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Exploring Sigma Coloring within Lexicographic Products of Innovative Graph Families

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Abstract The coloring strategy based on the open neighbourhood sum of any 2-adjacent vertices in a graph G need be distinct is known as sigma coloring (or) $\sigma-$ coloring. The smallest number of colors needed to produce the sigma coloring of G is called sigma chromatic number of G, denoted by $\sigma(G)$. The value of $\sigma(G)$ is tight. For any graph G we always have $\sigma(G) \leq \chi(G)$. A clear investigation is presented for the lexicographic product of $P_m[P_n], P_m[C_n], P_m[K_{1,n}], P_m[K_{1,n,n}], P_m[P_n \odot K_1]$ in this article. Additionally, the $\sigma-$chromatic number for each case is found and verified with chromatic number for bounds.

Keywords coloring, $\sigma-$coloring, chromatic number, $\sigma-$chromatic number, lexicographic product.

AMS 2010 subject classifications 05C15, 05C76

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1. Introduction

Graph coloring is an important part of graph theory & is an inevitable solution in scheduling, networking, resource allocation and assigning frequencies in wireless communication [10, 11]. Researchers defined & discussed a number of coloring concepts. The concept of σ-coloring, which belongs to the class of neighbourhood-distinguishing colorings, was proposed by Chartrand et al. in 2008 [3]. Later, Chartrand et al. determined the values of $\sigma(G)$ for the complete graph K_n, the complete multipartite graph $K_{n_1,n_2,\ldots,n_r}$ with $r \geq 2$, and the cycle C_n [4].

In 2023, Preethi K. Pillai et al. computed $\sigma-$ coloring of B_n, T_m, $S_{n,n-3}$, T_{nl}, L_{nl}, fusing of cycle & duplication of vertex in C_n [6] and also extended some graph operations in 2021 [8]. In 2024, C. Yogalakshmi et al. [1] studied $\sigma-$ coloring on mycielski operation & they proved that $\sigma(G) \leq \chi(G)$ for P_n, $K_{m,n}$, K_n, C_n, W_n and H_n. S. Divya et al. [16] discussed mycielski operation under some new graphs in 2026. Wenhui Ma et al. consider lexicographic operation of $P_m[P_n]$ for antimagicness in 2018[9], which make us think to use it under $\sigma-$ coloring. The objective of this paper is to find $\sigma(G)$ of lexicographic product of graphs $P_m[P_n], P_m[C_n], P_m[K_{1,n}], P_m[K_{1,n,n}], P_m[P_n \odot K_1]$.

Consider a simple connected graph G, and suppose that N denotes a set of positive integers. A coloring assigned to the vertices of G is represented by a mapping $\tau : V(G) \to N$. Such a coloring, written as $\tau : V(G) \to C$, is called a σ-coloring whenever $\tau(v) \neq \tau(u)$ for every pair of 2-adjacent vertices $u, v \in V(G)$. The description and

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terms associated with $\sigma-$coloring have been provided in [12, 13, 14, 15, 17] as well. The $\sigma-$chromatic number $\sigma(G)$ is the minimal number of colors required in a $\sigma-$coloring of a graph G. The lexicographic product $X[Y]$ of graphs[9] X and Y is a graph have the vertex set of X[Y] is the cartesian product $V(X) \times V(Y)$, and any two vertices (p, q) and (r, s) are adjacent in $X[Y]$ if and only if either p is adjacent with r in X or $p = q$ and q is adjacent with s in Y. This is the same as swapping each vertex of X with a copy of Y, then connecting the copies together with all possible bipartite graphs between them corresponding to the edges of X. It is not commutative: $X[Y] \neq Y[X]$.

Path P_n is a way to travel from one point to another point without repeating any vertices or edges, [2]. comprises n vertices placed in a circular sequence, where an edge joins every pair of consecutive vertices, including the pair formed by the first and last vertices. A star graph $K_{1,n}$ [2] is a tree with exactly one vertex which are connected by n leaves. A double star graph $K_{1,n,n}$ [2] is a tree with exactly one vertex which are connected by n leaves of P_2. $P_n \odot K_1$ represents the comb graph [5]. A path is joined by a pendant edge to create the graph.

1.1. Conceptual Foundation

This section presents foundational concepts and previously established results concerning sigma coloring.

In graph theory, a vertex's neighborhood $(N(v))$ consists of all the vertices directly connected to it. If we include the vertex itself in that group, it is called the closed neighborhood $(N[v])$. When two vertices have the exact same closed neighborhood—meaning they are connected to each other and share identical connections to the rest of the graph, then they are strong twins.[4]

Lemma 1.1.1 [4]

A connected graph H has $\sigma(H) = 1$ iff no 2-adjacent vertices have the same degree.

Lemma 1.1.2 [4]

For any σ-coloring c of H, the vertices u and v must receive different colors whenever they are 2-adjacent and satisfy $N[u] = N[v]$.

Lemma 1.1.3 [4]

Let $\sigma(X) \leq k$ if there exists a complete subgraph $Y \subseteq X$ with $|V(Y)| = k$ where every pair $u, v \in V(Y)$ has identical closed neighborhoods.

Lemma 1.1.4 [4]

A connected graph H of order n satisfies $\sigma(H) = n$ if it is complete graph.

2. Outcome of the Study

Results on $\sigma-$chromatic no. of lexicographic products for new types of graphs are presented in the below theorems.

Theorem 1.2.1

If $m, n \geq 2 \in \mathbb{N}$, $\sigma-$chromatic no. of. $P_m[P_n]$ is given by $\sigma(P_m[P_n]) = 4$

Proof:

Vertex Structure:

For clarity and consistency, throughtout this theorem, a $m-$ path P_m is represented by vertices of sequence $u_1, u_2, ..., u_m$ & a $n-$ path P_n is represented by vertices of sequence $v_1, v_2, ..., v_n$. In the graph $P_m[P_n]$, the vertex set is given by $V(P_m[P_n]) = \{(u_a, v_b); 1 \leq a \leq m, 1 \leq b \leq n\}$.

Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v_b)(1 \leq b \leq n)$
- **Vertical external vertices:** All vertices of the form $(u_a, v_1)(1 \leq a \leq m)$
- **Internal vertices:** All remaining vertices not included in the above two categories

The cardinality of $|V(P_m[P_n])|$ is $m \times n$ vertices

Coloring Construction:

Let $\tau : V(P_m[P_n]) \longrightarrow \{1, 2, 3, 4\}$ as:

Horizontal external vertices:-

$$\tau(u_1, v_b) = \begin{cases} 1, & \text{if } b \equiv 1 \pmod 4, \\ 2, & \text{if } b \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 1 \le b \le n.$$

Vertical external vertices:-

$$\tau(u_a, v_1) = \begin{cases} 1, & \text{if } a \equiv 1 \pmod 4, \\ 3, & \text{if } a \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 1 \le a \le m.$$

Internal vertices:-

$$\tau(u_a, v_b) = \begin{cases} 1, & \text{if } a \equiv 1 \pmod 4 \text{ and } b \equiv 1 \pmod 4, \\ 2, & \text{if } a \equiv 1 \pmod 4 \text{ and } b \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 5 \le a \le m,\ 2 \le b \le n.$$

$$\tau(u_a, v_b) = \begin{cases} 4, & \text{if } b \equiv 0 \pmod 2, \\ 3, & \text{if } b \equiv 1 \pmod 2, \end{cases} \qquad \text{for } 2 \le a \le m,\ a \not\equiv 1 \pmod 4,\ 2 \le b \le n.$$

By using modular concept in coloring construction of τ, the neighborhood color sums of every pair of adjacent vertices are distinct. Here, take τ is a $\sigma-$ coloring with $\sigma(P_m[P_n]) \le 4$. Consider $\sigma(P_m[P_n]) = 1$, then the horizontal external vertices (u_1, v_b) & $(u_1, v_{b+1})(2 \le b \le n-2)$ are of uniformity in degree & are adjacent resulting in identical neighborhood sum which disrupts the $\sigma-$ coloring requirement. According to lemma 1.1.1, $\sigma(P_m[P_n]) \ne 1$. Next, assume $\sigma(P_m[P_n]) = 2$ or 3, then for instance, if the graph $P_m[P_n]$ is colored using less than 4 colors. Then, for any $(u, v) \,\epsilon\, V(P_m[P_n])$, $\sigma(u) = \sigma(v)$ such that u & v being adjacent and may or may not equal in degree ends in identical neighborhood sum where the rule of $\sigma-$ coloring breaks. From the coloring of τ, it can be concluded that $\sigma(P_m[P_n]) = 4$

Theorem 1.2.2
If $m \ge 2, n \ge 3 \in \mathbb{N}$, $\sigma-$chromatic no. of. $P_m[C_n]$ is given by

$$\sigma(P_m[C_n]) = \begin{cases} 6, & \text{if } n = 3, \\ 5, & \text{if } n \equiv 1 \pmod 2, \\ 4, & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

Proof:
Vertex Structure:
The vertices of $m-$ path P_m be represented as $u_1, u_2, ..., u_m$ & the cycle C_n of length n has the vertices $v_1, v_2, ..., v_n$. In $P_m[C_n]$, vertex set is given by $V(P_m[C_n]) = \{(u_a, v_b); 1 \le a \le m, 1 \le b \le n\}$.
Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v_b)(1 \le b \le n)$
- **Vertical external vertices:** All vertices of the form $(u_a, v_1)(1 \le a \le m)$
- **Internal vertices:** All remaining vertices not included in the above two categories

The cardinality of $|V(P_m[C_n])|$ is $m \times n$ vertices.
Case(i): If $n = 3$, then it have two sub cases for 'value of m'
Sub case(i): $m = 2$
Coloring Construction:
Let $\tau : V(P_m[C_n]) \longrightarrow \{1, 2, 3, 4, 5, 6\}$ as:

$$\tau(u_1, v_1) = 3, \qquad \tau(u_1, v_2) = 1, \qquad \tau(u_1, v_3) = 2, \qquad \tau(u_2, v_1) = 6, \qquad \tau(u_2, v_2) = 4, \qquad \tau(u_2, v_3) = 5.$$

By using modular concept in coloring construction of τ, the neighborhood color sums of every pair of adjacent vertices are distinct. Suppose τ is a $\sigma-$ coloring with $\sigma(P_m[C_n]) \leq 6$. Let us examine the possibility of using fewer colors, Consider $\sigma(P_m[C_n]) = 1$, then the horizontal external vertices (u_1, v_b) & $(u_1, v_b + 1)(1 \leq b \leq n-1)$ are of uniformity in degree & are adjacent receives twin neighborhood sum which fails the $\sigma-$ coloring criterion. By lemma 1.1.1, it is evident that $\sigma(P_m[C_n]) \neq 1$. It is to be noted that all the vertices in $P_m[C_n]$ are uniformity in degree & adjacency structure implies that the graph is a complete graph. Then, for any $(u, v) \,\epsilon\, V(P_m[C_n])$, then the sigma chomatic number of u is equal to sigma chomatic number of of v such that u & v are adjacent and equal in degree results in identical neighborhood sum. Since, each vertex contributes one another during neighborhood sum. This results in that to satisfy the $\sigma-$ coloring criteria, the vertices must be colored with at least six different colors. Hence $\sigma(P_m[C_n]) = 6$.

Sub case(ii): For $m \geq 3$

Coloring Construction:

Let $\tau : V(P_m[C_n]) \longrightarrow \{1, 2, 3, 4, 5\}$ as below:

$$\tau(u_a, v_1) = \begin{cases} 2, & \text{if } a \equiv 1 \pmod 2, \\ 1, & \text{if } a \equiv 0 \pmod 2, \end{cases} \qquad \text{for } 1 \leq a \leq m.$$

$$\tau(u_a, v_2) = \begin{cases} 1, & \text{if } a \equiv 1 \pmod 2, \\ 4, & \text{if } a \equiv 0 \pmod 2, \end{cases} \qquad \text{for } 1 \leq a \leq m.$$

$$\tau(u_a, v_3) = \begin{cases} 3, & \text{if } a \equiv 1 \pmod 2, \\ 5, & \text{if } a \equiv 0 \pmod 2, \end{cases} \qquad \text{for } 1 \leq a \leq m.$$

Under the vertex coloring of τ,the neighborhood color sums of every pair of adjacent vertices are distinct. Take τ is a sigma coloring with $\sigma(P_m[C_n]) \leq 5$. Let us analyze the possibility of using fewer colors, consider $\sigma(P_m[C_n]) = 1$, then the vertical external vertices (u_a, v_1) & $(u_{a+1}, v_1)(1 \leq a \leq m-1)$ are of uniformity in degree & are adjacent receives identical neighborhood sum which defies the condition of $\sigma-$ coloring. Then by lemma 1.1.1, it is evident that $\sigma(P_m[C_n]) \neq 1$. In graph $P_m[C_n]$, if $2 \leq \sigma(P_m[C_n]) \leq 4$, then for instance, if the graph $P_m[C_n]$ is colored using less than 5 colors. Then, if there exist adjacent vertices $u, v \in V(P_m[C_n])$ such that $\sigma(u) = \sigma(v)$, they produce identical neighborhood sums, whether or not they have the same degree, thereby violating the condition of a σ-coloring. From the coloring of τ, it can be concluded that $\sigma(P_m[C_n]) = 5$

Case(ii): For n is odd, then it leads to two sub cases for 'value of m'.

Coloring Construction:

Let $\tau : V(P_m[C_n]) \longrightarrow \{1, 2, 3, 4, 5\}$ as below:

Sub case(i): For $m = 2, 3, 4$.

Horizontal external vertices:-

$\tau(u_1, v_1) = 2$;

$$\tau(u_1, v_b) = \begin{cases} 1, & \text{if } b \equiv 0 \pmod 2, \\ 3, & \text{if } b \equiv 1 \pmod 2, \end{cases} \qquad \text{for } 2 \leq b \leq n.$$

Vertical external vertices:-

$$\tau(u_a, v_1) = \begin{cases} 2, & \text{if } a \equiv 0 \pmod 4, \\ 1, & \text{if } a \not\equiv 0 \pmod 4, \end{cases} \qquad \text{for } 2 \leq a \leq m.$$

Internal vertices:-

$$\tau(u_a, v_b) = \begin{cases} 4, & \text{if } b \equiv 0 \pmod 2, \\ 5, & \text{if } b \equiv 1 \pmod 2, \end{cases} \qquad \text{for } 2 \leq a \leq m,\ 2 \leq b \leq n.$$

Sub case(ii): For $m \geq 5$.
The internal vertices are partitioned into two cases based on the value of $a \pmod 4$.
For the case $a \equiv 1 \pmod 4$:

$$\tau(u_a, v_b) = \begin{cases} 3, & \text{if } b \in \{1, 4\}, \\ 1, & \text{if } b = 2, \\ 2, & \text{if } b = 3, \\ 1, & \text{if } b \geq 5 \text{ and } b \text{ is odd}, \\ 2, & \text{if } b \geq 5 \text{ and } b \text{ is even}, \end{cases} \qquad \text{for } 1 \leq a \leq m,\ a \equiv 1 \pmod 4,\ 1 \leq b \leq n.$$

For the case $a \not\equiv 1 \pmod 4$:

$$\tau(u_a, v_b) = \begin{cases} 5, & \text{if } b \in \{1, 4\}, \\ 3, & \text{if } b = 2, \\ 4, & \text{if } b = 3, \\ 3, & \text{if } b \geq 5 \text{ and } b \text{ is even}, \\ 4, & \text{if } b \geq 5 \text{ and } b \text{ is odd}, \end{cases} \qquad \text{for } 1 \leq a \leq m,\ a \not\equiv 1 \pmod 4,\ 1 \leq b \leq n.$$

From the above, under the coloring construction of τ, the neighborhood color sums of every pair of adjacent vertices are distinct. Consider τ is a $\sigma-$ coloring with $\sigma(P_m[C_n]) \leq 5$. Assume $\sigma(P_m[C_n]) = 1$, then the vertical external vertices (u_a, v_1) & $(u_{b+1}, v_1)(1 \leq a \leq m-1)$ are of uniformity in degree & are adjacent receives same neighborhood sum. By lemma 1.1.1, it is evident that $\sigma(P_m[C_n]) \neq 1$. Next, let us take $2 \leq \sigma(P_m[C_n]) \leq 4$, then for instance, if the graph $P_m[C_n]$ is colored using less than 5 colors. Then, for any $(u, v)\ \epsilon\ V(P_m[C_n])$, then $\sigma(u) = \sigma(v)$ such that u & v are adjacent & may or may not equal in degree leads to identical neighborhood sum, which again contradicts the rule of $\sigma-$ coloring. By the coloring of τ given above, it can be concluded that $\sigma(P_m[C_n]) = 5$
Case(iii): For n is even.
Coloring Construction:
Let $\tau : V(P_m[C_n]) \longrightarrow \{1, 2, 3, 4\}$ as below:

$$\tau(u_a, v_b) = \begin{cases} 1, & \text{if } b \equiv 1 \pmod 2, \\ 2, & \text{if } b \equiv 0 \pmod 2, \end{cases} \qquad a \equiv 1 \pmod 4, \text{ for } 1 \leq a \leq m,\ 1 \leq b \leq n.$$

$$\tau(u_a, v_b) = \begin{cases} 3, & \text{if } b \equiv 1 \pmod 2, \\ 4, & \text{if } b \equiv 0 \pmod 2, \end{cases} \qquad a \not\equiv 1 \pmod 4, \text{ for } 1 \leq a \leq m,\ 1 \leq b \leq n.$$

From the above vertex color construction, the neighborhood color sums of every pair of adjacent vertices are distinct. Take τ is a $\sigma-$ coloring with $\sigma(P_m[C_n]) \leq 4$. Let us scrutinize the potential for minimizing the number of colors, for it consider $\sigma(P_m[C_n]) = 1$, then the horizontal vertical vertices (u_1, v_b) & $(u_1, v_{b+1})(1 \leq b \leq n-1)$ are of uniformity in degree & are adjacent receives identical neighborhood sum . By lemma 1.1.1, it is true $\sigma(P_m[C_n]) \neq 1$. Let us take the assumption that $2 \leq \sigma(P_m[C_n]) \leq 3$, then for instance, if the graph $P_m[C_n]$ is colored using less than 4 colors. Then, for any $(u, v) \epsilon V(P_m[C_n])$, then $\sigma(u) = \sigma(v)$ such that u & v are adjacent & may or may not equal in degree ends in equal neighborhood sum which breaks the rule of $\sigma-$ coloring. Considering the coloring of τ from above, it can be concluded that $\sigma(P_m[C_n]) = 4$

Theorem 1.2.3
If $m, n \geq 2 \in \mathbb{N}$, $\sigma-$chromatic no .of .$P_m[K_{1,n}]$ is given by $\sigma(P_m[K_{1,n}]) = 4$.

Proof:
Vertex Structure:
Let the vertices of $m-$ path represented as $u_1, u_2, ..., u_m$ & the star graph $K_{1,n}$ has v_1 as apex vertex & $v_2, v_3, ..., v_n$ are connected to it. Then the vertex set of $V(P_m[K_{1,n}]) = \{(u_a, v_b); 1 \leq a \leq m, 1 \leq b \leq n\}$.
Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v_b)(1 \leq b \leq n)$
- **Vertical external vertices:** All vertices of the form $(u_a, v_1)(1 \leq a \leq m)$
- **Internal vertices:** All remaining vertices not included in the above two categories

The cardinality of $|V(P_m[K_{1,n}])|$ is $m(n+1)$ vertices.
Coloring Construction:
Let $\tau : V(P_m[K_{1,n}]) \longrightarrow \{1, 2, 3, 4\}$ as:

$$\tau(u_a, v_b) = \begin{cases} 1, & \text{if } a \equiv 1 \pmod 2 \text{ and } b = 1, \\ 2, & \text{if } a \equiv 1 \pmod 2 \text{ and } 2 \leq b \leq n+1, \end{cases} \qquad \text{for } 1 \leq a \leq m.$$

$$\tau(u_a, v_b) = \begin{cases} 3, & \text{if } a \equiv 0 \pmod 2 \text{ and } b \equiv 1 \pmod 2, \\ 4, & \text{if } a \equiv 0 \pmod 2 \text{ and } b \equiv 0 \pmod 2, \end{cases} \qquad \text{for } 1 \leq a \leq m,\ 1 \leq b \leq n+1.$$

From the above procedure, the neighborhood color sums of every pair of adjacent vertices are distinct. Take τ is a sigma coloring with $\sigma(P_m[K_{1,n}]) \leq 4$. Suppose $\sigma(P_m[K_{1,n}]) = 1$, then the vertical external vertices (u_a, v_1) & $(u_{a+1}, v_1)(2 \leq a \leq m-2)$ being adjacent & have same degree receives equal neighborhood sum, which violates the condition required for a σ-coloring. Hence, it follows from Lemma 1.1.1 that $\sigma(P_m[K_{1,n}]) \neq 1$. In the structure of $P_m[K_{1,n}]$, when $2 \leq \sigma(P_m[K_{1,n}]) \leq 3$, then for instance, if the graph $P_m[K_{1,n}]$ is colored using less than 4 colors. Then, for any $(u, v) \epsilon V(P_m[K_{1,n}])$, there exist adjacent vertices u and v such that $\sigma(u) = \sigma(v)$ and may or may not equal in degree ends in identical neighborhood sum, where the rule breaks. By the coloring of τ assigned above, each vertex yields distinct neighborhood sum and it conclude that $\sigma(P_m[K_{1,n}]) = 4$.

Theorem 1.2.4

If $m, n \geq 2 \in \mathbb{N}$, the σ-chromatic number of $P_m[K_{1,n,n}]$ is given by $\sigma(P_m[K_{1,n,n}]) = 4$.

Proof:
Vertex Structure:
For convenience, throughout this theorem, an m-path is denoted by $u_1, u_2, \ldots, u_m$. Then, let $K_{1,n,n}$ be the double star graph whose apex vertex is v, and $v_1, v_2, \ldots, v_n$ are connected to it. Let $w_1, w_2, \ldots, w_n$ be the vertices connected to $v_1, v_2, \ldots, v_n$, respectively, in $K_{1,n,n}$. The vertex set of $P_m[K_{1,n,n}]$ is $V(P_m[K_{1,n,n}]) = \{(u_a, v), (u_a, v_b), (u_a, w_c) : 1 \leq a \leq m,\ 1 \leq b \leq n,\ 1 \leq c \leq n\}$.

Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form (u_1, v_b) for $1 \leq b \leq n$, and (u_1, w_c) for $1 \leq c \leq n$
- **Vertical external vertices:** All vertices of the form (u_a, v) for $(1 \leq a \leq m)$.

- **Internal vertices:** All remaining vertices not included in the above two categories

The cardinality of $|V(P_m[K_{1,n,n}])|$ is $m(2n+1)$ vertices.
Coloring Construction:
Let $\tau : V(P_m[K_{1,n,n}]) \longrightarrow \{1,2,3,4\}$ be defined as follows.

Horizontal external vertices:

$$\tau(u_1, v) = 4,$$
$$\tau(u_1, v_b) = 2, \quad \text{where } 1 \leq b \leq n,$$
$$\tau(u_1, w_c) = 1, \quad \text{where } 1 \leq c \leq n.$$

Vertical external vertices:

$$\tau(u_a, v) = \begin{cases} 4, & \text{if } a \equiv 1 \pmod 4, \\ 3, & \text{if } a \not\equiv 1 \pmod 4, \end{cases} \quad \text{for } 2 \leq a \leq m.$$

Internal vertices:

$$\tau(u_a, v_b) = \begin{cases} 2, & \text{if } a \equiv 1 \pmod 4, \\ 4, & \text{if } a \not\equiv 1 \pmod 4, \end{cases} \quad \text{for } 2 \leq a \leq m,\ 1 \leq b \leq n,$$

$$\tau(u_a, w_c) = \begin{cases} 1, & \text{if } a \equiv 1 \pmod 4, \\ 3, & \text{if } a \not\equiv 1 \pmod 4, \end{cases} \quad \text{for } 2 \leq a \leq m,\ 1 \leq c \leq n.$$

From the above, constructed coloring of τ, the neighborhood color sums of every pair of adjacent vertices are distinct. Here, τ is a sigma coloring with $\sigma(P_m[K_{1,n,n}]) \leq 4$. Let us inspect the potential for minimizing the number of colors, for it consider $\sigma(P_m[K_{1,n,n}]) = 1$, then the horizontal external vertices (u_a, v) & $(u_{a+1}, v)(2 \leq a \leq m-2)$ being adjacent & are uniformity in degree receives identical neighborhood sum, it defies the $\sigma-$ coloring rule. Therefore by lemma 1.1.1, $\sigma(P_m[K_{1,n,n}]) \neq 1$. Then taking the structure of $P_m[K_{1,n,n}]$ into account & $2 \leq \sigma(P_m[K_{1,n,n}]) \leq 3$, then for instance, if the graph $P_m[K_{1,n,n}]$ is colored using less than 4 colors. Then, for any $(u, v) \epsilon V(P_m[K_{1,n,n}])$, there exist adjacent vertices u and v such that $\sigma(u) = \sigma(v)$ and may or may not equal in degree results in neighborhood sum which again breaks the rule of $\sigma-$ coloring. Consider the coloring of τ assigned above successfully ensures that it can be concluded that $\sigma(P_m[K_{1,n,n}]) = 4$.

Theorem 1.2.5

If $m, n \geq 2 \in \mathbb{N}$, $\sigma-$chromatic no .of .$P_m[P_n \odot K_1]$ is given by $\sigma(P_m[P_n \odot K_1]) = 4$.

Proof:
Vertex Structure:
Let P_m be the path with vertices $u_1, u_2, \ldots, u_m$. Then, let $P_n \odot K_1$ be the comb graph with vertices $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$, where each w_i is a pendant vertex attached to v_i, respectively. The vertex set of $P_m[P_n \odot K_1]$ is $V(P_m[P_n \odot K_1]) = \{(u_a, v_b), (u_a, w_c) : 1 \leq a \leq m,\ 1 \leq b \leq n,\ 1 \leq c \leq n\}$.
Let categorize the vertices as follows:

- **Horizontal external vertices:** All vertices of the form $(u_1, v_b)(1 \leq b \leq n)$ and $(u_1, w_c)(1 \leq c \leq n)$.
- **Vertical external vertices:** All vertices of the form $(u_a, v_1)(1 \leq a \leq m)$.
- **Internal vertices:** All remaining vertices not included in the above two categories

Coloring Construction:
Let $\tau : V(P_m[P_n \odot K_1]) \longrightarrow \{1,2,3,4\}$ as follows
Horizontal external vertices:-

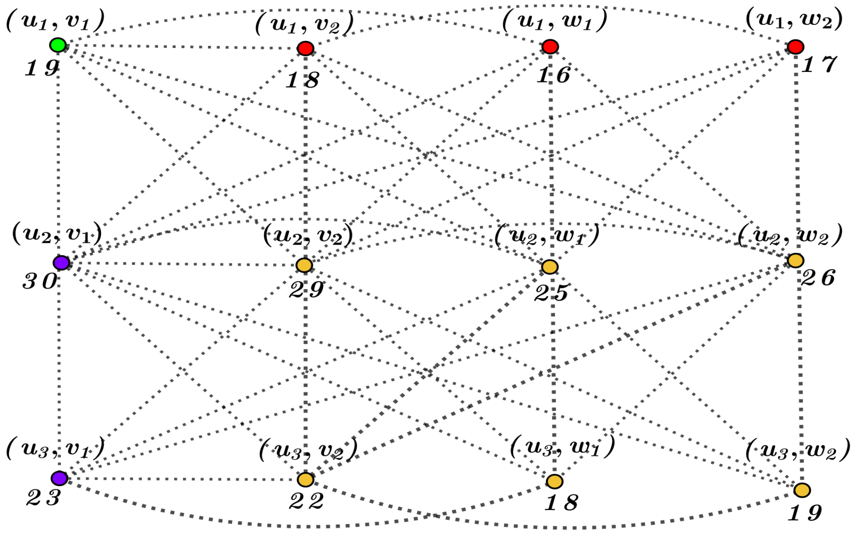

Figure 1. $P_3[P_2 \odot K_1]$

$\tau(u_1, v_b) = 2$; where $1 \leq b \leq n$;
$\tau(u_1, w_c) = 2$; where $1 \leq c \leq n$.
Vertical external vertices:-

$$\tau(u_a, v_1) = \begin{cases} 1, & \text{if } a \equiv 1 \pmod 4, \\ 3, & \text{if } a \not\equiv 1 \pmod 4, \end{cases} \qquad \text{for } 2 \leq a \leq m.$$

Internal vertices:-

$$\tau(u_a, v_b) = \begin{cases} 4, & \text{if } a \not\equiv 1 \pmod 4 \text{ and } b \equiv 0 \pmod 2, \\ 3, & \text{if } a \not\equiv 1 \pmod 4 \text{ and } b \equiv 1 \pmod 2, \end{cases} \qquad \text{for } 2 \leq a \leq m,\ 2 \leq b \leq n.$$

$\tau(u_a, w_c) = 4$ if $a \not\equiv 1 \pmod 4$; for $2 \leq a \leq m, 1 \leq c \leq n$
$\tau(u_a, v_b) = 2$ if $a \equiv 1 \pmod 4$; for $2 \leq b \leq m, 2 \leq b \leq n$
$\tau(u_a, w_c) = 2$ if $a \equiv 1 \pmod 4$; for $2 \leq a \leq m, 1 \leq c \leq n$

From the above, the neighborhood color sums of every pair of adjacent vertices are distinct. Here, τ is a $\sigma-$ coloring with $\sigma(P_m[P_n \odot K_1]) \leq 4$. Suppose let us take $\sigma(P_m[P_n \odot K_1]) = 1$, then the vertical external vertices (u_a, v_1) & $(u_{a+1}, v_1)(2 \leq a \leq m-2)$ being adjacent & are of uniformity in degree receives identical neighborhood sum, which goes against the $\sigma-$ coloring rule. So, by lemma 1.1.1, $\sigma(P_m[P_n \odot K_1]) \neq 1$. Then considering the structure of $P_m[P_n \odot K_1]$ into account & $2 \leq \sigma(P_m[P_n \odot K_1]) \leq 3$, then for instance, if the graph $P_m[P_n \odot K_1]$ is colored using less than 4 colors. Then, for any $(u, v)\ \epsilon\ V(P_m[P_n \odot K_1])$, then $\sigma(u) = \sigma(v)$ such that u & v are adjacent & may or may not equal in degree gets equal neighborhood sum, violates the rule of $\sigma-$ coloring. Taking the coloring of τ into consideration from above, it can be concluded that $\sigma(P_m[P_n \odot K_1]) = 4$. The sigma coloring of $P_3[P_2 \odot K_1]$ is explained in figure1, which clearly gives a idea about how 4 colors are assigned to avoid neighborhood sum conflicts.

3. Conclusion

Finally, it was acclaimed that sigma coloring properties are satisfied by lexicographic product of path with path, path with cycle, path with star, path with double star and path with comb graphs and also sigma chromatic number of its kind is obtained. It can be concluded that $\sigma(G) \leq \chi(G)$ under lexicographic of graphs, where G includes graphs as mentioned above.

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