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Power Series Algorithm for Solving Linear Fractional Partial Differential Equations

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Abstract A class of linear fractional partial differential equations with variable coefficients and their applications to tumor dynamics and thermal equations are considered in this study. Analytical solutions are obtained by applying the Aboodh residual power series (ARPS) approach, which proves to be simple and efficient in generating analytical solutions in the form of series, even for equations with non-constant coefficients. Symbolic computations are implemented in Mathematica, and all graphical illustrations are presented in Python, clearly demonstrating the convergence of solutions. This demonstrates that ARPS technique is an efficient and therefore reliable method for fractional partial differential models, offering broad potential applications in mathematical physics and biomedical engineering, including modeling cancer evolution, immune system behavior, and drug delivery systems.

Keywords Fractional cancer tumor models; Aboodh residual power series technique; Fractional Caputo derivative; Numerical series solution

AMS 2010 subject classifications 26A33, 35R11, 65M70

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1. Introduction

Malignant tumor's function behaves as dynamic biological systems in which cancer cells continuously proliferate and invade surrounding tissues, competing with healthy tissues for essential nutrients and oxygen. When these resources become scarce, portions of the tumor cells may undergo necrosis. The complex behavior of tumor in cases of linear and nonlinear make mathematics and modeling an effective tool to deal with the dynamicity of the tumors and their interaction to the suggested mechanical treatment [1]. About the treatment of cancer, if we want it effective, we need to suggest therapy that should be administrated with rate exceeds the malignant spreads and maintain control strict of the the tumor and its growth, see [2].

Research in cancer tumor that are presented by mathematical modeling have an important interest by researchers, because it allows the incorporation of the interactions of the system of immune with the growth of treatment responses and their analysis [3]. The microenvironment of the tumor contains different types of components, they have a basic role in the progression of the disease, mentioning hormones, cytokines, immune cells, chemokines, and fibroblasts [4]. It is crucial to understand all of these components and their interactions to design an efficient immunotherapies, and it presents a basic contributor in modeling tumors [5].

Mathematical modeling are considered to be basic for predicting and analyzing the behavior of tumors responded to the considered data. Fractional derivatives play a vital role in these models, because of their properties alternative

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to the classical integer order derivatives, they can represent the hereditary and the memory effects that appears in multi biological systems [6, 7]. Integer case differential equations fail to explain these factors, whereas fractional differential equations (FDEs) give wider explanations and offer more accurate mathematical results [8]. As a result, FDEs have obtained a great interest of researchers in modeling difficult biological systems [9]. For more details, you can see [10, 11, 12]. Aboodh residual power series method (ARPS) method is considered to be one of the most powerful methods in handling fractional differential equations. This method merges two mathematical techniques, the Aboodh integral transform and the residual power series method [13, 14]. The outcomes show that the ARPS is applicable and easy to deal with many types of fractional differential equations, that have different applications which were studied in many articles [15, 16, 17]. Similar methods and comprehensive research can be see in [18, 19, 20].

One of the advantages of ARPS method is that it depends on taking the limit after some computational steps, which takes shorter time during the calculations in comparison with residual power series method [21, 22, 23], and it needs no linearization or discretization, which increases its simplicity. For further details, the reader may refer to [24, 25, 26].

The remainder of this paper is organized as follows: Section 2 presents the fundamental preliminaries, including the notations, core definitions and key properties associated with the Aboodh transform, the fractional derivatives, and the power series framework required for the proposed approach. The detailed APRS procedure used to derive the analytical series solutions for fractional PDEs with variable coefficients is described in Section 3. Moreover, Section 4 presents two real illustrative examples that highlight the method's properties. Section 5 concludes the paper by demonstrating the main results and outlining possible areas for future work.

2. Basic Preliminaries

This section introduces key concepts related to fractional derivatives, together with the Aboodh transform, the power series framework, and several auxiliary properties needed for the proposed approach.

Definition 2.1
[27] The Aboodh Integral Transform of the function $\phi(y,\beta)$ with respect to β is given by

$$\mathcal{A}[\phi(y,\beta)] = \frac{1}{\mu}\int_0^\infty e^{-\mu\beta}\phi(y,\beta)\,d\beta = \Psi(y,\mu).$$

Theorem 2.1
[27] Suppose $\phi(y,\beta)$ is a function defined on the region $y \geq 0, 0 \leq \beta \leq y$, where y is constant, and is of exponential order. The Aboodh Integral Transform (AIT) with respect to β is given by

$$\mathcal{A}\left[\frac{\partial\phi(y,\beta)}{\partial\beta}\right] = \mu\Psi(y,\mu) - \frac{\phi(y,0)}{\mu}.$$

Definition 2.2
[28] The fractional derivative of $\phi(y,\beta)$ of order α in the Caputo sense with respect to the variable β is defined as follows:

$$D_\beta^\alpha\phi(y,\beta) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^\beta (\beta-s)^{n-\alpha-1}\frac{\partial^n}{\partial s^n}\phi(y,s)\,ds, & n-1<\alpha<n,\ n\in\mathbb{N}, \\ \dfrac{\partial^n\phi(y,\beta)}{\partial\beta^n}, & \alpha = n. \end{cases}$$

Lemma 2.1
[30] Assume that $\phi(y,\beta)$ is of exponential order function, piecewise continuous. Then the AIT is as $\mathcal{A}[\phi(y,\beta)] = \Psi(y,\mu)$. Hence:

$$\mathcal{A}\left[D_\beta^\alpha\phi(y,\beta)\right] = \mu^\alpha\Psi(y,\mu) - \mu^{\alpha-2}\phi(y,0), \quad 0<\alpha<1.$$

To prove the above equation from the definition of AIT,

$$\mathcal{A}\left[D_\beta^\alpha \phi(y,\beta)\right] = \mathcal{A}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^\beta (\beta-s)^{-\alpha}\frac{\partial}{\partial s}\phi(y,s)\,ds\right].$$

By convolution property, for the above equation,

$$\begin{aligned}\mathcal{A}\left[D_\beta^\alpha \phi(y,\beta)\right] &= \mathcal{A}\left[\frac{1}{\Gamma(1-\alpha)}\left(\beta^{-\alpha} * \frac{\partial}{\partial\beta}\phi(y,\beta)\right)\right]\\ &= \frac{1}{\Gamma(1-\alpha)}\mu\mathcal{A}[\beta^{-\alpha}]\mathcal{A}\left[\frac{\partial}{\partial\beta}\phi(y,\beta)\right]\\ &= \mu^{\alpha-1}\left(\mu\Psi(y,\mu) - \frac{\phi(y,0)}{\mu}\right)\\ &= \mu^\alpha\Psi(y,\mu) - \mu^{\alpha-2}\phi(y,0).\end{aligned}$$

Definition 2.3
[29] The power series in the following form:

$$\sum_{r=0}^{\infty} P_r(\beta-\beta_0)^r = P_0 + P_1(\beta-\beta_0) + P_2(\beta-\beta_0)^2 + \cdots$$

It is called a power series about $\beta = \beta_0$ where β is a variable and P_r are constants called the coefficients of the series.

Theorem 2.2
[27] Assume $\mathcal{A}[\phi(y,\beta)] = \Psi(y,\mu)$ can be denoted as the new form of the fractional Taylor's series formula. If

$$\left|\mu^2\mathcal{A}\left[D_\beta^{(\nu+1)\alpha}\phi(y,\beta)\right]\right| \le T, \quad 0<\mu\le s,\ 0<\alpha\le 1,$$

then the remainder $R_\nu(y,\mu)$ of the new form of multiple fractional Taylor's formula satisfies the following inequality:

$$|R_\nu(y,\mu)| \le \frac{T}{\mu^{(\nu+1)\alpha+2}}, \quad 0<\mu<s.$$

3. Algorithm to Solve Fractional PDEs with Variable Coefficients

The ARPS method steps for solving the problem are detailed as follows:

$$D_\beta^\alpha\phi(y,\beta) = L_y(\phi(y,\beta)) + V(y,\beta)\phi(y,\beta) + h(y,\beta), \tag{1}$$

with the initial condition

$$\phi(y,0) = f(y), \tag{2}$$

where L_y is a linear differential operator with respect to y, and $0<\alpha<1$, $V(y,\beta)$ and $h(y,\beta)$ are given analytic functions and $\phi(y,\beta)$ is the unknown function to be determined.

Step 1. Apply AIT to Eq.(1), with respect to β

$$\mathcal{A}\left[D_\beta^\alpha\phi(y,\beta)\right] = \mathcal{A}\left[L_y(\phi(y,\beta))\right] + \mathcal{A}\left[V(y,\beta)\phi(y,\beta)\right] + \mathcal{A}\left[h(y,\beta)\right]. \tag{3}$$

By utilizing Lemma (2.1) we get

$$\mu^\alpha\Psi(y,\mu) - \mu^{\alpha-2}\phi(y,0) = L_y(\Psi(y,\mu)) + \mathcal{A}\left[V(y,\beta)\phi(y,\beta)\right] + \mathcal{A}\left[h(y,\beta)\right]. \tag{4}$$

Step 2. Simplify Eq.(3) and use the condition in (2), to get

$$\Psi(y,\mu)=\frac{f(y)}{\mu^2}+\frac{L_y(\Psi(y,\mu))}{\mu^\alpha}+\frac{\mathcal{A}\left[\mathcal{A}^{-1}[K(y,\mu)]\mathcal{A}^{-1}[\Psi(y,\mu)]\right]}{\mu^\alpha}+\frac{H(y,\mu)}{\mu^\alpha}. \tag{5}$$

where $\mathcal{A}[V(y,\beta)]=K(y,\mu)$, $\mathcal{A}[h(y,\beta)]=H(y,\mu)$.

Step 3. Consider the solution of Eq.(4), to be of the form

$$\Psi(y,\mu)=\sum_{r=0}^{\infty}\frac{J_r(y)}{\mu^{r\alpha+2}},\quad \mu>0, \tag{6}$$

where

$$J_0(y)=\lim_{\mu\to\infty}\mu^2\Psi(y,\mu)=f(y).$$

The k-th truncated series of $\Psi(y,\mu)$ in Eq.(6)

$$\begin{aligned}\Psi_v(y,\mu)&=\sum_{r=0}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}},\quad \mu>0,\ v=1,2,3,\dots\\&=\frac{f(y)}{\mu^2}+\sum_{r=1}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}}.\end{aligned} \tag{7}$$

Step 4. Consider separately the Aboodh residual function (ARF) of Eq.(4) and the k-th truncated ARF, as

$$\begin{aligned}ARes(y,\mu)=\Psi(y,\mu)&-\frac{f(y)}{\mu^2}-\frac{L_y(\Psi(y,\mu))}{\mu^\alpha}\\&-\frac{\mathcal{A}\left[\mathcal{A}^{-1}[K(y,\mu)]\mathcal{A}^{-1}[\Psi(y,\mu)]\right]}{\mu^\alpha}-\frac{H(y,\beta)}{\mu^\alpha},\end{aligned}$$

and

$$\begin{aligned}ARes_v(y,\mu)=\Psi_v(y,\mu)&-\frac{f(y)}{\mu^2}-\frac{L_y(\Psi_v(y,\mu))}{\mu^\alpha}\\&-\frac{\mathcal{A}\left[\mathcal{A}^{-1}[K(y,\mu)]\mathcal{A}^{-1}[\Psi_v(y,\mu)]\right]}{\mu^\alpha}-\frac{H(y,\beta)}{\mu^\alpha}.\end{aligned} \tag{8}$$

Step 5. Replace the expansion of $\Psi_v(y,\mu)$ in (7) into (8).

$$\begin{aligned}ARes_v(y,\mu)=&\frac{f(y)}{\mu^2}+\sum_{r=1}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}}-\frac{f(y)}{\mu^2}\\&-\frac{1}{\mu^\alpha}L_y\left(\frac{f(y)}{\mu^2}+\sum_{r=1}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}}\right)\\&-\frac{1}{\mu^\alpha}\mathcal{A}\left[\mathcal{A}^{-1}[K(y,\mu)]\mathcal{A}^{-1}\left(\frac{f(y)}{\mu^2}+\sum_{r=1}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}}\right)\right]\\&-\frac{H(y,\beta)}{\mu^\alpha}.\end{aligned} \tag{9}$$

Step 6. Multiply both sides of Eq.(9) by $\mu^{v\alpha+2}$ and take the limit as $\mu \to \infty$,

$$\begin{aligned}\lim_{\mu\to\infty}\mu^{v\alpha+2}ARes_v(y,\mu) = \lim_{\mu\to\infty}\mu^{v\alpha+2}\Bigg(&\frac{f(y)}{\mu^2}+\sum_{r=1}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}}-\frac{f(y)}{\mu^2}\\&+\frac{1}{\mu^\alpha}L_y\left(\frac{f(y)}{\mu^2}+\sum_{r=1}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}}\right)\\&+\frac{1}{\mu^\alpha}\mathcal{A}\left[\mathcal{A}^{-1}[K(y,\mu)]\mathcal{A}^{-1}\left(\frac{f(y)}{\mu^2}+\sum_{r=1}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}}\right)\right]\\&+\frac{H(y,\beta)}{\mu^\alpha}\Bigg),\ v=1,2,3,\ldots\end{aligned} \tag{10}$$

Step 7. Using the following fact to find $J_v(y)$,

$$\lim_{\mu\to\infty}\mu^{v\alpha+2}ARes_v(y,\mu)=0, \tag{11}$$

where $v=1,2,3,\ldots$. To find $J_1(y)$, we substitute $\Psi_1(y,\mu)$ in the 1^{st} ARF,

$$\begin{aligned}ARes_1(y,\mu) = &\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}\right)-\frac{f(y)}{\mu^2}\\&-\frac{1}{\mu^\alpha}L_y\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}\right)\\&-\frac{1}{\mu^\alpha}\mathcal{A}\left[\mathcal{A}^{-1}[K(y,\mu)]\mathcal{A}^{-1}\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}\right)\right]\\&-\frac{H(y,\beta)}{\mu^\alpha}.\end{aligned} \tag{12}$$

Multiply both sides Eq.(12) by $\mu^{\alpha+2}$, we have

$$\begin{aligned}\mu^{\alpha+2}ARes_1(y,\mu) = \mu^{\alpha+2}\Bigg(&\frac{J_1(y)}{\mu^{\alpha+2}}-\frac{1}{\mu^\alpha}L_y\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}\right)\\&-\frac{1}{\mu^\alpha}\mathcal{A}\left[\mathcal{A}^{-1}[K(y,\mu)]\mathcal{A}^{-1}\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}\right)\right]\\&-\frac{H(y,\beta)}{\mu^\alpha}\Bigg).\end{aligned}$$

Now take the limit as $\mu\to\infty$, for the above equation, then we get

$$\begin{aligned}\lim_{\mu\to\infty}\mu^{\alpha+2}ARes_1(y,\mu) = \lim_{\mu\to\infty}\mu^{\alpha+2}\Bigg(&\frac{J_1(y)}{\mu^{\alpha+2}}-\frac{1}{\mu^\alpha}L_y\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}\right)\\&-\frac{1}{\mu^\alpha}\mathcal{A}\left[\mathcal{A}^{-1}[K(y,\mu)]\mathcal{A}^{-1}\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}\right)\right]\\&-\frac{H(y,\beta)}{\mu^\alpha}\Bigg)=0.\end{aligned}$$

Thus, we get

$$\Psi_1(y,\mu)=\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}.$$

To find $J_2(y)$, we substitute $\Psi_2(y,\mu)$ in the 2nd ARF and multiply both sides by $\mu^{2\alpha+2}$, then take the limit as $\mu \to \infty$, we obtain

$$\begin{aligned}\lim_{\mu\to\infty}\mu^{2\alpha+2}ARes_2(y,\mu) = \lim_{\mu\to\infty}\mu^{2\alpha+2}\Bigg(&\frac{J_1(y)}{\mu^{\alpha+2}}+\frac{J_2(y)}{\mu^{2\alpha+2}}\\ &-\frac{1}{\mu^{\alpha}}L_y\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}+\frac{J_2(y)}{\mu^{2\alpha+2}}\right)\\ &-\frac{1}{\mu^{\alpha}}\mathcal{A}\left[\mathcal{A}^{-1}[G(y,\beta)]\mathcal{A}^{-1}\left(\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}+\frac{J_2(y)}{\mu^{2\alpha+2}}\right)\right]\\ &-\frac{H(y,\beta)}{\mu^{\alpha}}\Bigg)=0.\end{aligned}$$

Thus, we get

$$\Psi_2(y,\mu)=\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}+\frac{J_2(y)}{\mu^{2\alpha+2}}.$$

Repeat the same steps to find $J_3(y),\ldots,J_v(y)$.

Step 8. Replace the obtained values of $J_v(y)$ into the k-th truncated series of $\Psi_k(y,\mu)$,

$$\Psi_k(y,\mu)=\frac{f(y)}{\mu^2}+\frac{J_1(y)}{\mu^{\alpha+2}}+\frac{J_2(y)}{\mu^{2\alpha+2}}+\sum_{r=3}^{v}\frac{J_r(y)}{\mu^{r\alpha+2}}.$$

Step 9. To derive the k-approximate solution of Eq.(1), apply the inverse of AIT on $\Psi_k(y,\mu)$ to obtain

$$\phi_v(y,\beta)=f(y)+J_1(y)\frac{\beta^{\alpha}}{\Gamma(\alpha+1)}+J_2(y)\frac{\beta^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+J_v(y)\frac{\beta^{v\alpha}}{\Gamma(v\alpha+1)}.$$

Moreover,

$$\phi(y,\beta)=\lim_{v\to\infty}\sum_{r=0}^{v}J_r(y)\frac{\beta^{v\alpha}}{\Gamma(v\alpha+1)}$$

$$\phi(y,\beta)=f(y)+J_1(y)\frac{\beta^{\alpha}}{\Gamma(\alpha+1)}+J_2(y)\frac{\beta^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots$$

4. Some Examples

In this section, the proposed algorithm is applied to a set of representative fractional differential equations, including cancer tumor models and heat-like equations. These examples are selected due to their significance in modeling real-world phenomena, particularly in biological and physical systems. The aim is to demonstrate the effectiveness, accuracy, and convergence behavior of the proposed method in handling both linear and variable-coefficient fractional partial differential equations.

Example 1. [19, 30] The clear killing ratio of the cancer cells is just time dependent

$$D_{\beta}^{\alpha}\phi(y,\beta)=D_y^2\phi(y,\beta)-\beta^2\phi(y,\beta), \tag{13}$$

with the initial condition

$$\phi(y,0)=e^{my}, \tag{14}$$

where $\beta>0$, $0\le y\le 1$, $0<\alpha<1$.

Solution. Firstly, take the AIT for Eq.(13) so we have

$$\mathcal{A}\left[D_{\beta}^{\alpha}\phi(y,\beta)\right]=\mathcal{A}\left[D_{y}^{2}\phi(y,\beta)\right]-\mathcal{A}\left[\beta^{2}\phi(y,\beta)\right], \tag{15}$$

simplify Eq.(15) and use the initial condition (14), to get

$$\Psi(y,\mu)=\frac{e^{my}}{\mu^{2}}+\frac{D_{y}^{2}\Psi(y,\mu)}{\mu^{\alpha}}-\frac{\mathcal{A}\left[\beta^{2}\mathcal{A}^{-1}[\Psi(y,\mu)]\right]}{\mu^{\alpha}}. \tag{16}$$

Assume the solution of Eq.(16) is defined by

$$\Psi(y,\mu)=\frac{e^{my}}{\mu^{2}}+\sum_{r=1}^{\infty}\frac{J_{r}(y)}{\mu^{r\alpha+2}},\quad \mu>0, \tag{17}$$

the k-th truncated is defined

$$\Psi_{v}(y,\mu)=\frac{e^{my}}{\mu^{2}}+\sum_{r=1}^{v}\frac{J_{r}(y)}{\mu^{r\alpha+2}},\quad \mu>0.$$

Consider the k-th truncated of ARF is

$$\begin{aligned}ARes_{v}(y,\mu)=\Psi_{v}(y,\mu)-\frac{e^{my}}{\mu^{2}}-\frac{D_{y}^{2}\Psi_{v}(y,\mu)}{\mu^{\alpha}}\\ +\frac{\mathcal{A}\left[\mathcal{A}^{-1}\left[\frac{\Gamma(6)}{\mu^{4}}\right]\mathcal{A}^{-1}[\Psi_{v}(y,\mu)]\right]}{\mu^{\alpha}}.\end{aligned} \tag{18}$$

To find $J_1(y)$, we substitute $\Psi_1(y,\mu)$ in the 1st ARF

$$\begin{aligned}ARes_{1}(y,\mu)=\frac{e^{my}}{\mu^{2}}+\frac{J_{1}(y)}{\mu^{\alpha+2}}-\frac{e^{my}}{\mu^{2}}-D_{y}^{2}\left(\frac{e^{my}}{\mu^{2}}+\frac{J_{1}(y)}{\mu^{\alpha+2}}\right)\\ +\frac{e^{my}\Gamma(6)}{\mu^{\alpha+4}}+\frac{J_{1}(y)\Gamma(\alpha+3)}{\mu^{2\alpha+4}\Gamma(\alpha+1)}.\end{aligned} \tag{19}$$

Multiply by $\mu^{\alpha+2}$ on both sides and take limit as $\mu\to\infty$ of Eq.(19)

$$\begin{aligned}\lim_{\mu\to\infty}\mu^{\alpha+2}ARes_{1}(y,\mu)=\lim_{\mu\to\infty}\mu^{\alpha+2}\Bigg(\frac{e^{my}}{\mu^{2}}+\frac{J_{1}(y)}{\mu^{\alpha+2}}-\frac{e^{my}}{\mu^{2}}-m^{2}\frac{e^{my}}{\mu^{\alpha+2}}\\ -D_{y}^{2}\left(\frac{J_{1}(y)}{\mu^{\alpha+2}}\right)+\frac{e^{my}\Gamma(6)}{\mu^{\alpha+4}}+\frac{J_{1}(y)\Gamma(\alpha+3)}{\mu^{2\alpha+4}\Gamma(\alpha+1)}\Bigg)=0.\end{aligned}$$

By solving the previous equation, we obtain

$$J_{1}(y)=m^{2}e^{my}.$$

To find $J_2(y)$, we substitute $\Psi_2(y,\mu)$ in the 2nd ARF

$$\begin{aligned}ARes_{2}(y,\mu)=\frac{e^{my}}{\mu^{2}}+\frac{m^{2}e^{my}}{\mu^{\alpha+2}}+\frac{J_{2}(y)}{\mu^{2\alpha+2}}-\frac{e^{my}}{\mu^{2}}-\frac{e^{my}}{\mu^{\alpha+2}}-\frac{m^{2}e^{my}}{\mu^{2\alpha+2}}\\ -D_{y}^{2}\left(\frac{J_{2}(y)}{\mu^{3\alpha+2}}\right)+\frac{1}{\mu^{\alpha}}\mathcal{A}\left[\mathcal{A}^{-1}\left[\frac{\Gamma(6)}{\mu^{4}}\right]\mathcal{A}^{-1}\left[\frac{e^{my}}{\mu^{2}}+\frac{m^{2}e^{my}}{\mu^{\alpha+2}}+\frac{J_{2}(y)}{\mu^{2\alpha+2}}\right]\right].\end{aligned} \tag{20}$$

Multiply by $\mu^{2\alpha+2}$ on both sides and take the limit as $\mu \to \infty$ of Eq.(20)

$$\lim_{\mu\to\infty} \mu^{2\alpha+2} ARes_2(y,\mu) = \lim_{\mu\to\infty} \mu^{2\alpha+2} \left(\frac{e^{my}}{\mu^2} + \frac{m^2 e^{my}}{\mu^{\alpha+2}} + \frac{J_2(y)}{\mu^{2\alpha+2}} - \frac{e^{my}}{\mu^2} - \frac{e^{my}}{\mu^{\alpha+2}} - \frac{m^2 e^{my}}{\mu^{2\alpha+2}} - m^4 \frac{e^{my}}{\mu^{2\alpha+2}} - D_y^2 \left(\frac{J_2(y)}{\mu^{3\alpha+2}} \right) + \frac{e^{my}\Gamma(6)}{\mu^{\alpha+4}} + \frac{\Gamma(\alpha+3) m^2 e^{my}}{\Gamma(\alpha+1)\mu^{2\alpha+2}} + \frac{\Gamma(3\alpha+3) J_2(y)}{\Gamma(2\alpha+1)\mu^{4\alpha+4}} \right) = 0.$$

By solving the previous equation, we obtain

$$J_2(y) = m^4 e^{my}.$$

To find $J_3(y)$, we substitute $\Psi_3(y,\mu)$ in the 3rd ARF

$$ARes_3(y,\mu) = \frac{e^{my}}{\mu^2} + \frac{m^2 e^{my}}{\mu^{\alpha+2}} + \frac{m^4 e^{my}}{\mu^{2\alpha+2}} + \frac{J_3(y)}{\mu^{3\alpha+2}} - \frac{e^{my}}{\mu^2} - \frac{1}{\mu^\alpha} D_y^2 \left(\frac{e^{my}}{\mu^2} + \frac{m^2 e^{my}}{\mu^{\alpha+2}} + \frac{m^4 e^{my}}{\mu^{2\alpha+2}} + \frac{J_3(y)}{\mu^{3\alpha+2}} \right) + \frac{1}{\mu^\alpha} \mathcal{A} \left[\mathcal{A}^{-1} \left[\frac{\Gamma(6)}{\mu^4} \right] \mathcal{A}^{-1} \left[\frac{e^{my}}{\mu^2} + \frac{m^2 e^{my}}{\mu^{\alpha+2}} + \frac{m^4 e^{my}}{\mu^{2\alpha+2}} + \frac{J_3(y)}{\mu^{3\alpha+2}} \right] \right]. \tag{21}$$

Multiply by $\mu^{3\alpha+2}$ on both sides and take limit as $\mu \to \infty$ of Eq.(21)

$$\lim_{\mu\to\infty} \mu^{3\alpha+2} ARes_3(y,\mu) = \lim_{\mu\to\infty} \mu^{3\alpha+2} \left(\frac{e^{my}}{\mu^2} + \frac{m^2 e^{my}}{\mu^{\alpha+2}} + \frac{m^4 e^{my}}{\mu^{2\alpha+2}} + \frac{J_3(y)}{\mu^{3\alpha+2}} - \frac{e^{my}}{\mu^2} - D_y^2 \left(\frac{e^{my}}{\mu^2} + \frac{m^2 e^{my}}{\mu^{\alpha+2}} + \frac{m^4 e^{my}}{\mu^{2\alpha+2}} + \frac{J_3(y)}{\mu^{3\alpha+2}} \right) \frac{1}{\mu^\alpha} + \frac{1}{\mu^\alpha} \mathcal{A} \left[\mathcal{A}^{-1} \left[\frac{\Gamma(6)}{\mu^4} \right] \mathcal{A}^{-1} \left[\frac{e^{my}}{\mu^2} + \frac{m^2 e^{my}}{\mu^{\alpha+2}} + \frac{m^4 e^{my}}{\mu^{2\alpha+2}} + \frac{J_3(y)}{\mu^{3\alpha+2}} \right] \right] \right) = 0.$$

By solving the previous equation, we obtain

$$J_3(y) = \frac{m^6 - 2}{2} e^{my}.$$

Repeat the previous steps to obtain

$$J_4(y) = \frac{m^2(m^6-8)}{6} e^{my}, \quad J_5(y) = \frac{m^4(m^6-20)}{24} e^{my}.$$

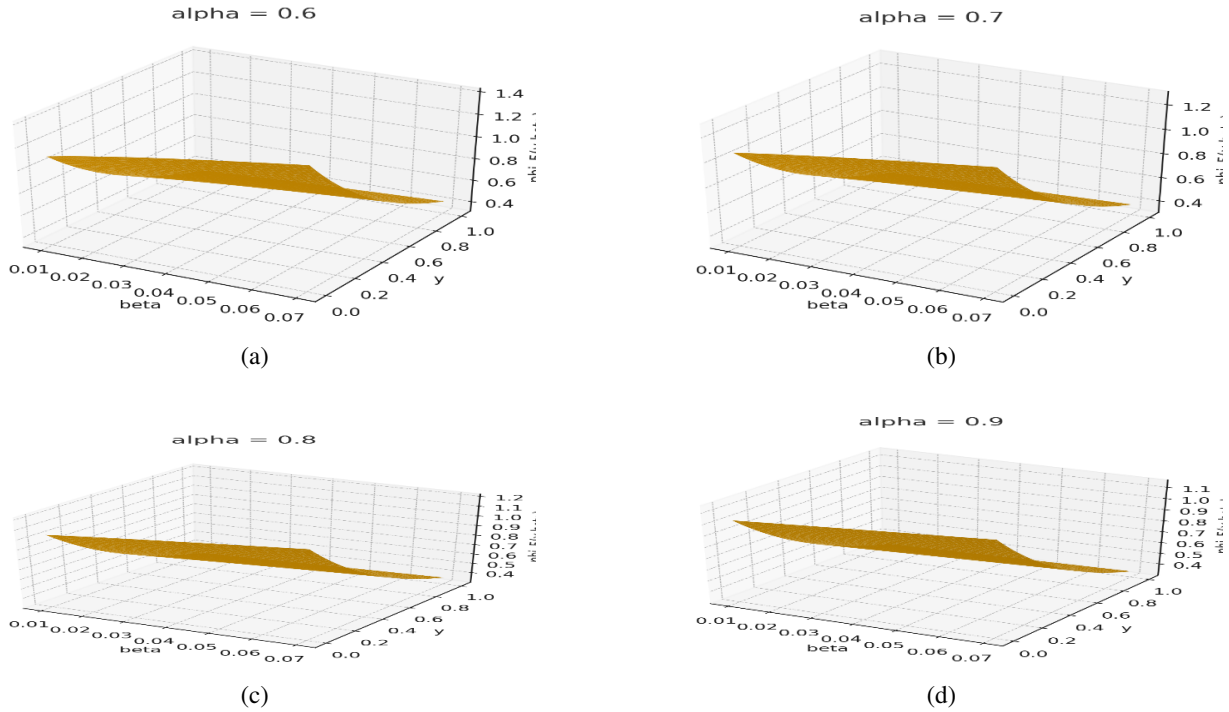
Replace the obtained values of $J_r(y)$, $r = 0, 1, \ldots, 5$ into $\Psi_5(y,\mu)$ to derive the fifth series solution of Eq.(17),

$$\Psi_5(y,\mu) = \frac{e^{my}}{\mu^2} + \frac{m^2 e^{my}}{\mu^{\alpha+2}} + \frac{m^4 e^{my}}{\mu^{2\alpha+2}} + \frac{1}{2}\frac{(m^6-2)e^{my}}{\mu^{3\alpha+2}} + \frac{1}{6}\frac{m^2(m^6-8)e^{my}}{\mu^{4\alpha+2}} + \frac{1}{24}\frac{m^4(m^6-20)e^{my}}{\mu^{5\alpha+2}}. \tag{22}$$

Step 6. Use the inverse of AIT on $\Psi(y,\mu)$ to obtain the solution $\phi_5(y,\beta)$

$$\phi_5(y,\beta) = e^{my} + \frac{m^2 e^{my}\beta^\alpha}{\Gamma(\alpha+1)} + \frac{m^4 e^{my}\beta^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(m^6-2)e^{my}\beta^{3\alpha}}{2\Gamma(3\alpha+1)} + \frac{m^2(m^6-8)e^{my}\beta^{4\alpha}}{6\Gamma(4\alpha+1)} + \frac{m^4(m^6-20)e^{my}\beta^{5\alpha}}{24\Gamma(5\alpha+1)}. \tag{23}$$

The graphical and contour representations of the 5^{th} approximation series solution $\phi_5(y,\beta)$ of Example 1 are presented in Figure 1, where $m=-1.2$ in all graphs: (a) $\alpha=0.6$, (b) $\alpha=0.7$, (c) $\alpha=0.8$, (d) $\alpha=0.9$.



(a) (b) (c) (d)

Figure 1. The graph of the solution $\phi_5(y,\beta)$, and several values α in Example 1 at $m=-1.2$ in all graphs (a) $\alpha=0.6$, (b) $\alpha=0.7$, (c) $\alpha=0.8$, (d) $\alpha=0.9$.

Example 2. [30, 36] solve the following initial value problem

$$D_\beta^\alpha \phi(y,\beta) = D_y^2\phi(y,\beta) - \frac{2}{y^2}\phi(y,\beta), \tag{24}$$

with the initial condition

$$\phi(y,0) = \frac{1}{y} + y^2, \tag{25}$$

where $\beta>0$, $0\le y\le 1$, $0<\alpha\le 1$. We will notice from this example that the fractional derivative does not affect the solution.

Solution. Firstly, apply AIT for Eq.(24)

$$\mathcal{A}\left[D_\beta^\alpha \phi(y,\beta)\right] = \mathcal{A}\left[D_y^2\phi(y,\beta)\right] - \frac{2}{y^2}\mathcal{A}[\phi(y,\beta)]. \tag{26}$$

Simplify (26) and use condition in (25)

$$\Psi(y,\mu) = \frac{1}{\mu^2 y} + \frac{y^2}{\mu^2} + \frac{D_y^2\Psi(y,\mu)}{\mu^\alpha} - \frac{2\Psi(y,\mu)}{y\mu^\alpha}. \tag{27}$$

Secondly, we define the solution of Eq.(27) to be given as

$$\Psi(y,\mu) = \sum_{r=0}^{\infty} \frac{J_r(y)}{\mu^{r\alpha+2}}, \quad \mu>0.$$

The k-th truncated of this series

$$\Psi_v(y,\mu) = \frac{1}{\mu^2 y} + \frac{y^2}{\mu^2} + \sum_{r=1}^{v} \frac{J_r(y)}{\mu^{r\alpha+2}}, \quad \mu > 0,\ v = 1, 2, \ldots. \tag{28}$$

Consider the k-th truncated of ARF is

$$ARes_v(y,\mu) = \Psi_v(y,\mu) - \frac{1}{\mu^2 y} - \frac{y^2}{\mu^2} - \frac{D_y^2 \Psi_v(y,\mu)}{\mu^\alpha} + \frac{2\Psi_v(y,\mu)}{y\mu^\alpha}. \tag{29}$$

Define the 1st ARF and substitute $\Psi_1(y,\mu)$

$$\begin{aligned} ARes_1(y,\mu) = {} & \frac{1}{\mu^2 y} + \frac{y^2}{\mu^2} + \frac{J_1(y)}{\mu^{\alpha+2}} - \frac{1}{\mu^2 y} - \frac{y^2}{\mu^2} \\ & - \frac{1}{\mu^\alpha} D_y^2 \left(\frac{1}{\mu^2 y} + \frac{y^2}{\mu^2} + \frac{J_1(y)}{\mu^{\alpha+2}} \right) + \frac{2}{y\mu^\alpha} \left(\frac{1}{\mu^2 y} + \frac{y^2}{\mu^2} + \frac{J_1(y)}{\mu^{\alpha+2}} \right). \end{aligned}$$

Using the fact

$$\lim_{\mu\to\infty} \mu^{\alpha+2} ARes_1(y,\mu) = 0, \tag{30}$$

Then, we have

$$\begin{aligned} \lim_{\mu\to\infty} \mu^{\alpha+2} \Bigg(& \frac{J_1(y)}{\mu^{\alpha+2}} - \frac{2}{\mu^{\alpha+2} y^3} - \frac{2}{\mu^{\alpha+2}} - D_y^2 \left(\frac{J_1(y)}{\mu^{2\alpha+2}} \right) \\ & + \frac{2}{\mu^{\alpha+2} y^2} + \frac{2y}{\mu^{\alpha+2}} + \frac{2J_1(y)}{y\mu^{2\alpha+2}} \Bigg) = 0. \end{aligned} \tag{31}$$

By solving the previous equation, we obtain

$$J_1(y) = 0. \tag{32}$$

By similar steps we have

$$J_2(y) = J_3(y) = J_4(y) = J_5(y) = 0.$$

Thus,

$$\Psi_5(y,\mu) = \Psi_6(y,\mu) = \cdots = \frac{1}{\mu^2 y} + \frac{y^2}{\mu^2}. \tag{33}$$

Apply the inverse of AIT, to get

$$\phi_5(y,\beta) = \phi_6(y,\beta) = \cdots = \frac{1}{y} + y^2. \tag{34}$$

The 3D graphics of the sixth approximate series solution $\phi(y,\beta)$ of Example 2 are clarified below in Figure 2. We mention that the fractional order is not effective.

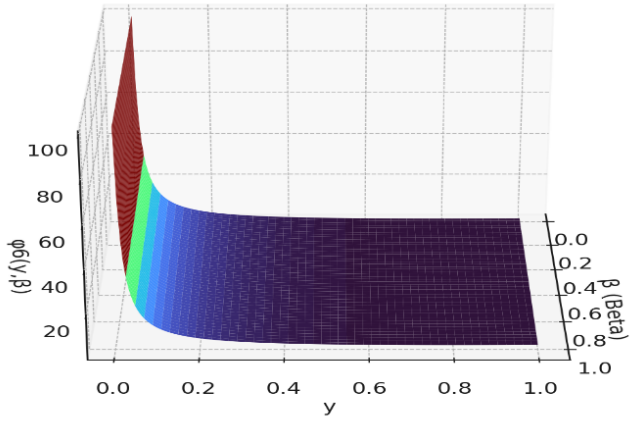


Figure 2. The graph of the solution $\phi(y,\beta)$ of the Example 2.

Example 3. [31] Consider the fractional heat-like equation

$$D_{\beta}^{\alpha}\phi(y,\beta)=\frac{1}{2}y^{2}D_{y}^{2}\phi(y,\beta),\quad 0<y\leq 1,\ 0<\beta\leq 1,\ 0<\alpha<1, \tag{35}$$

Subject to the boundary

$$\phi(0,\beta)=0,\quad \phi(1,\beta)=E_{\alpha}(\beta^{\alpha}),$$

with the following initial condition

$$\phi(y,0)=y^{2}. \tag{36}$$

Solution. Firstly, apply AIT for Eq.(35)

$$\mathcal{A}\left[D_{\beta}^{\alpha}\phi(y,\beta)\right]=\mathcal{A}\left[\frac{1}{2}y^{2}D_{y}^{2}\phi(y,\beta)\right]. \tag{37}$$

Simplify (37) and use condition in (36)

$$\Psi(y,\mu)=\frac{y^{2}}{\mu^{2}}+\frac{y^{2}D_{y}^{2}\Psi(y,\mu)}{2\mu^{\alpha}}. \tag{38}$$

Secondly, we define the solution of Eq.(38) to be given as

$$\Psi(y,\mu)=\frac{y^{2}}{\mu^{2}}+\sum_{r=1}^{\infty}\frac{J_{r}(y)}{\mu^{r\alpha+2}},\quad \mu>0.$$

The k-th truncated of this series

$$\Psi_{v}(y,\mu)=\frac{y^{2}}{\mu^{2}}+\sum_{r=1}^{v}\frac{J_{r}(y)}{\mu^{r\alpha+2}},\quad \mu>0,\ v=1,2,\ldots \tag{39}$$

Consider the k-th truncated of ARF is

$$ARes_{v}(y,\mu)=\Psi_{v}(y,\mu)-\frac{y^{2}}{\mu^{2}}-\frac{y^{2}D_{y}^{2}\Psi(y,\mu)}{2\mu^{\alpha}}. \tag{40}$$

Define the 1st ARF and substitute $\Psi_1(y,\mu)$

$$ARes_1(y,\mu) = \frac{J_1(y)}{\mu^{\alpha+2}} - \frac{y^2 D_y^2\left(\frac{y^2}{\mu^2} + \frac{J_1(y)}{\mu^{\alpha+2}}\right)}{2\mu^{\alpha}}.$$

Using the fact

$$\lim_{\mu\to\infty} \mu^{\alpha+2} ARes_1(y,\mu) = 0. \tag{41}$$

Then, we have

$$\lim_{\mu\to\infty} \mu^{\alpha+2}\left(\frac{J_1(y)}{\mu^{\alpha+2}} - \frac{y^2 D_y^2\left(\frac{y^2}{\mu^2} + \frac{J_1(y)}{\mu^{\alpha+2}}\right)}{2\mu^{\alpha}}\right) = 0. \tag{42}$$

By solving the previous equation, we obtain

$$J_1(y) = y^2. \tag{43}$$

By similar steps we have

$$J_2(y) = J_3(y) = J_4(y) = J_5(y) = \cdots = J_r(y) = y^2.$$

Thus,

$$\Psi_v(y,\mu) = \sum_{r=1}^{v} \frac{y^2}{\mu^{r\alpha+2}}. \tag{44}$$

Apply the inverse of AIT, to get

$$\phi_v(y,\beta) = y^2 \sum_{r=0}^{v} \frac{\beta^{r\alpha}}{\Gamma(r\alpha+1)}. \tag{45}$$

Moreover

$$\phi(y,\beta) = y^2 \lim_{v\to\infty} \sum_{r=0}^{v} \frac{\beta^{r\alpha}}{\Gamma(r\alpha+1)} = y^2 E_\alpha(\beta),$$

where $E_\alpha(\beta)$ is Mittag-Leffler function.

Table 1 show the numerical solution values and the corresponding consecutive errors (con.Err) of 17th. As obtained by ARPS method for several values of α, where the con.Err of $\phi_{17}(y,\beta)$ is $con.Err(y,\beta) = |\phi_{17}(y,\beta) - \phi_{15}(y,\beta)|$. From the results ARPS method expands accurate analytical solutions with fast convergence and minimal computation time. Its simplicity and efficiency make it a workable and attractive method for solving differential equations.

Table 1. The Numerical solutions and Con.Err obtained by the ARPS method for Example 3

β	Y	$\alpha = 0.75$ (V=17)	Con.Err.	$\alpha = 0.90$ (V=17)	Con.Err.	$\alpha = 1.00$ (V=17)	Con.Err.
0.25	0.3	0.13491	1.1782×10^{-17}	0.12186	7.7226×10^{-22}	0.11556	1.0163×10^{-24}
	0.6	0.53965	4.7127×10^{-7}	0.48744	3.0890×10^{-20}	0.46225	4.0650×10^{-24}
	0.9	1.21422	1.0603×10^{-16}	1.09675	6.9503×10^{-17}	1.04006	9.1463×10^{-24}
0.5	0.3	0.18196	4.9884×10^{-14}	0.15953	1.7042×10^{-16}	0.14839	6.7567×10^{-23}
	0.6	0.72782	1.9954×10^{-13}	0.63810	6.8169×10^{-17}	0.59354	2.7082×10^{-19}
	0.9	1.63759	4.4896×10^{-10}	1.43573	1.5338×10^{-15}	1.33546	6.0810×10^{-18}
0.75	0.3	0.24016	6.6573×10^{-12}	0.20709	5.9642×10^{-15}	0.19053	4.5015×10^{-16}
	0.6	0.96062	2.6629×10^{-10}	0.82837	7.2857×10^{-14}	0.76212	1.8006×10^{-16}
	0.9	2.16140	5.9916×10^{-11}	1.86383	5.3677×10^{-14}	1.7777	4.0513×10^{-16}
1.0	0.3	0.31373	2.1553×10^{-10}	0.26775	3.8243×10^{-9}	0.24465	4.5546×10^{-16}
	0.6	1.25491	8.6213×10^{-9}	1.07098	1.5297×10^{-12}	0.97858	1.8218×10^{-14}
	0.9	2.82355	1.9398×10^{-9}	2.40970	3.4419×10^{-12}	2.20181	4.0991×10^{-14}

Figure 3 provides a clear comparison of the manner of solution values under different fractional orders across a range of evaluation points. As the fractional order increases, the values exhibit a slight yet consistent decrease. This transition shows how precision fractional differential equations to this model can handle the behavior of system, resulting a powerful and efficient technique for solving difficult models that present phenomena in real-world.

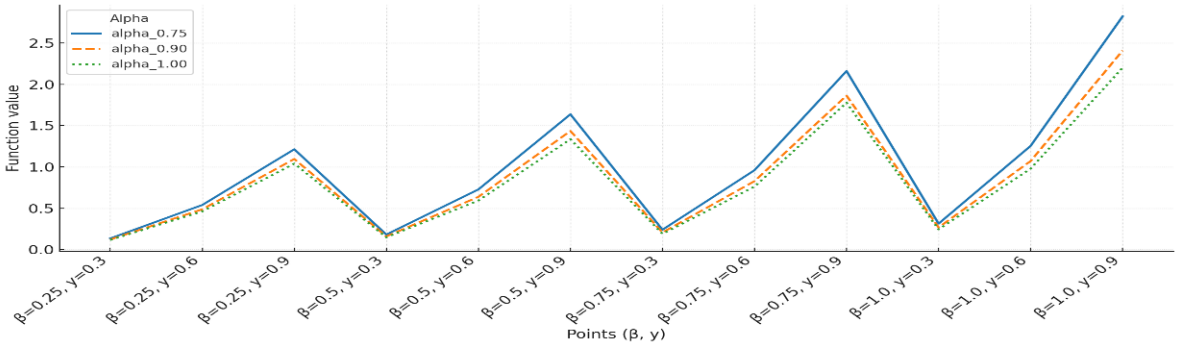


Figure 3. Comparative Analysis of Fractional approximate Solutions at Varying Points

Table 2 presents the comparison of absolute error gained by ARPS approach with the variational iteration method (VIM) [32], the Adomian decomposition method (ADM) [33], the Sinc–Legendre spectral collocation method (S-LCM) [34], and the Jacopi Tau method (JTM) [35] at $\alpha = 1$. From Table 2, It is clear that the absolute errors obtained by ARPS method for the equation are small, and it can be observed that these errors decrease as the number of iterations increases.

Table 2. Comparison of ARPS method with other Methods ($\alpha = 1$)

β	Y	VIM/ADM (r=17)	S-LCM (N=15)	S-LCM (N=25)	JTM (N=10)	JTM (N=15)	ARPS (V=15)	ARPS (V=17)
0.25	0.3	1.54×10^{-5}	1.09×10^{-6}	9.92×10^{-8}	2.26×10^{-16}	1.26×10^{-16}	1.12×10^{-17}	1.12×10^{-17}
	0.6	6.16×10^{-6}	2.96×10^{-6}	2.70×10^{-8}	4.20×10^{-16}	3.09×10^{-16}	4.49×10^{-17}	4.49×10^{-17}
0.5	0.3	2.86×10^{-6}	6.45×10^{-6}	5.56×10^{-8}	4.08×10^{-16}	1.93×10^{-16}	3.60×10^{-17}	3.60×10^{-17}
	0.6	1.20×10^{-6}	1.54×10^{-5}	1.00×10^{-7}	1.05×10^{-16}	1.02×10^{-16}	2.44×10^{-17}	2.44×10^{-17}
	0.9	1.38×10^{-5}	5.64×10^{-6}	4.08×10^{-8}	3.42×10^{-16}	2.34×10^{-16}	5.87×10^{-17}	5.87×10^{-17}
0.75	0.3	2.34×10^{-3}	1.08×10^{-5}	8.97×10^{-8}	2.06×10^{-16}	1.83×10^{-16}	4.94×10^{-17}	4.94×10^{-17}
	0.6	5.55×10^{-3}	1.27×10^{-5}	1.00×10^{-7}	4.19×10^{-16}	3.39×10^{-16}	8.57×10^{-17}	8.57×10^{-17}
	0.9	1.46×10^{-2}	4.16×10^{-5}	2.90×10^{-7}	1.16×10^{-15}	1.00×10^{-15}	2.53×10^{-16}	2.53×10^{-16}
1.0	0.3	4.65×10^{-3}	7.63×10^{-5}	2.73×10^{-7}	1.23×10^{-15}	1.00×10^{-15}	2.64×10^{-16}	2.64×10^{-16}
	0.6	1.84×10^{-2}	2.46×10^{-4}	7.81×10^{-7}	2.73×10^{-15}	2.50×10^{-15}	6.99×10^{-16}	6.99×10^{-16}
	0.9	4.18×10^{-2}	4.86×10^{-4}	1.58×10^{-6}	4.60×10^{-15}	4.08×10^{-15}	1.86×10^{-16}	1.86×10^{-16}

Figure 4 provides a clear comparison of the accuracy levels of several numerical methods used to solve complex differential equations. The methods compared include VIM/ADM, S-LCM in two configurations, JTM with different settings, and the ARPS approach.

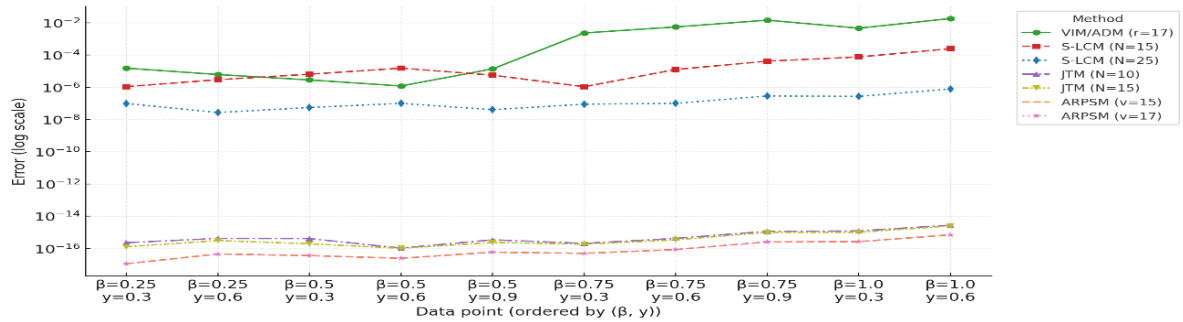


Figure 4. comparison of the different numerical methods

Figure 5 provide a clear visual understanding of how the function behaves for different values of the fractional order α. The visualizations were created using Python, ensuring precise and flexible representation.

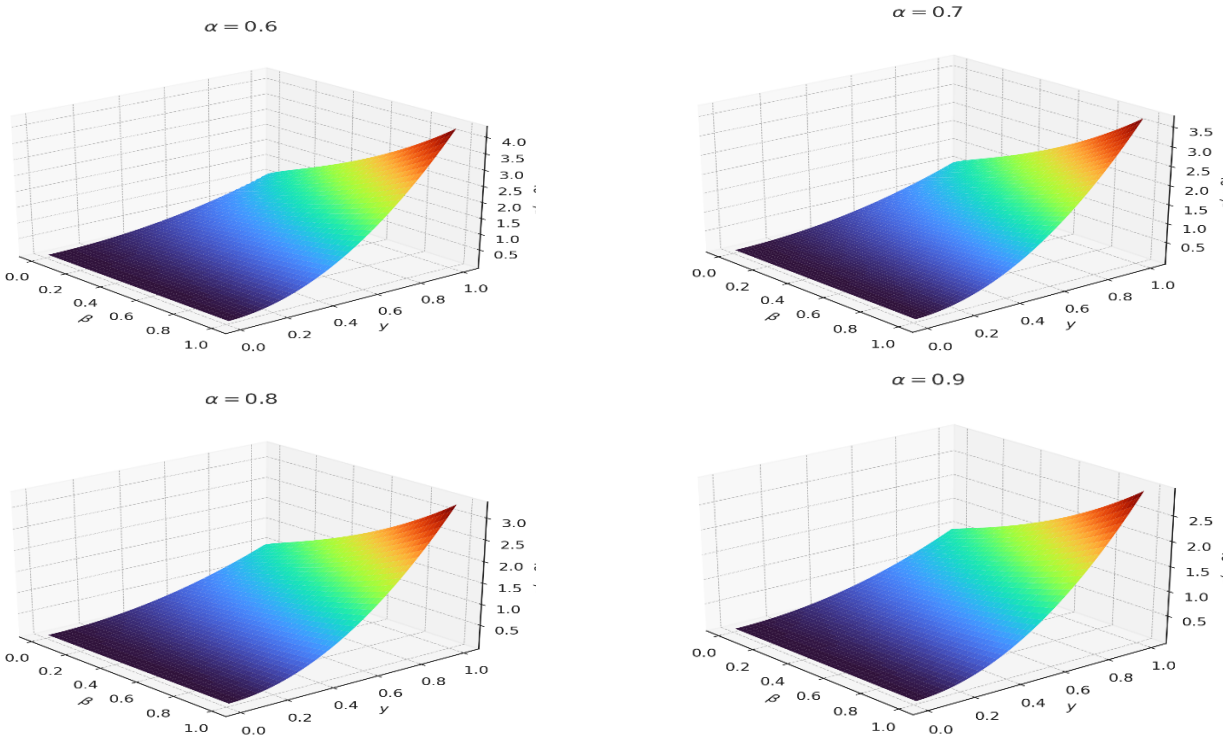


Figure 5. The graph of the solution $\phi(y, \beta)$ of the Example 3.

5. Conclusion

This paper introduced the ARPS technique that shows high efficiency and reliability when dealing with families of linear fractional PDEs, especially those with variable coefficients. Authors have faced many challenges in solving this kind of equations. The results show that the ARPS technique can present analytical series solutions with high accuracy, without needing linearity or discretization, that make the method more applicable ad efficient in comparison to other numerical methods. The computations performed in this research were done using

Mathematica software. The successful applications of the ARPS method in solving fractional tumor models that was shown in this article, is considered to be potential in mathematical biology and biomedical engineering, especially for describing tumor development, immune interactions, and drug transport mechanisms. Obtained results are promising for future works in bases on biological data and models of cancer tumor or other disease. This paper shows the validity of ARPS method in handling similar models.

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Author Contributions

All authors contributed equally to this work.

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