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Decision Making Using The Score Function And New Distance Measure: A Comparative Study.

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Abstract In this paper, we define a new metric (distance function) between the dual fuzzy set of hesitation and also between the weighted dual fuzzy set of hesitation. This new measure is made by adding the relative coefficient of variation to the previous measure [53], a comparison is obtained between the previous measure and the present measure. In addition, a comparison is presented between the score function and the distance measure to see which is more efficient and easier to use in decision-making, in the medical diagnosis application. We have proved the triangle inequality of the new normalized Hamming metric, the new Euclidean metric, and the new generalized metric between two dual hesitant fuzzy elements. We have used a Python program to write the code of distance measure calculations. The score function is very important for ordering the dual hesitant fuzzy elements, so it is very important in simplifying the calculations for decision-making.

Keywords Distance measure (distance function)- D.F.S.H- W.D.F.S.H.- hesitancy degree- score function - accuracy function- medical diagnosis - decision-making.

AMS 2010 subject classifications 46N30-46N60-30F45-03E72

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1. Introduction

The score function has been widely employed in addressing multi-criteria vague decision-making problems. In its simplest form, the score function is defined as the difference between the membership and non-membership functions [30]. Chen and Tan [2] were among the first to apply this concept in the context of intuitionistic fuzzy sets. Subsequently, De et al. [3] and Kharal [15] proposed three alternative formulations of the S.F. to more accurately capture the predisposition toward positive and negative outcomes. The first defines the score function as the membership degree minus the product of the non-membership and hesitation degrees. The second follows a similar structure but substitutes the product with the arithmetic mean of the non-membership and hesitation degrees. The third defines the score function as the arithmetic mean of the membership and non-membership degrees minus the hesitation degree.Wang [35] used the score function to evaluate the dominated degree, thereby reflecting the relative importance of the criteria in service quality assessment. and when explicitly accounting for hesitation, the S.F. is expressed as the membership degree minus the product of the hesitation and non-membership degrees, which coincides with the first definition introduced by De et al. [3] and Kharal [15].

Although Chen and Tan [2] formalized the score function within fuzzy decision-making, a related notion can be traced back to the cumulative prospect theory (CPT) developed by Tversky and Kahneman [33]. As emphasized

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by Grabisch et al. [13], CPT provides a decision-making model in which the net predisposition is computed in a straightforward manner as the difference between the positive and negative components, i.e. the membership degree minus the non-membership degree. In recent years, the score function has been successfully applied in a variety of domains, including similarity measures [1], [14], aggregation operators [45], [56], ranking procedures [43], [49], Choquet integrals [29], [47], preference relations [12], [45], programming models [20], [46], multiple-attribute decision making [21], [37], and group decision making [17], [36], [48]. Dual hesitant fuzzy sets provide a powerful means of representing uncertainty in real-world situations by simultaneously employing membership and non-membership degrees. They enable the collection of vague data and their subsequent use in distance-based decision-making tasks [53]. Distance measures, as fundamental constructs in system theory, have numerous applications in diverse fields such as image segmentation, decision making, and artificial intelligence. The distance function, or metric, was extended to Dual hesitant fuzzy sets by Su et al. [27] and has since been applied in a wide range of contexts.

The introduction of fuzzy sets by Zadeh [51] marked a seminal moment in uncertainty modeling, where membership degrees were proposed to capture the complexity of real-world phenomena. This idea had a profound influence and paved the way for subsequent generalizations. In particular, Torra and Narukawa [31,32] generalized the traditional vague set to allow multiple degrees of membership, leading to the development of the hesitant fuzzy set. This framework has since attracted significant scholarly attention. Xia and Xu [38] and Xia et al. [39] studied the mathematical representation of HFS and developed a series of powerful aggregation operators in hesitant environments. Xu and Xia [40] extended entropy and cross-entropy concepts to, deriving several meaningful conclusions. Furthermore, Xu and Xia [41,42] conducted systematic investigations into distance, similarity, and correlation measures in hesitant fuzzy contexts. Farhadinia [10] examined information measures for hesitant fuzzy sets and interval-valued fuzzy sets, while Li et al. [18,19] introduced the concept of hesitancy degree and established distance measures based on it. To broaden the applicability of hesitant fuzzy sets, Zeng et al. [55] proposed a novel distance measure with successful applications in pattern recognition. Collectively, distance and similarity measures for hesitant fuzzy sets have been widely utilized in pattern recognition, approximate reasoning, image segmentation, and medical diagnosis.

The flexibility of vague and hesitant fuzzy systems has also facilitated their use in medical applications. Emanuel et al. [7,8] introduced type-2 fuzzy theory into medical diagnosis, achieving promising results. Similarly, Molla et al. [23] applied Pythagorean fuzzy theory within the Preference Ranking Organization Method for Enrichment Evaluation (PROMETHEE), enhancing diagnostic decision-making.

In practical contexts, dual hesitant fuzzy sets DHFS have proven to be effective tools for managing uncertainty. Singh [26] developed a metric for DHFS and demonstrated its application in evaluating investment alternatives. Su et al. [27] further advanced this line of research by designing distance and similarity measures for DHFS with applications in pattern recognition. However, in certain decision-making scenarios, DHFS may not adequately convey all relevant information to the decision maker. To address this limitation, the weighted dual hesitant fuzzy set (WDHFS) and weighted dual hesitant fuzzy element (WDHFE) were introduced [54]. In this framework, aggregation operators such as the WDHFS averaging operator and the weighted DHFS geometric operator were developed to combine weighted dual hesitant fuzzy information. The concept of a feature vector of a WDHFE was also proposed, consisting of five intrinsic properties: upper bound, lower bound, average level function, standard deviation, and hesitancy degree. Based on this representation, a distance measure was designed to better preserve the original information contained in WDHFE.

Because DHFS incorporates flexible evaluations from experts, it has become a convenient and widely applied framework in decision making and medical diagnosis. Building upon this, Wenyi Zeng et al. [53] introduced several important properties of dual hesitant fuzzy elements (DHFE), including the average function, the variance function, and the degree of hesitancy. They further proposed both standard and weighted metrics for DHFE, with the latter allowing for the adjustment of contributions from different feature indices. Comparative analyses demonstrated the effectiveness of these new distance measures, particularly in capturing medical diagnostic information with greater accuracy. In [11], a novel metric for picture fuzzy sets was introduced, while in [24], a new distance measure for (DFHE) and (WDFHE) was proposed.

The relative coefficient of variation is a powerful tool in medical diagnosis and decision making, it expresses the degree of variability relative to the mean, making it easier to compare the consistency of tests—even when they use different units or measurement scales.So, we added it to the new distance measure in this paper.[16] In what follows, we will denote the dual fuzzy set of hesitation by D.F.S.H., the weighted dual fuzzy set of hesitation by W.D.F.S.H., the dual fuzzy element of hesitation by D.F.E.H., and the weighted dual fuzzy element of hesitation by W.DF.E.H. We also denote the score function by S.F.

The structure of the present paper is organized as follows: Part 2. provides the preliminaries of dual hesitant fuzzy sets (DHFS). Introduces key definitions, including the upper and lower bounds, mean function, standard deviation function, coefficient of variation, and hesitancy degree of DHFS.

Part 3. presents the main theoretical contributions and applications. A new distance measure has been formally defined, and several of its properties have been established through rigorous proofs. Furthermore, the effectiveness of the proposed metric is demonstrated in medical diagnosis and decision-making contexts. A comparative analysis between the distance measure and the score function is also provided, highlighting the relative simplicity and practical advantages of the score function. The following chart shows the algorithm used to calculate the distance function in medical diagnosis application:

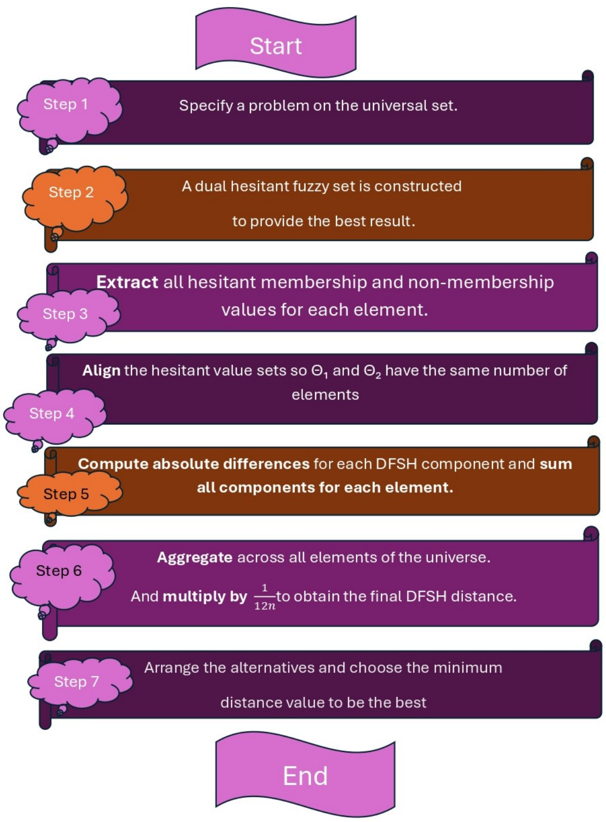

Figure 1. Chart of the algorithm used in calculating the distance function.

2. Preliminaries

Definition 1. Let $U = \{u_1, u_2, ..., u_n\}$ be the comprehensive set, let H denote the fuzzy set of hesitation (F.S.H) on U where

$$H = \{(u, \psi_H(u)) \mid u \in U\} \tag{2.1}$$

and $\psi_H(u) \subseteq [0,1]$ is the set of all possible degrees of membership of element $u \in U$ and let $\psi_H(u) = \psi(u)$ be the fuzzy element of hesitation (F.E.H.)

For vague sets of hesitation $\psi(u), \psi_1(u), \psi_2(u)$, we define the following properties [31,54]:
(1) $\psi^- = min\psi(u), \psi^+(u) = max\psi(u)$
(2) $\bar{\psi}(u) = \cup_{\phi\in\psi}\{1-\phi\}$
(3) $\psi_1(u) \cup \psi_2(u) = \cup_{r_1\in\psi_1(u), r_2\in\psi_2(u)} max\{r_1, r_2\}$
(4) $\psi_1(u) \cap \psi_2(u) = \cup_{r_1\in\psi_1(u), r_2\in\psi_2(u)} min\{r_1, r_2\}$
(5) $\psi^\lambda(u) = \cup_{r\in\psi}\{r^\lambda\}$
(6) $\lambda r(u) = \cup_{r\in\psi}\{1-(1-r)^\lambda\}, \lambda > 0$
(7) $\psi_1(u) \oplus \psi_2(u) = \cup_{r_1\in\psi_1(u), r_2\in\psi_2(u)}\{r_1 + r_2 - r_1 r_2\}$
(8) $\psi_1(u) \otimes \psi_2(u) = \cup_{r_1\in\psi_1(u), r_2\in\psi_2(u)}\{r_1 r_2\}$

Definition 2. Let $\psi(u)$ be a vague element of hesitation, the score function Υ of $\psi(u)$ is defined as follows:

$$\Upsilon(\psi(u)) = \frac{1}{\tau(\psi(u))}\sum_{r\in\psi(u)} r \tag{2.2}$$

where $\tau(\psi(u))$ is the length of $\psi(u)$ (the number of elements in $\psi(u)$).
If $\psi_1(u), \psi_2(u)$ are fuzzy hesitant elements, then
(i) If $\Upsilon(\psi_1(u)) \leq \Upsilon(\psi_2(u))$ then $\psi_1(u) \leq \psi_2(u)$
(ii) If $\Upsilon(\psi_1(u)) = \Upsilon(\psi_2(u))$ then $\psi_1(u) = \psi_2(u)$.

Definition 3. The D.F.S.H. is defined as follows:

$$\Theta = \{(u, \psi(u), \zeta(u)) \mid u \in U\} \tag{2.3}$$

where $\psi(u) \subseteq [0,1]$ and $\zeta(u) \subseteq [0,1]$ are the sets of all possible membership degree and nonmembership degree of the element $u \in U$ in the set Θ, respectively, $r \in \psi(u)$ and $g \in \zeta(u)$ satisfy the following conditions:
(1) $r \geq 0$, $1 \geq g$
(2) $r^+ = \cup_{r\in\psi(u)} \max\{r\},\ g^+ = \cup_{g\in\zeta(u)} \max\{g\}$
(3) $1 \geq r^+ + g^+ \geq 0$.

Consider $\theta = \{\psi(u), \zeta(u)\}$ to be the D.F.E.H.

Definition 4. Let $\theta = \{\psi(u), \zeta(u)\}$ be a dual vague element of hesitation, the score function Υ of θ is defined as follows:

$$\Upsilon(\theta) = \frac{1}{\tau(\psi(u))}\sum_{r\in\psi} r - \frac{1}{\tau(\zeta(u))}\sum_{y\in\zeta} y \tag{2.4}$$

The accuracy function k of θ is defined as follows:

$$k(\theta) = \frac{1}{\tau(\psi(u))}\sum_{r\in\psi} r + \frac{1}{\tau(\zeta(u))}\sum_{y\in\zeta} y \tag{2.5}$$

If $\theta_1(u), \theta_2(u)$ are D.F.E.H., then if $\Upsilon(\theta_1(u)) \leq \Upsilon(\theta_2(u))$ then $\theta_1(u) \leq \theta_2(u)$;if $\Upsilon(\theta_1(u)) = \Upsilon(\theta_2(u))$ then $\theta_1(u) = \theta_2(u)$.

Definition 5. Let $\theta^\alpha = \{\psi^\alpha(u), \zeta^\alpha(u)\}$ be a weighted dual hesitant fuzzy element, the score function Υ of θ is defined as follows:

$$\Upsilon(\theta^\alpha) = \sum_{r \in \psi^\alpha} \alpha_r r - \sum_{y \in \zeta^\alpha} \alpha_y y \tag{2.6}$$

The accuracy function k of θ^α is defined as follows:

$$k(\theta^\alpha) = \sum_{r \in \psi^\alpha} \alpha_r r + \sum_{y \in \zeta^\alpha} \alpha_y y \tag{2.7}$$

If $\theta_1^\alpha(u), \theta_2^\alpha(u)$ are W.F.E.H. that are hesitant, then if $\Upsilon(\theta_1^\alpha(u)) \leq \Upsilon(\theta_2^\alpha(u))$ then $\theta_1^\alpha(u) \leq \theta_2^\alpha(u)$; if $\Upsilon(\theta_1^\alpha(u)) = \Upsilon(\theta_2^\alpha(u))$ then $\theta_1^\alpha(u) = \theta_2^\alpha(u)$ if $k(\theta_1^\alpha) = k(\theta_2^\alpha)$ and $\theta_1^\alpha(u) \leq \theta_2^\alpha(u)$ if $k(\theta_1^\alpha) \leq k(\theta_2^\alpha)$

Definition 6. Let $O = (o_{ij})_(m \times n)$ be the collective intuitionistic fuzzy decision matrix; then we call $S = (s_{ij})_(m \times n)$ the score matrix of $O = (o_{ij})_(m \times n)$ where:

$$s_i = s(o_{ij}) = \psi_(ij) - \zeta_(ij), i = 1, 2, ...m; j = 1, 2, ...n \tag{2.8}$$

and $s(o_{ij})$ is called the score of o_{ij}

The score value of s_{ij} is defined as follows:

$$s_j(w) = \sum_{i=1}^{m} w_i s_{ij} \tag{2.9}$$

Definition 7.(Distance measure) The function $\delta(DHFS, DHFS)$ is called a distance measure or metric between E, F, if the following conditions hold:
(1) $1 \geq \delta(L, M) \geq 0$.
(2) $\delta(L, M) = 0$ if and only if $E = F$.
(3) $\delta(L, M) = \delta(M, L)$.
(4) $\delta(L, M) \leq \delta(L, N) + \delta(N, M)$.
where$L = \{(u, \psi_L(u), \zeta_L(u)); u \in U\}$, $M = \{(u, \psi_M(u), \zeta_M(u)); u \in U\}$,
$N = \{u, \psi_N(u), \zeta_N(u); u \in U\}$ are D.F.S.H. and $\delta(L, M) : DHFS \times DHFS \rightarrow [0, 1]$.

Definition 8. Let $\theta = \{\psi, \zeta\}$ then $\theta^+ = \{\theta_\psi^+, \theta_\zeta^+\}$
$= \{max\{r\}_{r \in \psi}, min\{u\}_{u \in \zeta}\}$, $\theta^- = \{\theta_\psi^-, \theta_\zeta^-\}$ =$\{min\{r\}_{r \in \psi}, max\{r\}_{u \in \zeta}\}$ are called the upper and lower bound of θ, respectively.

Definition 9. If $\theta = \{\psi, \zeta\}$ then

$$\phi(\theta) = \{\phi_\psi, \phi_\zeta\} = \left\{ \frac{1}{\tau(\psi)} \sum_{r \in \psi} r, \frac{1}{\tau(\zeta)} \sum_{u \in \zeta} u \right\} \tag{2.10}$$

and

$$b(\theta) = \{b_\psi, b_\zeta\} = \left\{ \sqrt{\frac{1}{\tau(\psi)} \sum_{r\in\psi} (r-\phi_\psi)^2}, \sqrt{\frac{1}{\tau(\zeta)} \sum_{u\in\zeta} (u-\phi_\zeta)^2} \right\} \tag{2.11}$$

and

$$c(\theta) = \{c_\psi, c_k\} = \left\{ \frac{\sqrt{\frac{1}{\tau(\psi)} \sum_{r\in\psi} (r-\phi_\psi)^2}}{\frac{1}{\tau(\psi)} \sum_{r\in\psi} r}, \frac{\sqrt{\frac{1}{\tau(\zeta)} \sum_{u\in\zeta} (u-\phi_\zeta)^2}}{\frac{1}{\tau(\zeta)} \sum_{u\in\zeta} u} \right\} \tag{2.12}$$

denote the mean function, the standard deviation, and the relative coefficient of variation of the F.E.H. respectively.
$\tau(\psi), \tau(\zeta)$ are the lengths of ψ, ζ (the number of elements in each)

$$\eta(\theta) = \{\eta_\psi, \eta_\zeta\} = \left\{ 1 - \frac{1}{\tau(\psi)}, 1 - \frac{1}{\tau(k)} \right\} \tag{2.13}$$

denotes the degree of hesitancy of θ.

3. The Fundamental Results

3.1. The New Distance Measure

Definition 10. If θ_1 and θ_2 are two D.F.E, then

$$\begin{aligned} \delta_1(\theta_1, \theta_2) = \frac{1}{12}\Big(& |\theta^+_{1\psi} - \theta^+_{2\psi}| + |\theta^-_{1\psi} - \theta^-_{2\psi}| + |\theta^+_{1\zeta} - \theta^+_{2\zeta}| + |\theta^-_{1\zeta} - \theta^-_{2\zeta}| \\ & + |\phi_{1\psi} - \phi_{2\psi}| + |\phi_{1\zeta} - \phi_{2\zeta}| + |b_{1\psi} - b_{2\psi}| + |b_{1\zeta} - b_{2\zeta}| \\ & + |\eta_{1\psi} - \eta_{2\psi}| + |\eta_{1\zeta} - \eta_{2\zeta}| + |c_{1\psi} - c_{2\psi}| + |c_{1\zeta} - c_{2\zeta}| \Big) \end{aligned} \tag{3.1}$$

$$\begin{aligned} \delta_2(\theta_1, \theta_2) = \Bigg[\frac{1}{12}\Big(& (\theta^+_{1\psi} - \theta^+_{2\psi})^2 + (\theta^-_{1\psi} - \theta^-_{2\psi})^2 + (\theta^+_{1\zeta} - \theta^+_{2\zeta})^2 + (\theta^-_{1\zeta} - \theta^-_{2\zeta})^2 \\ & + (\phi_{1\psi} - \phi_{2\psi})^2 + (\phi_{1\zeta} - \phi_{2\zeta})^2 + (b_{1\psi} - b_{2\psi})^2 + (b_{1\zeta} - b_{2\zeta})^2 \\ & + (\eta_{1\psi} - \eta_{2\psi})^2 + (\eta_{1\zeta} - \eta_{2\zeta})^2 + (c_{1\psi} - c_{2\psi})^2 + (c_{1\zeta} - c_{2\zeta})^2 \Big) \Bigg]^{1/2} \end{aligned} \tag{3.2}$$

$$\begin{aligned} \delta_3(\theta_1, \theta_2) = \Bigg[\frac{1}{12}\Big(& |\theta^+_{1\psi} - \theta^+_{2\psi}|^\lambda + |\theta^-_{1\psi} - \theta^-_{2\psi}|^\lambda + |\theta^+_{1\zeta} - \theta^+_{2\zeta}|^\lambda + |\theta^-_{1\zeta} - \theta^-_{2\zeta}|^\lambda \\ & + |\phi_{1\psi} - \phi_{2\psi}|^\lambda + |\phi_{1\zeta} - \phi_{2\zeta}|^\lambda + |b_{1\psi} - b_{2\psi}|^\lambda + |b_{1\zeta} - b_{2\zeta}|^\lambda \\ & + |\eta_{1\psi} - \eta_{2\psi}|^\lambda + |\eta_{1\zeta} - \eta_{2\zeta}|^\lambda + |c_{1\psi} - c_{2\psi}|^\lambda + |c_{1\zeta} - c_{2\zeta}|^\lambda \Big) \Bigg]^{\frac{1}{\lambda}} \end{aligned} \tag{3.3}$$

where $\lambda > 0$, $\delta_1(\theta_1, \theta_2)$ is the normalized Hamming metric, $\delta_2(\theta_1, \theta_2)$ is the normalized Euclidean metric, and $\delta_3(\theta_1, \theta_2)$ is the normalized generalized metric between θ_1 and θ_2.

Theorem 1. Let θ_1, θ_2 be two dual hesitant fuzzy elements then:
(i) Normalized Hamming metric $\delta_1(\theta_1, \theta_2)$
(ii) Normalized Euclidean metric$\delta_2(\theta_1, \theta_2)$
(iii) Normalized generalized metric $\delta_3(\theta_1, \theta_2)$
satisfy the conditions of distance measurement.

Proof. (i) The normalized Hamming metric $\delta_1(\theta_1, \theta_2)$ satisfies the following conditions:
(1) $1 \geq \delta_1(\theta_1, \theta_2) \geq 0$.
(2) $\delta_1(\theta_1, \theta_2) = 0$ if and only if $\theta_1 = \theta_2$.
(3) $\delta_1(\theta_1, \theta_2) = \delta_1(\theta_2, \theta_1)$.
(4)

$$\begin{aligned}\delta_1(\theta_1, \theta_3) =& \frac{1}{12}(|\theta^+_{1\psi} - \theta^+_{3\psi}| + |\theta^-_{1\psi} - \theta^-_{3\psi}| + |\theta^+_{1\zeta} - \theta^+_{3\zeta}| + |\theta^-_{1\zeta} - \theta^-_{3\zeta} + |\phi_{1\psi} - \phi_{3\psi}| \\ & + |\phi_{1\zeta} - \phi_{3\zeta}| + |b_{1\psi} - b_{3\psi}| + |b_{1\zeta} - b_{3\zeta}| + |\eta_{1\psi} - \eta_{3\psi}| + |\eta_{1\zeta} - \eta_{3k}| \\ & + |c_{1\psi} - c_{2\psi}| + |c_{1\zeta} - c_{2\zeta}|) \\ \leq & \frac{1}{12}(|\theta^+_{1\psi} - \theta^+_{2\psi}| + |\theta^-_{1\psi} - \theta^-_{2\psi}| + |\theta^+_{1\zeta} - \theta^+_{2\zeta}| + |\theta^-_{1\zeta} - \theta^-_{2\zeta} + |\phi_{1\psi} - \phi_{2\psi}| \\ & + |\phi_{1\zeta} - \phi_{2\zeta}| + |b_{1\psi} - b_{2\psi}| + |b_{1\zeta} - b_{2\zeta}| + |\eta_{1\psi} - \eta_{2\psi}| + |\eta_{1\zeta} - \eta_{2\zeta}| \\ & + |c_{1\psi} - c_{2\psi}| + |c_{1\zeta} - c_{2\zeta}|) \\ + & \frac{1}{12}(|\theta^+_{2\psi} - \theta^+_{3\psi}| + |\theta^-_{2\psi} - \theta^-_{3\psi}| + |\theta^+_{2\zeta} - \theta^+_{3\zeta}| + |\theta^-_{2\zeta} - \theta^-_{3\zeta} + |\phi_{2\psi} - \phi_{3\psi}| \\ & + |\phi_{2\zeta} - \phi_{3\zeta}| + |b_{2\psi} - b_{3\psi}| + |b_{2\zeta} - b_{3\zeta}| + |\eta_{2\psi} - \eta_{3\psi}| + |\eta_{2\zeta} - \eta_{3\zeta}| \\ & + |c_{1\psi} - c_{2\psi}| + |c_{1\zeta} - c_{2\zeta}|) \\ & = \delta_1(\theta_1, \theta_2) + \rho_1(\theta_2, \theta_3)\end{aligned}$$

(ii) The normalized Euclidean distance measure $\rho_2(\theta_1, \theta_2)$ also satisfies the axioms of the distance measure:
(1) $1 \geq \delta_2(\theta_1, \theta_2) \geq 0$.
(2) $\delta_2(\theta_1, \theta_2) = 0$ if and only if $\theta_1 = \theta_2$.
(3) $\delta_2(\theta_1, \theta_2) = \delta_1(\theta_2, \theta_1)$.
(4)

$$\begin{aligned}\delta_2(\theta_1, \theta_3) =& (\frac{1}{12}(d^+_{1\psi} - \theta^+_{3\psi})^2 + (\theta^-_{1\psi} - \theta^-_{3\psi})^2 + (\theta^+_{1\zeta} - \theta^+_{3\zeta})^2 + (\theta^-_{1\zeta} - \theta^-_{3\zeta})^2 + (\phi_{1\psi} - g_{3\psi})^2 \\ & + (g_{1\zeta} - g_{3\zeta})^2 + (b_{1\psi} - b_{3\psi})^2 + (b_{1\zeta} - b_{3\zeta})^2 + (\eta_{1\psi} - \eta_{3\psi})^2 + (\eta_{1\zeta} - \eta_{3\zeta})^2 \\ & (c_{1\psi} - c_{3\psi})^2 + (c_{1\zeta} - c_{3\zeta})^2)^{1/2} \\ \leq & (\frac{1}{12}(\theta^+_{1\psi} - \theta^+_{2\psi})^2 + (\theta^-_{1\psi} - \theta^-_{2\psi})^2 + (\theta^+_{1\zeta} - \theta^+_{2\zeta})^2 + (\theta^-_{1\zeta} - \theta^-_{2\zeta})^2 + (\phi_{1\psi} - \phi_{2\psi})^2 \\ & + (\phi_{1\zeta} - \phi_{2\zeta})^2 + (b_{1\psi} - b_{2\psi})^2 + (b_{1\zeta} - b_{2\zeta})^2 + (\eta_{1\psi} - \eta_{2\psi})^2 + (\eta_{1\zeta} - \eta_{2\zeta})^2 \\ & (c_{1\psi} - c_{2\psi})^2 + (c_{1\zeta} - c_{2\zeta})^2)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&+(\frac{1}{12}(\theta^{+}_{2\psi}-\theta^{+}_{3\psi})^2+(\theta^{-}_{2\psi}-\theta^{-}_{3\psi})^2+(\theta^{+}_{2\zeta}-\theta^{+}_{3\zeta})^2+(\theta^{-}_{2\zeta}-\theta^{-}_{3\zeta})^2+(\phi_{2\psi}-\phi_{3\psi})^2\\
&\quad+(\phi_{2\zeta}-\phi_{3\zeta})^2+(b_{2\psi}-b_{3\psi})^2+(b_{2\zeta}-b_{3\zeta})^2+(\eta_{2\psi}-\eta_{3\psi})^2+(\eta_{2\zeta}-\eta_{3\zeta})^2\\
&\quad(c_{2\psi}-c_{3\psi})^2+(c_{2\zeta}-c_{3\zeta})^2)^{1/2}))^{\frac{1}{2}}\\
&\quad=\rho_2(\theta_1,\theta_2)+\rho_2(\theta_2,\theta_3)
\end{aligned}
$$

(iii) The metric axioms hold for the normalized generalized metric $\rho_3(\theta_1,\theta_2)$ since:
(1) $1\geq\delta_3(\theta_1,\theta_2)\geq 0$.
(2) $\delta_3(\theta_1,\theta_2)=0$ if and only if $\theta_1=\theta_2$.
(3) $\delta_3(\theta_1,\theta_2)=\delta_1(\theta_2,\theta_1)$.
(4)

$$
\begin{aligned}
\delta_3(\theta_1,\theta_3)=&(\frac{1}{12}(|\theta^{+}_{1\psi}-\theta^{+}_{3\psi}|^\lambda+|\theta^{-}_{1\psi}-\theta^{-}_{3\psi}|^\lambda+|\theta^{+}_{1\zeta}-\theta^{+}_{3\zeta}|^\lambda+|\theta^{-}_{1\zeta}-\theta^{-}_{3\zeta}|^\lambda+|\phi_{1\psi}-\phi_{3\psi}|^\lambda\\
&+|\phi_{1\zeta}-\phi_{3\zeta}|^\lambda+|b_{1\psi}-b_{3\psi}|^\lambda+|b_{1\zeta}-b_{3\zeta}|^\lambda+|\eta_{1\psi}-\eta_{3\psi}|^\lambda+|\eta_{1\zeta}-\eta_{3\zeta}|^\lambda\\
&+|c_{1\psi}-c_{3\psi}|^\lambda+|c_{1\zeta}-c_{3\zeta}|^\lambda)^{1/\lambda}\\
\leq&(\frac{1}{12}(|\theta^{+}_{1\psi}-\theta^{+}_{2\psi}|^\lambda+|\theta^{-}_{1\psi}-\theta^{-}_{2\psi}|^\lambda+|\theta^{+}_{1\zeta}-\theta^{+}_{2\zeta}|^\lambda+|\theta^{-}_{1\zeta}-\theta^{-}_{2\zeta}|^\lambda+|\phi_{1\psi}-\phi_{2\psi}|^\lambda\\
&+|\phi_{1\zeta}-\phi_{2\zeta}|^\lambda+|b_{1\psi}-b_{2\psi}|^\lambda+|b_{1\zeta}-b_{2\zeta}|^\lambda+|\eta_{1\psi}-\eta_{2\psi}|^\lambda+|\eta_{1\zeta}-\eta_{2\zeta}|^\lambda\\
&+|c_{1\psi}-c_{2\psi}|^\lambda+|c_{1\zeta}-c_{2\zeta}|^\lambda))^{1/\lambda}\\
+&(\frac{1}{12}(|\theta^{+}_{2\psi}-\theta^{+}_{3\psi}|^\lambda+|\theta^{-}_{2\psi}-\theta^{-}_{3\psi}|^\lambda+|\theta^{+}_{2\zeta}-\theta^{+}_{3\zeta}|^\lambda+|\theta^{-}_{2\zeta}-\theta^{-}_{3\zeta}|^\lambda+|\phi_{2\psi}-\phi_{3\psi}|^\lambda\\
&+|\phi_{2\zeta}-\phi_{3\zeta}|^\lambda+|b_{2\psi}-b_{3\psi}|^\lambda+|b_{2\zeta}-b_{3\zeta}|^\lambda+|\eta_{2\psi}-\eta_{3\psi}|^\lambda+|\eta_{2\zeta}-\eta_{3\zeta}|^\lambda\\
&+|c_{2\psi}-c_{3\psi}|^\lambda+|c_{2\zeta}-c_{3\zeta}|^\lambda))^{1/\lambda}\\
&=\delta_3(\theta_1,\theta_2)+\delta_3(d\theta_2,\theta_3)
\end{aligned}
$$

□

The weighted normalized Hamming metric, the weighted normalized Euclidean metric, and the weighted normalized generalized metric between θ_1 and θ_2 are defined as follows:

$$
\begin{aligned}
\delta^1_\alpha(\theta_1,\theta_2)=&\alpha_1|\theta^{+}_{1\psi}-\theta^{+}_{2\psi}|+\alpha_2|\theta^{-}_{1\psi}-\theta^{-}_{2\psi}|+\alpha_3|\theta^{+}_{1\zeta}-\theta^{+}_{2\zeta}|+\alpha_4|\theta^{-}_{1\zeta}-\theta^{-}_{2\zeta}|+\alpha_5|\phi_{1\psi}-\phi_{2\psi}|\\
&+\alpha_6|\phi_{1\zeta}-\phi_{2\zeta}|+\alpha_7|b_{1\psi}-b_{2\psi}|+\alpha_8|b_{1\zeta}-b_{2\zeta}|+\alpha_9|\eta_{1\psi}-\eta_{2\psi}|+\alpha_{10}|\eta_{1\zeta}-\eta_{2\zeta}|\\
&+\alpha_{11}|c_{1\psi}-c_{2\psi}|+\alpha_{12}|c_{1\zeta}-c_{2\zeta}|
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\delta^2_\alpha(\theta_1,\theta_2)=&(\alpha_1(\theta^{+}_{1\psi}-\theta^{+}_{2\psi})^2+\alpha_2(\theta^{-}_{1\psi}-\theta^{-}_{2\psi})^2+\alpha_3(\theta^{+}_{1\zeta}-\theta^{+}_{2\zeta})^2+\alpha_4(\theta^{-}_{1\zeta}-\theta^{-}_{2\zeta})^2\\
&+\alpha_5(\phi_{1\psi}-\phi_{2\psi})^2+\alpha_6(\phi_{1\zeta}-\phi_{2\zeta})^2+\alpha_7(b_{1\psi}-b_{2\psi})^2\\
&+\alpha_8(b_{1\zeta}-b_{2\zeta})^2+\alpha_9(\eta_{1\psi}-\eta_{2\psi})^2+\alpha_{10}(\eta_{1\zeta}-\eta_{2\zeta})^2)^{1/2}\\
&+\alpha_{11}(c_{1\psi}-c_{2\psi})^2+\alpha_{12}(c_{1\zeta}-c_{2\zeta})^2)^{1/2}
\end{aligned}
\tag{3.5}
$$

$$\begin{aligned}\delta_\alpha^3(\theta_1,\theta_2) =&((\alpha_1|\theta_{1\psi}^+-\theta_{2\psi}^+|^\lambda+\alpha_2|\theta_{1\psi}^--\theta_{2\psi}^-|^\lambda+\alpha_3|\theta_{1\zeta}^+-\theta_{2\zeta}^+|^\lambda+\alpha_4|d_{1\zeta}^--d_{2\zeta}^-|^\lambda\\&+\alpha_5|\phi_{1\psi}-\phi_{2\psi}|^\lambda+\alpha_6|\phi_{1\zeta}-\phi_{2\zeta}|^\lambda+\alpha_7|b_{1\psi}-b_{2\psi}|^\lambda\\&+\alpha_8|b_{1\zeta}-b_{2\zeta}|^\lambda+\alpha_9|\eta_{1\psi}-\eta_{2\psi}|^\lambda+\alpha_{10}|\eta_{1\zeta}-\eta_{2\zeta}|^\lambda))^{1/\lambda}\\&+\alpha_{11}|c_{1\psi}-c_{2\psi}|^\lambda+\alpha_{12}|c_{1\zeta}-c_{2\zeta}|^\lambda))^{1/\lambda}\end{aligned}\tag{3.6}$$

where α_i are the weights.

Theorem 2. Let θ_1, θ_2 be two W.D.F.E.H then the weighted normalized Hamming metric, the weighted normalized Euclidean metric and the weighted normalized generalized metric satisfy the triangle inequality.
Proof.: The proof of this theorem is similar to theorem 1. □

Definition 11. Let $U = \{u_1, u_2, ..., u_n\}$ and Θ_1 and Θ_2 dual hesitant fuzzy sets on U then we can define the distance function between Θ_1 and Θ_2 as follow

$$\begin{aligned}\delta_1(\Theta_1,\Theta_2) =&\frac{1}{12n}\sum_{i=1}^{n}(|\theta_{1\psi}^+-\theta_{2\psi}^+|+|\theta_{1\psi}^--\theta_{2\psi}^-|+|\theta_{1\zeta}^+-\theta_{2\zeta}^+|+|\theta_{1\zeta}^--\theta_{2\zeta}^-+|\phi_{1\psi}-\phi_{2\psi}|\\&+|\phi_{1\zeta}-\phi_{2\zeta}|+|b_{1\psi}-b_{2\psi}|+|b_{1\zeta}-b_{2\zeta}|+|\eta_{1\psi}-\eta_{2\psi}|+|\eta_{1\zeta}-\eta_{2\zeta}|\\&+|c_{1\psi}-c_{2\psi}|+|c_{1\zeta}-c_{2\zeta}|)\end{aligned}\tag{3.7}$$

$$\begin{aligned}\delta_2(\Theta_1,\Theta_2) =&(\frac{1}{12n}\sum_{i=1}^{n}(\theta_{1\psi}^+-\theta_{2\psi}^+)^2+(\theta_{1\psi}^--\theta_{2\psi}^-)^2+(\theta_{1\zeta}^+-\theta_{2\zeta}^+)^2+(\theta_{1\zeta}^--\theta_{2\zeta}^-)^2+(\phi_{1\psi}-\phi_{2\psi})^2\\&+(\phi_{1\zeta}-\phi_{2\zeta})^2+(b_{1\psi}-b_{2\psi})^2+(b_{1\zeta}-b_{2\zeta})^2+(\eta_{1\psi}-\eta_{2\psi})^2+(\eta_{1\zeta}-\eta_{2\zeta})^2\\&+(c_{1\psi}-c_{2\psi})^2+(c_{1\zeta}-c_{2\zeta})^2)^{1/2}\end{aligned}\tag{3.8}$$

$$\begin{aligned}\delta_3(\Theta_1,\Theta_2) =&(\frac{1}{12n}\sum_{i=1}^{n}(|\theta_{1\psi}^+-\theta_{2\psi}^+|^\lambda+|\theta_{1\psi}^--\theta_{2\psi}^-|^\lambda+|\theta_{1\zeta}^+-\theta_{2\zeta}^+|^\lambda+|\theta_{1\zeta}^--d\theta_{2\zeta}^-|^\lambda+|\phi_{1\psi}-\phi_{2\psi}|^\lambda\\&+|\phi_{1\zeta}-\phi_{2\zeta}|^\lambda+|b_{1\psi}-b_{2\psi}|^\lambda+|b_{1\zeta}-b_{2\zeta}|^\lambda+|\eta_{1\psi}-\eta_{2\psi}|^\lambda+|\eta_{1\zeta}-\eta_{2\zeta}|^\lambda)\\&+|c_{1\psi}-c_{2\psi}|^\lambda+|c_{1\zeta}-c_{2\zeta}|^\lambda))^{1/\lambda}\end{aligned}\tag{3.9}$$

where $\lambda > 0$, $\delta_1(\Theta_1, \Theta_2)$ is the normalized Hamming metric, $\delta_2(\Theta_1, \Theta_2)$ is the normalized Euclidean metric, and $\delta_3(\Theta_1, \Theta_2)$ is the normalized generalized metric between D.F.S.H. Θ_1 and Θ_2 depending on the axioms of D.F.E.H.

Definition 12. Let Θ_1 and Θ_2 be D.F.S.H. on $U = \{u_1, u_2, ..., u_n\}$, the weighted normalized Hamming metric, the weighted normalized Euclidean metric and the weighted normalized generalized metric between Θ_1 and Θ_2 are defined, respectively, as follows:

$$\begin{aligned}\delta_\alpha^1(\Theta_1,\Theta_2) =& \frac{1}{n}\sum_{i=1}^{n}(\alpha_1|\theta_{1\psi}^+ - \theta_{2\psi}^+| + \alpha_2|\theta_{1\psi}^- - \theta_{2\psi}^-| + \alpha_3|\theta_{1\zeta}^+ - \theta_{2\zeta}^+| + \alpha_4|\theta_{1\zeta}^- - \theta_{2\zeta}^-| + \alpha_5|\phi_{1\psi} - \phi_{2\psi}|\\ &+ \alpha_6|\phi_{1\zeta} - \phi_{2\zeta}| + \alpha_7|b_{1\psi} - b_{2\psi}| + \alpha_8|b_{1\zeta} - b_{2\zeta}| + \alpha_9|\eta_{1\psi} - \eta_{2\psi}| + \alpha_{10}|\eta_{1\zeta} - \eta_{2\zeta}|\\ &+ \alpha_{11}|c_{1\psi} - c_{2\psi}| + \alpha_{12}|c_{1\zeta} - c_{2k\zeta}|)\end{aligned} \tag{3.10}$$

$$\begin{aligned}\delta_\alpha^2(\Theta_1,\Theta_2) =& (\frac{1}{n}\sum_{i=1}^{n}(\alpha_1(\theta_{1\psi}^+ - \theta_{2\psi}^+)^2 + \alpha_2(\theta_{1\psi}^- - \theta_{2\psi}^-)^2 + \alpha_3(\theta_{1\zeta}^+ - \theta_{2\zeta}^+)^2 + \alpha_4(\theta_{1\zeta}^- - \theta_{2\zeta}^-)^2\\ &+ \alpha_5(\phi_{1\psi} - \phi_{2\psi})^2 + \alpha_6(\phi_{1\zeta} - \phi_{2\zeta})^2 + \alpha_7(b_{1\psi} - b_{2\psi})^2\\ &+ \alpha_8(b_{1\zeta} - b_{2\zeta})^2 + \alpha_9(\eta_{1\psi} - \eta_{2\psi})^2 + \alpha_10(\eta_{1\zeta} - \eta_{2\zeta})^2)\\ &+ \alpha_{11}(c_{1\psi} - c_{2\psi})^2 + \alpha_{12}(c_{1\zeta} - c_{2\zeta})^2))^{1/2}\end{aligned} \tag{3.11}$$

$$\begin{aligned}\delta_\alpha^3(\Theta_1,\Theta_2) =& (\frac{1}{12n}\sum_{i=1}^{n}(\alpha_1|\theta_{1\psi}^+ - \theta_{2\psi}^+|^\lambda + \alpha_2|\theta_{1\psi}^- - \theta_{2\psi}^-|^\lambda + \alpha_3|\theta_{1\zeta}^+ - \theta_{2\zeta}^+|^\lambda + \alpha_4|\theta_{1\zeta}^- - \theta_{2\zeta}^-|^\lambda\\ &+ \alpha_5|\phi_{1\psi} - \phi_{2\psi}|^\lambda + \alpha_6|\phi_{1\zeta} - \phi_{2\zeta}|^\lambda + \alpha_7|b_{1\psi} - b_{2\psi}|^\lambda\\ &+ \alpha_8|b_{1\zeta} - b_{2\zeta}|^\lambda + \alpha_9|\eta_{1\psi} - \eta_{2\psi}|^\lambda + \alpha_{10}|\eta_{1\zeta} - \eta_{2\zeta}|^\lambda)\\ &+ \alpha_{11}|c_{1\psi} - c_{2\psi}|^\lambda + \alpha_{12}|c_{1\zeta} - c_{2\zeta}|^\lambda))^{1/\lambda}\end{aligned} \tag{3.12}$$

These definitions are based on the properties of D.F.S.H. where $0 \leq \alpha_i \leq 1$and $\sum_{i=1}^{12}\alpha_i = 1$.

3.2. Motivation Of The New Distance Measure

The previous distance measure in[53] is defined by the following equation:

$$\begin{aligned}\delta_1(\Theta_1,\Theta_2) =& \frac{1}{10n}\sum_{i=1}^{n}(|\theta_{1\psi}^+ - \theta_{2\psi}^+| + |\theta_{1\psi}^- - \theta_{2\psi}^-| + |\theta_{1\zeta}^+ - \theta_{2\zeta}^+| + |\theta_{1\zeta}^- - \theta_{2\zeta}^- + |\phi_{1\psi} - \phi_{2\psi}|\\ &+ |\phi_{1\zeta} - \phi_{2\zeta}| + |b_{1\psi} - b_{2\psi}| + |b_{1\zeta} - b_{2\zeta}| + |\eta_{1\psi} - \eta_{2\psi}| + |\eta_{1\zeta} - \eta_{2\zeta}|)\end{aligned} \tag{3.13}$$

Existing distance measures for dual hesitant fuzzy sets (3-14) incorporate the mean, variance, and hesitancy degree. However, these quantities are treated independently and fail to capture the interaction between central tendency and dispersion. In particular, variance is scale-dependent and does not adequately reflect relative uncertainty. For example, consider two hesitant fuzzy elements $q_1 = 0.1, 0.9andq_2 = 0.49, 0.51$. Both have the same mean, but q_1 exhibits significantly higher dispersion than q_2. Classical distance measures assign them nearly equal distances, failing to distinguish between highly inconsistent and highly consistent information. Moreover, the hesitancy degree only reflects the number of elements and ignores how widely the values are distributed. To overcome these limitations, we introduce the relative coefficient of variation, which normalizes the standard deviation by the mean and captures relative dispersion. This allows the proposed distance measure to distinguish between alternatives with similar averages but different levels of uncertainty. Consequently, the proposed distance provides a more robust and conceptually complete representation of dual hesitant fuzzy information, while remaining consistent with classical measures when dispersion is small.

3.3. The Applications And The Comparative Work

Example 1 [50, 53] In this application, we discuss how to make a decision using the score function and show how easy it is to do this. A car factory needs to decide the best global supplier for one of its most important parts used in collection operation. The characteristics considered here in the selection of five global suppliers $M_i, i = \{1, 2, 3, 4, 5\}$ are: (1) p_1 is the cost of the product; (2) p_2 is the quality of the product; (3) p_3 is the service performance of the supplier; (4) p_4 is the supplier's profile and (5) p_5 is a risk factor. $z = (0.2, 0.15, 0.2, 0.3, 0.15)^T$ is the weight vector of S_j. $\omega = (0.4, 0.2, 0.1, 0.3)^T$ is the weight vector of the experts group $S_l(l = 1, 2, 3, 4)$ which consists of four specialists from each strategic area. The five global suppliers M_1, M_2, M_3, M_4, M_5 will be evaluated using fuzzy intuition numbers by the specialists.
four intuitionistic fuzzy decision matrices are made by specialists, which denotes the satisfaction and dissatisfaction of specialists or experts $S_l(l = 1, 2, 3, 4)$ for the possible values of the supplier M_i with respect to the properties p_j.

The following tables show the W.D.F.S.H., made by experts:

Table 1. Intuitionistic fuzzy decision matrix S^1

	M_1	M_2	M_3	M_4	M_5
p_1	(0.8, 0.1)	(0.5, 0.1)	(0.5, 0.5)	(0.7, 0.1)	(0.8, 0.1)
p_2	(0.6, 0.2)	(0.8, 0.2)	(0.6, 0.3)	(0.6, 0.1)	(0.4, 0.5)
p_3	(0.7, 0.3)	(0.3, 0.4)	(0.3, 0.2)	(0.9, 0.1)	(0.7, 0.2)
p_4	(0.5, 0.2)	(0.7, 0.2)	(0.7, 0.3)	(0.4, 0.3)	(0.5, 0.2)
p_5	(0.4, 0.5)	(0.6, 0.2)	(0.8, 0.1)	(0.3, 0.4)	(0.3, 0.4)

Table 2. Intuitionistic fuzzy decision matrix S^2

	M_1	M_2	M_3	M_4	M_5
p_1	(0.7, 0.3)	(0.7, 0.2)	(0.4, 0.2)	(0.7, 0.3)	(0.5, 0.1)
p_2	(0.6, 0.1)	(0.6, 0.2)	(0.6, 0.3)	(0.7, 0.2)	(0.6, 0.2)
p_3	(0.8, 0.2)	(0.4, 0.4)	(0.7, 0.2)	(0.4, 0.2)	(0.5, 0.4)
p_4	(0.7, 0.1)	(0.6, 0.2)	(0.6, 0.1)	(0.8, 0.1)	(0.6, 0.3)
p_5	(0.5, 0.3)	(0.7, 0.3)	(0.5, 0.3)	(0.5, 0.4)	(0.7, 0.3)

Table 3. Intuitionistic fuzzy decision matrix S^3

	M_1	M_2	M_3	M_4	M_5
p_1	(0.7, 0.3)	(0.8, 0.2)	(0.8, 0.2)	(0.6, 0.1)	(0.7, 0.1)
p_2	(0.8, 0.1)	(0.5, 0.3)	(0.6, 0.1)	(0.5, 0.4)	(0.6, 0.2)
p_3	(0.6, 0.4)	(0.5, 0.2)	(0.7, 0.3)	(0.9, 0.1)	(0.8, 0.1)
p_4	(0.5, 0.2)	(0.6, 0.3)	(0.4, 0.2)	(0.5, 0.2)	(0.7, 0.3)
p_5	(0.6, 0.3)	(0.6, 0.4)	(0.8, 0.1)	(0.6, 0.3)	(0.5, 0.2)

Table 4. Intuitionistic fuzzy decision matrix S^4

	M_1	M_2	M_3	M_4	M_5
p_1	(0.9, 0.1)	(0.7, 0.3)	(0.8, 0.1)	(0.7, 0.2)	(0.8, 0.1)
p_2	(0.8, 0.1)	(0.7, 0.1)	(0.6, 0.2)	(0.6, 0.4)	(0.6, 0.3)
p_3	(0.5, 0.5)	(0.3, 0.1)	(0.6, 0.3)	(0.7, 0.2)	(0.8, 0.2)
p_4	(0.4, 0.6)	(0.8, 0.2)	(0.4, 0.1)	(0.5, 0.1)	(0.7, 0.2)
p_5	(0.3, 0.4)	(0.4, 0.2)	(0.5, 0.2)	(0.8, 0.2)	(0.6, 0.1)

When we use equation (2.8) we get the following tables:

Table 5. Score matrix S^1

	M_1	M_2	M_3	M_4	M_5
p_1	0.7	0.4	0.0	0.6	0.7
p_2	0.4	0.6	0.3	0.5	-0.1
p_3	0.4	-0.1	0.1	0.8	0.5
p_4	0.3	0.5	0.4	0.1	0.3
p_5	-0.1	0.4	0.7	-0.1	-0.1

Table 6. Score matrix S^2

	M_1	M_2	M_3	M_4	M_5
p_1	0.4	0.5	0.2	0.4	0.4
p_2	0.5	0.4	0.3	0.5	0.4
p_3	0.6	0.0	0.5	0.2	0.1
p_4	0.6	0.4	0.5	0.7	0.3
p_5	0.2	0.4	0.2	0.1	0.4

Table 7. Score matrix S^3

	M_1	M_2	M_3	M_4	M_5
p_1	0.4	0.6	0.6	0.5	0.6
p_2	0.7	0.2	0.5	0.1	0.4
p_3	0.2	0.3	0.4	0.8	0.7
p_4	0.3	0.3	0.2	0.3	0.4
p_5	0.3	0.2	0.7	0.3	0.3

Table 8. Score matrix S^4

	M_1	M_2	M_3	M_4	M_5
p_1	0.8	0.4	0.7	0.5	0.7
p_2	0.7	0.6	0.4	0.2	0.3
p_3	0.0	0.2	0.3	0.5	0.6
p_4	-0.2	0.6	0.3	0.4	0.5
p_5	-0.1	0.2	0.3	0.6	0.5

We use the following equation

$$s_{ij}(w) = \sum_{k=1}^{4} w_k s_{ij}^{(k)}$$

and the weight vector $\omega = (0.4, 0.2, 0.1, 0.3)^T$ to get the following results:

$$s_{11}(w) = \sum_{k=1}^{4} w_k s_{11}^{(k)} = w_1 s_{11}^{(1)} + w_2 s_{11}^{(2)} + w_3 s_{11}^{(3)} + w_4 s_{11}^{(4)}$$
$$= (0.4)(0.7) + (0.2)(0.4) + (0.1)(0.4) + (0.3)(0.8) = 0.64$$

$$s_{12}(w) = \sum_{k=1}^{4} w_k s_{12}^{(k)} = w_1 s_{12}^{(1)} + w_2 s_{12}^{(2)} + w_3 s_{12}^{(3)} + w_4 s_{12}^{(4)}$$
$$= (0.4)(0.4) + (0.2)(0.5) + (0.1)(0.6) + (0.3)(0.4) = 0.44$$

$$s_{13}(w) = \sum_{k=1}^{4} w_k s_{13}^{(k)} = w_1 s_{13}^{(1)} + w_2 s_{13}^{(2)} + w_3 s_{13}^{(3)} + w_4 s_{13}^{(4)}$$
$$= (0.4)(0.0) + (0.2)(0.2) + (0.1)(0.6) + (0.3)(0.7) = 0.31$$

$$s_{14}(w) = \sum_{k=1}^{4} w_k s_{14}^{(k)} = w_1 s_{14}^{(1)} + w_2 s_{14}^{(2)} + w_3 s_{14}^{(3)} + w_4 s_{14}^{(4)}$$
$$= (0.4)(0.6) + (0.2)(0.4) + (0.1)(0.5) + (0.3)(0.5) = 0.52$$

$$s_{15}(w) = \sum_{k=1}^{4} w_k s_{15}^{(k)} = w_1 s_{15}^{(1)} + w_2 s_{15}^{(2)} + w_3 s_{15}^{(3)} + w_4 s_{15}^{(4)}$$
$$= (0.4)(0.7) + (0.2)(0.4) + (0.1)(0.6) + (0.3)(0.7) = 0.63$$

$$s_{21}(w) = \sum_{k=1}^{4} w_k s_{21}^{(k)} = w_1 s_{21}^{(1)} + w_2 s_{21}^{(2)} + w_3 s_{21}^{(3)} + w_4 s_{21}^{(4)}$$
$$= (0.4)(0.4) + (0.2)(0.5) + (0.1)(0.7) + (0.3)(0.7) = 0.54$$

$$s_{22}(w) = \sum_{k=1}^{4} w_k s_{22}^{(k)} = w_1 s_{22}^{(1)} + w_2 s_{22}^{(2)} + w_3 s_{22}^{(3)} + w_4 s_{22}^{(4)}$$
$$= (0.4)(0.6) + (0.2)(0.4) + (0.1)(0.2) + (0.3)(0.6) = 0.52$$

$$s_{23}(w) = (0.4)(0.3) + (0.2)(0.3) + (0.1)(0.5) + (0.3)(0.4) = 0.35$$

$$s_{24}(w) = (0.4)(0.5) + (0.2)(0.5) + (0.1)(0.1) + (0.3)(0.2) = 0.37$$

$$s_{25}(w) = (0.4)(-0.1) + (0.2)(0.4) + (0.1)(0.4) + (0.3)(0.3) = 0.17$$

We show the values of the first and second rows, in the following table the remaining results can be obtained similarly.

Table 9. Collective score matrix S

	M_1	M_2	M_3	M_4	M_5
p_1	0.64	0.44	0.31	0.52	0.63
p_2	0.54	0.52	0.35	0.37	0.17
p_3	0.3	0.05	0.27	0.59	0.47
p_4	0.21	0.49	0.37	0.33	0.37
p_5	0	0.32	0.48	0.19	0.22

Now we use the weight row $z = (0.2, 0.15, 0.2, 0.3, 0.15)^T$ and the weight function of the score

$$s_j(z_i) = \sum_{i=1}^{5} z_i s_{ij} \quad j = 1, 2, 3, 4, 5$$

to obtain the following results:

$$s_1(z_i) = \sum_{i=1}^{5} z_i s_{i1} = z_1 s_{11} + z_2 s_{21} + z_3 s_{31} + z_4 s_{41} + z_5 s_{51}$$
$$= (0.2)(0.64) + (0.15)(0.54) + (0.2)(0.3) + (0.3)(0.21) + (0.15)(0.0) = 0.3320$$

$$s_2(z_i) = z_1 s_{12} + z_2 s_{22} + z_3 s_{32} + z_4 s_{42} + z_5 s_{52}$$

$$= (0.2)(0.44) + (0.15)(0.52) + (0.2)(0.05) + (0.3)(0.49) + (0.15)(0.32) = 0.3320$$

$$s_3(z_i) = z_1 s_{13} + z_2 s_{23} + z_3 s_{33} + z_4 s_{43} + z_5 s_{53}$$
$$= (0.2)(0.31) + (0.15)(0.35) + (0.2)(0.27) + (0.3)(0.37) + (0.15)(0.48) = 0.3515$$

$$s_4(z_i) = (0.2)(0.52) + (0.15)(0.37) + (0.2)(0.59) + (0.3)(0.33) + (0.15)(0.19) = 0.405$$

$$s_5(z_i) = (0.2)(0.63) + (0.15)(0.17) + (0.2)(0.47) + (0.3)(0.37) + (0.15)(0.22) = 0.3895$$

The score function of M_j is then shown in the following table:

Table 10. The score function of M_j

	s_1	s_2	s_3	s_4	s_5
M_j	0.3320	0.3710	0.3515	0.4050	0.3895

So we can arrange the W.D.F.S.H. as follows:

$$M_1 \leq M_3 \leq M_2 \leq M_5 \leq M_4 \tag{3.14}$$

Equation 3.14 shows that M_4 is the best global supplier because its intuitionistic fuzzy evaluations, after expert-weighted aggregation, give it the highest score on the most heavily weighted criteria—particularly supplier profile and service performance—resulting in the highest overall D.F.S.H. score among all suppliers. So, the best global supplier is M_4.

All the results in the previous application is confirmed by python coding.
The following implementation on decision-making in medical diagnosis.

Example 2 [53] Let $U = \{t_1, t_2, t_3, t_4, t_5\}$ be the universal set with D.F.S.H. $\Theta_1 = \{l_1, l_2, l_3, l_4, l_5\}$ and $\Theta_2 = \{M_1, M_2, M_3, M_4\}$ where l_1, l_2, l_3, l_4, l_5 are five diseases, such as $l_1(Malaria)$, $l_2(Typhoid)$, $l_3(Chestproblem)$,$l_4(Stomachproblem)$,$l_5(fever)$ and M_1, M_2, M_3, M_4 are four patients , the universal set U represents five symptoms $t_1(Headache)$, $t_2(Cough)$, $t_3(Chestpain)$,$t_4(stomachachpain)$.
,$t_5(Bodytempreture)$, in the following the distance measure $\delta(\Theta_1, \Theta_2)$between Θ_1 and Θ_2 will be calculated according to our novel distance measure. Decision making is taken according to the fact that the smaller the distance measure, the more likely the patient will be diagnosed with the disease. In the following tables,

Table 11. Disease data for symptoms

	t_1	t_2	t_3	t_4	t_5
l_1	{{0.6, 0.4, 0.3},{0.2, 0.0}}	{{0.7, 0.5, 0.3, 0.2},{0.3, 0.1}}	{{0.5, 0.3},{0.5, 0.4, 0.2}}	{{0.5, 0.4, 0.3, 0.2, 0.1},{0.5, 0.3}}	{{0.5, 0.4, 0.2, 0.1},{0.5, 0.4, 0.3}}
l_2	{{0.9, 0.8, 0.7},{0.1, 0.0}}	{{0.5, 0.3, 0.2, 0.1},{0.4, 0.3}}	{{0.2, 0.1},{0.7, 0.6, 0.5}}	{{0.6, 0.5, 0.3, 0.2, 0.1},{0.3, 0.2}}	{{0.4, 0.3, 0.2, 0.1},{0.6, 0.5, 0.4}}
l_3	{{0.6, 0.3, 0.1},{0.3, 0.2}}	{{0.9, 0.8, 0.7, 0.6},{0.1, 0.0}}	{{0.5, 0.3},{0.5, 0.4, 0.3}}	{{0.5, 0.4, 0.3, 0.2, 0.1},{0.5, 0.4}}	{{0.6, 0.4, 0.3, 0.2},{0.4, 0.3, 0.2}}
l_4	{{0.5, 0.4, 0.2},{0.5, 0.3}}	{{0.4, 0.3, 0.2, 0.1},{0.4, 0.3}}	{{0.4, 0.3},{0.6, 0.5, 0.4}}	{{0.9, 0.8, 0.7, 0.6, 0.5},{0.1, 0.0}}	{{0.5, 0.4, 0.2, 0.1},{0.5, 0.4, 0.3}}
l_5	{{0.3, 0.2, 0.1},{0.7, 0.6}}	{{0.5, 0.3, 0.2, 0.1},{0.5, 0.3}}	{{0.3, 0.2},{0.6, 0.4, 0.3}}	{{0.7, 0.6, 0.5, 0.3, 0.2},{0.2, 0.1}}	{{0.8, 0.7, 0.6, 0.5},{0.2, 0.1, 0.0}}

Table 12. Patient data for symptoms

	t_1	t_2	t_3	t_4	t_5
M_1	{{0.9, 0.7, 0.5},{0.1, 0.0}}	{{0.4, 0.3, 0.2, 0.1},{0.5, 0.4}}	{{0.4, 0.3},{0.5, 0.4, 0.2}}	{{0.6, 0.5, 0.4, 0.2, 0.1},{0.3, 0.2}}	{{0.4, 0.3, 0.2, 0.1},{0.5, 0.4, 0.3}}
M_2	{{0.5, 0.4, 0.2},{0.5, 0.3}}	{{0.5, 0.4, 0.3, 0.1},{0.4, 0.3}}	{{0.2, 0.1},{0.7, 0.6, 0.5}}	{{0.9, 0.8, 0.6, 0.5, 0.4},{0.1, 0.0}}	{{0.5, 0.4, 0.3, 0.2},{0.5, 0.4, 0.3}}
M_3	{{0.9, 0.7, 0.6},{0.1, 0.0}}	{{0.7, 0.4, 0.3, 0.1},{0.2, 0.1}}	{{0.3, 0.2},{0.5, 0.4, 0.3}}	{{0.6, 0.4, 0.3, 0.2, 0.1},{0.4, 0.3}}	{{0.6, 0.3, 0.2, 0.1},{0.4, 0.3, 0.2}}
M_4	{{0.8, 0.7, 0.5},{0.2, 0.1}}	{{0.6, 0.5, 0.4, 0.2},{0.4, 0.3}}	{{0.5, 0.3},{0.5, 0.4, 0.3}}	{{0.6, 0.4, 0.3, 0.2, 0.1},{0.4, 0.3}}	{{0.5, 0.4, 0.2, 0.1},{0.5, 0.4, 0.3}}

We used the Python program to compute the following metric when $\lambda = 0.5$, $\lambda = 1$, $\lambda = 2$.

Table 13. M_1 when $\lambda = 0.5$, $\lambda = 1$, $\lambda = 2$

M_1	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
l_1	0.03601762	0.07149409	0.11876774
l_2	**0.02192978**	**0.05430682**	**0.09506508**
l_3	0.08886402	0.15362159	0.24865306
l_4	0.05314720	0.11811907	0.21174163
l_5	0.08956279	0.15965574	0.26284623

Table 14. M_2 when $\lambda = 0.5$, $\lambda = 1$, $\lambda = 2$

M_2	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
l_1	0.08568493	0.14428164	0.22172978
l_2	0.04628080	0.11535905	0.21511797
l_3	0.10587763	0.16791103	0.25564072
l_4	**0.01012882**	**0.03387334**	**0.06892046**
l_5	0.07410045	0.12175801	0.18402569

Table 15. M_3 when $\lambda = 0.5$, $\lambda = 1$, $\lambda = 2$

M_3	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
l_1	**0.03758830**	**0.06440703**	**0.09801902**
l_2	**0.03693951**	0.06977840	0.10415730
l_3	0.05813835	0.12072951	0.21444210
l_4	0.08973059	0.15219970	0.23604621
l_5	0.09387574	0.16814634	0.26850910

Table 16. M_4 when $\lambda = 0.5$, $\lambda = 1$, $\lambda = 2$

M_4	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$
l_1	**0.02326248**	**0.05903791**	**0.12267776**
l_2	0.04120925	0.07724478	0.13184612
l_3	0.05035189	0.10380775	0.18599472
l_4	0.04759524	0.10663432	0.18964704
l_5	0.09759155	0.15477417	0.22570673

Based on the above results, we can diagnose M_1 with Typhoid l_2, M_2 with the Stomach problem l_4, and M_3, M_4 with Malaria l_1 as the following table shows:

Table 17. The diagnose by the distance measure

	diagnose
M_1	Typhoid l_2
M_2	Stomach problem l_4
M_3	Malaria l_1
M_4	Malaria l_1

Now we compare our distance measure with the previous distance measure in [53] when $\lambda = 1$: Table 12 shows the values of the old distance measure [53] for $\lambda = 1$

Table 18. The values of the old distance measure [53]

	M_1	M_2	M_3	M_4
l_1	0.0712167	0.121351	0.0586095	0.040090
l_2	0.050783	0.092528	0.0667410	0.067123
l_3	0.127008	0.144407	0.090505	0.0862206
l_4	0.0965736	0.030229	0.128719	0.0944345
l_5	0.1401883	0.1009896	0.148002	0.141718

From the comparison graphs, it can be observed that the values produced by the proposed distance measure (3.7) are generally slightly higher than those obtained using the classical distance measure[53]. This behavior is expected and can be explained by the structural enrichment of the new distance. Unlike classical distances, which mainly reflect differences in the central tendencies of dual hesitant fuzzy elements, the proposed distance incorporates additional informative components, including the mean function ϕ, the standard deviation b, the relative coefficient of variation c, and the hesitancy degree η. As a result, the new distance responds not only to differences in membership and non-membership degrees, but also to discrepancies in dispersion, stability, and hesitation between dual hesitant fuzzy sets. The graphical results indicate that when two alternatives exhibit similar average assessments but differ in their internal variability or degree of hesitation, the proposed distance becomes more sensitive to these differences. This enhanced sensitivity leads to slightly larger distance values in comparison with the classical measure, thereby providing a more informative and realistic evaluation of dissimilarity. Importantly, the overall ranking patterns in the graphs remain consistent with those obtained by the classical distance, demonstrating that the proposed measure preserves compatibility with existing distance formulations. At the same time, it offers improved discriminatory power in situations characterized by higher uncertainty or greater diversity in expert opinions. Therefore, the proposed distance measure achieves a balance between consistency and sensitivity, extending classical approaches by capturing richer information without causing abrupt changes in decision outcomes. This makes it particularly suitable for complex decision-making problems, such as medical diagnosis, where uncertainty and hesitation play a crucial role.

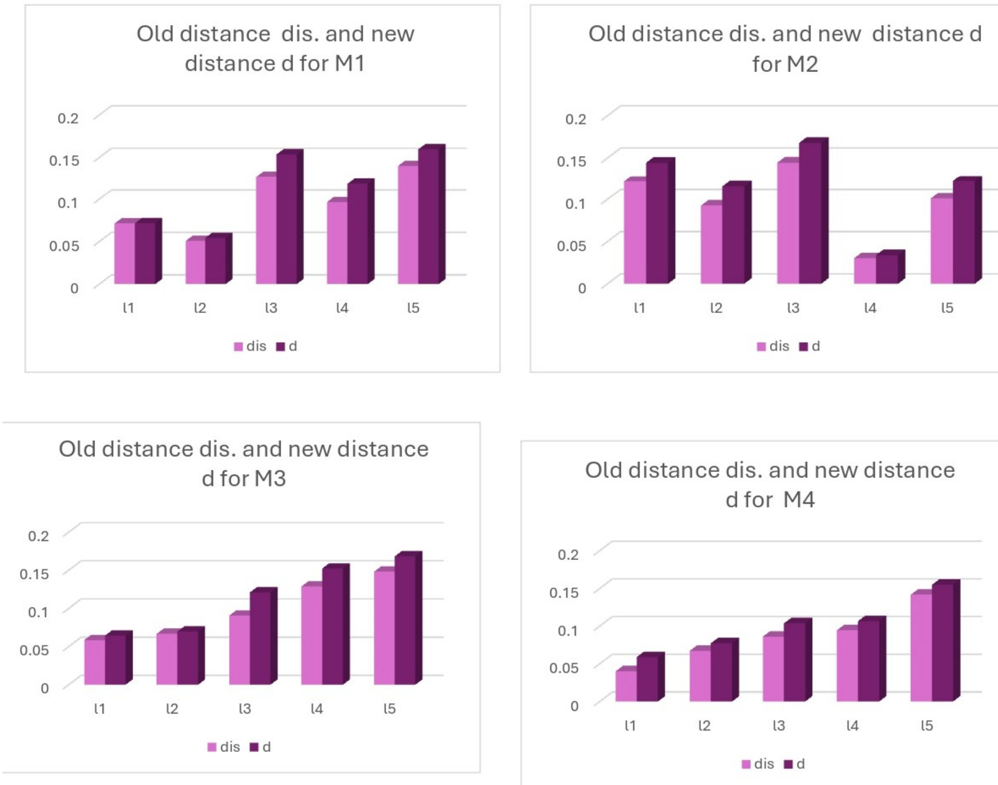

Figure 2. A comparison between old and new distances for patients

Now we apply the score in (2.4) to diagnose every patient:

Table 19. Score function of disease data for symptoms

	t_1	t_2	t_3	t_4	t_5
l_1	0.333	0.225	0.033	-0.100	-0.100
l_2	0.750	-0.075	-0.450	0.090	-0.250
l_3	0.083	0.700	0.000	-0.150	-0.075
l_4	-0.033	-0.100	-0.150	0.650	-0.100
l_5	-0.450	-0.125	-0.183	0.310	0.550

Table 20. Score function of patient data for symptoms

	t_1	t_2	t_3	t_4	t_5
M_1	0.650	-0.200	-0.016	0.110	-0.150
M_2	0.033	-0.025	-0.450	0.590	-0.050
M_3	0.683	0.225	-0.150	-0.030	0.000
M_4	0.517	0.075	0.000	-0.030	-0.100

The following table shows the distance between patients and disease according to the score functions of tables (11,12) where $D(l_i, M_j) = \sum_{k=1}^{5} |\Upsilon(l_i, t_k) - \Upsilon(t_k, M_j)|$ where $i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4, k = 1, 2, 3, 4, 5$

Table 21. Score function of patient data for symptoms

	l_1	l_2	l_3	l_4	l_5
M_1	1.0517	0.7783	1.9683	1.5067	2.2417
M_2	1.8400	1.5333	2.1567	0.4850	1.6633
M_3	0.7033	1.0367	1.4200	1.8217	2.4067
M_4	0.4367	1.1033	1.3533	1.5550	2.3400

Table 22. The diagnose by the score function

	diagnose
M_1	Typhoid l_2
M_2	Stomach problem l_4
M_3	Malaria l_1
M_4	Malaria l_1

The results in the above table gives the same result obtained by the distance measure in diagnose, we conclude that the score function is easier in calculations and efficient also in diagnose as well as distance measure We note that the score-function--based solution provides a simplified and efficient decision tool suitable for quick assessments, while the distance-function--based solution offers a richer and more comprehensive evaluation by preserving the intrinsic properties of dual hesitant fuzzy information. The consistency between the two approaches confirms the validity of the proposed distance measure, while the additional discriminatory power of the distance--based method demonstrates its advantage in complex and uncertainty-intensive decision-making environments.

4. Conclusion

From the results presented, we conclude that the proposed distance measure, indicated by δ, produces results that are larger than distance in[53], which confirms that the new distance measure clarifies the

data variance. This reflects the importance of the relative coefficient of variation function in calculating the distance function. We find that the new and old distances sometimes are equal, or the new function is larger, which means that the new function reveals data that are more dispersed as well as homogeneous data. Moreover, the auxiliary functions introduced provide valuable interpretive tools: the upper and lower bound functions capture the range of values of W.D.F.S.H., the mean function reflects the average opinion of decision makers, the standard deviation quantifies the degree of dispersion among their opinions, the relative coefficient of variation characterizes the relative variability by relating the mean to the standard deviation, and the hesitancy degree function represents the extent of indecision in the decision-making process. also when compared the solution by score function and distance function, we found that the score-function--based solution provides a simplified and efficient decision tool suitable for quick assessments, while the distance-function--based solution offers a richer and more comprehensive evaluation by preserving the intrinsic properties of dual hesitant fuzzy information. The consistency between the two approaches confirms the validity of the proposed distance measure, while the additional discriminatory power of the distance--based method demonstrates its advantage in complex and uncertainty-intensive decision-making environments
We guide the reader to interesting points in [4,5,6,9,22,25].

Declarations

Data availability:Data sets used and/or analyzed during the current study are available from the corresponding author on reasonable request.
Conflict of Interest: The authors declare that they have no known competing financial interests or personal relationships that could appear to influence the work reported in this article.
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