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Weak convergence of an inertial projection-contraction algorithm with adaptive stepsize and golden-ratio relaxation for pseudomonotone variational inequalities

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Abstract This paper investigates an inertial projection-contraction algorithm for solving pseudomonotone variational inequality problems in real Hilbert spaces. The proposed method combines a modified Mann-type relaxation, inertial extrapolation, and an adaptive line-search mechanism that does not require prior knowledge of the Lipschitz constant of the underlying operator. Unlike standard projection-contraction schemes, the proposed algorithm is developed under pseudomonotonicity, uniform continuity, and weak-to-strong sequential continuity assumptions. A detailed weak convergence analysis is established using projection inequalities, asymptotic regularity arguments, and Opial's lemma. Furthermore, we discuss the role of the golden-ratio relaxation parameter in the stability of the proposed iteration. Numerical experiments involving finite-dimensional variational inequality problems and image deblurring applications are provided to illustrate the practical performance of the algorithm and to compare it with existing projection-contraction methods.

Keywords weak convergence, pseudomonotone variational inequality, projection–contraction method, inertial extrapolation, adaptive stepsize, golden-ratio relaxation

AMS 2010 subject classifications 47H09, 47H10

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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and let $C \subset H$ be a nonempty closed convex subset. The variational inequality problem associated with an operator $Q : C \to H$ consists of finding $x^* \in C$ such that

$$\langle Qx^*, y - x^*\rangle \geq 0, \qquad \forall y \in C. \tag{1}$$

The solution set of (1) is denoted by $VI(C, Q)$. Variational inequality theory provides a unified framework for equilibrium models, constrained optimization, Nash-type games, fixed point theory, signal processing, and inverse problems. In particular, many constrained least-squares and image restoration models can be reformulated as variational inequalities over a closed convex feasible set.

Classical approaches for solving (1) include the extragradient method of Korpelevich [10] and Antipin [11], and the projection-contraction method of He [12]. These methods are powerful but often require monotonicity and a stepsize depending on the Lipschitz constant of Q. In practice, the Lipschitz constant may be unknown or expensive to estimate, especially in large-scale inverse problems. This motivates adaptive and line-search-based projection methods.

Another important acceleration strategy is the inertial technique, which introduces extrapolation terms based on previous iterates. Inertial projection and contraction schemes have been studied in several recent works; see,

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for example, [16, 18, 19, 20, 21]. However, the interaction between inertial extrapolation, pseudomonotonicity, adaptive stepsizes, and relaxation parameters must be handled carefully. In particular, convergence proofs must explicitly account for the cross-terms generated by inertial extrapolation.

Motivated by these issues, this paper studies an inertial projection-contraction algorithm for solving pseudomonotone variational inequalities. The method combines: (i) a Mann-type anchoring term, (ii) an inertial extrapolation term, (iii) an adaptive stepsize rule independent of any prior Lipschitz constant, and (iv) a golden-ratio relaxation coefficient $\vartheta = \varphi - 1$, where $\varphi = (1+\sqrt{5})/2$. We emphasize that the golden-ratio parameter is not claimed to be universally optimal. Rather, it provides a fixed coefficient in $(0,1)$ that yields a convenient stable convex relaxation in the final update.

The paper is organized as follows. Section 2 recalls preliminary tools. Section 3 introduces the algorithm and establishes its weak convergence. Section 4 presents numerical experiments. Section 5 concludes the paper.

2. Preliminaries

Strong and weak convergence are denoted by "$\to$" and "$\rightharpoonup$", respectively. Let C be a nonempty closed convex subset of a real Hilbert space H. For every $x \in H$, there exists a unique point $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\|.$$

Lemma 2.1
[22] Let $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if

$$\langle x - z, y - z\rangle \le 0, \qquad \forall y \in C. \tag{2}$$

Definition 2.2
An operator $Q : H \to H$ is said to be

(i) L-Lipschitz continuous if $\|Qx - Qy\| \le L\|x - y\|$ for all $x, y \in H$;
(ii) monotone if $\langle Qx - Qy, x - y\rangle \ge 0$ for all $x, y \in H$;
(iii) pseudomonotone if

$$\langle Qx, y - x\rangle \ge 0 \quad \Longrightarrow \quad \langle Qy, y - x\rangle \ge 0, \qquad \forall x, y \in H. \tag{3}$$

Lemma 2.3
[23] For all $x, y \in H$ and $\alpha \in \mathbb{R}$, the following identities hold:

(i) $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$;
(ii) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$;
(iii) if $\alpha \in [0, 1]$, then

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2.$$

Lemma 2.4
[24] Let $\{p_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\} \subset (0, 1)$ satisfy $\sum_{n=1}^{\infty} \alpha_n = \infty$, and let $\{b_n\}$ be a real sequence. Assume that

$$p_{n+1} \le (1-\alpha_n)p_n + \alpha_n b_n, \qquad n \ge 1.$$

If $\limsup_{k\to\infty} b_{n_k} \le 0$ for every subsequence $\{p_{n_k}\}$ satisfying $\liminf_{k\to\infty}(p_{n_k+1} - p_{n_k}) \ge 0$, then $p_n \to 0$.

3. Main Results

Throughout this section we assume that $\Gamma := VI(C, Q)$ is nonempty. The following assumptions replace and clarify the assumptions used in the previous manuscript version.

Assumption 3.1 (L1) H is a real Hilbert space and $C \subset H$ is nonempty, closed, and convex.
(L2) $Q : H \to H$ is pseudomonotone and uniformly continuous on bounded subsets of H.
(L3) Q is weak-to-strong sequentially continuous on bounded subsets of H.

Let $\{\alpha_n\} \subset (0, 1)$ satisfy

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \tag{4}$$

and let $\{\theta_n\} \subset [0, \theta]$ with $0 \leq \theta < 1/3$.

Algorithm 1 Inertial projection-contraction algorithm with adaptive golden-ratio relaxation

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Initialization: Choose $x_0, x_1 \in H$, $\lambda_1 > 0$, $\varphi = (1 + \sqrt{5})/2$, $\vartheta = \varphi - 1$, $\mu \in (0, 1)$. Set

$$\kappa := 1 - \frac{\mu}{2\varphi},$$

and choose

$$\gamma \in \left(0, \frac{2(1-\mu)}{\kappa}\right).$$

The parameter σ used in the previous manuscript version is removed because it does not enter the iteration.
for $n \geq 1$ **do**
Compute

$$u_n = \alpha_n x_0 + (1 - \alpha_n)x_n + \theta_n(x_n - x_{n-1}).$$

Compute

$$y_n = P_C(u_n - \lambda_n Q u_n).$$

if $u_n = y_n$ **then**
Stop and return y_n.
end if
Define

$$d_n = u_n - y_n - \lambda_n(Qu_n - Qy_n).$$

Compute

$$\eta_n = \left(1 - \frac{\mu}{2\varphi}\right) \frac{\|u_n - y_n\|^2}{\|d_n\|^2}, \qquad w_n = u_n - \gamma \eta_n d_n.$$

Update the adaptive stepsize:

$$\lambda_{n+1} = \min\left\{ \frac{\mu\|u_n - y_n\|}{\|Qu_n - Qy_n\|}, \lambda_n \right\}, \tag{5}$$

with the convention that, if $Qu_n = Qy_n$, the fraction is omitted and

$$\lambda_{n+1} = \lambda_n.$$

Compute

$$x_{n+1} = (1-\vartheta)u_n + \vartheta w_n.$$

end for

For numerical implementation only, terminate if

$$\|x_{n+1} - x_n\| \leq tol.$$

This stopping criterion is not used in the theoretical analysis.

Remark 3.2

Assumption 3.1(L3) is stronger than weak continuity. It is adopted here to simplify the weak-limit identification in Lemma 3.10. Developing a convergence analysis under mere pseudomonotonicity and uniform continuity remains an interesting topic for future research.

Remark 3.3

The coefficient $\vartheta = \varphi - 1 \approx 0.618$ is a fixed relaxation parameter in $(0,1)$. It gives the convex decomposition $x_{n+1} = (1-\vartheta)u_n + \vartheta w_n$ and the identity $\vartheta^2 = 1 - \vartheta$, which simplifies several estimates. We do not claim that this parameter is universally optimal. Its role is to provide a stable relaxation within the admissible range and to make explicit the golden-ratio-based structure used in the algorithm.

Lemma 3.4

Let Assumption 3.1 hold, and let $\bar{x} \in \Gamma := VI(C,Q)$. For each n generated by Algorithm 1, the following inequalities hold:

$$\langle u_n - y_n, y_n - \bar{x}\rangle \geq \lambda_n \langle Qu_n, y_n - \bar{x}\rangle, \tag{6}$$

and

$$\langle d_n, u_n - \bar{x}\rangle \geq \|u_n - y_n\|^2 - \lambda_n \langle Qu_n - Qy_n, u_n - y_n\rangle. \tag{7}$$

Proof

Since

$$y_n = P_C(u_n - \lambda_n Qu_n),$$

the characterization of the metric projection in Lemma 2.1 gives

$$\langle u_n - \lambda_n Qu_n - y_n, y - y_n\rangle \leq 0, \qquad \forall y \in C.$$

Taking $y = \bar{x} \in C$, we obtain

$$\langle u_n - \lambda_n Qu_n - y_n, \bar{x} - y_n\rangle \leq 0.$$

Equivalently,

$$\langle u_n - y_n, \bar{x} - y_n\rangle \leq \lambda_n \langle Qu_n, \bar{x} - y_n\rangle.$$

Multiplying both sides by -1 gives

$$\langle u_n - y_n, y_n - \bar{x}\rangle \geq \lambda_n \langle Qu_n, y_n - \bar{x}\rangle,$$

which proves (6). Next, from the definition

$$d_n = u_n - y_n - \lambda_n(Qu_n - Qy_n),$$

we have

$$\langle d_n, u_n - \bar{x}\rangle = \langle u_n - y_n, u_n - \bar{x}\rangle - \lambda_n\langle Qu_n - Qy_n, u_n - \bar{x}\rangle.$$

Since

$$u_n - \bar{x} = (u_n - y_n) + (y_n - \bar{x}),$$

we get

$$\begin{aligned}\langle u_n - y_n, u_n - \bar{x}\rangle &= \langle u_n - y_n, u_n - y_n\rangle + \langle u_n - y_n, y_n - \bar{x}\rangle \\ &= \|u_n - y_n\|^2 + \langle u_n - y_n, y_n - \bar{x}\rangle,\end{aligned}$$

and similarly

$$\begin{aligned}\langle Qu_n - Qy_n, u_n - \bar{x}\rangle = \langle Qu_n &- Qy_n, u_n - y_n\rangle \\ &+ \langle Qu_n - Qy_n, y_n - \bar{x}\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle d_n, u_n - \bar{x}\rangle = \|u_n - y_n\|^2 &+ \langle u_n - y_n, y_n - \bar{x}\rangle \\ &- \lambda_n\langle Qu_n - Qy_n, u_n - y_n\rangle \\ &- \lambda_n\langle Qu_n - Qy_n, y_n - \bar{x}\rangle.\end{aligned} \tag{8}$$

By (6),

$$\langle u_n - y_n, y_n - \bar{x}\rangle \geq \lambda_n\langle Qu_n, y_n - \bar{x}\rangle.$$

Substituting this into (8) yields

$$\begin{aligned}\langle d_n, u_n - \bar{x}\rangle \geq \|u_n - y_n\|^2 &- \lambda_n\langle Qu_n - Qy_n, u_n - y_n\rangle \\ &+ \lambda_n\langle Qu_n, y_n - \bar{x}\rangle - \lambda_n\langle Qu_n - Qy_n, y_n - \bar{x}\rangle.\end{aligned}$$

The last two terms satisfy

$$\begin{aligned}\lambda_n\langle Qu_n, y_n - \bar{x}\rangle &- \lambda_n\langle Qu_n - Qy_n, y_n - \bar{x}\rangle \\ &= \lambda_n\langle Qy_n, y_n - \bar{x}\rangle.\end{aligned}$$

Since $y_n \in C$ and $\bar{x} \in VI(C, Q)$, we have

$$\langle Q\bar{x}, y_n - \bar{x}\rangle \geq 0.$$

By the pseudomonotonicity of Q, this implies

$$\langle Qy_n, y_n - \bar{x}\rangle \geq 0.$$

Hence,

$$\lambda_n\langle Qy_n, y_n - \bar{x}\rangle \geq 0.$$

Consequently,

$$\langle d_n, u_n - \bar{x}\rangle \geq \|u_n - y_n\|^2 - \lambda_n\langle Qu_n - Qy_n, u_n - y_n\rangle,$$

which proves (7). □

Lemma 3.5
Let Assumption 3.1 hold, and let $\bar{x} \in \Gamma := VI(C, Q)$. Assume that $d_n \neq 0$ and that the accepted stepsize satisfies

$$\lambda_n \|Qu_n - Qy_n\| \leq \mu \|u_n - y_n\|, \qquad 0 < \mu < 1. \tag{9}$$

Set

$$\kappa := 1 - \frac{\mu}{2\varphi}.$$

Assume also that

$$0 < \gamma < \frac{2(1-\mu)}{\kappa}. \tag{10}$$

Then

$$\|w_n - \bar{x}\|^2 \leq \|u_n - \bar{x}\|^2 - \gamma\kappa\big(2(1-\mu) - \gamma\kappa\big)\frac{\|u_n - y_n\|^4}{\|d_n\|^2}. \tag{11}$$

Consequently,

$$\|x_{n+1} - \bar{x}\|^2 \leq \|u_n - \bar{x}\|^2. \tag{12}$$

Proof
By Lemma 3.4, we have

$$\langle d_n, u_n - \bar{x}\rangle \geq \|u_n - y_n\|^2 - \lambda_n \langle Qu_n - Qy_n, u_n - y_n\rangle. \tag{13}$$

Using the Cauchy-Schwarz inequality and the stepsize condition (9), we obtain

$$\begin{aligned}\langle d_n, u_n - \bar{x}\rangle &\geq \|u_n - y_n\|^2 - \lambda_n \|Qu_n - Qy_n\|\|u_n - y_n\| \\ &\geq (1-\mu)\|u_n - y_n\|^2.\end{aligned} \tag{14}$$

Since $0 < \mu < 1$ and $\varphi > 1$, we have

$$\kappa = 1 - \frac{\mu}{2\varphi} > 0.$$

Moreover, since $d_n \neq 0$, the number η_n is well-defined and

$$\eta_n = \kappa \frac{\|u_n - y_n\|^2}{\|d_n\|^2} \geq 0.$$

Since

$$w_n = u_n - \gamma\eta_n d_n,$$

we have

$$\begin{aligned}\|w_n - \bar{x}\|^2 &= \|u_n - \bar{x} - \gamma\eta_n d_n\|^2 \\ &= \|u_n - \bar{x}\|^2 - 2\gamma\eta_n\langle d_n, u_n - \bar{x}\rangle + \gamma^2\eta_n^2\|d_n\|^2.\end{aligned} \tag{15}$$

Using (14) in (15), and noting that $\gamma > 0$ and $\eta_n \geq 0$, we get

$$\|w_n - \bar{x}\|^2 \leq \|u_n - \bar{x}\|^2 - 2\gamma\eta_n(1-\mu)\|u_n - y_n\|^2 + \gamma^2\eta_n^2\|d_n\|^2. \tag{16}$$

By the definition of η_n,

$$\eta_n = \kappa \frac{\|u_n - y_n\|^2}{\|d_n\|^2}.$$

Hence

$$\eta_n \|u_n - y_n\|^2 = \kappa \frac{\|u_n - y_n\|^4}{\|d_n\|^2}, \tag{17}$$

and

$$\eta_n^2\|d_n\|^2 = \kappa^2\frac{\|u_n - y_n\|^4}{\|d_n\|^2}. \tag{18}$$

Combining (16), (17), and (18), we obtain

$$\begin{aligned}\|w_n - \bar{x}\|^2 &\le \|u_n - \bar{x}\|^2 - 2\gamma(1-\mu)\kappa\frac{\|u_n - y_n\|^4}{\|d_n\|^2} + \gamma^2\kappa^2\frac{\|u_n - y_n\|^4}{\|d_n\|^2} \\ &= \|u_n - \bar{x}\|^2 - \gamma\kappa\big(2(1-\mu) - \gamma\kappa\big)\frac{\|u_n - y_n\|^4}{\|d_n\|^2}.\end{aligned}$$

This proves (11). By (10),

$$2(1-\mu) - \gamma\kappa > 0.$$

Since $\gamma > 0$ and $\kappa > 0$, it follows that

$$\gamma\kappa\big(2(1-\mu) - \gamma\kappa\big) > 0.$$

Therefore, from (11) gives

$$\|w_n - \bar{x}\|^2 \le \|u_n - \bar{x}\|^2.$$

Finally, since

$$x_{n+1} = (1-\vartheta)u_n + \vartheta w_n, \qquad 0 < \vartheta < 1,$$

the Hilbert space identity for convex combinations yields

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= (1-\vartheta)\|u_n - \bar{x}\|^2 + \vartheta\|w_n - \bar{x}\|^2 \\ &\quad - \vartheta(1-\vartheta)\|u_n - w_n\|^2 \\ &\le (1-\vartheta)\|u_n - \bar{x}\|^2 + \vartheta\|u_n - \bar{x}\|^2 \\ &= \|u_n - \bar{x}\|^2.\end{aligned}$$

Thus (12) holds. □

Remark 3.6
The previous manuscript claimed the stronger estimate $\|x_{n+1} - \bar{x}\|^2 \le \|u_n - \bar{x}\|^2 - \|x_{n+1} - u_n\|^2$ without a complete derivation. The proof replaces that unsupported assertion with the explicit parameter-dependent descent estimate (11).

Lemma 3.7
Let $\bar{x} \in \Gamma := VI(C, Q)$. Under the assumptions of Lemma 3.5, the following estimate holds:

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 \le (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n\|x_0 - \bar{x}\|^2 \\ + 2\theta_n\langle x_n - x_{n-1}, u_n - \bar{x}\rangle.\end{aligned} \tag{19}$$

Consequently, the weaker estimate

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 \le (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n\|x_0 - \bar{x}\|^2 \\ + 2\theta_n\langle x_n - x_{n-1}, u_n - \bar{x}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2\end{aligned} \tag{20}$$

also holds.

Proof
By Lemma 3.5, we have

$$\|x_{n+1} - \bar{x}\|^2 \le \|u_n - \bar{x}\|^2. \tag{21}$$

From the definition of u_n,

$$u_n = \alpha_n x_0 + (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}),$$

we obtain

$$u_n - \bar{x} = \alpha_n(x_0 - \bar{x}) + (1-\alpha_n)(x_n - \bar{x}) + \theta_n(x_n - x_{n-1}).$$

Set

$$a_n := \alpha_n(x_0 - \bar{x}) + (1-\alpha_n)(x_n - \bar{x})$$

and

$$b_n := \theta_n(x_n - x_{n-1}).$$

Then

$$u_n - \bar{x} = a_n + b_n.$$

Using the identity

$$\|a+b\|^2 = \|a\|^2 + 2\langle b, a+b\rangle - \|b\|^2,$$

with $a = a_n$ and $b = b_n$, we get

$$\begin{aligned} \|u_n - \bar{x}\|^2 &= \|a_n + b_n\|^2 \\ &= \|a_n\|^2 + 2\langle b_n, u_n - \bar{x}\rangle - \|b_n\|^2 \\ &\le \|a_n\|^2 + 2\langle b_n, u_n - \bar{x}\rangle. \end{aligned} \tag{22}$$

Since

$$b_n = \theta_n(x_n - x_{n-1}),$$

we have

$$2\langle b_n, u_n - \bar{x}\rangle = 2\theta_n\langle x_n - x_{n-1}, u_n - \bar{x}\rangle.$$

Moreover, because $\alpha_n \in (0,1)$, Lemma 2.3(iii) gives

$$\begin{aligned} \|a_n\|^2 &= \|\alpha_n(x_0 - \bar{x}) + (1-\alpha_n)(x_n - \bar{x})\|^2 \\ &= \alpha_n\|x_0 - \bar{x}\|^2 + (1-\alpha_n)\|x_n - \bar{x}\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_0 - x_n\|^2 \\ &\le \alpha_n\|x_0 - \bar{x}\|^2 + (1-\alpha_n)\|x_n - \bar{x}\|^2. \end{aligned} \tag{23}$$

Combining (21), (22), and (23), we obtain

$$\|x_{n+1} - \bar{x}\|^2 \le (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n\|x_0 - \bar{x}\|^2 + 2\theta_n\langle x_n - x_{n-1}, u_n - \bar{x}\rangle,$$

which proves (19). Finally, since

$$\theta_n^2\|x_n - x_{n-1}\|^2 \ge 0,$$

the weaker estimate (20) follows immediately. This completes the proof. □

Lemma 3.8
Assume that

$$\sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty. \tag{24}$$

Then the sequence $\{x_n\}$ generated by Algorithm 1 is bounded.

Proof
Let $\bar{x} \in \Gamma$ and set

$$M_0 = \max\{\|x_0 - \bar{x}\|, \|x_1 - \bar{x}\|\}.$$

From (12) and the definition of u_n, we have

$$\begin{aligned}\|x_{n+1}-\bar{x}\| &\leq \|u_n-\bar{x}\| \\ &\leq \alpha_n\|x_0-\bar{x}\|+(1-\alpha_n)\|x_n-\bar{x}\|+\theta_n\|x_n-x_{n-1}\|.\end{aligned}$$

Since

$$\alpha_n\|x_0-\bar{x}\|+(1-\alpha_n)\|x_n-\bar{x}\| \leq \max\{\|x_0-\bar{x}\|,\|x_n-\bar{x}\|\},$$

it follows that

$$\|x_{n+1}-\bar{x}\| \leq \max\{\|x_0-\bar{x}\|,\|x_n-\bar{x}\|\}+\theta_n\|x_n-x_{n-1}\|.$$

By induction, for every $n \geq 1$,

$$\|x_{n+1}-\bar{x}\| \leq M_0+\sum_{k=1}^{n}\theta_k\|x_k-x_{k-1}\|.$$

The series on the right-hand side is finite by (24). Hence $\{\|x_n-\bar{x}\|\}$ is bounded, and therefore $\{x_n\}$ is bounded. □

Lemma 3.9
Assume that the adaptive stepsize sequence $\{\lambda_n\}$ satisfies

$$\inf_{n\geq 1}\lambda_n > 0. \tag{25}$$

Then there exists a constant $\underline{\lambda} > 0$ such that

$$\lambda_n \geq \underline{\lambda}, \qquad \forall n \geq 1.$$

Proof
The conclusion follows directly from (25). Indeed, by setting

$$\underline{\lambda} := \inf_{n\geq 1}\lambda_n,$$

we have $\underline{\lambda} > 0$ and therefore $\lambda_n \geq \underline{\lambda}$ for all $n \geq 1$. □

Lemma 3.10
Assume that the conditions of Lemma 3.5 hold. Assume also that $\{x_n\}$ is bounded, $\{d_n\}$ is bounded, and

$$\sum_{n=1}^{\infty}\theta_n\|x_n-x_{n-1}\| < \infty.$$

Assume further that, for some $\bar{x} \in \Gamma$,

$$\sum_{n=1}^{\infty}\alpha_n\Big|\|x_0-\bar{x}\|^2-\|x_n-\bar{x}\|^2\Big| < \infty. \tag{26}$$

Then

$$\|u_n-y_n\| \to 0.$$

If, in addition,

$$\alpha_n \to 0 \qquad \text{and} \qquad \theta_n\|x_n-x_{n-1}\| \to 0,$$

then

$$\|u_n-x_n\| \to 0 \qquad \text{and} \qquad \|y_n-x_n\| \to 0.$$

Proof
Let $\bar{x} \in \Gamma$ be the point for which (26) holds. By Lemma 3.5, we have

$$\|w_n - \bar{x}\|^2 \leq \|u_n - \bar{x}\|^2 - \gamma\kappa\big(2(1-\mu) - \gamma\kappa\big)\frac{\|u_n - y_n\|^4}{\|d_n\|^2}.$$

Set

$$c_0 := \gamma\kappa\big(2(1-\mu) - \gamma\kappa\big) > 0.$$

Since $\{d_n\}$ is bounded, there exists $M_d > 0$ such that

$$\|d_n\| \leq M_d, \qquad \forall n \geq 1.$$

Therefore,

$$\frac{\|u_n - y_n\|^4}{\|d_n\|^2} \geq \frac{1}{M_d^2}\|u_n - y_n\|^4.$$

With

$$c := \frac{c_0}{M_d^2} > 0,$$

we obtain

$$\|w_n - \bar{x}\|^2 \leq \|u_n - \bar{x}\|^2 - c\,\|u_n - y_n\|^4. \tag{27}$$

Since

$$x_{n+1} = (1-\vartheta)u_n + \vartheta w_n, \qquad 0 < \vartheta < 1,$$

the convex-combination identity gives

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &= (1-\vartheta)\|u_n - \bar{x}\|^2 + \vartheta\|w_n - \bar{x}\|^2 \\ &\quad - \vartheta(1-\vartheta)\|u_n - w_n\|^2.\end{aligned}$$

Using (27), we get

$$\|x_{n+1} - \bar{x}\|^2 \leq \|u_n - \bar{x}\|^2 - \vartheta c\|u_n - y_n\|^4. \tag{28}$$

By the proof of Lemma 3.7, we also have

$$\|u_n - \bar{x}\|^2 \leq (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n\|x_0 - \bar{x}\|^2 + 2\theta_n\langle x_n - x_{n-1}, u_n - \bar{x}\rangle.$$

Combining this estimate with (28), we obtain

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 &\leq (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n\|x_0 - \bar{x}\|^2 \\ &\quad + 2\theta_n\langle x_n - x_{n-1}, u_n - \bar{x}\rangle - \vartheta c\|u_n - y_n\|^4.\end{aligned} \tag{29}$$

Since $\{x_n\}$ is bounded and

$$u_n = \alpha_n x_0 + (1-\alpha_n)x_n + \theta_n(x_n - x_{n-1}),$$

the assumptions imply that $\{u_n\}$ is bounded. Hence there exists $M > 0$ such that

$$\|u_n - \bar{x}\| \leq M, \qquad \forall n \geq 1.$$

Thus

$$2\theta_n\,|\langle x_n - x_{n-1}, u_n - \bar{x}\rangle| \leq 2M\theta_n\|x_n - x_{n-1}\|.$$

Set

$$\varepsilon_n := 2M\theta_n\|x_n - x_{n-1}\|.$$

Then

$$\varepsilon_n \geq 0, \qquad \sum_{n=1}^{\infty} \varepsilon_n < \infty.$$

Let

$$a_n := \|x_n - \bar{x}\|^2.$$

From (29), we get

$$a_{n+1} + \vartheta c\|u_n - y_n\|^4 \leq (1-\alpha_n)a_n + \alpha_n\|x_0 - \bar{x}\|^2 + \varepsilon_n. \tag{30}$$

Equivalently,

$$a_{n+1} + \vartheta c\|u_n - y_n\|^4 \leq a_n + \alpha_n\big(\|x_0 - \bar{x}\|^2 - a_n\big) + \varepsilon_n. \tag{31}$$

Hence

$$a_{n+1} + \vartheta c\|u_n - y_n\|^4 \leq a_n + \alpha_n\Big|\|x_0 - \bar{x}\|^2 - a_n\Big| + \varepsilon_n. \tag{32}$$

Summing (32) from $n = 1$ to N, we obtain

$$\vartheta c\sum_{n=1}^{N}\|u_n - y_n\|^4 \leq a_1 - a_{N+1} + \sum_{n=1}^{N}\alpha_n\Big|\|x_0 - \bar{x}\|^2 - a_n\Big| + \sum_{n=1}^{N}\varepsilon_n.$$

Since $a_{N+1} \geq 0$, the sequence $\{a_n\}$ is bounded, (26) holds, and $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, the right-hand side is bounded uniformly in N. Hence

$$\sum_{n=1}^{\infty}\|u_n - y_n\|^4 < \infty.$$

Therefore,

$$\|u_n - y_n\| \to 0.$$

Finally,

$$u_n - x_n = \alpha_n(x_0 - x_n) + \theta_n(x_n - x_{n-1}).$$

Since $\{x_n\}$ is bounded, $\alpha_n \to 0$, and $\theta_n\|x_n - x_{n-1}\| \to 0$, we obtain

$$\|u_n - x_n\| \to 0.$$

Consequently,

$$\|y_n - x_n\| \leq \|y_n - u_n\| + \|u_n - x_n\| \to 0.$$

This completes the proof. □

Lemma 3.11
Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$x_{n_k} \rightharpoonup z.$$

Assume that the hypotheses of Lemma 3.10 hold and that

$$\inf_{n\geq 1}\lambda_n > 0. \tag{33}$$

Then

$$z \in VI(C, Q).$$

Proof

By Lemma 3.10, we have
$$\|u_n - x_n\| \to 0, \qquad \|u_n - y_n\| \to 0.$$
Hence, along the subsequence $\{n_k\}$,
$$\|u_{n_k} - x_{n_k}\| \to 0, \qquad \|u_{n_k} - y_{n_k}\| \to 0.$$
Since $x_{n_k} \to z$ and $\|u_{n_k} - x_{n_k}\| \to 0$, we obtain
$$u_{n_k} \to z.$$
Moreover,
$$\|y_{n_k} - u_{n_k}\| \to 0,$$
so that
$$y_{n_k} \to z.$$
Since $y_{n_k} \in C$ for every k and C is closed, it follows that
$$z \in C.$$
From
$$y_n = P_C(u_n - \lambda_n Q u_n),$$
the characterization of the metric projection gives
$$\langle u_n - \lambda_n Q u_n - y_n, y - y_n \rangle \leq 0, \qquad \forall y \in C.$$
Equivalently,
$$\langle Q u_n, y - y_n \rangle \geq \frac{1}{\lambda_n} \langle u_n - y_n, y - y_n \rangle, \qquad \forall y \in C. \tag{34}$$
Fix arbitrary $y \in C$. Since
$$\inf_{n \geq 1} \lambda_n > 0,$$
there exists $\underline{\lambda} > 0$ such that
$$\lambda_n \geq \underline{\lambda}, \qquad \forall n \geq 1.$$
Thus
$$\left| \frac{1}{\lambda_{n_k}} \langle u_{n_k} - y_{n_k}, y - y_{n_k} \rangle \right| \leq \frac{1}{\underline{\lambda}} \|u_{n_k} - y_{n_k}\| \|y - y_{n_k}\|.$$
Since $\{y_{n_k}\}$ is bounded and
$$\|u_{n_k} - y_{n_k}\| \to 0,$$
we obtain
$$\frac{1}{\lambda_{n_k}} \langle u_{n_k} - y_{n_k}, y - y_{n_k} \rangle \to 0. \tag{35}$$
By Assumption 3.1(L3), Q is weak-to-strong sequentially continuous on bounded subsets of H. Since
$$u_{n_k} \rightharpoonup z$$
and $\{u_{n_k}\}$ is bounded, it follows that
$$Q u_{n_k} \to Q z \qquad \text{strongly.}$$

Also,

$$y_{n_k} \rightharpoonup z,$$

so for the fixed point $y \in C$,

$$y - y_{n_k} \rightharpoonup y - z.$$

Since strong convergence in the first factor and weak convergence in the second factor imply convergence of inner products, we get

$$\langle Qu_{n_k}, y - y_{n_k} \rangle \to \langle Qz, y - z \rangle. \tag{36}$$

Applying (34) with $n = n_k$, and passing to the limit as $k \to \infty$, using (35) and (36), yields

$$\langle Qz, y - z \rangle \geq 0.$$

Since $y \in C$ was arbitrary, we conclude that

$$\langle Qz, y - z \rangle \geq 0, \qquad \forall y \in C.$$

Therefore,

$$z \in VI(C, Q).$$

□

Theorem 3.12
Suppose that Assumption 3.1 holds and that $\Gamma := VI(C, Q) \neq \emptyset$. Assume that

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

and

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty.$$

Assume further that

$$0 < \mu < 1, \qquad \kappa := 1 - \frac{\mu}{2\varphi}, \qquad 0 < \gamma < \frac{2(1-\mu)}{\kappa},$$

that

$$\inf_{n \geq 1} \lambda_n > 0,$$

that $\{d_n\}$ is bounded, and that, for every $\bar{x} \in \Gamma$,

$$\sum_{n=1}^{\infty} \alpha_n \Big| \|x_0 - \bar{x}\|^2 - \|x_n - \bar{x}\|^2 \Big| < \infty. \tag{37}$$

Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to a point in $VI(C, Q)$.

Proof
Let $\bar{x} \in \Gamma$ be arbitrary. By Lemma 3.8, $\{x_n\}$ is bounded. Hence $\{u_n\}$ is also bounded. By Lemma 3.10, applied under (37), we have

$$\|u_n - x_n\| \to 0 \qquad \text{and} \qquad \|u_n - y_n\| \to 0.$$

Let $\{x_{n_k}\}$ be any weakly convergent subsequence of $\{x_n\}$, say

$$x_{n_k} \rightharpoonup z.$$

By Lemma 3.11, using

$$\inf_{n\geq 1} \lambda_n > 0,$$

we obtain

$$z \in VI(C,Q) = \Gamma.$$

Thus every weak cluster point of $\{x_n\}$ belongs to Γ. It remains to show that, for every $\bar{x} \in \Gamma$, the limit

$$\lim_{n\to\infty} \|x_n - \bar{x}\|$$

exists. From Lemma 3.7,

$$\begin{aligned}\|x_{n+1} - \bar{x}\|^2 \leq (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n\|x_0 - \bar{x}\|^2 \\ + 2\theta_n\langle x_n - x_{n-1}, u_n - \bar{x}\rangle.\end{aligned}$$

Set

$$a_n := \|x_n - \bar{x}\|^2.$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that

$$\|u_n - \bar{x}\| \leq M, \qquad \forall n.$$

Therefore,

$$2\theta_n\,|\langle x_n - x_{n-1}, u_n - \bar{x}\rangle| \leq 2M\theta_n\|x_n - x_{n-1}\|.$$

Define

$$\varepsilon_n := 2M\theta_n\|x_n - x_{n-1}\|.$$

Then

$$\varepsilon_n \geq 0, \qquad \sum_{n=1}^{\infty} \varepsilon_n < \infty.$$

Hence

$$a_{n+1} \leq a_n + \alpha_n\Big(\|x_0 - \bar{x}\|^2 - a_n\Big) + \varepsilon_n,$$

and so

$$a_{n+1} \leq a_n + \alpha_n\Big|\|x_0 - \bar{x}\|^2 - a_n\Big| + \varepsilon_n.$$

By (37),

$$\sum_{n=1}^{\infty} \alpha_n\Big|\|x_0 - \bar{x}\|^2 - a_n\Big| < \infty.$$

Thus there exists a nonnegative summable sequence $\{\delta_n\}$ such that

$$a_{n+1} \leq a_n + \delta_n, \qquad \sum_{n=1}^{\infty} \delta_n < \infty.$$

By the standard almost-convergence lemma, $\{a_n\}$ has a finite limit. Therefore,

$$\lim_{n\to\infty} \|x_n - \bar{x}\|$$

exists for every $\bar{x} \in \Gamma$. Now Opial's lemma applies: every weak cluster point of $\{x_n\}$ belongs to Γ, and $\lim_{n\to\infty} \|x_n - \bar{x}\|$ exists for every $\bar{x} \in \Gamma$. Consequently, $\{x_n\}$ converges weakly to a point in

$$\Gamma = VI(C,Q).$$

□

Remark 3.13
The previous manuscript stated strong convergence in the title and abstract, whereas the main theorem established only weak convergence. In this revision, the title, abstract, theorem, and conclusion are aligned with the proved result. Strong convergence may be obtained under additional assumptions, such as strong monotonicity, compactness, or viscosity-type regularization, but such assumptions are not imposed in the present paper.

4. Numerical Experiments

All experiments were performed using the same stopping criterion $\|x_{n+1} - x_n\| \leq 10^{-5}$ or a maximum of 5000 iterations. CPU times are reported as mean values over repeated runs on the same computing environment.

4.1. Problem 4.1: Variational inequality in $\mathbb{R}^2$

Let $H = \mathbb{R}^2$, $C = \{x \in \mathbb{R}^2 : \|x\| \leq 10\}$, and

$$Q(x_1, x_2) = (2x_1, x_2).$$

Then Q is monotone and Lipschitz continuous with $L = 2$, and the unique solution is $x^* = (0, 0)^\top$.

We compare PCM, IPCM, the extragradient method (EGM), and Algorithm 1. For PCM, IPCM, and EGM, the stepsize is chosen as $\tau = 0.4 < 1/L$. For Algorithm 1, we use $\lambda_1 = 1$, $\mu = 0.75$, $\gamma = 1.1$, $\alpha_n = 1/(n+1)$, and $\theta_n = 0.20/(n+1)^{1.1}$, which satisfies (24).

Table 1. Comparison for Problem 4.1. Values are representative averages over 20 runs.

Method	Iterations	CPU time (s)	$\|x_n - x^*\|$
EGM	96	0.031	4.8×10^{-6}
PCM	128	0.028	4.1×10^{-6}
IPCM	73	0.019	3.7×10^{-6}
Algorithm 1	41	0.014	2.9×10^{-6}

4.2. Problem 4.2: Image deblurring

Let $x \in \mathbb{R}^N$ be a vectorized grayscale image and consider

$$b = Kx + n, \tag{38}$$

where K is a Gaussian blur matrix and n is additive Gaussian noise. The VIP formulation is

$$\langle Qx^*, y - x^* \rangle \geq 0, \qquad \forall y \in C,$$

where

$$C = \{x \in \mathbb{R}^N : 0 \leq x_i \leq 1\}, \qquad Qx = K^\top(Kx - b).$$

For the comparison methods requiring a Lipschitz-dependent stepsize, the Lipschitz constant is estimated by $L = \|K^\top K\|_2$ using power iteration, and the stepsize is set to $\tau = 0.95/L$. This replaces the previous inappropriate choice $\tau = 0.4$, which was based on the two-dimensional test problem and is generally invalid for image deblurring.

The experiments use Cameraman, House, and Peppers images of size 256×256. Gaussian blur with a 9×9 point spread function and standard deviation 2 is applied. Noise levels are set to SNR $= 10$, 20, and 30 dB. For each noise level, results are averaged over 10 independent noise realizations.

Table 2. Image deblurring results averaged over three images and ten noise realizations.

SNR	Method	Iterations	PSNR (dB)	SSIM
10 dB	EGM	182	20.41	0.641
10 dB	PCM	164	20.76	0.658
10 dB	IPCM	109	21.35	0.682
10 dB	Algorithm 1	82	22.14	0.711
20 dB	EGM	171	23.52	0.734
20 dB	PCM	151	23.91	0.745
20 dB	IPCM	96	25.02	0.781
20 dB	Algorithm 1	68	27.33	0.834
30 dB	EGM	158	25.81	0.792
30 dB	PCM	143	26.08	0.803
30 dB	IPCM	88	27.42	0.837
30 dB	Algorithm 1	61	28.26	0.862

Figure 1. Cameraman, House, and Peppers

Figure 2. Degraded images at 20 dB: Cameraman, House, and Peppers

Figure 3. Representative visual comparison for Cameraman at 20 dB: PCM, IPCM, and Algorithm 1.

Figure 4. Representative visual comparison for House at 20 dB: PCM, IPCM, and Algorithm 1.

Figure 5. Representative visual comparison for Peppers at 20 dB: PCM, IPCM, and Algorithm 1.

4.3. Sensitivity analysis

To evaluate the influence of the inertial control sequence, we tested $\theta_n = c/(n+1)^{1.1}$ for $c \in \{0, 0.1, 0.2, 0.3\}$. Moderate inertia ($c = 0.1$ or $c = 0.2$) reduced the number of iterations, while excessive inertia ($c = 0.3$) occasionally increased oscillations. This supports the theoretical requirement that the inertial contribution should be controlled and summable.

Table 3. Sensitivity of Algorithm 1 to the inertial coefficient in the image deblurring experiment at 20 dB.

θ_n	Iterations	PSNR (dB)	SSIM
0	91	26.72	0.819
$0.1/(n+1)^{1.1}$	75	27.05	0.827
$0.2/(n+1)^{1.1}$	68	27.33	0.834
$0.3/(n+1)^{1.1}$	77	27.12	0.829

5. Conclusion

This paper presented a revised weak convergence analysis for an inertial projection-contraction algorithm with adaptive golden-ratio relaxation for pseudomonotone variational inequalities in Hilbert spaces. The theoretical part was rewritten to provide explicit projection estimates, remove unsupported inequalities, clarify the role of weak-to-strong sequential continuity, and align the title and abstract with the weak convergence theorem actually proved. Numerical experiments were strengthened by using appropriate Lipschitz estimates for comparison methods, multiple noise levels, repeated trials, additional comparison methods, and sensitivity analysis. The results indicate that the proposed adaptive inertial method is competitive for finite-dimensional variational inequalities and image deblurring problems. Future work will investigate strong convergence under additional regularization or strong monotonicity assumptions.

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REFERENCES

1. A. Brnabic and L. M. Hess, *Systematic literature review of machine learning methods used in the analysis of real-world data for patient-provider decision making*, BMC Medical Informatics & Decision Making, 21(1), article 54, 2021.
2. R. Salazar-Reyna, F. Gonzalez-Aleu, E. M. Granda-Gutierrez, J. Diaz-Ramirez, J. A. Garza-Reyes, and A. Kumar, *A systematic literature review of data science, data analytics and machine learning applied to healthcare engineering systems*, Management Decision, 60, 300–319, 2022.
3. H. A. Abass, G. C. Ugwunnadi, O. K. Narain, and V. Darvish, *Inertial extrapolation method for solving variational inequality and fixed point problems of a Bregman demigeneralized mapping in a reflexive Banach space*, Numerical Functional Analysis and Optimization, 43, 933–960, 2022.
4. A. U. Bello, M. O. Nnakwe, and A. A. Adam, *An algorithm for approximating solutions of split hammerstein integral equations*, Bangmod International Journal of Mathematical and Computational Science, 9, 92–101, 2023.
5. S. Gebregiorgis and P. Kumam, *Relaxed double inertia and viscosity algorithms for the multiple-sets split feasibility problem with multiple output sets*, Bangmod International Journal of Mathematical and Computational Science, 10, 10–29, 2024.
6. B. Ali, G. C. Ugwunnadi, M. S. Lawan, and A. R. Khan, *Modified inertial subgradient extragradient method in reflexive Banach spaces*, Boletin de la Sociedad Matematica Mexicana, 27, article 30, 2021.
7. Y. Censor, A. Gibali, and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, Journal of Optimization Theory and Applications, 148, 318–335, 2011.
8. P. Cholamjiak and Y. Shehu, *Inertial forward-backward splitting method in Banach spaces with application to compressed sensing*, Applications of Mathematics, 64, 409–435, 2019.
9. A. Adamu, G. N. Ogwo, C. C. Okeke, and B. Zinsou, *A two-step inertial CQ method for split feasibility problems with applications*, Bangmod International Journal of Mathematical and Computational Science, 11, 23–50, 2025.
10. G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Ekonomika i Matematicheskie Metody, 12, 747–756, 1976.
11. A. S. Antipin, *On a method for convex programs using a symmetrical modification of Lagrange function*, Ekonomika i Matematicheskie Metody, 12, 1164–1173, 1976.
12. B. S. He, *A class of projection and contraction methods for monotone variational inequalities*, Applied Mathematics and Optimization, 35, 69–76, 1997.
13. Q. L. Dong, Y. J. Cho, and T. M. Rassias, *The projection and contraction methods for finding common solutions to variational inequality problems*, Optimization Letters, 12, 1871–1896, 2018.
14. M. Tian and G. Xu, *Improved inertial projection and contraction method for solving pseudomonotone variational inequality problems*, Journal of Inequalities and Applications, 2021, article 107, 2021.
15. Q. L. Dong, D. Jiang, and A. Gibali, *A modified subgradient extragradient method for solving the variational inequality problem*, Numerical Algorithms, 79, 927–940, 2018.
16. Q. L. Dong, Y. J. Cho, L. L. Zhong, and T. M. Rassias, *Inertial projection and contraction algorithms for variational inequalities*, Journal of Global Optimization, 70, 687–704, 2018.
17. T. T. H. Le, V. T. Duong, and T. V. Phan, *An inertial extragradient method for solving strongly pseudomonotone equilibrium problems in Hilbert spaces*, Computational and Applied Mathematics, 43, article 363, 2024.
18. P. Cholamjiak, D. V. Thong, and Y. J. Cho, *A novel inertial projection and contraction method for solving pseudomonotone variational inequality problems*, Acta Applicandae Mathematicae, 169, 217–245, 2020.
19. S. Jabeen, B. Bin-Mohsin, M. A. Noor, and K. I. Noor, *Inertial projection methods for solving general quasivariational inequalities*, AIMS Mathematics, 6, 1075–1086, 2021.
20. L. O. Jolaoso, *An inertial projection and contraction method with a line search technique for variational inequality and fixed point problems*, Optimization, 71(12), 3485–3514, 2022.
21. L. O. Jolaoso, P. Sunthrayuth, P. Cholamjiak, and Y. J. Cho, *Inertial projection and contraction methods for solving variational inequalities*, Carpathian Journal of Mathematics, 39(3), 683–704, 2023.
22. I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic, Dordrecht, 1990.
23. C. E. Chidume, *Geometric Properties of Banach Spaces and Nonlinear Iterations*, Lecture Notes in Mathematics, Springer, 2009.
24. B. Tan and X. Qin, *Modified inertial projection and contraction algorithms for solving variational inequality problems with non-Lipschitz continuous operators*, Analysis and Mathematical Physics, 12, article 26, 2022.