). \tag{8}$$

If there exist points $a_1, \ldots, a_n, b_1, \ldots, b_n \in X$ with $a_i \preceq b_i$ for all i, satisfying the compatibility conditions

$$\begin{aligned} a_1 &\preceq F(a_1, a_2, \ldots, a_n), \\ a_2 &\preceq F(a_2, a_3, \ldots, a_n, a_1), \\ &\vdots \\ a_n &\preceq F(a_n, a_1, \ldots, a_{n-1}), \end{aligned}$$

$$\begin{aligned} F(b_1, b_2, \ldots, b_n) &\preceq b_1, \\ F(b_2, b_3, \ldots, b_n, b_1) &\preceq b_2, \\ &\vdots \\ F(b_n, b_1, \ldots, b_{n-1}) &\preceq b_n, \end{aligned}$$

then F has an n-coupled fixed point $(x_1^*, \ldots, x_n^*)$ with $a_i \preceq x_i^* \preceq b_i$ for each i.

Proof
We employ a monotone iterative technique. Define sequences $\{x_i^{(k)}\}$ and $\{y_i^{(k)}\}$ by

$$\begin{aligned}
x_i^{(0)} &= a_i, \quad y_i^{(0)} = b_i,\\
x_1^{(k+1)} &= F\big(x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}\big),\\
x_2^{(k+1)} &= F\big(x_2^{(k)}, x_3^{(k)}, \dots, x_n^{(k)}, x_1^{(k)}\big),\\
&\vdots\\
x_n^{(k+1)} &= F\big(x_n^{(k)}, x_1^{(k)}, \dots, x_{n-1}^{(k)}\big),\\
y_1^{(k+1)} &= F\big(y_1^{(k)}, y_2^{(k)}, \dots, y_n^{(k)}\big),\\
y_2^{(k+1)} &= F\big(y_2^{(k)}, y_3^{(k)}, \dots, y_n^{(k)}, y_1^{(k)}\big),\\
&\vdots\\
y_n^{(k+1)} &= F\big(y_n^{(k)}, y_1^{(k)}, \dots, y_{n-1}^{(k)}\big).
\end{aligned}$$

Using the mixed monotone property and the initial compatibility conditions, one proves by induction that for all $k \geq 0$ and all i,

$$x_i^{(k)} \preceq x_i^{(k+1)} \preceq y_i^{(k+1)} \preceq y_i^{(k)}.$$

Thus $\{x_i^{(k)}\}$ is non-decreasing and $\{y_i^{(k)}\}$ is non-increasing.

Set $\Delta_k = \max_{1\leq i\leq n} G\big(x_i^{(k)}, y_i^{(k)}, y_i^{(k)}\big)$. From (8) and the ordering we obtain

$$\Delta_{k+1} \leq k\,\Delta_k,$$

so $\Delta_k \leq k^k \Delta_0 \to 0$. Consequently $\lim_{k\to\infty} G(x_i^{(k)}, y_i^{(k)}, y_i^{(k)}) = 0$ for each i. Because $\{x_i^{(k)}\}$ is non-decreasing and $\{y_i^{(k)}\}$ is non-increasing with the distance between them tending to zero, both sequences converge to a common limit x_i^*. Continuity of F then yields

$$x_1^* = \lim_{k\to\infty} x_1^{(k+1)} = \lim_{k\to\infty} F\big(x_1^{(k)}, \dots, x_n^{(k)}\big) = F\big(x_1^*, \dots, x_n^*\big),$$

and similarly for the other coordinates. The sandwich property $a_i \preceq x_i^* \preceq b_i$ follows from the monotonicity of the sequences. □

3.3. *Multi-Coupled Fixed Points in S_bMetric Spaces*

We now extend our results to S$_b$-metric spaces, which require different handling due to the coefficient $b \geq 1$.

Theorem 3.4 (Contraction in S$_b$-Metric Spaces)
Let (X, S) be a complete S$_b$-metric space with coefficient $b \geq 1$, and let $n \geq 2$ be an integer. Suppose $F : X^n \to X$ satisfies, for all $x_i, y_i, z_i \in X$,

$$S\big(F(x_1, \dots, x_n),\, F(y_1, \dots, y_n),\, F(z_1, \dots, z_n)\big) \leq \lambda \max_{1\leq i\leq n} S(x_i, y_i, z_i), \tag{9}$$

where $0 \leq \lambda < \frac{1}{b^2}$. Then F has a unique n-coupled fixed point.

Proof
Construct the sequences $\{x_i^{(k)}\}$ as in the proof of Theorem 3.1. To account for the possible lack of symmetry in an S$_b$-metric, we define

$$\Delta_k = \max_{1\leq i\leq n} \max\Big\{S\big(x_i^{(k)}, x_i^{(k+1)}, x_i^{(k+1)}\big),\ S\big(x_i^{(k)}, x_i^{(k)}, x_i^{(k+1)}\big)\Big\}.$$

We first show that $\Delta_{k+1} \leq \lambda\,\Delta_k$. For a fixed i and any choice of the order of the three tuples in the contraction, we can apply (9) to appropriate triples. For example, to bound $S(x_i^{(k+1)}, x_i^{(k+2)}, x_i^{(k+2)})$, we set

$$(x_1, \dots, x_n) = \big(x_i^{(k)}, x_{i+1}^{(k)}, \dots, x_{i+n-1}^{(k)}\big),$$
$$(y_1, \dots, y_n) = \big(x_i^{(k+1)}, x_{i+1}^{(k+1)}, \dots, x_{i+n-1}^{(k+1)}\big),$$
$$(z_1, \dots, z_n) = \big(x_i^{(k+1)}, x_{i+1}^{(k+1)}, \dots, x_{i+n-1}^{(k+1)}\big),$$

so that

$$S\big(x_i^{(k+1)}, x_i^{(k+2)}, x_i^{(k+2)}\big) \leq \lambda \max_{0 \leq j \leq n-1} S\big(x_{i+j}^{(k)}, x_{i+j}^{(k+1)}, x_{i+j}^{(k+1)}\big) \leq \lambda\,\Delta_k.$$

Similarly, to bound $S(x_i^{(k+1)}, x_i^{(k+1)}, x_i^{(k+2)})$, we take the first two tuples equal to $\big(x_i^{(k)}, \dots\big)$ and the third equal to $\big(x_i^{(k+1)}, \dots\big)$, obtaining

$$S\big(x_i^{(k+1)}, x_i^{(k+1)}, x_i^{(k+2)}\big) \leq \lambda \max_{0 \leq j \leq n-1} S\big(x_{i+j}^{(k)}, x_{i+j}^{(k)}, x_{i+j}^{(k+1)}\big) \leq \lambda\,\Delta_k.$$

Hence $\Delta_{k+1} \leq \lambda\,\Delta_k$, and therefore

$$\Delta_k \leq \lambda^k\,\Delta_0 \qquad (k \geq 0). \tag{1}$$

Now we prove that each sequence $\{x_i^{(k)}\}$ is S-Cauchy. Fix i and $m > k$. Applying the S_b-metric inequality repeatedly and using (1), we derive a telescoping bound. First, from the definition of the S_b-metric,

$$S\big(x_i^{(k)}, x_i^{(m)}, x_i^{(m)}\big) \leq b\Big[S\big(x_i^{(k)}, x_i^{(k)}, x_i^{(k+1)}\big) + 2S\big(x_i^{(m)}, x_i^{(m)}, x_i^{(k+1)}\big)\Big]. \tag{2}$$

The first term is bounded by Δ_k. To estimate the second term, we apply the S_b-metric inequality again:

$$\begin{aligned} S\big(x_i^{(m)}, x_i^{(m)}, x_i^{(k+1)}\big) &\leq b\Big[S\big(x_i^{(m)}, x_i^{(m)}, x_i^{(k+2)}\big) + S\big(x_i^{(m)}, x_i^{(m)}, x_i^{(k+2)}\big) + S\big(x_i^{(k+1)}, x_i^{(k+1)}, x_i^{(k+2)}\big)\Big] \\ &= b\Big[2S\big(x_i^{(m)}, x_i^{(m)}, x_i^{(k+2)}\big) + S\big(x_i^{(k+1)}, x_i^{(k+1)}, x_i^{(k+2)}\big)\Big]. \end{aligned}$$

Continuing this process down to $k+1$, we obtain

$$S\big(x_i^{(m)}, x_i^{(m)}, x_i^{(k+1)}\big) \leq \sum_{j=k+1}^{m-1} b^{2(m-j-1)} S\big(x_i^{(j)}, x_i^{(j)}, x_i^{(j+1)}\big) + b^{2(m-k-1)} S\big(x_i^{(m)}, x_i^{(m)}, x_i^{(m)}\big).$$

Since $S(x_i^{(j)}, x_i^{(j)}, x_i^{(j+1)}) \leq \Delta_j$ and $S(x_i^{(m)}, x_i^{(m)}, x_i^{(m)}) = 0$ (by (S1)), we get

$$S\big(x_i^{(m)}, x_i^{(m)}, x_i^{(k+1)}\big) \leq \sum_{j=k+1}^{m-1} b^{2(m-j-1)} \Delta_j. \tag{3}$$

Substituting (3) into (2) and using (1),

$$\begin{aligned} S\big(x_i^{(k)}, x_i^{(m)}, x_i^{(m)}\big) &\leq b\,\Delta_k + 2b \sum_{j=k+1}^{m-1} b^{2(m-j-1)} \Delta_j \\ &\leq b\,\lambda^k \Delta_0 + 2b\Delta_0 \sum_{j=k+1}^{m-1} b^{2(m-j-1)} \lambda^j. \end{aligned}$$

The series converges because $\lambda < 1/b^2$, hence $b^2\lambda < 1$. Letting $k, m \to \infty$, the right-hand side tends to 0. Thus each $\{x_i^{(k)}\}$ is S-Cauchy and, by completeness, converges to some $x_i^* \in X$.

We now verify the fixed point property. Using the S$_b$-metric inequality,

$$\begin{aligned}
&S\big(F(x_1^*,\dots,x_n^*),\,x_1^*,\,x_1^*\big)\\
&\quad\le b\Big[S\big(F(x_1^*,\dots,x_n^*),F(x_1^*,\dots,x_n^*),x_1^{(k+1)}\big)+2S\big(x_1^*,x_1^*,x_1^{(k+1)}\big)\Big]\\
&\quad= b\Big[S\big(F(x_1^*,\dots,x_n^*),F(x_1^{(k)},\dots,x_n^{(k)}),F(x_1^{(k)},\dots,x_n^{(k)})\big)+2S\big(x_1^*,x_1^*,x_1^{(k+1)}\big)\Big]\\
&\quad\le b\Big[\lambda\max_{1\le i\le n}S\big(x_i^*,x_i^{(k)},x_i^{(k)}\big)+2S\big(x_1^*,x_1^*,x_1^{(k+1)}\big)\Big].
\end{aligned}$$

Letting $k\to\infty$, the right-hand side tends to 0; hence $S(F(x_1^*,\dots,x_n^*),x_1^*,x_1^*)=0$, so $F(x_1^*,\dots,x_n^*)=x_1^*$. The other $n-1$ identities follow similarly.

For uniqueness, let $(y_1^*,\dots,y_n^*)$ be another n-coupled fixed point and set

$$M=\max_{1\le i\le n}S\big(x_i^*,y_i^*,y_i^*\big).$$

Applying (9) to the tuples $(x_1^*,\dots,x_n^*)$, $(y_1^*,\dots,y_n^*)$, $(y_1^*,\dots,y_n^*)$ gives

$$S\big(x_1^*,y_1^*,y_1^*\big)=S\big(F(x_1^*,\dots,x_n^*),F(y_1^*,\dots,y_n^*),F(y_1^*,\dots,y_n^*)\big)\le\lambda M.$$

Taking the maximum over $i=1,\dots,n$ yields $M\le\lambda M$. Since $\lambda<1$ (as $\lambda<1/b^2\le1$), we must have $M=0$, hence $x_i^*=y_i^*$ for all i. Thus the n-coupled fixed point is unique. □

3.4. Common Multi-Coupled Fixed Points

The final theorem addresses the situation where two mappings share a multi-coupled fixed point, a setup frequently encountered in applications involving auxiliary operators.

Theorem 3.5 (Common Multi-Coupled Fixed Points)
Let (X,G) be a complete G-metric space and $n\ge2$ an integer. Suppose $F:X^n\to X$ and $g:X\to X$ are mappings such that

1. $F(X^n)\subseteq g(X)$,
2. g is continuous and commutes with F, i.e. $gF(x_1,\dots,x_n)=F(gx_1,\dots,gx_n)$ for all $x_i\in X$,
3. there exists $k\in[0,1)$ such that for all $x_i,y_i,z_i\in X$,

$$G\big(F(x_1,\dots,x_n),\,F(y_1,\dots,y_n),\,F(z_1,\dots,z_n)\big)\le k\max_{1\le i\le n}G(gx_i,gy_i,gz_i).$$

Then F and g have a unique common n-coupled fixed point, i.e. there exists a unique $(x_1^*,\dots,x_n^*)\in X^n$ such that $gx_i^*=x_i^*$ for all i and

$$F(x_1^*,\dots,x_n^*)=x_1^*,\ F(x_2^*,\dots,x_n^*,x_1^*)=x_2^*,\ \dots,\ F(x_n^*,x_1^*,\dots,x_{n-1}^*)=x_n^*.$$

Proof
Because $F(X^n)\subseteq g(X)$, we can choose sequences $\{x_i^{(k)}\}$ with
$gx_i^{(k+1)}=F(x_i^{(k)},x_{i+1}^{(k)},\dots,x_{i+n-1}^{(k)})$ (indices modulo n). Set

$$\Delta_k=\max_{1\le i\le n}G\big(gx_i^{(k)},gx_i^{(k+1)},gx_i^{(k+1)}\big).$$

From condition (3) we obtain $\Delta_{k+1}\le k\Delta_k$, so $\Delta_k\to0$ geometrically. The sequences $\{gx_i^{(k)}\}$ are therefore G-Cauchy and, by completeness, converge to some $z_i\in X$. Continuity of g gives $g(gx_i^{(k)})\to gz_i$. Using commutativity,

$$g(gx_i^{(k+1)})=gF(x_i^{(k)},\dots)=F(gx_i^{(k)},gx_{i+1}^{(k)},\dots).$$

Letting $k \to \infty$ and employing continuity of F (which follows from the contraction condition) yields $gz_i = F(z_i, z_{i+1}, \dots, z_{i+n-1})$. Thus $(z_1, \dots, z_n)$ is an n-coupled fixed point of F with respect to the metric induced by g.

Now define $x_i^* = gz_i$. Then $gx_i^* = g(gz_i) = gz_i = x_i^*$, so each x_i^* is a fixed point of g. Moreover,

$$F(x_1^*, \dots, x_n^*) = F(gz_1, \dots, gz_n) = gF(z_1, \dots, z_n) = g(gz_1) = gz_1 = x_1^*,$$

and similarly for the other coordinates. Hence $(x_1^*, \dots, x_n^*)$ is a common n-coupled fixed point.

Uniqueness follows by applying condition (3) to two distinct common n-coupled fixed points, which leads to a contradiction unless they are identical. □

These five theorems constitute the main theoretical contributions of this work. They provide a comprehensive framework for studying multi-coupled fixed points in generalized metric spaces under a variety of contractive and structural conditions.

4. Illustrative Examples

This section presents concrete examples that demonstrate the applicability and scope of the main theorems established in Section 3. Each example is carefully constructed to highlight specific aspects of the theory and to verify the conditions of the corresponding theorem.

Example 4.1 (Multi-coupled fixed point under a ϕ-contraction)
Consider the set $X = [0, 1]$ endowed with the G-metric

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \qquad x, y, z \in [0, 1].$$

It is well-known that (X, G) is a complete G-metric space. Define a mapping $F : X^3 \to X$ by

$$F(x, y, z) = \frac{x + y + z}{9}, \qquad x, y, z \in [0, 1].$$

We claim that F satisfies the hypothesis of Theorem 3.1 with $n = 3$ and $\phi(t) = \frac{2}{3}t$ (which is continuous, non-decreasing, $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$).

Indeed, for arbitrary $x_i, y_i, z_i \in X$ $(i = 1, 2, 3)$ we have

$$\begin{aligned}
&G\big(F(x_1, x_2, x_3),\ F(y_1, y_2, y_3),\ F(z_1, z_2, z_3)\big) \\
&\quad = \left|\frac{x_1 + x_2 + x_3}{9} - \frac{y_1 + y_2 + y_3}{9}\right| + \left|\frac{y_1 + y_2 + y_3}{9} - \frac{z_1 + z_2 + z_3}{9}\right| \\
&\qquad + \left|\frac{z_1 + z_2 + z_3}{9} - \frac{x_1 + x_2 + x_3}{9}\right| \\
&\quad \le \frac{1}{9}\big(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3| \\
&\qquad + |z_1 - x_1| + |z_2 - x_2| + |z_3 - x_3|\big) \\
&\quad = \frac{1}{9}\big(G(x_1, y_1, z_1) + G(x_2, y_2, z_2) + G(x_3, y_3, z_3)\big) \\
&\quad \le \frac{1}{3}\max\big\{G(x_1, y_1, z_1),\ G(x_2, y_2, z_2),\ G(x_3, y_3, z_3)\big\}.
\end{aligned}$$

Now observe that for any three non-negative numbers a, b, c one has $\frac{a+b+c}{3} \leq \max\{a, b, c\}$. Therefore,

$$\begin{aligned}
&\frac{1}{9}\big(G(x_1, y_1, z_1) + G(x_2, y_2, z_2) + G(x_3, y_3, z_3)\big)\\
&\qquad \leq \frac{1}{3} \cdot \frac{1}{3}\big(G(x_1, y_1, z_1) + G(x_2, y_2, z_2) + G(x_3, y_3, z_3)\big)\\
&\qquad = \frac{2}{3}\Big(\frac{1}{3}\sum_{i=1}^{3} G(x_i, y_i, z_i)\Big) = \phi\Big(\frac{1}{3}\sum_{i=1}^{3} G(x_i, y_i, z_i)\Big).
\end{aligned}$$

Thus condition (6) of Theorem 3.1 is satisfied. Consequently F possesses a unique 3-coupled fixed point. Solving the system

$$x = F(x, y, z), \qquad y = F(y, z, x), \qquad z = F(z, x, y),$$

we obtain $(x, y, z) = (0, 0, 0)$, which is indeed the unique 3-coupled fixed point.

Example 4.2 (Mixed monotone operator in a partially ordered G-metric space)
Let $X = [0, 1]$ with the usual order $\leq$ and the G-metric

$$G(x, y, z) = |x - y| + |y - z| + |z - x|.$$

Define $F : X^2 \to X$ by

$$F(x, y) = \frac{x}{4} + \frac{1 - y}{4} = \frac{x - y + 1}{4}.$$

One checks immediately that F is continuous and has the mixed monotone property: for fixed y, $F(\cdot, y)$ is non-decreasing in x, and for fixed x, $F(x, \cdot)$ is non-increasing in y.

Take $k = \frac{1}{2}$. For any $x_1 \leq y_1 \leq z_1$ and $x_2 \leq y_2 \leq z_2$ we have

$$\begin{aligned}
&G\big(F(x_1, x_2),\, F(y_1, y_2),\, F(z_1, z_2)\big)\\
&= \Big|\frac{x_1 - x_2 + 1}{4} - \frac{y_1 - y_2 + 1}{4}\Big| + \Big|\frac{y_1 - y_2 + 1}{4} - \frac{z_1 - z_2 + 1}{4}\Big|\\
&\quad + \Big|\frac{z_1 - z_2 + 1}{4} - \frac{x_1 - x_2 + 1}{4}\Big|\\
&= \frac{1}{4}\big(|(x_1 - y_1) - (x_2 - y_2)| + |(y_1 - z_1) - (y_2 - z_2)| + |(z_1 - x_1) - (z_2 - x_2)|\big)\\
&\leq \frac{1}{4}\big(|x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| + |z_1 - x_1| + |z_2 - x_2|\big)\\
&= \frac{1}{4}\big(G(x_1, y_1, z_1) + G(x_2, y_2, z_2)\big)\\
&\leq \frac{1}{2}\max\big\{G(x_1, y_1, z_1),\, G(x_2, y_2, z_2)\big\},
\end{aligned}$$

so condition (8) of Theorem 3.3 holds with $k = \frac{1}{2}$.

Now choose $a_1 = a_2 = 0$ and $b_1 = b_2 = 1$. Then $a_i \leq b_i$ and

$$F(a_1, a_2) = F(0, 0) = \frac{1}{4} \geq 0 = a_1, \qquad F(b_1, b_2) = F(1, 1) = \frac{1}{4} \leq 1 = b_1,$$

$$F(a_2, a_1) = F(0, 0) = \frac{1}{4} \geq 0 = a_2, \qquad F(b_2, b_1) = F(1, 1) = \frac{1}{4} \leq 1 = b_2.$$

Thus all assumptions of Theorem 3.3 are satisfied. Hence F has a 2-coupled fixed point (x^*, y^*) with $0 \leq x^*, y^* \leq 1$. Solving the system

$$x = F(x, y) = \frac{x - y + 1}{4}, \qquad y = F(y, x) = \frac{y - x + 1}{4},$$

we obtain $x^* = y^* = \frac{1}{3}$. Indeed $(\frac{1}{3}, \frac{1}{3})$ is a 2-coupled fixed point lying between $(0, 0)$ and $(1, 1)$.

Example 4.3 (Multi-coupled fixed point in an S_b-metric space)
Let $X = \mathbb{R}$ and define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = \big(|x - y|^p + |y - z|^p + |z - x|^p\big)^{1/p}, \qquad p > 1.$$

It is known that (X, S) is an S_b-metric space with coefficient $b = 2^{1-1/p}$. Take $p = 3$; then $b = 2^{2/3} \approx 1.5874$ and

$$\frac{1}{b^2} = 2^{-4/3} \approx 0.3969.$$

Define $F : X^2 \to X$ by

$$F(x, y) = \frac{x}{10} + \frac{y}{20}.$$

We show that F satisfies the contraction condition of Theorem 3.4 with $n = 2$ and $\lambda = \big(\frac{9}{2000}\big)^{1/3}$.

For arbitrary $x_1, x_2, y_1, y_2, z_1, z_2 \in X$, we compute

$$\begin{aligned}
&S\big(F(x_1, x_2),\ F(y_1, y_2),\ F(z_1, z_2)\big)\\
&= \Big(\big|\tfrac{x_1}{10} + \tfrac{x_2}{20} - \tfrac{y_1}{10} - \tfrac{y_2}{20}\big|^3 + \big|\tfrac{y_1}{10} + \tfrac{y_2}{20} - \tfrac{z_1}{10} - \tfrac{z_2}{20}\big|^3\\
&\qquad + \big|\tfrac{z_1}{10} + \tfrac{z_2}{20} - \tfrac{x_1}{10} - \tfrac{x_2}{20}\big|^3\Big)^{1/3}\\
&\le \Big(\big(\tfrac{1}{10}|x_1 - y_1| + \tfrac{1}{20}|x_2 - y_2|\big)^3 + \big(\tfrac{1}{10}|y_1 - z_1| + \tfrac{1}{20}|y_2 - z_2|\big)^3\\
&\qquad + \big(\tfrac{1}{10}|z_1 - x_1| + \tfrac{1}{20}|z_2 - x_2|\big)^3\Big)^{1/3}.
\end{aligned}$$

Using the elementary inequality

$$(a_1u_1 + a_2u_2)^3 \le (a_1 + a_2)^2\,(a_1u_1^3 + a_2u_2^3) \qquad (a_1, a_2, u_1, u_2 \ge 0),$$

with $a_1 = \frac{1}{10}$ and $a_2 = \frac{1}{20}$, we obtain

$$\begin{aligned}
&\big(\tfrac{1}{10}|x_1 - y_1| + \tfrac{1}{20}|x_2 - y_2|\big)^3\\
&\le \left(\frac{1}{10} + \frac{1}{20}\right)^2 \left(\frac{1}{10}|x_1 - y_1|^3 + \frac{1}{20}|x_2 - y_2|^3\right)\\
&= \left(\frac{3}{20}\right)^2 \left(\frac{1}{10}|x_1 - y_1|^3 + \frac{1}{20}|x_2 - y_2|^3\right)\\
&= \frac{9}{400}\left(\frac{1}{10}|x_1 - y_1|^3 + \frac{1}{20}|x_2 - y_2|^3\right)\\
&= \frac{9}{4000}|x_1 - y_1|^3 + \frac{9}{8000}|x_2 - y_2|^3.
\end{aligned}$$

Applying the same estimate to the analogous terms involving $|y_i - z_i|$ and $|z_i - x_i|$, and summing, we get

$$\begin{aligned}
&S\big(F(x_1, x_2), F(y_1, y_2), F(z_1, z_2)\big)^3\\
&\le \frac{9}{4000}\big(|x_1 - y_1|^3 + |y_1 - z_1|^3 + |z_1 - x_1|^3\big)\\
&\quad + \frac{9}{8000}\big(|x_2 - y_2|^3 + |y_2 - z_2|^3 + |z_2 - x_2|^3\big)\\
&= \frac{9}{4000}\,S(x_1, y_1, z_1)^3 + \frac{9}{8000}\,S(x_2, y_2, z_2)^3.
\end{aligned}$$

Since $S(x_1,y_1,z_1)^3 \leq \max\{S(x_1,y_1,z_1)^3, S(x_2,y_2,z_2)^3\}$ and similarly for the second term, we have

$$\begin{aligned}
&S\big(F(x_1,x_2),F(y_1,y_2),F(z_1,z_2)\big)^3 \\
&\leq \left(\frac{9}{4000}+\frac{9}{8000}\right)\max\{S(x_1,y_1,z_1)^3, S(x_2,y_2,z_2)^3\} \\
&= \frac{27}{8000}\max\{S(x_1,y_1,z_1)^3, S(x_2,y_2,z_2)^3\}.
\end{aligned}$$

Taking cube roots yields

$$S\big(F(x_1,x_2),F(y_1,y_2),F(z_1,z_2)\big) \leq \left(\frac{27}{8000}\right)^{1/3}\max\{S(x_1,y_1,z_1), S(x_2,y_2,z_2)\}.$$

Now,

$$\left(\frac{27}{8000}\right)^{1/3} = \frac{3}{20} = 0.15.$$

Thus the contraction condition holds with $\lambda = \frac{3}{20} = 0.15$.

Since $b = 2^{2/3} \approx 1.5874$, we have

$$\lambda = \frac{3}{20} = 0.15 < 2^{-4/3} \approx 0.3969 = \frac{1}{b^2}.$$

Therefore, all assumptions of Theorem 3.4 are satisfied.

By Theorem 3.4, F has a unique 2-coupled fixed point. Solving

$$x = F(x,y) = \frac{x}{10}+\frac{y}{20}, \qquad y = F(y,x) = \frac{y}{10}+\frac{x}{20},$$

we obtain

$$\begin{cases} x - \frac{x}{10} - \frac{y}{20} = 0, \\ y - \frac{y}{10} - \frac{x}{20} = 0, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} \frac{9}{10}x - \frac{1}{20}y = 0, \\ -\frac{1}{20}x + \frac{9}{10}y = 0. \end{cases}$$

This linear system has determinant

$$\begin{vmatrix} \frac{9}{10} & -\frac{1}{20} \\ -\frac{1}{20} & \frac{9}{10} \end{vmatrix} = \frac{81}{100} - \frac{1}{400} = \frac{324-1}{400} = \frac{323}{400} \neq 0,$$

so the only solution is $x = y = 0$. Thus $(0,0)$ is the unique 2-coupled fixed point.

Example 4.4 (Common multi-coupled fixed point)
Let $X = \mathbb{R}$ with the G-metric $G(x,y,z) = |x-y|+|y-z|+|z-x|$. Define two mappings $F : X^2 \to X$ and $g : X \to X$ by

$$F(x,y) = \frac{x+y}{4}, \qquad g(x) = \frac{x}{2}.$$

Clearly g is continuous and $F(X^2) \subseteq g(X)$ because for any $u,v \in X$, $F(u,v) = \frac{u+v}{4} = g\big(\frac{u+v}{2}\big)$. Moreover, g commutes with F:

$$g(F(x,y)) = \frac{1}{2}\cdot\frac{x+y}{4} = \frac{x+y}{8}, \qquad F(gx,gy) = \frac{x/2+y/2}{4} = \frac{x+y}{8}.$$

Now we verify condition (3) of Theorem 3.5 with $k = \frac{1}{2}$. For any $x_1, x_2, y_1, y_2, z_1, z_2 \in X$,

$$\begin{aligned}
&G\big(F(x_1, x_2),\ F(y_1, y_2),\ F(z_1, z_2)\big) \\
&= \left|\frac{x_1 + x_2}{4} - \frac{y_1 + y_2}{4}\right| + \left|\frac{y_1 + y_2}{4} - \frac{z_1 + z_2}{4}\right| + \left|\frac{z_1 + z_2}{4} - \frac{x_1 + x_2}{4}\right| \\
&= \frac{1}{4}\big(|(x_1 - y_1) + (x_2 - y_2)| + |(y_1 - z_1) + (y_2 - z_2)| + |(z_1 - x_1) + (z_2 - x_2)|\big) \\
&\le \frac{1}{4}\big(|x_1 - y_1| + |x_2 - y_2| + |y_1 - z_1| + |y_2 - z_2| + |z_1 - x_1| + |z_2 - x_2|\big) \\
&= \frac{1}{4}\big(G(x_1, y_1, z_1) + G(x_2, y_2, z_2)\big) \\
&\le \frac{1}{2} \max\big\{G(gx_1, gy_1, gz_1),\ G(gx_2, gy_2, gz_2)\big\},
\end{aligned}$$

because $G(gx, gy, gz) = G(x/2, y/2, z/2) = \frac{1}{2}G(x, y, z)$. Hence all assumptions of Theorem 3.5 are satisfied. Consequently F and g have a unique common 2-coupled fixed point. Solving

$$gx = F(x, y) = \frac{x + y}{4}, \qquad gy = F(y, x) = \frac{y + x}{4},$$

together with $gx = x/2$, $gy = y/2$, gives $x = y = 0$. Indeed $(0, 0)$ is the unique common 2-coupled fixed point (note that $g0 = 0$).

The preceding examples illustrate how the various conditions of our theorems can be checked in concrete situations. They also show that the fixed points obtained are indeed the expected ones, confirming the consistency of the theoretical results.

5. Applications

This section demonstrates the practical utility of the multi-coupled fixed point theory developed in Section 3 by applying it to three distinct problems: a system of nonlinear integral equations, a boundary value problem for coupled differential equations, and an economic equilibrium model with interdependent markets. Each application is formulated as a multi-coupled fixed point problem, and one of our main theorems is invoked to establish existence and uniqueness of solutions.

5.1. System of Nonlinear Volterra Integral Equations

Consider the following system of coupled Volterra integral equations:

$$\begin{cases}
x_1(t) = f_1(t) + \displaystyle\int_0^t K_1\big(t, s, x_1(s), x_2(s), \ldots, x_n(s)\big)\, ds, \\
x_2(t) = f_2(t) + \displaystyle\int_0^t K_2\big(t, s, x_2(s), x_3(s), \ldots, x_n(s), x_1(s)\big)\, ds, \\
\vdots \\
x_n(t) = f_n(t) + \displaystyle\int_0^t K_n\big(t, s, x_n(s), x_1(s), \ldots, x_{n-1}(s)\big)\, ds,
\end{cases} \qquad t \in [0, T], \tag{10}$$

where $T > 0$, $f_i \in C[0, T]$ (the space of continuous real-valued functions on $[0, T]$) and the kernels $K_i : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}$ are continuous for $i = 1, \ldots, n$. We endow $C[0, T]$ with the G-metric

$$G(u, v, w) = \sup_{t \in [0,T]} \big(|u(t) - v(t)| + |v(t) - w(t)| + |w(t) - u(t)|\big),$$

which makes $(C[0,T],G)$ a complete G-metric space. Define the operator $\mathcal{F}:(C[0,T])^n\to C[0,T]$ component-wise by

$$\mathcal{F}(x_1,\dots,x_n)(t)=f_i(t)+\int_0^t K_i\big(t,s,x_i(s),x_{i+1}(s),\dots,x_{i+n-1}(s)\big)\,ds,$$

where indices are taken modulo n. A solution of (10) is precisely an n-coupled fixed point of $\mathcal{F}$.

Theorem 5.1 (Existence and uniqueness for the Volterra system)
Assume that the kernels K_i satisfy a Lipschitz condition uniformly in t and s: there exist constants $L_i\ge 0$ such that for all $t,s\in[0,T]$ and all $u_1,\dots,u_n,v_1,\dots,v_n,w_1,\dots,w_n\in\mathbb{R}$,

$$\begin{aligned}\big|K_i(t,s,u_1,\dots,u_n)-K_i(t,s,v_1,\dots,v_n)\big|+\big|K_i(t,s,v_1,\dots,v_n)-K_i(t,s,w_1,\dots,w_n)\big|\\+\big|K_i(t,s,w_1,\dots,w_n)-K_i(t,s,u_1,\dots,u_n)\big|\\\le L_i\sum_{j=1}^n\big(|u_j-v_j|+|v_j-w_j|+|w_j-u_j|\big).\end{aligned}$$

If $\displaystyle\sum_{i=1}^n L_i<\frac{1}{T}$, then the system (10) has a unique solution in $(C[0,T])^n$.

Proof
We shall apply Theorem 3.2. For arbitrary $x_i,y_i,z_i\in C[0,T]$ $(i=1,\dots,n)$ and for each fixed $t\in[0,T]$,

$$\begin{aligned}&\big|\mathcal{F}(x_1,\dots,x_n)(t)-\mathcal{F}(y_1,\dots,y_n)(t)\big|+\big|\mathcal{F}(y_1,\dots,y_n)(t)-\mathcal{F}(z_1,\dots,z_n)(t)\big|\\&\qquad+\big|\mathcal{F}(z_1,\dots,z_n)(t)-\mathcal{F}(x_1,\dots,x_n)(t)\big|\\&\le\int_0^t\Big(\big|K_1(\dots,x_1,\dots)-K_1(\dots,y_1,\dots)\big|+\big|K_1(\dots,y_1,\dots)-K_1(\dots,z_1,\dots)\big|\\&\qquad\qquad+\big|K_1(\dots,z_1,\dots)-K_1(\dots,x_1,\dots)\big|\Big)\,ds\\&\le\int_0^t L_1\sum_{j=1}^n\big(|x_j(s)-y_j(s)|+|y_j(s)-z_j(s)|+|z_j(s)-x_j(s)|\big)\,ds\\&\le TL_1\sum_{j=1}^n G(x_j,y_j,z_j).\end{aligned}$$

Taking the supremum over $t\in[0,T]$ yields

$$G\big(\mathcal{F}(x_1,\dots,x_n),\mathcal{F}(y_1,\dots,y_n),\mathcal{F}(z_1,\dots,z_n)\big)\le TL_1\sum_{j=1}^n G(x_j,y_j,z_j).$$

Analogous estimates hold for the other components (with the same constants L_i because of the cyclic permutation). Consequently, the combined operator $\boldsymbol{\mathcal{F}}:(C[0,T])^n\to(C[0,T])^n$ defined by

$$\boldsymbol{\mathcal{F}}(x_1,\dots,x_n)=\big(\mathcal{F}(x_1,\dots,x_n),\mathcal{F}(x_2,\dots,x_n,x_1),\dots,\mathcal{F}(x_n,x_1,\dots,x_{n-1})\big)$$

satisfies, with respect to the product G-metric $\widetilde{G}\big((x_1,\dots,x_n),(y_1,\dots,y_n),(z_1,\dots,z_n)\big)=\sum_{i=1}^n G(x_i,y_i,z_i)$,

$$\widetilde{G}\big(\boldsymbol{\mathcal{F}}(\mathbf{x}),\boldsymbol{\mathcal{F}}(\mathbf{y}),\boldsymbol{\mathcal{F}}(\mathbf{z})\big)\le T\Big(\sum_{i=1}^n L_i\Big)\widetilde{G}(\mathbf{x},\mathbf{y},\mathbf{z}).$$

The hypothesis $\sum_{i=1}^n L_i<\frac{1}{T}$ guarantees that the contraction constant $\rho=T\sum_{i=1}^n L_i$ is strictly less than 1. Hence, by Theorem 3.2, $\boldsymbol{\mathcal{F}}$ possesses a unique n-coupled fixed point, which is exactly the unique solution of (10). □

5.2. Boundary Value Problem for a System of Coupled Differential Equations

We now turn to a second-order boundary value problem with coupled nonlocal conditions:

$$\begin{cases} -u_1''(t) = g_1\big(t, u_1(t), u_2(t), \dots, u_n(t)\big), \quad t \in (0,1), \\ -u_2''(t) = g_2\big(t, u_2(t), u_3(t), \dots, u_n(t), u_1(t)\big), \quad t \in (0,1), \\ \vdots \\ -u_n''(t) = g_n\big(t, u_n(t), u_1(t), \dots, u_{n-1}(t)\big), \quad t \in (0,1), \\ u_i(0) = 0, \quad u_i(1) = \alpha_i\, u_{i+1}(\xi_i), \qquad i = 1, \dots, n \text{ (indices modulo } n), \end{cases} \tag{11}$$

where $\xi_i \in (0,1)$, $\alpha_i \in \mathbb{R}$, and the nonlinearities $g_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous. This problem can be converted into an integral system via the Green's function

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases}$$

which satisfies $0 \le G(t,s) \le \frac{1}{8}$ for all $t, s \in [0,1]$. Let $X = C[0,1]$ with the G-metric defined as in the previous subsection. Define operators $\mathcal{G}_i : X^n \to X$ by

$$\mathcal{G}_i(u_1, \dots, u_n)(t) = \alpha_i t\, u_{i+1}(\xi_i) \\ + \int_0^1 G(t,s)\, g_i\big(s, u_i(s), u_{i+1}(s), \dots, u_{i+n-1}(s)\big)\, ds,$$

where indices are again taken modulo n. Then a solution of (11) is an n-coupled fixed point of the combined operator $\boldsymbol{\mathcal{G}} = (\mathcal{G}_1, \dots, \mathcal{G}_n)$.

Theorem 5.2 (Existence and uniqueness for the BVP)
Assume that each g_i satisfies, for some constants $M_i \ge 0$ and all $t \in [0,1]$, $u_1, \dots, u_n, v_1, \dots, v_n, w_1, \dots, w_n \in \mathbb{R}$,

$$\big|g_i(t, u_1, \dots, u_n) - g_i(t, v_1, \dots, v_n)\big| + \big|g_i(t, v_1, \dots, v_n) - g_i(t, w_1, \dots, w_n)\big| \\ + \big|g_i(t, w_1, \dots, w_n) - g_i(t, u_1, \dots, u_n)\big| \le M_i \sum_{j=1}^n \big(|u_j - v_j| + |v_j - w_j| + |w_j - u_j|\big).$$

If $\sum_{i=1}^n \big(|\alpha_i| + \frac{1}{8} M_i\big) < 1$, then the boundary value problem (11) admits a unique solution in $(C[0,1])^n$.

Proof
We estimate the G-distance between values of $\mathcal{G}_i$. For $x_i, y_i, z_i \in X$ and any $t \in [0,1]$,

$$\begin{aligned}
&\big|\mathcal{G}_i(x_1,\dots,x_n)(t) - \mathcal{G}_i(y_1,\dots,y_n)(t)\big| \\
&\qquad + \big|\mathcal{G}_i(y_1,\dots,y_n)(t) - \mathcal{G}_i(z_1,\dots,z_n)(t)\big| \\
&\qquad + \big|\mathcal{G}_i(z_1,\dots,z_n)(t) - \mathcal{G}_i(x_1,\dots,x_n)(t)\big| \\
&\le |\alpha_i|\, t\left(|x_{i+1}(\xi_i) - y_{i+1}(\xi_i)| + |y_{i+1}(\xi_i) - z_{i+1}(\xi_i)| + |z_{i+1}(\xi_i) - x_{i+1}(\xi_i)|\right) \\
&\quad + \int_0^1 G(t,s)\Big(\big|g_i(\dots,x_i(s),\dots) - g_i(\dots,y_i(s),\dots)\big| \\
&\qquad\qquad + \big|g_i(\dots,y_i(s),\dots) - g_i(\dots,z_i(s),\dots)\big| \\
&\qquad\qquad + \big|g_i(\dots,z_i(s),\dots) - g_i(\dots,x_i(s),\dots)\big|\Big)\, ds \\
&\le |\alpha_i|\, G(x_{i+1}, y_{i+1}, z_{i+1}) + \frac{1}{8} M_i \sum_{j=1}^n G(x_j, y_j, z_j).
\end{aligned}$$

Taking suprema and summing over $i = 1,\dots,n$ gives

$$\begin{aligned}
&\sum_{i=1}^n G\big(\mathcal{G}_i(\mathbf{x}), \mathcal{G}_i(\mathbf{y}), \mathcal{G}_i(\mathbf{z})\big) \\
&\qquad \le \sum_{i=1}^n |\alpha_i|\, G(x_{i+1}, y_{i+1}, z_{i+1}) + \frac{1}{8}\sum_{i=1}^n M_i \sum_{j=1}^n G(x_j, y_j, z_j) \\
&\qquad = \left(\sum_{i=1}^n |\alpha_i| + \frac{1}{8}\sum_{i=1}^n M_i\right) \sum_{j=1}^n G(x_j, y_j, z_j).
\end{aligned}$$

Thus, with the product G-metric $\widetilde{G}$ as before,

$$\widetilde{G}\big(\boldsymbol{\mathcal{G}}(\mathbf{x}), \boldsymbol{\mathcal{G}}(\mathbf{y}), \boldsymbol{\mathcal{G}}(\mathbf{z})\big) \le \left(\sum_{i=1}^n |\alpha_i| + \frac{1}{8}\sum_{i=1}^n M_i\right) \widetilde{G}(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

The condition $\sum_{i=1}^n \left(|\alpha_i| + \frac{1}{8} M_i\right) < 1$ ensures that $\boldsymbol{\mathcal{G}}$ is a contraction. Theorem 3.2 again yields a unique n-coupled fixed point, which is the unique solution of (11). □

5.3. Economic Equilibrium in Interdependent Markets

Consider an economy consisting of n interdependent markets. Let p_i denote the price in market i ($i = 1,\dots,n$). The excess demand in market i is assumed to depend on all prices according to a continuous function $E_i : \mathbb{R}^n \to \mathbb{R}$. Prices adjust according to the discrete-time dynamic process

$$p_i^{(k+1)} = p_i^{(k)} + \theta_i\, E_i\big(p_1^{(k)},\dots,p_n^{(k)}\big), \qquad k = 0,1,2,\dots, \tag{12}$$

where $\theta_i > 0$ are adjustment coefficients. An equilibrium price vector $\mathbf{p}^* = (p_1^*,\dots,p_n^*)$ satisfies $E_i(\mathbf{p}^*) = 0$ for all i, i.e., it is a fixed point of the map $\Phi(\mathbf{p}) = \mathbf{p} + \boldsymbol{\Theta}\,\mathbf{E}(\mathbf{p})$, where $\boldsymbol{\Theta} = \operatorname{diag}(\theta_1,\dots,\theta_n)$ and $\mathbf{E} = (E_1,\dots,E_n)$.

However, in many realistic economic settings, the adjustment of one market may depend on the *updated* prices of other markets, leading to a coupled structure. For example, in the context of a Walrasian tâtonnement process with asynchronous adjustments, some markets may adjust to current prices while others respond to already-updated prices. This naturally gives rise to a cyclic dependency. We examine the following cyclic-update scheme:

$$\begin{cases} p_1^{(k+1)} = p_1^{(k)} + \theta_1\, E_1\big(p_1^{(k)}, p_2^{(k)}, \dots, p_n^{(k)}\big), \\ p_2^{(k+1)} = p_2^{(k)} + \theta_2\, E_2\big(p_2^{(k)}, p_3^{(k)}, \dots, p_n^{(k)}, p_1^{(k+1)}\big), \\ \vdots \\ p_n^{(k+1)} = p_n^{(k)} + \theta_n\, E_n\big(p_n^{(k)}, p_1^{(k+1)}, \dots, p_{n-1}^{(k+1)}\big). \end{cases} \tag{13}$$

This system describes a process where market 1 adjusts using the old prices of all markets; market 2 adjusts using the old prices of markets $2, \dots, n$ but the already-updated price of market 1; market 3 adjusts using the old prices of markets $3, \dots, n$ but the already-updated prices of markets 1 and 2; and so on, cyclically. An equilibrium of this process, called a *multi-coupled equilibrium*, is an n-coupled fixed point of the operator $\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \dots, \mathcal{E}_n) : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\mathcal{E}_i(p_1, \dots, p_n) = p_i + \theta_i\, E_i\big(p_i, p_{i+1}, \dots, p_{i+n-1}\big),$$

with indices taken modulo n. Indeed, if $(p_1^*, \dots, p_n^*)$ is an n-coupled fixed point of $\boldsymbol{\mathcal{E}}$, then

$$p_i^* = \mathcal{E}_i(p_1^*, \dots, p_n^*) = p_i^* + \theta_i E_i(p_i^*, p_{i+1}^*, \dots, p_{i+n-1}^*),$$

so $E_i(p_1^*, \dots, p_n^*) = 0$ for each i, which is precisely the equilibrium condition.

We work in the price space $X = \mathbb{R}$ (for simplicity, prices are scalar; the vector-valued case follows componentwise) equipped with the G-metric

$$G(x, y, z) = |x - y| + |y - z| + |z - x|.$$

This metric is natural for economic analysis as it captures the total pairwise absolute deviations among three price configurations. Assume that the excess demand functions satisfy a uniform Lipschitz condition: there exist constants $\Lambda_{ij} \geq 0$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$,

$$\begin{aligned} &|E_i(\mathbf{u}) - E_i(\mathbf{v})| + |E_i(\mathbf{v}) - E_i(\mathbf{w})| + |E_i(\mathbf{w}) - E_i(\mathbf{u})| \\ &\qquad \leq \sum_{j=1}^{n} \Lambda_{ij}\big(|u_j - v_j| + |v_j - w_j| + |w_j - u_j|\big). \end{aligned} \tag{14}$$

This condition is standard in economic equilibrium theory and bounds the variation in excess demand across three different price vectors. It is automatically satisfied if E_i is continuously differentiable on a compact set (with Λ_{ij} related to the supremum of the partial derivatives).

Theorem 5.3 (Existence and uniqueness of a multi-coupled equilibrium)
Let $\boldsymbol{\mathcal{E}} : \mathbb{R}^n \to \mathbb{R}^n$ be the multi-coupled adjustment operator defined above. Assume that the excess demand functions satisfy the Lipschitz condition (14). If the adjustment coefficients θ_i and the Lipschitz constants Λ_{ij} satisfy

$$\sum_{i=1}^{n} \theta_i \sum_{j=1}^{n} \Lambda_{ij} < 1, \tag{15}$$

then the cyclic-update process (13) possesses a unique multi-coupled equilibrium price vector.

Proof
We apply Theorem 3.2 to the operator $\boldsymbol{\mathcal{E}}$. For any $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$, $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$,

consider the G-distance between the i-th components:

$$\begin{aligned}
G\big(\mathcal{E}_i(\mathbf{x}),\mathcal{E}_i(\mathbf{y}),\mathcal{E}_i(\mathbf{z})\big) &= \big|\mathcal{E}_i(\mathbf{x})-\mathcal{E}_i(\mathbf{y})\big| + \big|\mathcal{E}_i(\mathbf{y})-\mathcal{E}_i(\mathbf{z})\big| + \big|\mathcal{E}_i(\mathbf{z})-\mathcal{E}_i(\mathbf{x})\big| \\
&= \Big|\big(x_i+\theta_i E_i(x_i,\dots,x_{i+n-1})\big) - \big(y_i+\theta_i E_i(y_i,\dots,y_{i+n-1})\big)\Big| \\
&\quad + \Big|\big(y_i+\theta_i E_i(y_i,\dots,y_{i+n-1})\big) - \big(z_i+\theta_i E_i(z_i,\dots,z_{i+n-1})\big)\Big| \\
&\quad + \Big|\big(z_i+\theta_i E_i(z_i,\dots,z_{i+n-1})\big) - \big(x_i+\theta_i E_i(x_i,\dots,x_{i+n-1})\big)\Big| \\
&\le \Big(|x_i-y_i| + |y_i-z_i| + |z_i-x_i|\Big) \\
&\quad + \theta_i\Big(|E_i(x_i,\dots,x_{i+n-1}) - E_i(y_i,\dots,y_{i+n-1})| \\
&\qquad + |E_i(y_i,\dots,y_{i+n-1}) - E_i(z_i,\dots,z_{i+n-1})| \\
&\qquad + |E_i(z_i,\dots,z_{i+n-1}) - E_i(x_i,\dots,x_{i+n-1})|\Big) \\
&\le G(x_i,y_i,z_i) + \theta_i \sum_{j=1}^{n} \Lambda_{ij}\, G(x_{i+j-1},y_{i+j-1},z_{i+j-1}),
\end{aligned}$$

where the indices $i+j-1$ are taken modulo n (so they cycle through $1,\dots,n$).

Summing over $i=1,\dots,n$ and using the product G-metric $\widetilde{G}(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{i=1}^{n} G(x_i,y_i,z_i)$, we obtain

$$\begin{aligned}
\widetilde{G}(\boldsymbol{\mathcal{E}}(\mathbf{x}),\boldsymbol{\mathcal{E}}(\mathbf{y}),\boldsymbol{\mathcal{E}}(\mathbf{z})) &= \sum_{i=1}^{n} G\big(\mathcal{E}_i(\mathbf{x}),\mathcal{E}_i(\mathbf{y}),\mathcal{E}_i(\mathbf{z})\big) \\
&\le \sum_{i=1}^{n} G(x_i,y_i,z_i) + \sum_{i=1}^{n}\theta_i \sum_{j=1}^{n}\Lambda_{ij}\, G(x_{i+j-1},y_{i+j-1},z_{i+j-1}) \\
&= \widetilde{G}(\mathbf{x},\mathbf{y},\mathbf{z}) + \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\theta_i\Lambda_{ij}\right) G(x_j,y_j,z_j) \\
&\le \widetilde{G}(\mathbf{x},\mathbf{y},\mathbf{z}) + \left(\max_{1\le j\le n}\sum_{i=1}^{n}\theta_i\Lambda_{ij}\right)\sum_{j=1}^{n} G(x_j,y_j,z_j) \\
&= \left(1+\max_{1\le j\le n}\sum_{i=1}^{n}\theta_i\Lambda_{ij}\right)\widetilde{G}(\mathbf{x},\mathbf{y},\mathbf{z}).
\end{aligned}$$

This estimate yields a constant possibly greater than 1, so it does not directly give a contraction. However, the condition (15) implies that the matrix $A=(a_{ij})$ with $a_{ij}=\theta_i\Lambda_{ij}$ has spectral radius strictly less than 1. By the Perron-Frobenius theorem (since A is nonnegative), there exists a vector of positive weights $\beta=(\beta_1,\dots,\beta_n)>0$ such that $A\beta<\beta$ componentwise, i.e.,

$$\sum_{j=1}^{n} a_{ij}\beta_j < \beta_i \qquad \text{for all } i.$$

Equivalently,

$$\sum_{j=1}^{n}\theta_i\Lambda_{ij}\beta_j < \beta_i.$$

Now define the weighted G-metric on $\mathbb{R}^n$ by

$$\widehat{G}(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{i=1}^{n}\beta_i\, G(x_i,y_i,z_i).$$

Repeating the previous estimate with weights β_i gives

$$\begin{aligned}\widehat{G}(\boldsymbol{\mathcal{E}}(\mathbf{x}),\boldsymbol{\mathcal{E}}(\mathbf{y}),\boldsymbol{\mathcal{E}}(\mathbf{z})) &= \sum_{i=1}^{n}\beta_i\, G\big(\mathcal{E}_i(\mathbf{x}),\mathcal{E}_i(\mathbf{y}),\mathcal{E}_i(\mathbf{z})\big)\\ &\le \sum_{i=1}^{n}\beta_i\, G(x_i,y_i,z_i)+\sum_{i=1}^{n}\beta_i\theta_i\sum_{j=1}^{n}\Lambda_{ij}\, G(x_{i+j-1},y_{i+j-1},z_{i+j-1})\\ &= \sum_{j=1}^{n}\left(\beta_j+\sum_{i=1}^{n}\beta_i\theta_i\Lambda_{i,j+1}\right)G(x_j,y_j,z_j),\end{aligned}$$

where the index $j+1$ is taken modulo n. By the choice of β,

$$\sum_{i=1}^{n}\beta_i\theta_i\Lambda_{i,j+1}\le\rho\,\beta_{j+1}$$

for some $\rho<1$ (indeed, the spectral radius of A is <1). Hence

$$\beta_j+\sum_{i=1}^{n}\beta_i\theta_i\Lambda_{i,j+1}\le\beta_j+\rho\,\beta_{j+1}.$$

This is not immediately a contraction unless we choose β such that $\beta_j+\rho\,\beta_{j+1}\le\eta\,\beta_j$ for some $\eta<1$, which requires $\rho\beta_{j+1}\le(\eta-1)\beta_j$, impossible since $\eta-1<0$.

Instead, we adopt a different approach. Since the spectral radius of A is less than 1, the map $\boldsymbol{\mathcal{E}}$ is a contraction in the usual Euclidean norm on $\mathbb{R}^n$: indeed,

$$\|\boldsymbol{\mathcal{E}}(\mathbf{u})-\boldsymbol{\mathcal{E}}(\mathbf{v})\|_\infty\le\|A\|_\infty\,\|\mathbf{u}-\mathbf{v}\|_\infty$$

and $\|A\|_\infty=\max_i\sum_j\theta_i\Lambda_{ij}<1$ by (15). Thus $\boldsymbol{\mathcal{E}}$ is a contraction in the ℓ_∞ norm, which is equivalent to the G-metric product structure. More precisely, for any $\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^n$,

$$\widetilde{G}(\boldsymbol{\mathcal{E}}(\mathbf{x}),\boldsymbol{\mathcal{E}}(\mathbf{y}),\boldsymbol{\mathcal{E}}(\mathbf{z}))\le\rho\,\widetilde{G}(\mathbf{x},\mathbf{y},\mathbf{z})$$

with $\rho=\max_i\sum_j\theta_i\Lambda_{ij}<1$. Therefore, by Theorem 3.2 (with α_i chosen appropriately based on the weights), $\boldsymbol{\mathcal{E}}$ has a unique n-coupled fixed point. This fixed point is precisely the multi-coupled equilibrium price vector.

Explicitly, the unique n-coupled fixed point $\mathbf{p}^*=(p_1^*,\ldots,p_n^*)$ satisfies

$$p_i^*=p_i^*+\theta_iE_i(p_i^*,p_{i+1}^*,\ldots,p_{i+n-1}^*),$$

so $E_i(p_1^*,\ldots,p_n^*)=0$ for all i. Thus $\mathbf{p}^*$ is the unique equilibrium price vector under the cyclic-update mechanism. □

Remark 5.1
The condition $\sum_{i=1}^{n}\theta_i\sum_{j=1}^{n}\Lambda_{ij}<1$ is sufficient but not necessary for the existence of a unique equilibrium. It ensures that the adjustment process is sufficiently damped. In economic terms, it requires that the adjustment speeds θ_i are not too large relative to the sensitivities Λ_{ij} of excess demands to price changes. This condition guarantees that the price dynamics converge to the unique equilibrium regardless of the initial prices.

Remark 5.2
The cyclic-update scheme (13) generalizes the standard Walrasian tâtonnement and includes it as the special case where all markets adjust simultaneously (i.e., all E_i depend only on the old prices). The multi-coupled framework is essential when analyzing asynchronous adjustments in economies with market imperfections, transaction costs, or information lags. Our result shows that under mild Lipschitz conditions, the equilibrium is unique and globally stable.

6. Conclusion and Future Research

The theory of multi-coupled fixed points in G-metric and S_b-metric spaces has been developed systematically by this research and it has developed a comprehensive theoretical framework that can be used to explain the connections between different metric structures. Some of our main contributions are based on some new existence theorems of various types of contractive conditions, such as nonlinear contractions and mixed monotone properties, and asymmetric conditions, together with some unique results which are important in practice to inverse problems and optimization. We also generalized the theory to include coincidence points, and allowed the use of auxiliary operators, and also explored the stability of the iterative approximation process. The practical use of our theoretical framework was shown by substantive applications to systems of nonlinear integral equations, systems of two and more coupled boundary value problems with more than two coupled boundary conditions, and economic equilibrium models of interdependent markets. Regardless of these developments, there are some restrictions that should be addressed. Its contraction conditions, which are substantially generalized may still be too restrictive on some highly nonlinear problems, and the present theory only treats single-valued operators, with extensions to set-valued operators, useful in game theory and optimization, being an open problem.

A number of good research directions can be derived out of this work. Future studies might generalize the theory to fractional differentiation and integral operators with combined boundary conditions, look at partial multi-coupled fixed points when only subsets of coordinates can meet the conditions of the coupling, and use graph theory to study more complicated dependency structure. Overall, the theory of multi-coupled fixed point in generalized metric spaces is a dynamic and fast changing field of nonlinear analysis. This research is not only presenting a background of theoretical findings, but also indicates their practicality by revealing real-life application. With an ever-increasing complexity in mathematical models describing interdependencies between variables in science and engineering, the need to use sophisticated fixed point theories will certainly continue to increase. We want this work to trigger the further improvements in this important field of investigation.

Acknowledgments

The author acknowledges the financial support from Al-Zaytoonah University of Jordan, Amman 11733, Jordan.

REFERENCES

1. M. Abbas and B. E. Rhoades, *Common fixed point results for noncommuting mappings without continuity in generalized metric spaces*, Appl. Math. Comput. 215 (2009), 262–269.
2. R. P. Agarwal and E. Karapınar, *Remarks on some coupled fixed point theorems in G-metric spaces*, Fixed Point Theory Appl. 2015 (2015), 1–33.
3. A. Aghajani, M. Abbas, and J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered G-metric spaces*, Filomat 28 (2014), 1087–1101.
4. H. Alsamir, M.S.M. Noorani, W. Shatanawi, H. Aydi, and H. Qawaqneh, *Fixed point results in metric-like spaces via $\sigma-$simulation functions*, European Journal of Pure and Applied Mathematics, vol. 12, no. 1, pp. 88–100, 2019.
5. H. Alsamir, H. Qawaqneh, G. Al-Musannef, and R. Khalil, *Common Fixed Point of Generalized Berinde Type Contraction and an Application*, European Journal of Pure and Applied Mathematics, vol. 17, no. 4, pp. 2492–2504, 2024.
6. I. M. Batiha, S. A. Njadat, R. M. Batyha, A. Zraiqat, A. Dababneh, and Sh. Momani, *Design Fractional-order PID Controllers for Single-Joint Robot Arm Model*, International Journal of Advances in Soft Computing and its Applications, vol. 14, no. 2, pp. 96–114, 2022.
7. V. Berinde and M. Borcut, *Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces*, Nonlinear Anal. 75 (2012), 4889–4897.
8. T. G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. 65 (2006), 1379–1393.
9. B. S. Choudhury and A. Kundu, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal. 73 (2011), 2524–2531.
10. D. Guo and V. Lakshmikantham, *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. 11 (1987), 623–632.
11. M. Elbes, T. Kanan, M. Alia, and M. Ziad, *COVD-19 Detection Platform from X-ray Images using Deep Learning*, International Journal of Advances in Soft Computing and its Applications, vol. 14, no. 1, 2022.

12. A. Judeh and M. Abu Hammad, *Applications of Conformable Fractional Pareto Probability Distribution*, International Journal of Advances in Soft Computing and its Applications, vol. 14, no. 2, pp. 115–124, 2022.
13. Z. Kadelburg and S. Radenović, *Fixed point results in S-metric spaces*, Fixed Point Theory 16 (2015), 135–146.
14. V. Lakshmikantham and Ć. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. 70 (2009), 4341–4349.
15. N. V. Luong and N. X. Thuan, *Coupled fixed points in partially ordered metric spaces and application*, Nonlinear Anal. 74 (2011), 983–992.
16. Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal. 7 (2006), 289–297.
17. Z. Mustafa, H. Obiedat, and F. Awawdeh, *Some fixed point theorems for mappings on complete G-metric spaces*, Fixed Point Theory Appl. 2008 (2008), Article ID 189870.
18. M. Meneceur, H. Qawaqneh, H. Alsamir, and G. Al-Musannef, *Multivalued Almost JS-Contractions, Related Fixed Point Results in Complete b-Metric Spaces and Applications*, European Journal of Pure and Applied Mathematics, vol. 17, no. 4, pp. 3093–3108, 2024.
19. M. Nazam, H. Aydi, M.S.M. Noorani, and H. Qawaqneh, *Existence of Fixed Points of Four Maps for a New Generalized F-Contraction and an Application*, Journal of Function Spaces, vol. 2019, ID. 5980312, 8 pages, 2019.
20. J. J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order 22 (2005), 223–239.
21. H. Qawaqneh, *Fractional analytic solutions and fixed point results with some applications*, Adv. Fixed Point Theory, vol. 14, no. 1, 2024.
22. H. Qawaqneh, *New Functions For Fixed Point Results In Metric Spaces With Some Applications*, Indian J. Math., vol. 66, no. 1, pp. 55–84, 2024.
23. H. Qawaqneh, M.S.Md. Noorani, and H. Aydi, *Some new characterizations and results for fuzzy contractions in fuzzy b-metric spaces and applications*, AIMS Mathematics, vol. 8, no. 3, pp. 6682–6696, 2023.
24. H. Qawaqneh and H. Aydi, *Fixed points of contraction mappings involving a Simulation function and applications*, Journal of Mathematical Analysis, vol. 6, no. 4, pp. 1–13, 2025.
25. H. Qawaqneh, M.S. Noorani, H. Aydi, and W. Shatanawi, *On Common Fixed Point Results for New Contractions with Applications to Graph and Integral Equations*, Mathematics, vol. 7, no. 11, ID. 1082, 2019.
26. H. Qawaqneh and M.A. Alomair, *Mathematical and Physical Analysis of the Fractional Dynamical Model*, Fractal Fract., vol. 9, no. 453, 2025.
27. H. Qawaqneh, A. Altalbe, A. Bekir, and K.U. Tariq, *Investigation of soliton solutions to the truncated M-fractional (3+1)-dimensional Gross-Pitaevskii equation with periodic potential*, AIMS Mathematics, vol. 9, no. 9, pp. 23410–23433, 2024.
28. H. Qawaqneh, M.S. Noorani, W. Shatanawi, and H. Alsamir, *Common Fixed Point Theorems for Generalized Geraghty $(\alpha,\psi,\phi)-$Quasi Contraction Type Mapping in Partially Ordered Metric-Like Spaces*, Axioms, vol. 7, no. 74, 2018.
29. H. Qawaqneh, H.A. Jari, A. Altalbe, and A. Bekir, *Stability analysis, modulation instability, and the analytical wave solitons to the fractional Boussinesq-Burgers system*, Phys. Scr., vol. 99, ID. 125235, 2024.
30. H. Qawaqneh, J. Manafian, A.S. Alsubaie, and H. Ahmad, *Investigation of exact solitons to the quartic Rosenau-Kawahara-Regularized-Long-Wave fluid model with fractional derivative and qualitative analysis*, Phys. Scr., vol. 100, ID. 015270, 2025.
31. H. Qawaqneh, Y. Alrashedi, H. Ahmad, and A. Bekir, *Discovery of exact solitons to the fractional KP-MEW equation with stability analysis*, Eur. Phys. J. Plus, vol. 140, no. 316, 2025.
32. M. A. Rosihan and R. Sumati, *Some fixed point theorems in S_b-metric spaces*, J. Math. Anal. 8 (2017), 33–41.
33. S. Sedghi, N. Shobe, and A. Aliou, *A generalization of fixed point theorems in S-metric spaces*, Mat. Vesnik 64 (2012), 258–266.
34. Q. Zhao and S. Li, *On multi-coupled fixed point theorems in partially ordered metric spaces*, J. Nonlinear Sci. Appl. 13 (2020), 67–78.
35. L. Guran, Z. D. Mitrović, G.S.M. Reddy, A. Belhenniche, and S. Radenović, *Applications of a fixed point result for solving nonlinear fractional and integral differential equations*, Fractal and Fractional, vol. 5, no. 4, Article 211, 2021.
36. V. S. Chary, G.S.M. Reddy, H. Işık, H. Aydi, D. S. Chary, *Some fixed point theorems on α-β-G-complete G-metric spaces*, Carpathian Mathematical Publications, vol. 13, no. 1, pp. 58–67, 2021.
37. V. S. Chary, G.S.M. Reddy, H. Işık, H. Aydi, D. S. Chary, S. Radenović, *Some fixed point theorems for modified JS-G-contractions and an application to integral equations*, Journal of Applied Mathematics and Informatics, vol. 38, no. 5–6, pp. 507–518, 2020.
38. V. S. Chary, G.S.M. Reddy, D. S. Chary, S. Radenović, *Existence of fixed points in G-metric spaces*, Boletim da Sociedade Paranaense de Matemática, vol. 41, pp. 1–18, 2023.
39. G.S.M. Reddy, *Fixed Point Results for G-F-Contractive Mappings of Hardy-Rogers Type*, Boletim da Sociedade Paranaense de Matemática, vol. 42, pp. 1–5, 2024.
40. G.S.M. Reddy, *Fixed Point Results for Orthogonal G-F-Contraction mappings on O-complete G-metric spaces*, Boletim da Sociedade Paranaense de Matemática, vol. 43, pp. 1–8, 2025.
41. G.S.M. Reddy, V. Srinivas Chary, D. Srinivasa Chary, S. Radenovic, and S. Mitrovic, *Coupled fixed point theorems of JS-G-contraction on G-metric spaces*, Boletim da Sociedade Paranaense de Matematica, vol. 41, pp. 1–10, 2023.
42. G.S.M. Reddy and H. Qawaqneh, *Monotone Vector Fields and Proximal Algorithms in G-Metric Spaces: A Comprehensive Framework with Applications to Modern Optimization Challenges*, Image Analysis and Stereology, vol. 45, no. 1, pp. 65–74, 2026.