STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **15**, June 2026, pp 4699–4717.
Published online in International Academic Press (www.IAPress.org)

Adaptive Liu Estimator Modified for Estimating Regression Coefficients in the Presence of Multicollinearity

Salih Muayad Al Bakal*

Department of Operation research and Intelligent Technique, University of Mosul, Iraq

Abstract Multicollinearity among predictor variables remains a major challenge in regression analysis. This issue arises when predictors are highly correlated, leading to inflated variances of ordinary least squares (OLS) estimators and unstable coefficient estimates. Several remedial methods have been proposed to mitigate Multicollinearity, including ridge regression, the Liu estimator, and principal component regression. A critical factor determining the performance of shrinkage estimators such as the Liu estimator is the selection of an appropriate shrinkage parameter, denoted by d. This study proposes a novel method for estimating the optimal value of d. The performance of the proposed estimator was evaluated through Monte Carlo simulations under varying levels of Multicollinearity severity and sample size. The method was also applied to a real-world dataset. Results demonstrate that the proposed estimator achieves a substantially lower mean squared error (MSE) and mean absolute error (MAE) compared to existing estimators, indicating superior estimation accuracy and stability.

Keywords Ordinary least squares estimator, Multicollinearity, Liu Estimator, Shrinkage parameter, Modified Liu Estimator

DOI: 10.19139/soic-2310-5070-3319

1. Introduction

Multicollinearity among predictor variables remains a major challenge in multivariate regression analysis. When predictors exhibit high intercorrelations, ordinary least squares (OLS) estimators although unbiased—suffer from inflated variances, resulting in unstable coefficient estimates and unreliable predictions. To address this issue, several remedial methods have been proposed in the academic literature.

Ridge regression, originally developed by [1], is an early biased estimation technique that mitigates variance inflation by adding a non-negative penalty parameter k to the diagonal elements of the $\mathbf{X}'\mathbf{X}$ matrix. [2] proposed an alternative approach based on principal component analysis (PCA), arguing that replacing the original predictors with orthogonal principal components alleviates the effects of Multicollinearity. More recently, [24] employed principal component regression and compared its performance with ridge regression in addressing Multicollinearity in secondary macroeconomic data obtained from the World Bank, the International Monetary Fund (IMF), and the Nigerian Debt Management Office.

[3] introduced a biased estimator known as the Liu estimator as an extension of ridge regression. This approach applies double shrinkage to OLS coefficients through a tuning parameter $d \in (0, 1)$. The estimation accuracy of the Liu estimator critically depends on the appropriate selection of d. [3] initially proposed an optimal estimator denoted d opt. Subsequently, [4] developed a method for selecting d from a set of candidate values, an approach later refined by [5].

During the late 1990s, the Liu estimator framework was further extended. [6] proposed a generalized biased estimator and demonstrated its superiority through mean squared error (MSE) matrix comparisons. [4] introduced

*Correspondence to: Salih Muayad Al Bakal (Email: salih.mooaed@uomosul.edu.iq). Department of Operation research and Intelligent Technique, University of Mosul, Iraq.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

a composite estimator that integrates ridge regression with a two-parameter Liu estimator. Their method minimizes the MSE matrix and establishes explicit links between shrinkage parameters and Multicollinearity diagnostics such as the variance inflation factor (VIF). [7] developed an almost unbiased Liu estimator (AULE) for multicollinear models, deriving optimal d values based on the covariance matrix of the estimators.
Recent contributions have continued to refine shrinkage parameter selection. [8] proposed several modified Liu estimators and evaluated their sensitivity to d using simulation studies and real-data applications. [9] conducted a comprehensive review of existing methodologies for estimating the shrinkage parameter in Liu-type estimators within linear regression frameworks. [14] introduced a jackknife gamma Liu-type estimator (JGLTE) and its modified version (MJGLTE) for datasets affected by Multicollinearity. [12] adapted the jackknife technique for Poisson regression using the Liu estimator. [16] proposed an almost unbiased Liu estimator that balances the high variance of OLS with the excessive bias of the traditional Liu estimator, achieving reduced bias while maintaining reasonable variance. Finally, [17] generalized the Liu-type estimator through a D-vector formulation incorporating two parameters (k and d), demonstrating that the proposed estimator yields a lower mean squared error than OLS under Multicollinearity.

2. Statistical Methodology

The Liu estimator is employed to mitigate Multicollinearity among predictor variables. It represents a generalization of the ridge regression estimator and was first proposed by [3]. The procedure takes the same line as ridge regression but it introduces an extra parameter d to make the parameters more stable. The general linear regression model describing the relationship between the predictor variables and the response variable is expressed as follows:

$$y = X\beta + e \tag{1}$$

Where y is an $n \times 1$ vector which includes observations of the response variable, X is an $n \times p$ dimension matrix of the columns of the predictive variables, e is an $n \times 1$ dimension vector the disturbances vector which Cov(e) = $\sigma^2 I_n$.
The formula for Liu estimator takes the following form:

$$\widehat{\beta}_{(Liu)} = (X'X + I_p)^{-1}(X'X + dI)\widehat{\beta}_{(OLS)} \tag{2}$$

Equation (2) can be written in Canonical form as follows [18]:

$$X'X = \Omega\Lambda\Omega',\ \Lambda = diag(\lambda_1, \dots, \lambda_p)$$

with orthogonal Ω So:
$\theta = \Omega'\beta$, $\hat{\theta} = \Omega'\hat{\beta}$ and $Z = X\Omega$
Then $Var\left(\hat{\beta}\right)$ of OLS is:

$$Var\left(\widehat{\beta}\right) = \sigma^2(X'X)^{-1}$$

By canonical form, then $Var(\widehat{\theta})$ of OLS is:

$$Var\left(\widehat{\theta}\right) = \sigma^2\Omega\Lambda\Omega'$$

So, for j-th coefficient:

$$Var\left(\widehat{\theta}_j\right) = \sigma^2\lambda_j^{-1}$$

Where $\Omega\Omega^{'}$=I_p, $Var\left(\widehat{\theta}_j\right)$ can also be written as follows:

$$Var\left(\widehat{\theta}_j\right) = \frac{\sigma^2}{\lambda_i} \tag{3}$$

The Liu estimator can be written in equation (2) in canonical form:

$$\widehat{\theta}_{Liu(d)} = (\Omega\Lambda\Omega' + I_p)^{-1}(\Omega\Lambda\Omega' + dI)\widehat{\theta}_{OLS}$$

For j-th coefficient:

$$\widehat{\theta}_{j(Liu)(d)} = (\lambda_j + 1)^{-1}(\lambda_j + d)\widehat{\theta}_{j(OLS)}$$

by simplification:

$$\widehat{\theta}_{j(Liu)(d)} = \frac{(\lambda_j + d)}{(\lambda_j + 1)}\widehat{\theta}_{j(OLS)} \tag{4}$$

The j-th component bias of the Liu estimator can be written as follows:

$$Bias_{j(Liu)(d)=}E(\widehat{\theta}_{j(Liu)}) - \theta_j$$

Substituting from equation (4), It will produce:

$$Bias_{j(Liu)(d)} = \frac{(\lambda_j + d)}{(\lambda_j + 1)}\widehat{\theta}_{j(OLS)} - \theta_j$$

by simplification

$$Bias_{j(Liu)(d)} = \left(\frac{d-1}{\lambda_j + 1}\right)\theta_j \tag{5}$$

The variance of j-th coefficient is:

$$var(\widehat{\theta}_{j(Liu)(d)}) = var\left(\frac{\lambda_j + d}{\lambda_j + 1}\theta_j\right)$$

$$var(\widehat{\theta}_{j(Liu)(d)}) = \left(\frac{\lambda_j + d}{\lambda_j + 1}\right)^2 var(\widehat{\theta}_j) \tag{6}$$

Substituting equation (3) into equation (6), It will produce:

$$var(\widehat{\theta}_{j(Liu)(d)}) = \frac{\sigma^2\,(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} \tag{7}$$

Thus, the Total MSE is:

$$MSE_{(d)} = \sum_{j=1}^{p} Bias^2_{j(Liu)(d)} + var(\widehat{\theta}_{j(Liu)(d)}) \tag{8}$$

Substituting equation (5) and equation (7) into equation (8), It will produce:

$$MSE_{(d)} = \sum_{j=1}^{p}\left(\left(\frac{d-1}{\lambda_j + 1}\right)^2 {\theta_j}^2 + \frac{\sigma^2\,(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2}\right)$$

In simpler terms [20]:

$$MSE_{(d)} = \sigma^2\sum_{j=1}^{p}\frac{(\lambda_j + d)^2}{\lambda_j(\lambda_j + 1)^2} + (1-d)^2\sum_{j=1}^{p}\frac{{\theta_j}^2}{(\lambda_j + 1)^2} \tag{9}$$

3. The Gap of Liu Estimator

One of the methods that have shown high effectiveness in dealing with Multicollinearity among predictive variables and also in giving correct estimates of the parameters is the Liu estimator. However, its performance is highly dependent on the appropriate selection of the shrinkage parameter d. Much of the existing literature focuses on deriving an optimal value of d that minimizes MSE in the presence of Multicollinearity. The estimation of a single shrinkage parameter d can reveal some weaknesses in the estimation results. To solve this problem, [19] described the standard Liu estimator, which uses one shrinkage parameter d as not being able to provide optimal shrinkage in all directions simultaneously, as the level of Multicollinearity differs in different directions. In turn, the application of a single parameter of shrinkage to address Multicollinearity with different degrees of strength undermines the performance of the estimation and has impact on the MSE.

4. The conceptual basis

4.1. The standard Shrinkage of Liu estimator

In the [3] estimator, a single scalar parameter d is used to uniformly shrink all regression coefficients. Because the shrinkage factor applied to coefficient j depends on both d and the corresponding eigenvalue λ_j of the matrix, the estimator can be expressed as follows:

$$f_i(d) = \frac{\lambda_j + d}{\lambda_j + 1} \tag{10}$$

Formula (10) is derived under the premise that all coefficients $\hat{\theta}_j$ are adjusted by a single shrinkage parameter d. The magnitude of some of the eigenvalues λ_j is small, which indicates high-instability directions of severe multicollinearity, whereas the magnitude of others is large, which indicates stable and well-conditioned directions. This is a theoretical weakness of the single-parameter method.

Moreover, employing a single shrinkage parameter necessitates a trade-off: a value of d that sufficiently stabilizes coefficients affected by strong Multicollinearity may simultaneously introduce unwanted bias in directions where Multicollinearity is weak [4].

4.2. Variable-wise Shrinkage

This weakness of using a single value of d caused the extension of the estimation of the shrinkage parameter by creating a vector D, which consists of many values of d expressed as follows:

$$D = (d_1, \ldots, d_p)$$

Therefore, equation (10) can be written in the following form:

$$f_i(d_j) = \frac{\lambda_j + d_j}{\lambda_j + 1} \tag{11}$$

The main advantage of formula (11) is that it addresses strong Multicollinearity. (Small d_j) only, leaving weak Multicollinearity untouched by (big d_j, where $d_j \simeq 1$).

The theoretical argument can be summarized as follows: using a single value of d is too rigid to adequately accommodate variations in Multicollinearity severity across different eigen-directions. In other words, a single d value cannot be optimal for all coefficients simultaneously. However, employing a shrinkage parameter vector D with distinct values for each direction enables more balanced shrinkage that is proportional to the magnitude of the corresponding eigenvalues.

5. The motivation for finding a vector D

From Equation (10), when a single value of d is used to minimize the mean squared error (MSE), one must find a value of d that simultaneously balances all p terms in the equation. Since λ_1 is typically small while λ_p is large, the optimal d for term 1 differs from the optimal d for term p. Consequently, identifying a single d value that achieves this balance proves difficult motivating the development of a shrinkage parameter vector D.
From equation (4), the parameters of the Liu estimator can be estimated using the vector D for different d_j values, where $D = (d_1, \ldots, d_p)$ for each eigenvalue. This is as follows:

$$\hat{\theta}_{j(Liu)(d)} = daig\left(\frac{\lambda_1 + d_1}{\lambda_1 + 1}, \ldots, \frac{\lambda_p + d_p}{\lambda_p + 1}\right)\hat{\theta}_{OLS}$$

Therefore, the total MSE can be found as follows:

$$MSE_{(D)} = \sum_{j=1}^{p}\left(\frac{\sigma^2\left(\lambda_j + d_j\right)^2}{\lambda_j\left(\lambda_j + 1\right)^2} + \frac{\left(1 - d_j\right)^2 {\theta_j}^2}{\left(\lambda_j + 1\right)^2}\right) \tag{12}$$

6. Proposed Vector D

Variance inflation factors (VIFs) are diagnostic measures used to detect multicollinearity among predictor variables. [11, 21, 22] demonstrated that VIFs quantify the severity of multicollinearity between predictors. Thus, this study suggests a vector D, which contains the $d_j$values obtained through the discovery of a correlation with the VIF values. The VIF values indicate multicollinearity between variables. The foundation of this relationship can be described in the following way:

$$Var\left(\hat{\beta}\right) = \sigma^2\left(X'X\right)^{-1}$$

So, the variance of the j-th Coefficient is:

$$Var\left(\hat{\beta}_j\right) = \sigma^2 C_{jj} \tag{13}$$

The diagonal elements C_{jj} for centered predictors can be write:

$$X'X = S$$

Were $S_{jj} = \sum_{j=1}^{p}\left(X_{ij} - \overline{X}_j\right)^2$
One of the fundamental implications of regression theory states that [10]:

$$C_{jj} = \frac{1}{S_{jj}(1 - R^2)} \tag{14}$$

Where R^2 is the coefficient of determination when the variable X_j is regressed on the other variables.
The values of VIF parameters can be expressed in the following format [11]:

$$VIF_j = \frac{1}{(1 - R^2)} \tag{15}$$

Substituting equation (15) into equation (14) yields:

$$C_{jj} = \frac{VIF}{S_{jj}} \tag{16}$$

The values of C_{jj} of equation (16) can be substituted into equation (13). It will produce:

$$Var\left(\hat{\beta}_j\right) = \sigma^2\frac{VIF}{S_{jj}} \tag{17}$$

The comparison in equation (3) can be compared with equation (16). It will produce:

$$\frac{\sigma^2}{S_{jj}} VIF \propto \frac{\sigma^2}{\lambda_j}$$

Neglecting the standard constants concern to the standard transformation, the following relationship will result:

$$VIF \propto \frac{1}{\lambda_j}$$

That mean:

$$\lambda_j \propto \frac{1}{VIF} \tag{18}$$

When strong multicollinearity is present, the VIF is large and the corresponding eigenvalue λ_j is small.
When $\lambda_j \ll 1$, the following approximation is possible:

$$\lambda_j + 1 \simeq 1,\ \lambda_j + d_j \simeq d_j \tag{19}$$

Equation (12) for component j has the following form:

$$MSE_{(d_j)} = \frac{\sigma^2 \left(\lambda_j + d_j\right)^2}{\lambda_j \left(\lambda_j + 1\right)^2} + \frac{\left(1 - d_j\right)^2 {\theta_j}^2}{\left(\lambda_j + 1\right)^2} \tag{20}$$

By substituting equations (19) into equation (20):

$$MSE_{(d_j)} = \frac{\left(d_j\right)^2}{1} \sigma^2 VIF + \left(1 - d_j\right)^2 {\theta_j}^2 \tag{21}$$

From equation (21), it is observed that the value of $MSE_{(d_j)}$ depends on the value of d_j as well as on the VIF limit.
In order to minimize the approximation, equation (21) will be derived with respect to d_j as follows:

$$g'\left(d_j\right) = 2 d_j \sigma^2 VIF + 2\left(1 - d_j\right)(-1) {\theta_j}^2$$

$$g'\left(d_j\right) = 2\left[d_j \sigma^2 VIF + \left(1 - d_j\right)(-1) {\theta_j}^2\right] \tag{22}$$

By making equation (22) equal to zero:

$$\left(1 - d_j\right) {\theta_j}^2 = d_j \sigma^2 VIF$$

By simplification:

$$d_j = \frac{{\theta_j}^2}{\sigma^2 VIF + {\theta_j}^2} \tag{23}$$

From equation (23), it is observed that MSE in equation (21) depends on d_j, which depends on three values ${\theta_j}^2$, σ^2 and VIF.
In practice, a rule can be derived from standard multicollinearity diagnostic measures.
Common derivations of the shrinkage factor rely on two assumptions. First, variables are standardized to enable meaningful comparison of coefficients. Second, the values ${\theta_j}^2$ and σ^2 will probably have a limited effect on the value of d_j. That is, the formula for d_j could be as follows:

$$d_j = \frac{1}{1 + VIF_j} \tag{24}$$

Equation (24) can be adopted by conducting a simulation study that demonstrates the sensitivity of the proposed d_j based on different signal-to-noise ratios to prove its robustness. This will be done in the next section. Equation

(24) can be considered an approximation that supports the choice of a d_j value, which reduces MSE proportionally to the strength of the Multicollinearity between variables, depending on the VIF index.
The formula shown in equation (24) can possess features that indicate it can be an optimal value within the vector D. These features are:
1- When there is weak Multicollinearity, the VIF value will be small, therefore:

$$d_j = \frac{1}{1 + small} \simeq 1$$

Therefore, there will almost no shrinkage.
2- When there is strong Multicollinearity, the VIF value will be large, and therefore d_j will become small, which will lead to a strong shrinkage of that coefficient.
Therefore, the Liu estimator using vector D, (Liu_D), can be written as follows:

$$\hat{\beta}_{(Liu_D)} = (X'X + I_p)^{-1} (X'X + DI_p)\, \hat{\beta}_{(OLS)}$$

by canonical form:

$$\hat{\theta}_{Liu_{d_j}} = (\Omega\Lambda\Omega' + I_p)^{-1} (\Omega\Lambda\Omega' + DI)\, \hat{\theta}_{OLS}$$

For j-th coefficient:

$$\hat{\theta}_{j\left(Liu_{d_j}\right)} = (\lambda_j + 1)^{-1} (\lambda_j + d_j)\, \hat{\theta}_{j(OLS)}$$

7. Simulation Study signal-to-noise ratios

To assess the influence of both signal and noise on the proposed Liu_D estimator via the shrinkage parameters d_j, a simulation study was conducted using mean squared error (MSE) and mean absolute error (MAE), to evaluate estimator performance. The simulation design is based on the signal-to-noise ratio (SNR), defined as follows:

$$SNR = \frac{Var(X\beta)}{Var(\epsilon)} = \frac{Var(X\beta)}{\sigma^2} \tag{25}$$

Several studies have employed the signal-to-noise ratio (SNR) to evaluate estimator performance across varying noise levels [27, 28].
The simulation assesses estimator stability across varying signal-to-noise ratio (SNR) levels by systematically altering the noise-to-signal relationship. Three SNR levels will be assumed: a low level at values (0.25, 0.5), a medium level at (1), and a high level at (6, 10). Values of β (0.5, 1, -1.5, 2) were assumed for four variables, and (-2.5, 3, 3.5, -2.5) for eight variables. Multicollinearity was also assumed at two levels (0.9, 0.999). The data generation process was repeated 1,000 times. The (MSE) and (MAE) were computed in each iteration and then averaged across all replications. The software (RStudio v.4.4.1) was used in all simulation and real data studies in this paper.

7.1. Evaluating results

After conducting the simulations, the results were summarized in Tables 1 and 2. The results indicate that the Liu_D estimator achieved lower (MSE) and (MAE) than all competing estimators. Furthermore, the (MSE) of the Liu_D estimator remained stable across varying SNR levels. The stability of the Liu_Destimator with respect to SNR variations is illustrated in Figure 1; due to the large number of simulation scenarios, only eight representative cases are plotted (n=50,ρ=0.9,p=4), (n=50,ρ=0.999,p=4), (n=500,ρ=0.9,p=4), (n=500,ρ=0.999,p=4),(n=50,ρ=0.9,p=8),(n=50,ρ=0.999,p=8), (n=500,ρ=0.9,p=8), (n=500,ρ=0.999,p=8).

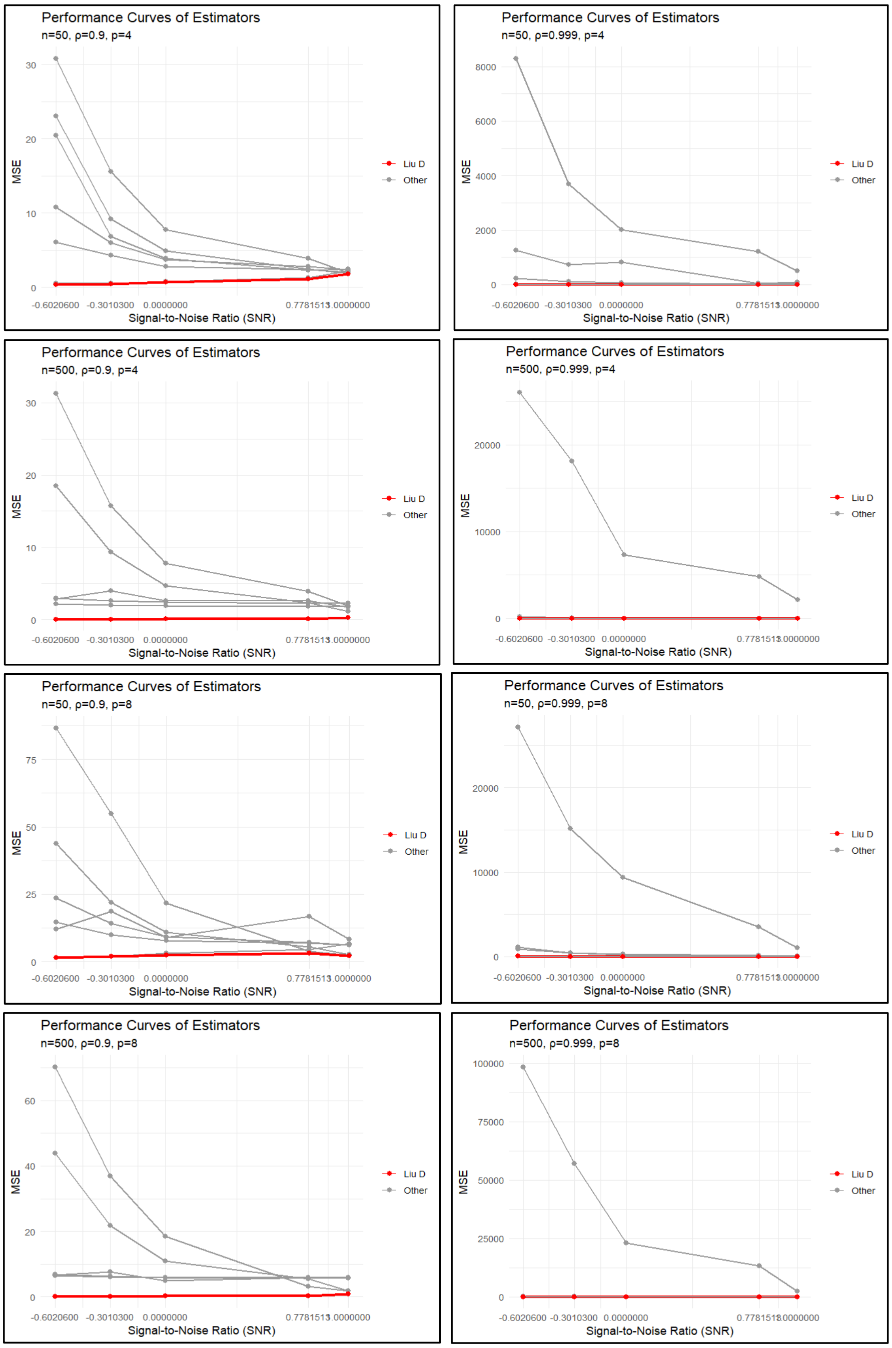

Figure 1. MSE vs. SNR curve.

7.2. Monte Carlo simulation

The Monte Carlo simulation method was used in the study to compare the proposed method [15]. From equation (1), the term e was generated, with $i.i.d\ N\left(0, \sigma^2\right)$, and $\sigma^2 = 0.5$ and 0.9. Similarly, $B = (B_1, \ldots, B_p)$

Table 1. MSE and MAE of different estimators of SNR, p=4

	OLS		Liu-GCV		Jack-Liu		PC		Liu-Opt.		Elastic Net		Liu_D	
	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
							n=50, $\rho = 0.9$							
0.25	30.83	1.645	20.42	1.663	10.77	1.587	6.098	1.272	0.539	2.288	23.06	3.673	0.370	1.399
0.5	15.55	1.136	6.820	1.148	6.006	1.122	4.304	1.072	0.511	1.365	9.156	2.475	0.476	0.985
1	7.747	0.806	3.920	0.811	3.681	0.792	2.790	0.881	0.749	0.873	4.882	1.737	0.671	0.706
6	3.904	0.553	2.254	0.556	2.773	0.540	2.344	0.776	1.278	0.577	2.429	1.251	1.089	0.491
10	1.923	0.400	2.441	0.404	2.413	0.382	2.071	0.716	2.172	0.419	1.890	1.111	1.902	0.371
							n=50, $\rho = 0.999$							
0.25	17.50	12.66	1246	12.8	8301	22.54	228.9	7.180	5.293	5.946	19.54	3.384	5.798	1.239
0.5	8.794	9.027	730.8	9.156	3681	14.79	118.4	5.211	5.487	2.026	8.350	2.358	5.847	1.125
1	4.339	6.542	813.2	6.607	2010	11.12	59.77	3.764	5.836	1.234	3.973	1.554	5.934	1.058
6	2.198	4.466	38.39	4.529	1219	8.076	29.33	2.719	6.512	1.031	2.024	1.114	6.143	1.007
10	1.094	3.124	90.18	3.184	507.9	5.450	15.32	2.015	7.631	0.962	1.382	0.950	6.536	0.978
							n=100, $\rho = 0.9$							
0.25	31.05	1.104	15.73	1.111	5.591	1.035	3.805	1.025	0.371	1.440	18.83	3.463	0.140	1.023
0.5	15.56	0.771	5.728	0.776	3.702	0.733	2.707	0.874	0.252	0.916	8.543	2.337	0.202	0.718
1	7.809	0.542	1.227	0.545	2.865	0.518	2.279	0.783	0.360	0.607	5.318	1.835	0.328	0.508
6	3.875	0.394	4.540	0.398	2.483	0.377	2.007	0.740	0.640	0.435	2.159	1.198	0.569	0.374
10	1.949	0.275	2.666	0.275	2.359	0.267	1.909	0.712	1.135	0.303	1.186	0.884	1.052	0.264
							n=100, $\rho = 0.999$							
0.25	17.74	8.947	216.1	9.015	1493	23.73	106.3	4.986	4.421	3.126	15.55	3.167	5.245	1.337
0.5	8.766	6.141	173.1	6.172	7125	17.09	52.15	3.590	4.739	1.585	7.357	2.153	5.316	1.157
1	4.357	4.247	8.202	4.264	4097	12.22	27.12	2.659	5.354	1.189	4.210	1.629	5.483	1.063
6	2.178	3.171	67.97	3.173	1576	8.185	15.91	2.069	6.386	1.013	1.701	1.069	5.813	0.995
10	1.100	2.153	6.555	2.163	1074	6.586	8.473	1.542	8.050	0.910	1.049	0.830	6.462	0.940
							n=200, $\rho = 0.9$							
0.25	31.12	0.773	3.510	0.775	4.067	0.739	2.806	0.868	0.165	0.967	19.08	3.487	0.055	0.742
0.5	15.70	0.552	2.660	0.552	3.068	0.533	2.285	0.779	0.108	0.647	9.032	2.427	0.089	0.531
1	7.797	0.384	2.977	0.385	2.601	0.376	2.051	0.735	0.173	0.442	5.203	1.841	0.157	0.372
6	3.886	0.268	1.644	0.268	2.351	0.264	1.908	0.717	0.315	0.313	2.654	1.300	0.289	0.262
10	1.961	0.190	1.894	0.191	2.255	0.189	1.836	0.716	0.591	0.220	1.248	0.901	0.560	0.188
							n=200, $\rho = 0.999$							
0.25	17.42	6.372	15.12	6.397	16.8	24.06	56.21	3.713	3.242	2.544	15.70	3.160	4.386	1.449
0.5	8.766	4.441	47.37	4.445	12.31	21.97	28.15	2.690	3.755	1.633	7.995	2.282	4.516	1.204
1	4.360	3.093	4.052	3.090	51.58	13.25	14.97	2.018	4.626	1.187	4.698	1.747	4.760	1.042
6	2.178	2.171	32.04	2.179	41.41	12.21	8.156	1.533	6.009	0.957	2.234	1.189	5.253	0.949
10	1.098	1.543	2.032	1.550	16.92	8.364	5.194	1.233	8.152	0.855	1.008	0.806	6.243	0.891
							n=500, $\rho = 0.9$							
0.25	31.32	0.481	2.889	0.481	2.962	0.474	2.172	0.764	0.037	0.583	18.48	3.468	0.018	0.473
0.5	15.71	0.346	4.005	0.346	2.582	0.342	1.993	0.730	0.037	0.405	9.325	2.450	0.032	0.340
1	7.806	0.241	2.568	0.241	2.395	0.240	1.880	0.725	0.065	0.282	4.634	1.711	0.061	0.237
6	3.908	0.169	2.627	0.169	2.272	0.168	1.826	0.721	0.124	0.200	2.342	1.213	0.117	0.166
10	1.945	0.118	1.773	0.118	2.245	0.117	1.825	0.719	0.239	0.143	1.168	0.869	0.231	0.116
							n=500, $\rho = 0.999$							
0.25	17.41	3.929	190.5	3.932	2605	30.26	23.33	2.484	1.777	2.514	16.33	3.260	2.962	1.541
0.5	8.696	2.836	27.40	2.833	1809	24.28	12.17	1.827	2.370	1.683	8.018	2.272	3.126	1.178
1	4.349	2.004	3.041	2.007	7296	17.31	7.309	1.463	3.367	1.166	3.990	1.588	3.461	0.983
6	2.190	1.356	9.317	1.359	4788	12.62	4.387	1.150	4.866	0.855	2.017	1.126	4.124	0.849
10	1.096	0.962	2.120	0.963	2140	8.858	3.149	0.959	7.106	0.717	1.018	0.810	5.445	0.760

was generated according to $\sum_{j=1}^{p} \beta_j^2 = 1$, such that $B_1 = B_2 = \ldots = B_p$. The explanatory variables $X_i^{'} = (x_{i1}, x_{i2} \ldots, x_{in})$ have been generated from the following formula:

$$x_{ij} = (1-\rho)^{1/2}\, w_{ij} + \rho w_{ip}, \quad {\scriptstyle i=1,\ldots,n, \quad j=1,\ldots,p} \tag{26}$$

Where w_{ij} are an independent standard normal pseudo-random number.
The parameter ρ represents the degree of multicollinearity among predictor variables. To evaluate estimator performance, four levels of multicollinearity were examined: $= 0.90,\ 0.95, 0.99$ and 0.999. The simulation design components are described as follows:

Table 2. MSE and MAE of different estimators of SNR, p=8

	OLS		Liu-GCV		Jack-Liu		PC		Liu-Opt.		Elastic Net		Liu_D	
	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
							n=50, $\rho = 0.9$							
0.25	43.69	2.046	12.00	2.073	23.43	2.269	14.67	1.899	1.522	2.776	86.72	7.450	2.653	1.698
0.5	21.89	1.458	18.66	1.480	14.15	1.625	10.00	1.503	1.971	1.780	54.85	5.803	2.907	1.241
1	10.93	1.018	9.019	1.034	9.225	1.135	7.859	1.219	2.997	1.185	21.54	3.720	3.310	0.894
6	5.425	0.744	16.64	0.754	7.025	0.779	6.777	1.047	4.515	0.837	3.671	1.521	4.163	0.687
10	2.715	0.505	8.196	0.512	6.202	0.532	6.190	0.939	6.701	0.647	2.284	1.237	5.818	0.526
							n=50, $\rho = 0.999$							
0.25	46.18	22.27	1144	22.60	2718	37.66	925.6	15.81	30.31	9.707	51.26	5.747	34.39	2.256
0.5	23.03	15.76	421.2	15.93	1515	27.01	453.2	11.08	31.19	3.436	31.08	4.401	34.65	2.051
1	11.43	11.19	257.2	11.29	9397	21.70	232.7	7.977	32.31	2.255	12.37	2.820	34.81	1.956
6	5.797	7.863	105.7	8.000	3504	14.56	117.9	5.724	34.46	1.927	2.726	1.278	35.37	1.893
10	1.932	4.612	79.65	4.686	1031	7.840	42.50	3.436	40.92	1.794	1.282	0.898	37.45	1.853
							n=100, $\rho = 0.9$							
0.25	43.77	1.386	6.345	1.393	11.39	1.363	9.672	1.446	0.695	1.889	83.46	7.164	0.992	1.258
0.5	21.61	0.966	6.302	0.970	8.252	0.976	7.547	1.191	0.804	1.228	46.89	5.508	1.115	0.893
1	10.76	0.687	7.217	0.692	6.916	0.694	6.650	1.028	1.317	0.857	21.63	3.608	1.367	0.649
6	5.493	0.492	5.111	0.493	6.307	0.493	6.114	0.940	2.082	0.622	3.402	1.472	1.914	0.471
10	1.835	0.282	6.653	0.282	5.861	0.283	5.785	0.869	4.368	0.408	1.895	1.102	3.978	0.306
							n=100, $\rho = 0.999$							
0.25	45.88	14.75	542.0	14.88	4064	41.64	412.1	10.57	23.99	5.162	54.97	5.839	30.42	2.405
0.5	22.83	10.53	944.9	10.59	2405	30.65	203.9	7.550	25.22	2.963	29.83	4.364	30.57	2.099
1	11.60	7.449	134.0	7.477	1150	20.22	102.2	5.387	27.52	2.206	12.75	2.790	31.07	1.938
6	5.719	5.285	252.7	5.309	5889	15.64	55.5	3.969	31.05	1.832	2.135	1.175	31.90	1.810
10	1.916	3.102	19.71	3.116	1704	8.985	22.90	2.507	40.95	1.652	1.373	0.937	35.39	1.753
							n=200, $\rho = 0.9$							
0.25	43.63	0.942	12.79	0.945	8.282	0.932	7.502	1.184	0.312	1.276	77.55	6.994	0.331	0.903
0.5	21.79	0.677	5.791	0.680	6.964	0.675	6.556	1.025	0.350	0.891	39.09	5.061	0.403	0.653
1	10.81	0.482	5.738	0.483	6.326	0.481	6.112	0.936	0.578	0.653	18.36	3.458	0.552	0.469
6	5.407	0.343	6.686	0.344	5.951	0.343	5.828	0.890	0.921	0.475	3.244	1.447	0.841	0.340
10	1.831	0.197	6.090	0.197	5.791	0.197	5.702	0.868	2.144	0.297	2.237	1.187	2.004	0.207
							n=200, $\rho = 0.999$							
0.25	46.01	10.45	1263	10.49	9579	50.07	202.3	7.504	16.37	4.569	48.96	5.534	24.57	2.620
0.5	23.01	7.370	224.0	7.377	4206	33.65	102.1	5.367	18.43	3.039	27.63	4.225	24.92	2.154
1	11.54	5.218	82.74	5.235	1995	24.67	53.55	3.884	21.72	2.185	11.73	2.798	25.51	1.847
6	5.796	3.658	13.90	3.670	5368	15.08	28.91	2.830	26.65	1.707	2.235	1.188	26.90	1.684
10	1.923	2.136	33.32	2.138	4001	12.68	13.50	1.841	39.04	1.444	1.489	0.979	31.91	1.579
							n=500, $\rho = 0.9$							
0.25	43.77	0.607	6.559	0.607	6.922	0.604	6.428	0.991	0.099	0.830	70.31	6.700	0.079	0.595
0.5	21.82	0.427	7.533	0.427	6.283	0.426	6.010	0.913	0.121	0.604	36.87	4.812	0.111	0.420
1	10.87	0.303	4.947	0.304	5.957	0.303	5.790	0.887	0.193	0.448	18.50	3.422	0.175	0.300
6	5.459	0.211	5.782	0.211	5.825	0.211	5.706	0.877	0.324	0.313	3.176	1.416	0.301	0.209
10	1.815	0.125	5.780	0.125	5.727	0.125	5.645	0.876	0.849	0.198	1.804	1.082	0.814	0.127
							n=500, $\rho = 0.999$							
0.25	46.18	6.431	273.7	6.431	9847	53.54	77.36	4.680	8.521	4.361	49.08	5.610	15.71	2.628
0.5	23.05	4.599	109.1	4.605	5715	38.76	42.36	3.479	10.87	2.979	24.61	3.938	16.17	2.059
1	11.55	3.236	24.46	3.245	2312	24.26	25.06	2.617	14.41	2.087	12.74	2.838	16.93	1.678
6	5.736	2.312	24.92	2.316	1337	21.13	14.91	1.952	19.31	1.509	2.175	1.176	18.52	1.435
10	1.921	1.328	18.33	1.329	2457	10.40	8.660	1.326	31.45	1.157	1.261	0.905	24.94	1.263

Error variance: $\sigma^2 = 0.5,\ 0.9$
Sample size: $n = 50,\ 100,\ 200, 500$
Predictor variables: $p = 4,\ 8$
Multicollinearity: $\rho = 0.90,\ 0.95,\ 0.99, 0.999$
The generation process was repeated 1000 times on the generated data according to the combination of values σ^2, n, p, and ρ. MSE and mean absolute error (MAE) were calculated for each iteration according to the following

two formulas:

$$\left\{ \begin{array}{c} MSE_i\left(\hat{\beta}\right) = \left(\hat{\beta} - \beta\right)^{\prime}\left(\hat{\beta} - \beta\right) \\ MAE_i\left(\hat{\beta}\right) = \frac{1}{p}\sum_{j=1}^{p}\left|\hat{\beta}_j - \beta\right| \end{array} \right. \quad i = 1, 2, \ldots, 1000 \tag{27}$$

The simulation compared the following estimators: (i) the proposed Liu_D estimator employing the shrinkage parameter vector D; (ii) ordinary least squares (OLS) as the baseline unbiased estimator; (iii) the Liu estimator with shrinkage parameter selected via generalized cross-validation (Liu-GCV); and (iv) the jackknife Liu estimator (Jack-Liu) using the optimal scalar parameter $\hat{d}_{Liu}$ proposed by [3].

$$\hat{d}_{Liu} = \frac{\sum_{j=1}^{p}\lambda_j\left(\hat{\theta}_j^2 - \hat{\sigma}^2\right)}{\sum_{j=1}^{p}\left(\hat{\sigma}^2 + \lambda_j\hat{\theta}_j^2\right)}$$

The principal component (PC) estimator addresses multicollinearity by transforming predictors into orthogonal (uncorrelated) principal components.

The (Elastic Net) method was also used for comparison with the Liu_D estimator as a supporting method for the proposed estimator [23]. Finally, the Liu-Opt estimator was included, which employs direction-specific shrinkage parameters d_j derived according to [17].

$$d_j = \frac{\hat{\sigma}^2 - \alpha_j^2 k_j}{\alpha_j^2 + \frac{\hat{\sigma}^2}{\lambda_j}}$$

7.3. Performance evaluation mechanism

The accuracy of the proposed method was evaluated by computing the mean squared error (MSE) for each estimator relative to the Liu_D estimator employing the shrinkage parameter vector d. For each simulation iteration, the minimum MSE achieved by the proposed method was recorded alongside the corresponding MSE values of all competing estimators, accounting for variations in Multicollinearity level (ρ), sample size (n), error variance (σ^2), and number of predictor variables (p). To provide a comprehensive performance assessment, comparisons were also conducted using mean absolute error (MAE) alongside bias and variance decomposition for each estimator.

7.4. Discussion of results

Tables 3 and 4 show that the Liu estimator with the proposed shrinkage parameter vector D outperformed all competing estimators, consistently achieving the lowest mean squared error (MSE). The results also indicate that MSE increases with higher Multicollinearity (ρ) and a larger number of predictors (p). Conversely, MSE decreases as sample size (n) increases for all estimators, with this effect most pronounced when $p = 8$.

Moreover, the Liu estimator employing the proposed shrinkage parameter vector D demonstrated superior performance relative to all competing estimators, consistently achieving the smallest mean absolute error (*MAE*) across all simulation scenarios.

Tables 5 and 6 present bias and variance values for each estimator to evaluate the performance of the $\mathrm{Liu_D}$ estimator. Both bias and variance varied across estimators. Through the variance values of the Liu_D estimator containing the $\mathrm{Liu_D}$ estimator, the variance value may reduce by (60.60%) in the case of P=4 in Table 5, and compared to the other estimators. As for P=8 in Table 6, the $\mathrm{Liu_D}$ estimator has decreased variance with (42.42%), compared to the other estimators as well. These ratios were computed as the percentage of simulation replications (out of 100) in which the $\mathrm{Liu_D}$ estimator exhibited smaller variance than each competing estimator. Although the $\mathrm{Liu_D}$ estimator achieved only modest bias reduction relative to other estimators a finding consistent with Abdelwahab et al. (2024)—it attained the lowest mean squared error (MSE) across all simulation scenarios. Smaller variance values for the $\mathrm{Liu_D}$ estimator are highlighted in bold in Tables 5 and 6.

In summary, the simulation results presented in Tables 3 to 6 demonstrate that the $\mathrm{Liu_D}$ estimator with the proposed shrinkage parameter vector D outperformed all competing estimators according to mean squared error (MSE), mean absolute error (MAE), bias, and variance criteria.

Table 3. MSE and MAE of different estimators, p=4

	OLS		Liu-GCV		Jack-Liu		PC		Liu-Opt.		Elastic Net		Liu_D	
ρ	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
$\sigma^2 = 0.5$														
n=50														
0.90	3.181	0.066	1.034	0.066	0.883	0.061	0.833	0.130	0.427	0.483	0.125	0.431	**0.115**	**0.050**
0.95	3.395	0.093	2.340	0.171	1.979	0.074	1.029	0.131	0.778	0.517	0.219	0.429	**0.206**	**0.031**
0.99	5.016	0.377	16.49	0.324	423.8	0.284	5.046	0.099	1.080	0.326	0.573	0.422	**0.282**	**0.086**
0.999	4.506	1.627	65.16	1.597	56.23	0.524	50.27	0.239	0.180	0.108	1.751	0.414	**0.049**	**0.038**
n=100														
0.90	3.787	0.043	1.747	0.091	1.706	0.045	0.180	0.140	0.322	0.596	0.162	0.415	**0.083**	**0.033**
0.95	3.290	0.074	1.784	0.074	1.476	0.066	0.212	0.113	0.506	0.434	0.133	0.417	**0.131**	**0.065**
0.99	4.930	0.051	12.90	0.255	4.069	0.064	2.912	0.098	1.120	0.366	0.319	0.414	**0.287**	**0.035**
0.999	3.683	0.810	70.54	0.968	42.11	0.739	63.98	0.626	0.412	0.239	0.916	0.409	**0.106**	**0.064**
n=200														
0.90	3.693	0.037	0.092	0.056	0.094	0.034	0.093	0.152	0.187	1.033	0.049	0.408	**0.048**	**0.030**
0.95	4.026	0.043	0.392	0.046	0.376	0.043	0.158	0.101	0.333	0.754	0.092	0.408	**0.085**	**0.041**
0.99	4.630	0.065	3.092	0.104	2.149	0.069	0.775	0.124	0.955	0.450	0.258	0.406	**0.242**	**0.064**
0.999	3.893	0.265	18.33	0.279	200.6	0.187	7.119	0.235	0.743	0.163	0.396	0.404	**0.188**	**0.062**
n=500														
0.90	3.698	0.017	0.029	0.021	0.029	0.018	0.160	0.157	0.085	0.545	0.260	0.408	**0.022**	**0.016**
0.95	3.878	0.031	0.797	0.025	0.761	0.031	0.137	0.107	0.170	0.738	0.254	0.402	**0.043**	**0.025**
0.99	4.495	0.047	0.755	0.056	0.580	0.040	0.983	0.079	0.645	0.427	0.253	0.401	**0.162**	**0.037**
0.999	3.711	0.155	1.379	0.153	134.9	0.119	1.181	0.089	1.250	0.263	0.352	0.400	**0.314**	**0.046**
$\sigma^2 = 0.9$														
n=50														
0.90	4.550	0.061	2.412	0.160	2.684	0.047	0.684	0.198	0.847	0.722	0.947	0.775	**0.226**	**0.037**
0.95	4.919	0.110	9.749	0.122	6.403	0.065	0.809	0.153	1.419	0.405	1.054	0.818	**0.375**	**0.064**
0.99	6.037	0.299	14.62	0.283	9.558	0.237	1.632	0.206	1.872	0.296	0.903	0.757	**0.489**	**0.097**
0.999	4.824	0.070	1.124	0.061	1.103	0.045	0.875	0.153	0.962	0.551	0.873	0.745	**0.258**	**0.038**
n=100														
0.90	2.996	0.041	2.203	0.072	2.134	0.041	0.166	0.148	0.562	1.055	0.880	0.751	**0.148**	**0.039**
0.95	4.357	0.070	2.862	0.108	2.449	0.068	0.348	0.117	0.941	0.535	0.877	0.750	**0.243**	**0.055**
0.99	4.120	0.106	11.26	0.200	6.934	0.257	1.612	0.181	2.096	0.352	0.861	0.743	**0.537**	**0.064**
0.999	3.329	0.522	24.08	0.734	15.28	0.302	23.79	0.269	0.854	0.081	0.844	0.736	**0.220**	**0.036**
n=200														
0.90	3.749	0.030	0.299	0.047	0.288	0.030	0.144	0.161	0.354	0.752	0.841	0.733	**0.030**	**0.091**
0.95	3.851	0.033	0.199	0.037	0.205	0.030	0.166	0.119	0.644	0.652	0.841	0.733	**0.027**	**0.164**
0.99	3.719	0.070	0.821	0.155	0.516	0.042	0.712	0.074	1.806	0.547	0.835	0.731	**0.031**	**0.458**
0.999	3.250	0.253	18.54	0.185	4.893	0.232	0.679	0.121	1.398	0.082	0.828	0.727	**0.054**	**0.354**
n=500														
0.90	3.882	0.026	0.171	0.026	0.174	0.027	0.125	0.141	0.158	0.672	0.822	0.723	**0.041**	**0.025**
0.95	4.402	0.013	0.103	0.017	0.104	0.012	0.169	0.114	0.314	0.647	0.822	0.723	**0.079**	**0.012**
0.99	4.132	0.040	1.956	0.042	1.412	0.037	0.715	0.082	1.137	0.676	0.820	0.722	**0.286**	**0.036**
0.999	4.130	0.188	35.08	0.222	10.17	0.193	4.451	0.061	2.271	0.370	0.816	0.734	**0.571**	**0.044**

8. Real Data

This section will use two sets of real-world data to demonstrate the effectiveness of the proposed method.

8.1. Real Data 1: Longley data

To evaluate the performance of the proposed method for determining the shrinkage parameter vector D, it was applied to real-world data specifically, the Longley dataset [13]. The dataset comprises a response variable (total employment) and six predictor variables, of which five were retained for analysis. The year variable was excluded because it is a time variable, not a numerical one. Table 7 describes the five retained predictor variables.

Longley published his data in 1967. The data describe the relationship between the dependent variable total employment and five predictor variables. Comprising 16 annual observations of U.S. economic indicators from 1947 to 1962, the Longley dataset exhibits severe Multicollinearity among predictors. Pairwise correlation coefficients were computed to quantify these interrelationships, as presented in Table 8.

Table 4. MSE and MAE of different estimators, p=8

	OLS		Liu-GCV		Jack-Liu		PC		Liu-Opt.		Elastic Net		Liu_D	
ρ	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE	MSE	MAE
$\sigma^2 = 0.5$														
n=50														
0.90	5.236	0.126	1.408	0.210	1.452	0.080	0.450	0.169	1.135	1.008	0.337	0.462	**0.077**	**0.298**
0.95	5.392	0.208	3.235	0.236	2.554	0.173	2.064	0.301	1.633	0.570	0.429	0.457	**0.154**	**0.429**
0.99	5.170	0.700	2.893	0.872	2.724	4.869	2.860	0.547	2.562	0.465	0.300	0.436	**0.141**	**0.265**
0.999	4.050	2.500	188.895	2.550	69.595	1.914	161.632	1.323	0.561	0.173	0.292	0.430	**0.107**	**0.146**
n=100														
0.90	2.986	0.107	0.736	0.158	0.690	0.108	0.778	0.149	0.741	0.919	0.295	0.434	**0.192**	**0.105**
0.95	4.802	0.145	0.667	0.146	0.609	0.154	0.536	0.148	1.164	0.790	0.299	0.434	**0.298**	**0.137**
0.99	4.755	0.415	8.653	0.279	78.911	0.269	5.079	0.282	2.539	0.593	0.278	0.421	**0.149**	**0.132**
0.999	4.088	2.596	130.395	2.606	90.529	0.918	41.603	0.870	1.126	0.495	0.289	0.425	**0.287**	**0.159**
n=200														
0.90	3.582	0.093	0.481	0.094	0.466	0.096	0.271	0.130	0.422	1.120	0.270	0.413	**0.108**	**0.086**
0.95	3.930	0.072	0.340	0.089	0.217	0.071	0.239	0.150	0.788	1.114	0.270	0.413	**0.199**	**0.070**
0.99	3.740	0.348	8.917	0.434	3.788	5.444	1.205	0.262	2.009	0.458	0.262	0.408	**0.108**	**0.207**
0.999	3.952	1.688	57.340	1.818	49.243	14.558	15.960	0.633	2.143	0.367	0.256	0.403	**0.141**	**0.101**
n=500														
0.90	3.797	0.047	0.104	0.046	0.102	0.047	0.113	0.115	0.214	1.104	0.256	0.404	**0.054**	**0.045**
0.95	4.499	0.074	0.257	0.073	0.234	0.073	0.136	0.118	0.393	1.023	0.257	0.404	**0.099**	**0.072**
0.99	4.144	0.175	2.226	0.172	4.846	0.170	0.713	0.166	1.332	0.730	0.254	0.402	**0.235**	**0.156**
0.999	4.334	1.373	33.508	1.382	6.604	1.030	6.217	0.442	3.105	0.729	0.252	0.400	**0.240**	**0.143**
$\sigma^2 = 0.9$														
n=50														
0.90	3.380	0.171	0.566	0.267	0.562	0.162	0.688	0.139	1.792	0.925	1.109	0.837	**0.481**	**0.136**
0.95	5.304	0.201	2.084	0.192	1.902	0.184	1.393	0.281	2.913	1.060	1.090	0.831	**0.762**	**0.176**
0.99	4.153	0.667	11.761	0.661	9.392	0.525	8.965	0.408	4.889	0.613	1.320	0.796	**1.275**	**0.216**
0.999	2.190	2.245	52.340	2.496	32.914	1.080	51.822	0.947	1.040	0.188	0.891	0.752	**0.271**	**0.102**
n=100														
0.90	4.109	0.113	0.506	0.117	0.475	0.115	0.465	0.130	1.221	1.296	0.948	0.778	**0.318**	**0.105**
0.95	4.028	0.122	0.643	0.164	0.588	0.146	0.619	0.123	2.206	0.736	0.948	0.778	**0.564**	**0.114**
0.99	4.609	0.309	7.072	0.324	131.373	0.471	2.818	0.277	4.613	0.597	0.891	0.754	**0.679**	**0.173**
0.999	3.157	0.288	3.718	0.228	2.000	1.132	3.718	0.226	4.673	0.705	0.850	0.735	**0.673**	**0.196**
n=200														
0.90	3.825	0.059	0.425	0.061	0.429	0.057	0.299	0.156	0.784	1.047	0.868	0.741	**0.201**	**0.056**
0.95	4.051	0.093	1.526	0.088	1.224	0.084	0.365	0.124	1.395	1.191	0.869	0.742	**0.353**	**0.080**
0.99	4.098	0.297	2.377	0.297	338.385	0.238	1.291	0.233	3.433	0.598	0.868	0.741	**0.867**	**0.203**
0.999	3.790	0.787	6.786	0.763	3.218	11.637	5.639	0.689	3.816	0.468	0.827	0.725	**0.764**	**0.176**
n=500														
0.90	3.600	0.055	0.304	0.055	0.298	0.054	0.181	0.112	0.373	1.141	0.826	0.725	**0.096**	**0.052**
0.95	4.162	0.066	0.532	0.067	0.475	0.067	0.430	0.096	0.720	1.068	0.826	0.726	**0.181**	**0.066**
0.99	3.909	0.154	1.737	0.162	0.862	0.121	0.939	0.126	2.419	0.946	0.828	0.726	**0.608**	**0.118**
0.999	3.995	0.551	26.138	0.533	7.543	0.290	3.783	0.323	5.631	0.556	0.811	0.718	**0.613**	**0.197**

As shown in Table 8, strong correlations exist between the GNP.deflator and Armed.Forces variables (*r* = 0.99), between the GNP.deflator and Population variables (*r* = 0.98), and between the Armed Forces and Population variables (*r* = 0.99). Multicollinearity is also present though less severe in moderate correlations between the GNP. deflator and Bunemployed variables (*r* = 0.69), between Bunemployed and Armed Forces (*r* = 0.60), and between Bunemployed and Population (*r* = 0.62). The condition number was computed as 500.685, confirming severe Multicollinearity in the dataset. Variance inflation factors (VIF_S) and their corresponding shrinkage parameters d_j are presented in Table 9.

Table 9 shows that all variance inflation factor values VIF_j very large, indicating Multicollinearity between the predictive variables [21]. Also, it is noted that the strength of this Multicollinearity differs according to the relationships of the predictive variables, based on the VIF_j values. As shown in Table 9, the shrinkage parameters d_j vary inversely with VIF_j: larger VIF_j values indicating stronger Multicollinearity correspond to smaller

Table 5. Bias and Variance of different estimators, P=4

	Liu-GCV		Jack-Liu		PC		Liu-Opt.		Elastic Net		Liu_D	
						$\sigma^2= 0.5$						
						n=50						
ρ	Bias	Var.	Bias	Var.	Bias	Var.	Bias	Var.	Bias	Var.	Bias	Var.
0.90	0.6659	0.5903	0.6154	0.5039	0.5977	0.4762	0.0840	0.4195	0.0004	0.1241	0.0160	**0.1149**
0.95	1.0026	1.3347	0.9226	1.1275	0.6716	0.5778	0.0241	0.7775	0.0014	0.2174	0.0137	**0.2054**
0.99	2.6626	9.4031	13.4918	241.8024	1.4707	2.8826	0.0103	1.0796	0.0041	0.5690	0.0104	**0.2817**
0.999	5.2858	37.2291	4.9099	32.1241	4.6422	28.7233	0.0101	0.1799	0.0048	1.7459	0.0101	**0.0489**
						n=100						
0.90	0.8724	0.9857	0.8628	0.9620	0.2866	0.0979	0.0449	0.3195	0.0003	0.1618	0.0064	**0.0833**
0.95	0.8858	0.9990	0.8067	0.8254	0.3049	0.1190	0.0308	0.5050	0.0004	0.1332	0.0077	0.1305
0.99	2.3516	7.3714	1.3242	2.3153	1.1184	1.6611	0.0060	1.1196	0.0008	0.3178	0.0079	**0.2865**
0.999	5.4989	40.3016	4.2487	24.0667	5.2369	36.5593	0.0050	0.4116	0.0011	0.9149	0.0050	**0.1063**
						n=200						
0.90	0.1989	0.0520	0.2013	0.0534	0.2018	0.0527	0.0565	0.1836	0.0002	0.0489	0.0042	**0.0480**
0.95	0.4101	0.2234	0.4017	0.2145	0.2599	0.0901	0.0233	0.3324	0.0003	0.0923	0.0039	**0.0848**
0.99	1.1518	1.7658	0.9600	1.2271	0.5765	0.4430	0.0035	0.9550	0.0009	0.1576	0.0041	0.2419
0.999	2.8031	10.4720	9.2737	114.6428	1.7468	4.0679	0.0025	0.7428	0.0009	0.3949	0.0025	**0.1882**
						n=500						
0.90	0.1141	0.0161	0.1146	0.0163	0.2626	0.0908	0.0539	0.0819	0.0003	0.0291	0.0018	0.0217
0.95	0.5886	0.4503	0.5756	0.4299	0.2443	0.0769	0.0219	0.1693	0.0003	0.0206	0.0016	0.0429
0.99	0.5689	0.4315	0.4984	0.3312	0.6491	0.5614	0.0029	0.6445	0.0008	0.0599	0.0015	0.1621
0.999	0.7688	0.7880	7.6048	77.0965	0.7116	0.6750	0.0010	1.2496	0.0023	0.1716	0.0010	0.3139
						$\sigma^2= 0.9$						
						n=50						
0.90	1.0262	1.3590	1.0819	1.5132	0.5488	0.3831	0.0666	0.8422	0.0019	0.3689	0.0123	**0.2256**
0.95	2.0457	5.5644	1.6571	3.6571	0.6173	0.4279	0.0218	1.4182	0.0022	0.4074	0.0165	**0.3748**
0.99	2.5032	8.3541	2.0239	5.4618	0.8365	0.9323	0.0102	1.8717	0.0021	1.4172	0.0103	**0.4886**
0.999	0.7051	0.6269	0.6990	0.6144	0.6134	0.4986	0.0837	0.9547	0.0055	4.5990	0.0149	**0.2582**
						n=100						
0.90	0.9739	1.2547	0.9586	1.2150	0.2723	0.0922	0.0661	0.5576	0.0008	0.2036	0.0069	0.1477
0.95	1.1082	1.6338	1.0249	1.3984	0.3889	0.1971	0.0276	0.9400	0.0014	0.3400	0.0078	0.2426
0.99	2.2010	6.4203	1.7242	3.9614	0.8358	0.9131	0.0055	2.0960	0.0020	0.8867	0.0070	**0.5366**
0.999	3.2136	13.762	2.5602	8.7336	3.1938	13.593	0.0050	0.8537	0.0037	2.9732	0.0050	**0.2198**
						n=200						
0.90	0.3579	0.1706	0.3512	0.1644	0.2492	0.0823	0.0520	0.3517	0.0004	0.0940	0.0042	0.0914
0.95	0.2918	0.1135	0.2966	0.1173	0.2712	0.0922	0.0264	0.6437	0.0007	0.1657	0.0043	0.1641
0.99	0.5934	0.4687	0.4708	0.2943	0.5528	0.4059	0.0029	1.8063	0.0018	0.4460	0.0034	0.4577
0.999	2.8193	10.595	1.4483	2.7956	0.5396	0.3883	0.0025	1.3977	0.0041	1.4912	0.0025	**0.3539**
						n=500						
0.90	0.2718	0.0976	0.2741	0.0992	0.2385	0.0677	0.0504	0.1555	0.0003	0.0370	0.0017	**0.0408**
0.95	0.2103	0.0588	0.2115	0.0595	0.2688	0.0963	0.0224	0.3135	0.0004	0.0672	0.0017	0.0793
0.99	0.9162	1.1168	0.7784	0.8065	0.5542	0.4083	0.0032	1.1366	0.0011	0.2005	0.0017	**0.2858**
0.999	3.8781	20.046	2.0888	5.8157	1.3812	2.5435	0.0010	2.2715	0.0028	0.5390	0.0010	**0.5706**

d_j values, thereby imposing greater shrinkage in the Liu estimator. Conversely, smaller VIF_j values indicating weaker Multicollinearity correspond to larger d_j values, resulting in milder shrinkage.
After applying the Longley data to the Liu_Destimator, regression coefficients and their Standard errors of estimators ($SE(\widehat{\beta})$) were computed. Mean squared error (MSE) values for all estimators were also calculated and compared against the Liu_Destimator. The results, presented in Table 10, indicate that the Liu_Destimator achieved the lowest MSE (0.00202) among all competing estimators, demonstrating superior estimation performance.

Boxplots of mean squared error (MSE) ratios were used to facilitate visual comparison of estimator performance. As shown in Figure 2, the Liu_D estimator consistently exhibits the lowest MSEratio, positioned at the bottom of the plot.

Table 6. Bias and Variance of different estimators for p=8

	Liu-GCV		Jack-Liu		PC		Liu-Opt.		Elastic Net		Liu_D	
ρ	Bias	Var.	Bias	Var.	Bias	Var.	Bias	Var.	Bias	Var.	Bias	Var.
						$\sigma^2 = 0.5$						
						n=50						
0.90	0.8109	0.7502	0.8237	0.7740	0.4592	0.2396	0.1278	1.1189	0.0010	0.3293	0.0253	0.2974
0.95	1.2327	1.7150	1.0957	1.3538	0.9816	1.1009	0.0573	1.6294	0.0017	0.5305	0.0294	**0.4277**
0.99	1.1620	1.5430	1.1275	1.4528	1.1554	1.5253	0.0082	2.5618	0.0047	1.2034	0.0083	**0.2648**
0.999	9.3891	100.74	5.6991	37.115	8.6850	86.202	0.0072	0.5612	0.5178	12.991	**0.0072**	**0.1464**
						n=100						
0.90	0.5862	0.3923	0.5676	0.3683	0.6031	0.4141	0.0830	0.7343	0.0007	0.1590	0.0189	0.1912
0.95	0.5582	0.3553	0.5336	0.3245	0.5002	0.2856	0.0307	1.1629	0.0009	0.3116	0.0170	0.2978
0.99	2.0100	4.6125	6.0715	42.048	1.5395	2.7087	0.0074	2.5393	0.0033	0.6531	0.0097	**0.1492**
0.999	7.8009	69.540	6.4999	48.280	4.4062	22.187	0.0036	1.1255	0.5663	9.0894	**0.0036**	**0.2875**
						n=200						
0.90	0.4768	0.2536	0.4694	0.2457	0.3557	0.1443	0.0855	0.4152	0.0004	0.0716	0.0104	0.1078
0.95	0.3987	0.1808	0.3185	0.1155	0.3337	0.1273	0.0280	0.7868	0.0005	0.1402	0.0101	0.1993
0.99	2.0414	4.7502	1.3305	2.0177	0.7499	0.6425	0.0043	2.0093	0.0032	0.3106	0.0067	**0.1075**
0.999	5.1731	30.578	4.7940	26.261	2.7291	8.5118	0.0018	2.1427	0.0133	0.2245	0.0018	**0.1410**
						n=500						
0.90	0.2199	0.0552	0.2181	0.0543	0.2309	0.0595	0.0710	0.2088	0.0004	0.0259	0.0045	0.0541
0.95	0.3470	0.1371	0.3308	0.1247	0.2518	0.0722	0.0369	0.3917	0.0005	0.0510	0.0050	0.0988
0.99	1.0195	1.1862	1.5047	2.5823	0.5770	0.3798	0.0023	1.3324	0.0040	0.1078	0.0033	**0.2346**
0.999	3.9546	17.868	1.7556	3.5216	1.7033	3.3157	0.0008	3.1052	0.0180	0.0380	0.0009	**0.2396**
						$\sigma^2 = 0.9$						
						n=50						
0.90	0.5227	0.2930	0.5193	0.2928	0.5668	0.3672	0.1352	1.7742	0.0015	1.2104	0.0369	0.4796
0.95	0.9879	1.1077	0.9432	1.0128	0.8093	0.7383	0.0351	2.9121	0.0017	2.2008	0.0361	0.7610
0.99	2.3431	6.2709	2.0936	5.0086	2.0454	4.7813	0.0078	4.8885	0.0103	6.6962	0.0079	**1.2752**
0.999	4.9424	27.913	3.9194	17.552	4.9178	27.637	0.0072	1.0396	0.3163	20.408	**0.0072**	**0.2711**
						n=100						
0.90	0.4870	0.2684	0.4721	0.2525	0.4662	0.2478	0.1163	1.2079	0.0014	0.5142	0.0244	0.3177
0.95	0.5480	0.3429	0.5240	0.3134	0.5382	0.3298	0.0264	2.2049	0.0021	0.9939	0.0153	0.5642
0.99	1.8167	3.7716	7.8322	70.029	1.1468	1.5030	0.0075	4.6131	0.0064	1.7304	0.0095	**0.6785**
0.999	1.3174	1.9827	0.9662	1.0662	1.3173	1.9823	0.0063	4.6726	0.0104	3.4528	0.0089	**0.6733**
						n=200						
0.90	0.4461	0.2256	0.4482	0.2279	0.3743	0.1591	0.0822	0.7772	0.0006	0.2315	0.0111	0.2010
0.95	0.8444	0.8133	0.7560	0.6524	0.4127	0.1943	0.0293	1.3946	0.0007	0.4531	0.0101	0.3534
0.99	1.0536	1.2674	12.572	180.30	0.7761	0.6884	0.0026	3.4328	0.0040	2.1903	0.0066	0.8674
0.999	1.7795	3.6191	1.2254	1.7162	1.6223	3.0077	0.0018	3.8157	0.0083	1.7418	0.0018	**0.7640**
						n=500						
0.90	0.3769	0.1619	0.3734	0.1589	0.2918	0.0959	0.0739	0.3678	0.0005	0.0838	0.0046	0.0958
0.95	0.4989	0.2831	0.4713	0.2526	0.4481	0.2290	0.0342	0.7189	0.0007	0.1645	0.0045	0.1813
0.99	0.9006	0.9264	0.6342	0.4596	0.6621	0.5009	0.0021	2.4193	0.0072	0.8620	0.0035	0.6075
0.999	3.4927	13.939	1.8762	4.0227	1.3287	2.0176	0.0009	5.6308	0.0111	0.3780	0.0009	0.6133

Table 7. scribe the five predictive variables used for Longly Dataset

Variable Name	Description
GNP.deflator	The inflation deflator for GNP.
GNP	Gross National Product
Bunemployed	Unemployed persons
Armed.Forces	Armed Forces employment
Population	Population, in thousands

8.2. *Real Data 2: Body Fat Dataset*

The proposed method for determining the shrinkage parameter vector D was applied to a second real-world dataset the Body Fat dataset [26]. The dataset comprises 14 predictor variables and a response variable (body fat percentage); a subset of six predictors was selected for analysis. Table 11 describes the selected predictor variables.

Table 8. Correlation matrix between predictive variables of Longly Dataset

	GNP.deflator	GNP	Bunemployed	Armed.Forces	Population
GNP.deflator	1	0.36	0.69	0.99	0.98
GNP		1	-0.18	0.45	0.46
Bnemployed			1	0.60	0.62
Armed.Forces				1	0.99
Population					1

Table 9. Values of variance inflation factors and d_j Values of Longly Dataset

	VIF_1	VIF_2	VIF_3	VIF_4	VIF_5
VIF_j	5209.481	306.539	37.783	39.908	2825.304
	d_1	d_2	d_3	d_4	d_5
d_j	0.000191	0.003251	0.025814	0.024401	0.000353

Table 10. Estimated regression coefficient values, standard of estimators ($SE(\widehat{\beta})$) and MSE of Longly Dataset

variables	Estimation	Estimation methods						
		OLS	Liu-GCV	Jack-Liu	PC	Liu-Opt.	Elastic Net	Liu_D
GNP.deflator	$\hat{\beta}_j$	0.217	-0.013	-0.375	0.003	0.111	0.071	0.219
	$SE\left(\hat{\beta}_j\right)$	0.105	0.104	1.283	0.0002	0.109	0.085	0.101
GNP	$\hat{\beta}_j$	-0.010	0.059	7.432	0.0402	-0.004	0.033	-0.010
	$SE\left(\hat{\beta}_j\right)$	0.006	0.021	3.377	0.0022	0.006	0.018	0.006
Unemployed	$\hat{\beta}_j$	-0.013	-0.005	-0.240	-0.008	-0.013	-0.008	-0.013
	$SE\left(\hat{\beta}_j\right)$	0.002	0.003	0.442	0.0002	0.002	0.003	0.003
Armed.Forces	$\hat{\beta}_j$	-0.005	-0.005	-0.338	-0.004	-0.045	-0.005	-0.005
	$SE\left(\hat{\beta}_j\right)$	0.003	0.002	0.164	0.003	0.003	0.002	0.0034
Population	$\hat{\beta}_j$	0.4528	-0.272	-3.303	0.0025	0.521	0.001	0.451
	$SE\left(\hat{\beta}_j\right)$	0.067	0.210	3.028	0.00013	0.086	0.221	0.064
MSE		0.35903	0.07425	0.61255	0.16048	0.0494	0.16847	0.00202

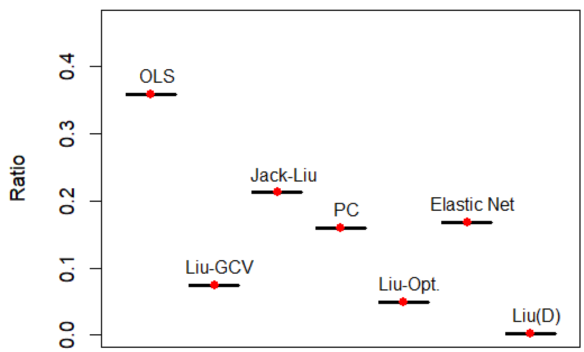

Figure 2. Boxplots of MSE ratios of Longly Dataset.

The dataset, sourced from [26], comprises 252 observations. It describes the relationship between multiple predictor variables and the response variable body fat percentage, computed using the Siri equation. The

Table 11. Describe the five predictive variables used in the Body Fat Dataset

Variable Name	Description
Abdomen	Abdominal circumference
Hip	Hip circumference
Weight	Weight
Chest	Chest circumference
Neck	Neck circumference
Age	A person's age in years

predictors exhibit moderate Multicollinearity. Pairwise correlation coefficients quantifying these interrelationships are presented in Table 12.

Table 12. Correlation matrix between predictive variables of Body Fat Dataset

	Abdomen	Hip	Weight	Chest	Neck	Age
Abdomen	1	0.87	0.88	0.91	0.75	0.23
Hip		1	0.94	0.82	0.73	-0.05
Weight			1	0.89	0.83	-0.01
Chest				1	0.78	0.17
Neck					1	0.11
Age						1

The correlations among the six predictor variables are presented in Table 12. Strong correlations exist between Abdomen and Hip (*r* = 0.87), Abdomen and Weight (*r* = 0.88), Abdomen and Chest (*r* = 0.91), and moderate correlation between Abdomen and Neck (*r* = 0.75). Similarly, Hip correlates strongly with Weight (*r* = 0.94) and Chest (*r* = 0.82), and moderately with Neck (*r* = 0.73). Additional strong correlations were observed between Weight and Chest (*r* = 0.89), Weight and Neck (*r* = 0.83), and Chest and Neck (*r* = 0.78). The condition number was computed as 173.818, indicating moderate Multicollinearity though less extreme than in the Longley dataset. Variance inflation factors (VIF_j) and their corresponding shrinkage parameters d_j are shown in Table 13.

Table 13. Values of variance inflation factors and d_j Values of Body Fat Dataset

	VIF_1	VIF_2	VIF_3	VIF_4	VIF_5	VIF_6
VIF_j	704.900	765.925	213.635	980.605	587.119	21.355
	d_1	d_2	d_3	d_4	d_5	d_6
d_j	0.00141	0.00130	0.00465	0.00102	0.00170	0.04473

The VIF_j values show variation depending on the relationships between the predictive variables. These values indicate Multicollinearity between the predictive variables. Table 13 also shows the values of the vector D, which includes the d_j values. It is observed that these values also vary depending on the magnitude of the Multicollinearity between the predictive variables. A large VIF_j value indicates a moderate Multicollinearity compared to Longly's dataset, which necessitates a greater reduction in the Liu estimator, resulting in a smaller corresponding d_j value. Thus, as the VIF_j value increases, the d_j value decreases, leading to a greater reduction in the Liu estimator, and vice versa.

Table 13 presents the shrinkage parameter vector D, whose elements d_j vary inversely with VIF_j: larger VIF_j values indicating stronger Multicollinearity correspond to smaller $d_j$values, thereby imposing greater shrinkage in the Liu estimator. Conversely, smaller VIF_j values correspond to larger $d_j$values and milder shrinkage.

Following the same approach used with the Longly data, the Liu_D estimator and the six comparison estimators were applied to the Body Fat data. The estimated regression coefficients and Standard errors of estimators ($SE(\widehat{\beta})$) were calculated. The MSE values for each estimator were also determined. As shown in Table 14, the Liu_D estimator obtained the lowest MSE value, which was 0.00133, lower than all other six estimators.

Table 14. Estimated regression coefficient values, standard of estimators ($SE(\widehat{\beta})$) and MSE of Body Fat Dataset

variables	Estimation	Estimation methods						
		OLS	Liu-GCV	Jack-Liu	PC	Liu-Opt.	Elastic Net	Liu_D
Abdomen	$\hat{\beta}_j$	1.072	0.998	10.803	0.996	2.516	0.955	1.071
	$SE\left(\hat{\beta}_j\right)$	0.080	0.084	1.046	0.084	0.118	0.077	0.081
Hip	$\hat{\beta}_j$	-0.360	-0.069	-0.533	0.015	-5.180	-0.047	-0.360
	$SE\left(\hat{\beta}_j\right)$	0.078	0.131	0.948	0.126	1.891	0.108	0.078
Weight	$\hat{\beta}_j$	-0.013	-0.106	-3.228	-0.154	-0.036	-0.095	-0.013
	$SE\left(\hat{\beta}_j\right)$	0.022	0.041	1.290	0.034	0.023	0.039	0.022
Chest	$\hat{\beta}_j$	-0.101	0.016	0.244	0.006	1.218	0.011	-0.101
	$SE\left(\hat{\beta}_j\right)$	0.087	0.096	0.898	0.097	0.117	0.085	0.087
Neck	$\hat{\beta}_j$	-0.821	-0.473	-1.181	-0.012	5.572	-0.470	-0.821
	$SE\left(\hat{\beta}_j\right)$	0.180	0.218	0.554	0.025	0.424	0.208	0.180
Age	$\hat{\beta}_j$	-0.003	-0.002	-0.028	-0.007	-0.554	0.008	-0.003
	$SE\left(\hat{\beta}_j\right)$	0.028	0.027	0.320	0.027	0.043	0.023	0.028
MSE		20.3022	0.0838	19.242	19.593	0.00149	19.2743	0.00133

Figure 3 displays boxplots of mean squared error (MSE) ratios for the Body Fat dataset. The Liu_D estimator occupies the lowest position in the plot, reflecting its superior performance relative to the other six estimators.

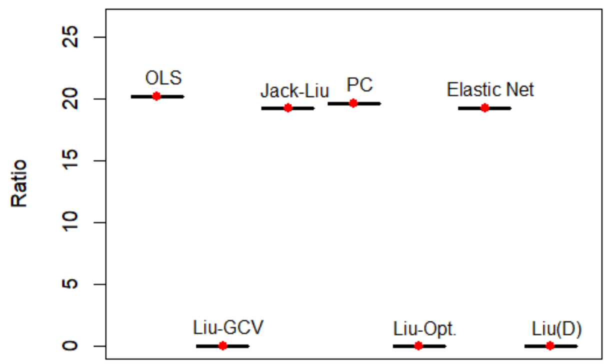

Figure 3. Boxplots of MSE ratios of Body Fat Dataset.

9. Conclusion

The current paper suggests the use of a vector D with more than one value, based on the variance inflation factors VIF_j , to address the different strength of Multicollinearity of the predictive variables in the Liu estimator. Simulation studies across varying signal-to-noise ratios (SNR) demonstrated that the Liu_D estimator maintained stable performance despite changes in SNR, sample size, number of predictors (p), and Multicollinearity intensity evident from the stability plots presented in Figure 1.

Monte Carlo simulations demonstrated that the Liu_Destimator outperforms competing methods ordinary least squares (OLS), Liu-GCV, jackknife Liu (Jack-Liu), principal component regression (PC), Liu-Opt, and elastic net by achieving lower mean squared error (MSE) and mean absolute error (MAE). The Liu_D estimator primarily reduces variance with only modest increases in bias, a bias-variance trade-off consistent with findings reported in prior literature. Analysis of real-world datasets (Longley and Body Fat) further confirmed the superior performance of the Liu_Destimator in mitigating Multicollinearity. Thus, using a vector D where the d_j values are different and proportional to the degree of Multicollinearity leads to better estimates and reduction in mean squared error since the multiple d_j values balance the different Multicollinearity of the predictive variables.

REFERENCES

1. A. E. Hoerl and R. W. Kennard, *Ridge regression: Biased estimation for nonorthogonal problems*, Technometrics, vol. 12, no. 1, pp. 55-67, 1970.
2. H. Hotelling, *Analysis of a complex of statistical variables into principal components*, Journal of educational psychology, vol. 24, no. 6, p. 417, 1933.
3. L. Kejian, *A new class of biased estimate in linear regression*, Communications in Statistics-Theory and Methods, vol. 22, no. 2, pp. 393-402, 1993.
4. M. R. Özkale and S. Kaçıranlar, *Superiority of the r-d class estimator over some estimators by the mean square error matrix criterion*, Statistics & probability letters, vol. 77, no. 4, pp. 438-446, 2007.
5. A. S. Adegoke, E. Adewuyi, K. Ayinde, and A. F. Lukman, *A comparative study of some robust ridge and Liu estimators*, Science World Journal, vol. 11, no. 4, pp. 16-20, 2016.
6. S. Kaçiranlar, S. Sakallioğlu, F. Akdeniz, G. P. Styan, and H. J. Werner, *A new biased estimator in linear regression and a detailed analysis of the widely-analysed dataset on Portland cement*, Sankhyā: The Indian Journal of Statistics, Series B, pp. 443-459, 1999.
7. F. Akdeniz and H. Erol, *Mean squared error matrix comparisons of some biased estimators in linear regression*, Communications in Statistics-Theory and Methods, vol. 32, no. 12, pp. 2389-2413, 2003.
8. I. Babar, H. Ayed, S. Chand, M. Suhail, Y. A. Khan, and R. Marzouki, *Modified Liu estimators in the linear regression model: An application to Tobacco data*, Plos one, vol. 16, no. 11, p. e0259991, 2021.
9. N. S. I. Alsharabi, R. R. Al-Mola, R. S. Yonan, and Z. Y. Algamal, *Employing Several Methods to Estimate the Generalized Liu Parameter in Multiple Linear Regression Model*, Int. J. Adv. Sci. Eng. Inf. Technol, vol. 12, no. 6, pp. 2386-2390, 2022.
10. D. N. Gujarati, *Basic Econometrics*, McGraw-Hill Higher Education, 4th edition, 2003.
11. D. C. Montgomery, E. A. Peck, and G. G. Vining, *Introduction to Linear Regression Analysis*, John Wiley & Sons, 5th edition, 2012.
12. Z. Y. Algamal and Y. Asar, *Liu-type estimator for the gamma regression model*, Communications in Statistics-Simulation and Computation, vol. 49, no. 8, pp. 2035-2048, 2020.
13. J. W. Longley, *An appraisal of least squares programs for the electronic computer from the point of view of the user*, Journal of the American Statistical association, vol. 62, no. 319, pp. 819-841, 1967.
14. A. M. Algboory, A. N. Alkhateeb, and Z. Y. Algamal, *A Modified Jackknife Liu-Type Estimator for the Gamma Regression*, Statistics, Optimization and Information Computing, vol. 14, no. 5, pp. 2142-2154, 2025.
15. A. N. Alkhateeb and Z. Y. Algamal, *Jackknifed Liu-type Estimator in Poisson Regression Model*, Journal of the Iranian Statistical Society, vol. 19, no. 01, pp. 21-37, 2020.
16. F. Erdugan, *An almost unbiased Liu-type estimator in the linear regression model*, Communications in Statistics-Simulation and Computation, vol. 53, no. 7, pp. 3081-3093, 2024.
17. R. A. Farghali, *Generalized Liu-type estimator for linear regression*, International Journal of Research and Reviews in Applied Sciences, vol. 38, no. 1, pp. 52-63, 2019.
18. I. Dawoud, M. R. Abonazel, and F. A. Awwad, *Modified Liu estimator to address the multicollinearity problem in regression models: A new biased estimation class*, Scientific African, vol. 17, p. e01372, 2022.
19. S. Arumairajan and P. Wijekoon, *Optimal generalized biased estimator in linear regression model*, Open Journal of Statistics, vol. 5, no. 05, p. 403, 2015.
20. M. Qasim, M. Amin, and M. Amanullah, *On the performance of some new Liu parameters for the gamma regression model*, Journal of Statistical Computation and Simulation, vol. 88, no. 16, pp. 3065-3080, 2018.
21. M. H. Kutner, C. J. Nachtsheim, J. Neter, and W. Li, *Applied Linear Statistical Models*, McGraw-Hill, 5th edition, 2004.
22. R. Salmerón, C. B. García, and J. García, *Variance inflation factor and condition number in multiple linear regression*, Journal of statistical computation and simulation, vol. 88, no. 12, pp. 2365-2384, 2018.
23. H. Zou and T. Hastie, *Regularization and variable selection via the elastic net*, Journal of the Royal Statistical Society: Series B (Statistical Methodology), vol. 67, no. 2, pp. 301–320, 2005.
24. K. C. Arum, S. C. Ndukwe, H. E. Oranye, and O. B. Sule, *Comparative Analysis of Ridge and Principal Component Regression in Addressing Multicollinearity*, FUDMA Journal of Sciences, vol. 9, no. 1, pp. 240–245, 2025.
25. M. M. Abdelwahab, M. R. Abonazel, A. T. Hammad, and A. M. El-Masry, *Modified Two-Parameter Liu Estimator for Addressing Multicollinearity in the Poisson Regression Model*, Axioms, vol. 13, no. 1, p. 46, 2024.
26. R. W. Johnson, *Fitting percentage of body fat to simple body measurements*, Journal of Statistics Education, vol. 4, no. 1, 1996.
27. T. Piotrowski and I. Yamada, *Performance of the stochastic MV-PURE estimator in highly noisy setting*, Journal of the Franklin Institute, vol. 351, pp. 3339–3350, 2014.
28. K. H. Yuan, Z. Zhang, and L. Wang, *Signal-to-noise ratio in estimating and testing the mediation effect: Structural equation modeling versus path analysis with weighted composites*, Psychometrika, vol. 89, no. 3, pp. 974–1006, 2024.