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Optimizing Healthcare Service Capacity: A Cost-Benefit Queueing Model for Reducing Patient Waiting Time

Sami Kadhim Althabhawi*

Department of Computer Science, University of Hilla, Babylon, Hillah, Iraq

Abstract :: Patient waiting times in healthcare services are a problem plaguing most cities worldwide, particularly in developing and least developed countries. Queueing theory plays a crucial role in improving these services and reducing waiting times in other sectors. Previous studies have suggested increasing the number of centers as a solution to reduce waiting times, a solution that can be completely overcome by simply increasing the number of centers. However, this approach may lead to an unnecessary increase in costs. This research paper presents a practical case study of a widespread problem in government hospitals with limited resources and inefficient resource allocation. It offers verifiable recommendations that provide decision-makers with significant flexibility through practical alternatives. This study was conducted in a government medical clinic in a suburb of Najaf, which was classified as having a Queueing system of type I/Type II (M/M / S) based on its recorded data. The chi-square test was used to analyze the data and determine the type of Queueing system, while the POM-OM software for Windows version 3.1 was used to calculate and evaluate the performance of the system. The clinic is part of a government hospital; therefore, consultations and treatment are free. The waiting time is the main factor in determining the optimal number of centers. A three-part diagram was used to illustrate the relationships between waiting costs, service costs, and total costs. The proposed solution reduced system waiting times by 85% and total system cost by 54%. This study provides healthcare managers with valuable information, including waiting system schedules, cost charts, peak times, and system costs per center. These findings facilitate effective planning and optimal, sustainable decision-making.

Keywords Optimization, Queueing Lines, Waiting cost, Quantitative methods, Measuring Performance Quality, Healthcare

AMS 2010 subject classifications 60K25, 90B22

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1. Introduction

The problem of Queueing lines is pertinent to many aspects of daily life. Queues form in most places where services are provided to customers in various sectors, for example, at airline counters, during student registration for limited courses, in completing transactions that require successive procedures or visits to multiple centers, in processing customer orders at banks [1,2,3], and in serving customers at fast-food restaurants to meet demand effectively. It is essential to determine the optimal service capacity. In all these aspects, waiting time represents a high cost, in addition to the cost of the service itself. These costs must be taken into account when evaluating the system's efficiency. Queueing theory offers many flexible mathematical models applicable to a wide range of contexts. Given their direct connection to health and the continuation of life, waiting lines of patients are among the most important applications of Queueing theory. According to studies in this field have varied in terms of direction, methodologies, and health characteristics in government institutions in many cities around the world.

*Correspondence to: Sami Kadhim Althabhawi (Email: sami_kadhim@hilla-unc.edu.iq). Department of Computer Science, University of Hilla, Babylon, Hillah, Iraq.

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1.1. Literature review

Studies that applied Queueing theory varied in the way they presented and analyzed results. A study conducted in Bangladesh developed two models of Queueing to manage and optimize services. The results were presented in a diagram showing the relationship between the number of service centers and the cost of waiting. The solution was chosen directly based on the lowest cost as compared to the planned model. The study used a trade-off (balance) model, considering continuity in the total number of customers, ranging from 0% to 100% at peak times [3, 4].
Numerous studies have employed software methods to analyze the system metrics used in the study setting, such as those conducted in India, Tehran, and other cities [4, 5]. The ARENA software was used to calculate Queueing system performance indicators, and the data were then simulated [5]. The results showed a reduction in waiting time and queue length. Additionally, the SPSS statistical software was applied to analyze the results [6]. The POM-QM software was used to determine system performance metrics and display probabilistic plots to determine system occupancy rates in the event of an increase in patient numbers [7]. In another study, QTS-Plus software was used to calculate metrics and analyze results [4]. The results indicated a reduction in waiting time and waiting list length. TORA software was also used to extract performance metrics and compare data collected in two Pakistani hospitals to determine the optimal number of service centers and staff needed to achieve the shortest waiting time, and to calculate the system [7]. Another study [8] implemented the $M/G/1$ model, in which the waiting system comprised two servers: the Human Resources department and the Nursing department. Matrix engineering analysis methods were used to assess the system's stability. A theoretical study was conducted at an Indonesian hospital using the Inventive Problem Solving (TRIZ) method. Two scenarios were evaluated according to criteria such as waiting time, number of patients served, and service completion time. The study also applied the Six Sigma model [9].
Researchers claim that questionnaires and surveys are valuable for analyzing waiting queues, providing insight into the problem from the service seeker's perspective. The results that follow must be studied and analyzed because they contribute to solving the waiting problem. The following results should be studied and analyzed as they contribute to solving the waiting problem. For example, a study conducted in Nigeria [10] at 12 hospitals collected data through observation, using patient arrival times, service delivery times, waiting times, and service duration. The study concluded that creating multiple service channels would reduce waiting times. Some studies that addressed the problem of waiting queues assumed that the distribution of arrival times was Poisson and that the distribution of service times was exponential, without using a test [11, 12]. Other researchers considered testing the distribution of arrival and service times necessary to understand the waiting line model. These times were tested using the chi-square test to test the distribution type [13]. While a research paper was presented on the method of testing by chi-square, the type of data for which this test can be used, and the computational steps required for the test [14].
Researchers [15], presented a study that compared different available Queueing models, compared the performance measures of each, and then simulated the selection of the optimal Queueing model that improves system efficiency [16]. A research paper [17] used two-stage probabilistic programming, which enables the formulation of iterative models for each queue. The paper also reviewed different types of healthcare facilities that aim to reduce expenses, improve the quality of healthcare, and balance the number of patients with available resources. Applications of Queueing theory are not limited to improving medical services. It is also used to organize air travel and reduce flight delays and ticket bookings. Furthermore, in various aspects of industrial production, such as book manufacturing companies that analyze production processes by analyzing queues and service times in each queue, this theory improves work efficiency by reducing bottlenecks [18, 19, 20].
Many studies have relied on financial cost as a criterion for determining the number of service points. The study [21]used an effective model for scheduling patient bed allocation and determining staffing levels in the $M/M/S/N$ model, where N is the maximum capacity of the system. Where N is the maximum capacity of the system.

1.2. Significance of the Study

There is a pressing need for more applied research based on field data regarding the use of quantitative methods and waiting list management in health centers and hospitals with limited and under-resourced resources. This is a problem that plagues healthcare systems in general, and in Iraq in particular. The significance of this research paper lies in its practical and empirical approach, which relies on total cost analysis, considering the waiting cost as the primary factor- a less common methodology. The paper offers practical, multifaceted recommendations for addressing this problem.

1.3. Research problem

Patients at Al-Sajjad Public Hospital in a district of Najaf Governorate, which has limited service capacity, face long wait times and queues at the internal medicine clinic. This study applies queuing theory to reduce patient waiting times, which significantly affect patients before they see a doctor. It aims to determine the optimal number of service centers to operate at the lowest cost by analyzing the relationship between service and queuing costs and evaluating the performance measures resulting from the proposed model. The goal is to reduce waiting times and lower the overall system cost to the lowest possible level. The study also provides recommendations and suggestions to help decision-makers improve the services provided .

1.4. Research methodology

Data were collected by recording patient arrival times and waiting list lengths daily over two months (2024-2025). Excluding part-time workdays and public holidays, due to the numerous public and religious holidays in Iraq, data were collected for 27 full working days. The data were categorized into 11 daily time periods, and the arrival times of 2,063 patients, recorded using a stopwatch, were compared with clinic records in tables used to test the distribution of arrival and service times. The chi-square test, shown in the test tables, indicated that the arrival time distribution followed a Poisson distribution, while the service time distribution followed an exponential distribution. Based on these results, the most suitable waiting list model for the clinic was determined. The cost-to-service ratio was used to determine the number of service centers.

2. Characteristics of the Queueing lines system

Any waiting line system can be described through five characteristics that highlight the components of the system [22, 23, 24, 25, 26].

The first classification, based on the basic structural features, has attracted many scholars, classifying them into five types.

The second classification, based on the Input population of Queueing Lines, introduces two models: the Finite Q.L-model and the infinite Q.L-model.

The third classification is based on the priority of service delivery, referring to the method used to indicate the unit to which the service is provided. Four methods can be identified: (FCFS) First-come, First-served, (LIFO); Last-in, First-out, (PS); Priority system, (RS), and Random system [22,27]

The fourth classification is based on the probability distributions of arrival time and service time (Probability Distribution). The arrival time and the time for performing the service are considered random values subject to one of the following types of probability distributions: Poisson distribution, It is denoted by the constant (M); Erlang of order (k), symbolized by (Ek), and general or arbitrary is symbolized by (G). The arrival of units often follows the Poisson distribution in its general form,

$$p_{(x)} = \frac{(\lambda)^x}{x!} e^{-\lambda}; x = 0, 1, 2, 3 \tag{1}$$

where $p\,(x)$ is the probability that x visitors arrive at the system per unit of time, and (λ) is the arrival rate, which means the number of units arriving per unit of time.

Similarly, the service time is either a constant time or follows a probability distribution, which is formulated as:

$$p_n\,(t) = \mu e^{-\mu t} \tag{2}$$

3. The main models of Queueing lines according to Kendall's notations

The British mathematician D. J. Kendall used three letters to represent the nature of a system according to the aforementioned criteria, and A. M. Law later added two more. Thus, five letters are now used to represent any waiting system: (a/b/c):(/d/e/f). A sixth symbol (f) is used to represent the type of community (the source of the connecting units), which is either ∞ or n [23, 24, 26].
(a). symbolizes the type of distribution followed by the arrival time.
(b). symbolizes the type of distribution that represents the service time.
(c). symbolizes the number of service centers. If it is a single center, it is placed at number 1.
(d). symbolizes the method of selecting the unit that is served first.
(e). symbolizes the capacity of the system. The number that the system allows to be entered is set as n or ∞.
(f). symbolizes the type of population (the source of the arrived units), which is either ∞ or n.

According to Kendall's notations, the Queueing models are classified into five types. The present study focused on two models, given their properties, conditions, and mathematical formulations. The first model represents the system under analysis, whereas the second constitutes the proposed model for optimization.

3.1. The first model

$M/M/1/FCFS/\infty/\infty$ (abbreviated as M/M/1). It is a Queueing system in which the probability distribution of arrival times is subject to the Poisson distribution, and the distribution of service times is exponential [24, 26]

3.1.1. Conditions of the first model: The Queueing Line may be regarded as the first type if the following conditions are met [21, 23].

i. The Poisson probability distribution describes the arrival, and the source population is infinite.
ii. The method of providing the service is FCFS.
iii. The service time is different for each and is independent of the one preceding it.
iv. The service rate μ is faster than the arrival rate λ, or $(\frac{\lambda}{\mu} < 1)$
v. $\frac{\lambda}{\mu} = \rho$ is called the utilization factor for a system, it means the percentage of time the system is busy. If $\frac{\lambda}{\mu} \geq 1$, in this case, the queue grows continuously and the system becomes unstable.
vi. The capacity of the system and the source population are unlimited.
vii. The following two features do not exist in the system: bypassing the queue or leaving the queue after joining [10, 27].

3.1.2. Mathematical equations of the first model: If the model's assumptions are verified, the balance equations that govern the model are:

i. Utility factor or utilization factor for $\mu > \lambda$,
ii. The probability that there is no unity in the system

$$P_0 = 1 - \frac{\lambda}{\mu} \tag{3}$$

iii. The probability that there are n units in the system at any time

$$P_n = p_0(\lambda/\mu)^n \tag{4}$$

iv. Average number of units in the system

$$L_S = \frac{\lambda}{\mu - \lambda} \tag{5}$$

v. Average time spent by units in the system

$$W_S = \frac{1}{\mu - \lambda} \tag{6}$$

vi. The average number of units in the waiting

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \tag{7}$$

vii. Average time a unit spends in line

$$w_q = \lambda/\mu(\mu - \lambda) \tag{8}$$

viii. The probability that the number of units in the system (n) is greater than a certain [6, 23]

$$P_{(n>k)} = \left(\frac{\lambda}{\mu}\right)^{k+1} \tag{9}$$

3.2. *The second model* $M/M/S/FCFS/\infty/\infty$

This model is denoted by Kendall's notations $(M/M/S)$. It follows the same assumptions (conditions) as in the first model, but the model contains parallel service centers. Utilization factors and stability conditions are among the influential factors that are expressed as follows $S\mu > \lambda$ or $\lambda/S\mu < 1$ where (S) is the number of service centers.

The model's performance metrics and mathematical relationships have different forms in terms of mathematical expression, depending on the research papers and references [18, 19, 23].

The appropriate form was adopted, and it was verified that the main formula for the metrics (P_0) has a value in the program (POM-QM) equal to the value of the mathematical formula manually, as well as the value of L_q. The same applies to the formulas for the other measures.

i. The probability that there is no unit

$$P_0 = \left[\left\{\sum_{n=0}^{S-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right\} + \frac{1}{S!}\left(\frac{\lambda}{\mu}\right)^S \left(\frac{S\mu}{(S\mu - \lambda)}\right)\right]^{-1} \tag{10}$$

ii. The average number of units in the waiting

$$L_q = \frac{\frac{\lambda}{S\mu}\left(\frac{\lambda}{\mu}\right)^S}{S!\left(1 - \frac{\lambda}{S\mu}\right)^2} p_0 \tag{11}$$

iii. The probability that there are (n) units in the system at any time.

$$P_n = \begin{cases} \frac{(n\rho)^n}{n!} p_0, & 1 \le n \le S \\ \frac{S^S \rho^n}{S!} p_0, & n \ge S \end{cases} \quad where\ \rho = \frac{\lambda}{S\mu} \tag{12}$$

iv. Average number of units in the system

$$L_S = L_q + \frac{\lambda}{\mu} \tag{13}$$

v. Average time spent by units in the system

$$W_S = \frac{\rho p_s}{\lambda\,(1 - \rho)^2} + \frac{S\rho}{\lambda} \tag{14}$$

vi. Average time a unit spends in line [23, 27]

$$w_q = \frac{L_S}{\lambda} \tag{15}$$

For the rest of the models, we will only mention their general formula because they are not needed in this research.

3.3. *The third model*

$(M/M/1/GD/k/\infty)$.

3.4. *The fourth model*

$(M/D/1/GD/\infty/\infty)$: or $(M/D/1)$.

3.5. *The fifth model*

$(M/G/1/GD/\infty/\infty)$: it is abbreviated as $(M/G/1)$.

4. Characteristics of the waiting line model in the medical clinic under study

For the purpose of determining the type of waiting queue pattern that the clinic in question is subject to, The cases that arrived at the clinic were monitored and distributed over approximately two months at a frequency, excluding holidays and part-time work, as these were outside the scope of the study.

According to the clinic records, the patients were classified according to the time of arrival into 11 periods of 30 minutes, ranging from 8:30 AM to 2 PM, and placed in a frequency table for the purpose of entering them into the POM program for quantitative analysis.

4.1. *Arrival times*

4.1.1. Finding the rate of arrival (λ): The POM-QM program windows were used to calculate the average number of patient arrivals distributed over the specified periods. Thus, we used the statistical results window to input clinic data and populate the frequency tables. As shown in Table 1, the calculated mean is 6.764 ≈7 patients per period.

Table 1. Numbers of patients per period and frequencies.

x_i	F_i	% Percentage	$F_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$(x_i - \bar{x})^2 F_i$
1	3	0.010344828	3	-5.7639	33.2225	99.6676
2	8	0.027586207	16	-4.7639	22.6947	181.5579
3	25	0.086206897	75	-3.7639	14.1669	354.1736
4	37	0.127586207	148	-2.7639	7.6391	282.6483
5	37	0.127586207	185	-1.7639	3.1113	115.1197
6	43	0.148275862	258	-0.7639	0.5835	25.0924
7	41	0.141379310	287	0.2361	0.0557	2.2855
8	39	0.134482759	312	1.2361	1.5279	59.5898
9	23	0.079310345	207	2.2361	5.0001	115.0033
10	14	0.048275862	140	3.2361	10.4723	146.6128
11	13	0.044827586	143	4.2361	17.9445	233.2791
12	8	0.027586207	96	5.2361	27.4167	219.3339
13	7	0.024137931	91	6.2361	38.8889	272.2226
14	4	0.013793103	56	7.2361	52.3611	209.4446
15	2	0.006896552	30	8.2361	67.8333	135.6667
16	1	0.003448276	16	9.2361	85.3055	85.3055
Total	305	1.051724138	2063		388.2247	2537.0033
Mean		**6.763934426**				

Based on the frequency column in Table 1, near-average frequencies ≅7 are generally high. Frequencies involving 14, 15, or 16 patients, or even just one patient, are rare. This can be determined by calculating the standard deviation of the number of patients during the periods shown in Table 1, using the following equation.

$$standarddeviation(S.D) = \sqrt{\frac{(x_i - \bar{x})^2 F_i}{n}} \tag{16}$$

$S.D(\sigma) = \sqrt{\frac{2537.0033}{305}} = 2.884$. The mean number of patients in each period falls within the following range $\bar{x} \pm \sigma = 7 \mp 2.88, 4 \leq \bar{x} \leq 10$. The study focuses on critical periods, specifically those with peak patient numbers. Initial clinic data indicate that the peak occurs between 9:30 and 10:00 AM, while later hours have lower-than-average patient numbers, which is not relevant to the research problem.
Assuming a 20% increase in the number of patients, the calculations were repeated in the same Table 1, where the total number of patients was 2477, the total number of frequencies was 366, and the new mean was 6.768. This falls within the calculated mean and does not affect the optimal solution.

4.1.2. Testing the type of probability distribution of arrival time : To identify the appropriate model for the clinic under study, it is essential to determine and understand the distribution governing both arrival and service times. For the purpose of determining the pattern of arrival periods and service rate distribution.
Hypothesis: The arrival times distribution was assumed to follow a Poisson probability distribution (H_0), This was verified by applying the chi-square (χ^2) test [12, 13] by using Table 2, and the probability function (Poisson's law) Eq. (1) where $P_{(x)}$ is the probability function of the frequency of the number in the period, and λ is the rate of arrival per unit time, and x_i the number of patients in period i.

The chi-square was calculated using a table containing the following columns: $x_i F_I$: the frequency in the period, $(x_i)!$: the factorial of the number E_i: Expected frequencies per number of patients.

$$\text{Expected frequency } E_i = P_{(x)} * \sum F_i \tag{17}$$

and the last column is

$$chi - square\ (\chi^2) = \frac{Actual\ frequency}{Expected\ frequency} = \frac{(F_i - E_i)^2}{E_i} \tag{18}$$

Table 2. Calculating Chi-square for arrival times.

Patients (x_i)	Frequency (F_i)	$F_i x_i$	$x_i!$	$p(x)$	$E_i = p(x)\sum F_i$	$F_i - E_i$	$(F_i - E_i)^2$	$\frac{(F_i - E_i)^2}{E_i}$
1	3	3	1	0.0078	2.38197	0.6180	0.3820	0.16035
2	8	16	2	0.0264	8.05583	-0.0558	0.0031	0.00039
3	25	75	6	0.0596	18.16320	6.8368	46.7418	2.57343
4	37	148	24	0.1007	30.71397	6.2860	39.5141	1.28652
5	37	185	120	0.1362	41.54986	-4.5499	20.7013	0.49823
6	43	258	720	0.1536	46.84055	-3.8405	14.7498	0.31489
7	41	287	5040	0.1484	45.26135	-4.2614	18.1591	0.40121
8	39	312	40320	0.1255	38.26847	0.7315	0.5351	0.01398
9	23	207	362880	0.0943	28.76088	-5.7609	33.1878	1.15392
10	14	140	3628800	0.0638	19.45386	-5.4539	29.7446	1.52898
11	13	143	39916800	0.0392	11.96236	1.0376	1.0767	0.09001
12	8	96	479001600	0.0221	6.74278	1.2572	1.5806	0.23441
13	7	91	6227020800	0.0115	3.50832	3.4917	12.1918	3.47512
14	4	56	87178291200	0.0056	1.69502	2.3050	5.3129	3.13444
15	2	30	1307674368000	0.0025	0.76434	1.2357	1.5269	1.99761
16	1	16	20922789888000	0.0011	0.32313	0.6769	0.4582	1.41790
Sum	**305**	**2063**						**18.2813878**
Mean	**6.76393**							

Patients' arrival rate in one period $= \frac{2063}{305} = 6.764$ patients per period.

Patients' arrival rate per unit of time $\lambda = 0.2254$ patients per minute.

To verify the results from the program and the direct substitution in the mathematical formulas, we find the results, for example, row 8,

$x_i\ = 8, f_i\ = 39$, Eq. (1) $\rightarrow P_{(8)} = \frac{(2.764)^{-6.8991} * (6.764)^8}{8\,!} = 0.1255$

Also expected frequency by Eq. (17) $E_i = p\,(x) * \sum F_i = 0.1284 * 299 = 38.2684$. Actual frequency $(F_i - E_i)^2 = (39 - 38.2684)^2 = (0.7315)^2 = 0,5351$.

From Eq. (18) $Chi - Square\ (\chi^2) = (F_i - E_i)^2 / E_i = \frac{Actual\ frequency}{Expected\ frequency} = \frac{0.5351}{38.2684} = 0.01398$.

Table 2 shows findings χ^2 value is 18.28139.

To find (χ^2) tabular chi-square from the approved tables, we calculated the degree of freedom or factor V according to the following rule. [18]

$$V = M - C - 1 \tag{19}$$

where M is the number of periods or categories, C is the number of parameters, (λ) is a parameter, and the level of significance (negligible error) is 0.05.

$$V = M - C - 1 = 16 - 1 - 1 = 14.$$

From the general table for$\chi\ ^2$.It was found that

$$\chi^2{}_{(0.05)} = \chi^2\,(\mathcal{V}; 0.05) = \chi\ ^2\,(14; 0.05) = 23.685.$$

Since this tabulated $\chi\ ^2$ value is greater than the calculated value in Table 2, which is 18.28139, we accepted the hypothesis (H_0) [14, 18], which means the distribution of arrival times in the clinic follows a Poisson distribution. Therefore, it is assigned Kendall's symbol (M).

Regarding the increase imposed in The previous paragraph, the chi-square test for frequencies was recalculated with the increase mentioned in Table 2, resulting in (19.227), which is also less than the tabulated value and confirms the hypothesis regarding the type of distribution.

4.2. Mean service time

4.2.1. Finding the Mean Number of Patients Served in the Clinic in Unit Time (μ): To find the average service time in the clinic, times are divided into one-minute intervals, and frequencies are counted. We obtained it using the POM-QM Windows program.Table3 was obtained, showing a mean of 4.159.

Table 3. Table 3. Service time data with frequencies .

service time	**midpoint** x_i	F_i	$x_i * F_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$(x_i - \bar{x})^2 * F_i$
0.5–<1.5	1	98	98	-3.1604	9.988128	978.83656
1.5–<2.5	2	87	174	-2.1604	4.667328	406.05755
2.5–<3.5	3	71	213	-1.1604	1.346528	95.6034994
3.5–<4.5	4	58	232	-0.1604	0.025728	1.49223328
4.5–<5.5	5	44	220	0.8396	0.704928	31.016839
5.5–<6.5	6	36	216	1.8396	3.384128	121.828614
6.5–<7.5	7	28	196	2.8396	8.063328	225.773188
7.5–<8.5	8	24	192	3.8396	14.74253	353.820676
8.5–<9.5	9	18	162	4.8396	23.42173	421.591107
9.5–<10.5	10	14	140	5.8396	34.10093	477.412994
10.5–<11.5	11	6	66	6.8396	46.78013	280.680769
11.5–<12.5	12	5	60	7.8396	61.45933	307.296641
12.5–<13.5	13	4	52	8.8396	78.13853	312.554113
13.5–<14.5	14	3	42	9.8396	96.81773	290.453184
Sum		496	2063		383.64	4304.41797
Mean	4.1593					

It is clear from Table 3 that the mean of service time (α) for one patient is equal to α =2063/496= 4.15927 minutes. Therefore, the average number of patients served per minute is μ = 1/α = 1/4.15927 = 0.2404. The frequency distribution shows that a large proportion, greater than 63%, of patients are examined in less time than the average (4.159).
The standard deviation for service times was calculated to be 0.8794, which is relatively small. This indicates that the physician provides services at a consistent pace and that service times are regular. This results in regular waiting times, making the system more predictable. Simulations can be performed to estimate the number of patients waiting in the clinic and the probability of such a scenario occurring. The resulting system performance metrics in the POM-QM application are more realistic. Service times remain close to the arithmetic mean of 4 minutes, falling within the following range: $3 \leq \alpha \leq 5$. Service times rarely exceed 10 minutes because the clinic is primarily for examinations and does not perform lengthy procedures. If such delays occur, they do not affect the clinic's normal operations.

4.2.2. Testing the type of probability distribution of service times The null hypothesis (H_0) assumes that the service times follow a negative exponential distribution, whereas the alternative hypothesis (H_1) assumes that they do not.The hypotheses were tested using the Chi-square $\chi^{\ 2}$ goodness-of-fit test based on the probability function $P_{(x)}$ as given in Eq. (2), [12, 14].

Table 4. Calculating Chi-square for service times.

Period time	**mid point** x_i	F_i	$F_i x_i$	$p(x_i)$	$E_i = p(x_i)\sum F_i$	$F_i - E_i$	$(F_i - E_i)^2$	$\chi^2 = \frac{(F_i - E_i)^2}{E_i}$
0.5–<1.5	1	98	98	0.18903	93.75873655	4.24126	17.988	0.191857488
1.5–<2.5	2	87	174	0.148637	73.723739	13.2763	176.26	2.390805304
2.5–<3.5	3	71	213	0.116875	57.9699545	13.03	169.78	2.928794533
3.5–<4.5	4	58	232	0.0919	45.58254465	12.4175	154.19	3.382724651
4.5–<5.5	5	44	220	0.072262	35.84215987	8.15784	66.55	1.856761867
5.5–<6.5	6	36	216	0.056821	28.18316603	7.81683	61.103	2.168063491
6.5–<7.5	7	28	196	0.044679	22.16079752	5.8392	34.096	1.538585675
7.5–<8.5	8	24	192	0.035132	17.42532923	6.57467	43.226	2.480658771
8.5–<9.5	9	18	162	0.027625	13.70176765	4.29823	18.475	1.34835167
9.5–<10.5	10	14	140	0.021722	10.77388177	3.22612	10.408	0.966024973
10.5–<11.5	11	6	66	0.01708	8.471646234	-2.4716	6.109	0.721115464
11.5–<12.5	12	5	60	0.01343	6.661367877	-1.6614	2.7601	0.414350817
12.5–<13.5	13	4	52	0.01056	5.237921977	-1.2379	1.5325	0.29256847
13.5–<14.5	14	3	42	0.008304	4.118647573	-1.1186	1.2514	0.303830898
Sum		496	2063					20.98449407

Based on Table 4, the chi-square sum of the observation values $\sum \chi^2$ was found to be 20.984494.
The tabulated value was extracted from the general table for the chi-square test [18] as follows.
Recalling Eq.(19), where $M = 14$ is the number of periods used in the table, and $C = 1$ (Number of parameters), then $V = M - C - 1 = 14 - 1 - 1 = 12$ and the significance level $a = 0.05$. The tabulated value is $\chi^2(v; a) = (12; .05) = 21.026$, and compared to the value calculated from Table 4 (for observation), which is 20.9845, it is less than the tabular value χ^2. Therefore, we accept the previous hypothesis H_0, which is that the distribution of service times follows the negative exponential distribution according to the Kendall symbols D.G., symbolized by M. [7, 14, 23]

4.3. Choosing the model that matches the clinic model

After finding the type of arrival ti mes distribution, λ, according to Kendall symbols, we symbolized it as M, as did the service rate μ. It followed an exponential distribution and was denoted by M.The number of service centers in the clinic under study was 1. The general method for performing the service is FCFS; the clinic receives an indefinite number of patients, and the source population of the clinic is infinite.

Thus, all the necessary elements for classifying the waiting system in the clinic under study have been identified.[12], representing the first model according to Kendall's Notations symbolized by $M/M/1/FCFS/\infty/\infty$ and abbreviated as $(M/M/1)$.
The present clinic model was applied in the next sections. $M/M/S$ was applied in the proposed model because it has $S = 2$, two service centers. The system's performance measures were then analyzed. For the other models, because they are not related to the clinic model, only their codes were mentioned.

5. Calculation of the cost of operating the clinic

Four employees: a doctor, two health workers, and a patient receptionist. Average salaries are shown in Table5.

Table 5. Average monthly salaries for clinic employees (thousand IQD).

No.	**Job title**	**Number**	**Monthly salary**	**Total salary**
1	Specialist doctor	1	1500	1500
2	Nurse	2	600	1200
3	Information Officer	1	450	450
Sum				3150

The monthly salary cost was 3,150,000 IQD. Other costs for operating the clinic have been determined by the hospital administration, as shown in Table 6.

Table 6. Total costs of operating the clinic.

Total cost of clinic/month(thousand) =	Medical Equipment	Furniture of midicen	Monthly salaries of employees	Total cost (thousand)
	58.264	90.36	3150	3298.621
Total cost of clinic/day $=$	109.954 thousand IQD			
Total cost of clinic/day $=$	109954.037 IQD			
Total cost of clinic IQD/minute	305.427 IQD			

On the other hand, in addition to the psychological damage caused by waiting, there is a financial or approximate loss due to waiting time, estimated as a loss of patient productivity due to waiting, taking into account the assumption that approximately half of the patients are accompanied by someone and according to their average daily income:
600 + $600 * 0.50 = 900$ thousand. = 900000/10800 = 83 IQD. Average cost of waiting /minute

5.1. Finding performance measures for the system

The POM-QM Windows operating system software was used to enter the clinic's data. $\lambda = 0.2254$ is the average number of patients arriving at the clinic, as well as the average number of patients served per unit of time $\mu = 0.2404$, the number of centers $S = 1$, the cost of the service $Cs = 305$ IQD per minute, and the waiting cost is 83 IQD per minute.

Waiting Lines Results

Waiting Lines /M/M/1

Parameter	Value	Parameter	Value	Seconds
Arrival rate(lambda)	.2254	Average number in the queue(Lq)	14.0891	
Service rate(mu)	.2404	Average number in the system(Ls)	15.0267	
Number of servers	1	Average time in the queue(Wq)	62.507	3750.417
Server cost $/time	305	Average time in the system(Ws)	66.6667	4000.001
Waiting cost $/time	83	Cost (Labor + # waiting*wait cost)	1474.393	
		Cost (Labor + # in system*wait cost)	1552.214	

Figure 1. Performance measures of the single system.

Figure 1 shows the values of all the measures of the system. From (POM-QM), they are listed below.

i. The system's utilization rate is 94%. Thus, the doctor is occupied 94% of the time, suggesting the staff operate at maximum capacity.

ii. The average number of patients in line.

$$L_q \cong 14 \; patients$$

iii. The average number of patients in the system.

$$L_s \cong 15 \; patients$$

iv. The average waiting time in line.

$$w_q \cong 63 \; minutes$$

It is considered a large number compared to the duration of service. This means that the patient waits ≅63 minutes before seeing the doctor, which is a long time.

In light of the system metrics resulting from using the POM-QM program and analyzing the results.

5.2. *Checking the results of the POM-QM program*

Verifying the values that appeared in the POM-QM program outputs according to the known mathematical formulas for those Performance metrics for a single service center system.

i. Average numbers in the system. According to Eq. (5)

$$L_S = \frac{0.2254}{0.2404 - 0.2254} = 15.0267$$

ii. The average number of units in the waiting. According to Eq. (8)

$$w_q = \frac{1}{\mu} L_S = \frac{L_S}{\mu} = \frac{15.0267}{o.2404} = 62.051$$

iii. The cost of waiting in the queue

$$C_{(wq)} = L_q * C_w \tag{20}$$

iv. The cost of waiting in the system

$$C_{(ws)} = L_S * C_w \tag{21}$$

v. The Total Cost of the queue waiting for one center service [27]

$$T_{Cq} = L_q * C_W + Cs \tag{22}$$

$$T_{Cq} = 14.089 * 83 + 305 \cong 1474 \; IQD/M.$$

vi. The Total Cost of waiting in the system

$$T_{Cq} = L_q * C_W + Cs \tag{23}$$

$$T_{cs} = 15.026 * 83 + 305 \cong 1552 \;\; IQD/M.$$

5.3. Analysis and simulation of the current system model

In the fourth column of Fig. 2A, the probability of the clinic being occupied is 0.9376, meaning that 94% of the time a patient is present. This indicates overcrowding, reflecting a lack of flexibility in physician and staff schedules and an inability to handle emergencies, which can lead to longer waiting times. The occupancy percentage decreases as patient arrivals increase. On the other hand, Fig. 2B. shows that decision-makers can simulate the expected number of patients in the clinic and the percentage of those who increase or decrease it, meaning they know the probability of these changes. This enables them to make well-thought-out operational decisions.

A

k	Prob (num in sys = k)	Prob (num in sys <= k)	Prob (num in sys >k)
0	.0624	.0624	.9376
1	.0585	.1209	.8791
2	.0549	.1758	.8242
3	.0514	.2272	.7728
4	.0482	.2754	.7246
5	.0452	.3206	.6794
6	.0424	.363	.637
7	.0397	.4028	.5972
8	.0373	.44	.56
9	.0349	.475	.525
10	.0328	.5077	.4923
11	.0307	.5384	.4616
12	.0288	.5672	.4328
13	.027	.5942	.4058

B

14	.0253	.6196	.3804
15	.0237	.6433	.3567
16	.0223	.6656	.3344
17	.0209	.6864	.3136
18	.0196	.706	.294
19	.0183	.7243	.2757
20	.0172	.7415	.2585
21	.0161	.7577	.2423
22	.0151	.7728	.2272
23	.0142	.787	.213
24	.0133	.8003	.1997
25	.0125	.8127	.1873
26	.0117	.8244	.1756
27	.011	.8354	.1646
28	.0103	.8456	.1544
29	.0096	.8553	.1447
30	.009	.8643	.1357

Figure 2. Analysis and simulation of the current system model.

Sensitivity analysis was performed using POM-QM software and performance measures were calculated for multiple service center scenarios. Table 7 shows that, for two centers (in shows that decision-makers can simulate the expected number of patients in the clinic and the percentage of those who increase or decrease it, meaning they know the probability of these changes. This enables him to make well-thought-out operational decisions. column (3), the system occupancy or efficiency rate is 0.469, and the number of patients in the system does not exceed two. But for three centers(0.312), approximately 69% of the time, the centers are idle, up to five parallel centers.

Table 7. Comparison of performance metrics for multi-center models.

Number of service centers	**1**	**2**	**3**	**4**	**5**
Average server utilization	0.937	0.469	0.312	0.234	0.188
Average server number in queue (Lq)	14.09	0.264	0.035	0.005	0
Average number in the system (Ls)	15.03	1.202	0.973	0.943	0.938
Average time in the queue (Wq)	62.51	1.172	0.156	0.022	0.003
Average time in the system (Ws)	66.67	5.332	4.316	4.182	4.163

6. Comparison between system measures in the case of multiple service centers

The mathematical formulas for the first and second proposed models, mentioned in sections 3.3 and 4.1, were manually verified and found to be fully consistent with the software output. By comparing all system measures, we concluded that with the increase in the number of centers S, the queue length L_q, L_s and waiting time W_q, W_s continue to decrease. We observe that if there are two centers, there are two or one patient in the queue. But if more than two centers are operational, either there is no patient or one of the service centers is out of order. Therefore, both waiting costs and service costs determine the optimal number of service centers.

7. Finding the optimal number of centers that achieves the lowest total cost

Increasing the number of service centers reduces waiting times and queue lengths. However, this approach may result in some centers becoming unnecessarily out of service. Therefore, this study uses the cost variables of waiting costs and operating costs to determine the optimal number of centers that minimizes waiting times and reduces overall costs.

7.1. Calculating waiting line cost

The waiting cost in subsection 5.2 [16, 23] was calculated according to Eqs. (20) - (22) for one service center

$$S = 1 \rightarrow \; based\ on\ waiting\ \ in\ queue\ cost\ \ T_{cq}\ \cong 1474\ IQ^{D}/M$$
$$\rightarrow \; based\ on\ waiting\ in\ the\ system\ \ T_{cs}\ \cong 1552\ \ IQD/M$$
$$S = 2 \rightarrow from\ Eq\ \ (13)\ \ \ the\ cost\ \ based\ on\ waiting\ cost\ in\ queue$$

With reference to Table 7, column 3:

$$C_{wq} = 0.2641 * 83 = 21.920\ IQD/M$$

Total cost based on waiting cost in queue using Eq.(22):

$$T_{Cq} = 21.920 + 305 * 2 \cong 632\ IQD/M$$

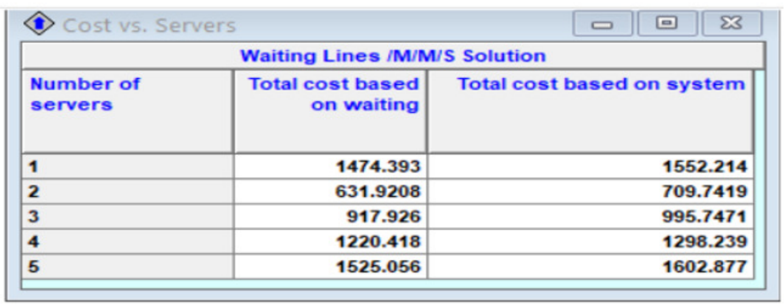
In Eq. (14), the cost is based on waiting cost in the system [23, 25].and

$$C_{ws} = [1.201 * 83] = 99.68\ IQD/M$$

Total cost is based on waiting cost in system by using Eq. (23):

$$T_{Cs} = [99.68] + [305 * 2]$$

$\cong$ 710 IQD/M Also, the total cost is calculated as in Fig. 3 for $S \geq 1$


Cost vs. Servers

Waiting Lines /M/M/S Solution

Number of servers	Total cost based on waiting	Total cost based on system
1	1474.393	1552.214
2	631.9208	709.7419
3	917.926	995.7471
4	1220.418	1298.239
5	1525.056	1602.877

Figure 3. Total cost of waiting and the system with a multi-service center (QID).

These results were based on data, some of which was from clinic records and some of which was subject to change based on the hospital director's assessment. Therefore, to determine the consistency of the results within certain limits, calculations were performed on hypothetical data with a 20% increase in the estimated clinic cost, using the same tables. This resulted in a change in the service cost from 305 to 366 dinars per minute. Similarly, the cost of waiting, with an increased number of companions or a loss of patient productivity allowance, increased by the same percentage, from 83 to 92 dinars per minute. The system's performance indicators were recalculated with this increase, and the total system cost in the proposed solution was 843 dinars per minute instead of the 710 shown in Fig. 3, which remains the lower cost. This increase resulted in a slight increase of one minute in waiting time and one additional patient on the waiting list. It was found that these increases did not significantly affect the distribution pattern or the results.

7.2. Choosing the optimal number of service centers (number of clinics

To analyze the contents of Fig. 3, which illustrates the total cost based on the length of the waiting line in the second column and the system in the third column, the number of service centers was calculated as follows for $S = 1, 2, 3, 4, 5$. We can see that the lowest total cost was 710 IQD per minute, as achieved in the second row $S = 2$. Therefore, the optimal solution is to add only one more center.
The same result can be obtained by representing the three costs: waiting cost, service cost, and total cost in a unified diagram, as in Fig.4, which illustrates the intersection point of the three costs.

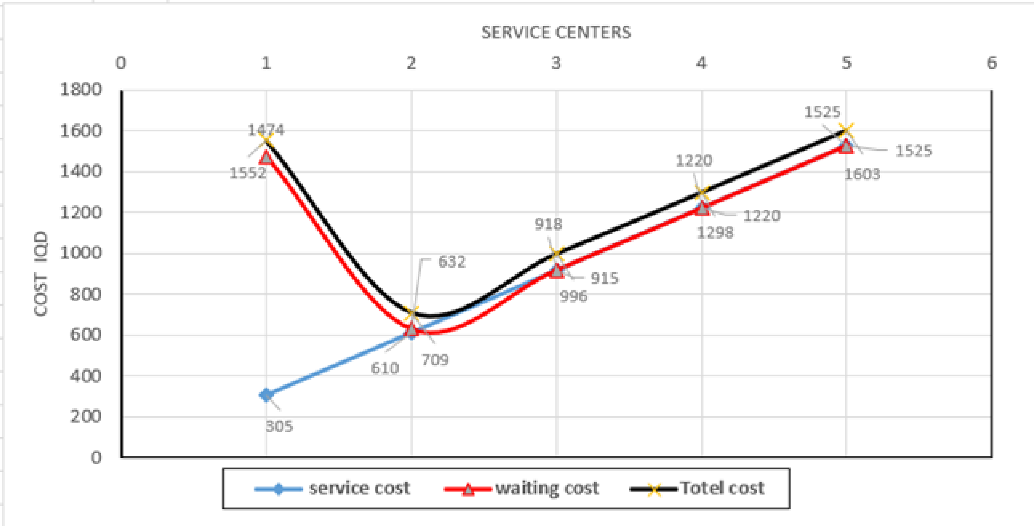


Figure 4. Analysis and simulation of the current system model.

Moreover, it shows that the lowest cost, 710 IQD/minute, occurs at S = 2, which is the optimal solution. We can also find the costs for any number of centers directly for the clinic under study. To determine the optimal number of service centers, the cost of the service, the waiting cost, and the total system cost were combined To illustrate the relationship between them, as shown in Fig.4, Based on the total system cost curve, we conclude that the proposed solution is the optimal number of service centers in the clinic, with the lowest cost of 710 IQD/minute.
These results were based on data, some from clinic records and some subject to increase or decrease based on the hospital director's assessment. Therefore, to determine the consistency of the results within certain limits, calculations were performed with a 20% increase; when added to the data in Fig. 4, it does not affect the optimal solution because the solution point remains the second center point.

8. Conclusions

1. The health problems in government hospitals in many cities are likely due to administrative shortcomings in resource management, rather than a lack of resources. Therefore, despite the availability of resources in Iraqi hospitals, these problems persist.
2. A quantitative analysis program (POM-QM) was implemented using data recorded from observations and clinic records. The program provides organized, concise, and clear results. Tables illustrate system performance metrics, including waiting list length and waiting time.
3. The first waiting line model,$M/M/1/FCFS/\infty/\infty$, was selected for the clinic. Upon verifying its conditions, specifically the distribution of arrival and service times using the chi-square test, it was found that it meets the conditions of both the first- and second-waiting-line models, as well as the FCFO service delivery method.

4. Costs, being an important factor in the analysis and finding the optimal solution, were used to plot the waiting cost, service cost, and total cost in a single chart that clearly shows the relationship and indicates the optimal number of centers, as shown in Fig.4
5. The performance measures for the first model were calculated, as shown in Fig. 1. The clinic utilization (occupancy) rate is high at 95%, and patients experience a long waiting time of approximately 63 minutes before seeing a doctor.
6. The proposed solution, according to Table 7 of the performance metrics for the second model of the proposed Queueing system, reduces the total cost of the system from 1552 IQD/ M to 710 IQD/M and reduces the waiting time in the system by approximately 80%.

9. Recommendations

1. The results of this research, in terms of performance metrics, total cost, waiting cost, and when comparing the total cost with the number of service centers, indicate the need to establish an additional service center. The implementation of the first proposed solution follows this mechanism:

   (a) The final study is submitted to the hospital administration after approval for publication, and they are advised on its implementation.
   (b) After establishing the second service center and providing the necessary medical staff and clinic equipment, both centers are operational. Data is collected for both centers, and performance metrics for the system are calculated.
   (c) The results of the proposed solution, collected over two months, are compared with the previously recorded data for each center.
   (d) The performance indicators of the proposed system are evaluated and monitored based on waiting line length and total system cost using a waiting line management application such as POM-QM or other similar applications.

2. If opening a new center in an increasingly overcrowded clinic proves impossible, we recommend implementing the evening public clinic system (known in Iraq). This system would complement Iraq's daily healthcare services for a nominal fee. It would help alleviate the problem of long waiting times in busy morning clinics. This requires hospital administration to assign doctors and staff to work in these public clinics on a rotating basis.
3. The researcher recommends that decision-makers in government health institutions adopt quantitative methods, with the help of operations research specialists, and use software applications to improve the performance of the health sector.
4. .If opening a new center at a clinic experiencing increasing overcrowding is not feasible, and given the surplus of doctors at some health centers, the clinic's statistical records can be used to identify peak times and days with long waiting times. The researcher recommends redistributing these doctors so that one doctor is assigned to specific days to support overcrowded clinics.
5. Based on the fourth recommendation, the clinic can be restructured to adopt a two-tiered service system. The following mechanism would be employed:

   (a) Specialist physicians would be assigned specific days of the week, and a junior physician would be assigned to each day. A weekly schedule would be issued by the hospital administration.
   (b) The junior physician would examine general and MRI cases that do not require much time. Cases requiring more detailed diagnosis and time-consuming procedures would be referred to the senior physician. This approach reduces waiting times and alleviates the workload on the specialist physician, freeing them from being burdened with simpler cases.

10. Future work

We propose developing an electronic application to manage waiting lists at major specialty clinics in a large government hospital. Patients will be able to book appointments through the application for one of the specialties listed on the application page and view their estimated arrival time. The system consists of parallel waiting lines for each specialty, each with two consecutive service centers. At the first center, any available general practitioner examines the patient and treats general ailments, after which the patient leaves the system. At the second center, if the case requires it, the patient is referred to a specialist and their name is added to the second-stage waiting list. From the list of patients already registered for a specific specialty, a waiting list for that clinic is created. In both stages, based on the service hours at that clinic, the estimated arrival time for each patient is determined.

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