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A Novel Compound G-Family: Statistical Characteristics and Applications to Aircraft Windshield Service Durations and Survival Time Data

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Abstract The aim of this scientific research is to present a thorough comprehension of the innovative G families of Statistical models incorporating different copulas, elucidating their statistical characteristics, and illustrating their practical utility across various real lifetime data scenarios. Researchers, statisticians, and practitioners stand to gain valuable insights from the findings of this study, as it equips them with the ability to model intricate relationships and employ multivariate data analysis for informed decision-making. The versatility of compound classes of continuous Statistical models makes them instrumental in simulating diverse phenomena. They can be effectively utilized to represent the frequency of accidents within a defined time frame, the duration until system failure, or the financial losses associated with a transaction. The adaptability of these compound models is a significant advantage, enabling them to address a wide array of situations, including those challenging for other models to adequately capture. Furthermore, these models frequently exhibit ease of application and understanding. A significant advantage of utilizing compound classes of continuous Statistical models lies in their capacity to represent the interdependence of two or more continuous random variables. This research introduces and investigates a novel compound G class of Statistical models, derived from the truncated-Poisson distribution following a zero, which subsequently generates Weibull G classes, thereby establishing the foundation for this new compound G family. The statistical properties of this family are meticulously scrutinized, thereby enhancing comprehension of its mathematical foundations.

Keywords Truncated Poisson G Class, Compounding Models, Generated Weibull Model, Farlie-Gumbel-Morgenstern, Reliability Data

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1. Introduction

Compound classes of continuous Statistical models are used to model various forms of data such as reliability, insurance and engineering data. These categories provide flexible models of interpretation and comprehension of complicated phenomena in these areas. These compound classes can be used effectively by means of combining different Statistical models to reflect some different patterns and features observed in real-life data. Compound models are quite often applied in the modeling process of the service and survival time in the case of reliability analysis. Compound classes of Statistical models and can therefore be flexibly used to model heavy-tailed, over-dispersion and zero-inflation features that are common in insurance data sets. Compound classes of models are used in the engineering scenario to provide simulation of several types of data, such as longevity, strength and

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degradation data. The models allow studying mechanisms underlying data patterns. Using the concept of truncation after zero, we develop and study a new group of continuous G statistical models, the zero-truncated-Poisson (ZTRP) probabilistic model. This new G family is introduced because it is useful in mathematical and statistical modeling. To explain this, consider a main system made up of N subsystems (smaller parts), where each smaller part works independently and has a parameter called "a." The probabilistic mass function (PMF) of N can then be determined as follows:

$$\Pr_a(N=n) = \frac{\exp(-a)a^n}{n![-\exp(-a)+1]}\Big|_{(n=1,2,\ldots)}.$$

Noting that for the ZTRP random variable (RV), the mean (expected value) $E(N|a)$ and the corresponding variance $U(N|a)$ are obtained as

$$E(N \mid a) = \frac{a}{C_a},$$

and

$$\Pr(N \mid a) = \frac{1}{C_a}(a+a^2) - \frac{1}{C_a^2}a^2,$$

where

$$C_a = 1-\exp(-a)$$

An acceptable failure rate can be determined by distributing the failures of each tiny subsystem according to the generated Weibull (GW) model, which was recently supplied and studied by Cordeiro et al[11]., (2017). The applicable cumulative model function (CDF) for the GW-G family is represented by the following equation [4]:

$$Q_{b,c,\xi}(x) = 1-\exp[-\Omega_{c,\xi}(x)]^b\Big|_{(x\in\mathbb{R})}. \tag{1}$$

where the function $\Omega_{(c,\xi)}(x) = \left[\frac{W_\xi(x)}{\bar{W}_\xi(x)}\right]^c\Big|_{x\in\mathbb{R}}$, $W_\xi(x)$: the CDF of the original model with the common parameters vector ξ, $\bar{W}_\xi(x)$ refers to the survival function (SF) original baseline model, $\frac{dW_\xi(x)}{dx} = w_\xi(x)$ is the original baseline PDF and $b, c > 0$ is a shape parameters. Following Alizadeh et al[6]., (2019), Abid et al[3]., (2021), Ramos et al[29]., (2015), Abdullah and Masmoudi[1]. (2023), Aryal and Yousof[9]. (2017), Korkmaz et al[24]., (2018), and Yousof et al[37] [39]., (2018 and 2020). The CDF of the generated Weibull-Poisson-G (GWP-G) model may be written as

$$F_{\xi U}(x) = C_a^{-1}\left[1-\exp\left(-a\left\{1-\exp\left[-\Omega_{(c,\xi)}(x)\right]\right\}^b\right)\right]\Big|_{x\in\mathbb{R}}, \tag{2}$$

where $\xi U = (a,b,c)$ is the parameter vector of the GWP-G model. Hence, the PDF of 2 can be reduced to

$$f_{\xi U}(x) = \frac{abc\left(C_a^{-1}w_\xi(x)W_\xi(x)^{(c-1)}\exp\left[-\Omega_{(c,\xi)}(x)\right]\right)}{\bar{W}_\xi(x)^{(c+1)}\left\{1-\exp\left[-\Omega_{(c,\xi)}(x)\right]\right\}^{(1-b)}}\underbrace{\left(\exp\left(-a\left\{1-\exp\left[-\Omega_{(c,\xi)}(x)\right]\right\}^b\right)\right)\Big|_{x\in\mathbb{R}}}_{A_o(x)}. \tag{3}$$

Table 1 presents specific instances of the GWP-G family. Furthermore, Table 1 plays a role in promoting the dissemination of knowledge within the scientific community. It serves as a reference point for discussions, comparisons, special cases, and further exploration of the GWP-G family's potential applications. Researchers can draw inspiration from these specific cases to adapt the model to their specific needs, fostering a more extensive and nuanced utilization of the GWP-G family in various fields of study. The construction of novel bivariate GWP-G classes necessitates the utilization of widely-used copulas, including the Farlie, Gumbel, and Morgenstern copula (FAGM), the modified FAGM copula, As Fisher (1997) observed, these copulas are of considerable significance to statisticians because they underpin the development of bivariate model classes and provide a framework for examining scale-invariant measures of dependence. Copulas are especially valuable when assessing the relationship between two variables, as they facilitate the separation of the impacts of dependence from those of the marginal

Table 1. Some reduced models, reduced CDFs and their corresponding authors from the new family

a	b	c	Reduced- Model	Reduced-CDF	Author
1	b	c	Simi GWP-G	$C_1^{-1}\left[1-\exp\left(-\left\{1-\exp\left[-\Omega_{c,\|\xi}(x)\right]\right\}^b\right)\right]$	New
1	1	c	Simi WP-G	$C_1^{-1}\left[1-\exp\left(-\left\{1-\exp\left[-\Omega_{c,\|\xi}(x)\right]\right\}\right)\right]$	New
1	b	1	Simi PGE-G	$C_1^{-1}\left[1-\exp\left(-\left\{1-\exp\left[-\Omega_{1,\|\xi}(x)\right]\right\}^b\right)\right]$	New
1	b	2	Simi PBX-G	$C_1^{-1}\left[1-\exp\left(-\left\{1-\exp\left[-\Omega_{2,\|\xi}(x)\right]\right\}^b\right)\right]$	New
1	1	2	Quasi RP-G	$C_1^{-1}\left[1-\exp\left(-\left\{1-\exp\left[-\Omega_{2,\|\xi}(x)\right]\right\}\right)\right]$	Abooelmagd et al., (2019)
a	1	2	RP-G	$C_a^{-1}\left[1-\exp\left(-a\left\{1-\exp\left[-\Omega_{2,\|\xi}(x)\right]\right\}\right)\right]$	Abooelmagd et al., (2019)
a	b	2	BXP-G	$C_a^{-1}\left[1-\exp\left(-a\left\{1-\exp\left[-\Omega_{2,\|\xi}(x)\right]\right\}^b\right)\right]$	Abooelmagd et al., (2019)
1	1	1	Simi PE-G	$C_1^{-1}\left[1-\exp\left(-\left\{1-\exp\left[-\Omega_{1,\|\xi}(x)\right]\right\}\right)\right]$	Yousof et al., (2020)
a	1	c	WP-G	$C_a^{-1}\left[1-\exp\left(-a\left\{1-\exp\left[-\Omega_{c,\|\xi}(x)\right]\right\}\right)\right]$	Yousof et al., (2020)
a	1	1	GP-G	$C_a^{-1}\left[1-\exp\left(-a\left\{1-\exp\left[-\Omega_{1,\|\xi}(x)\right]\right\}\right)\right]$	Yousof et al., (2020)

models. The fact that multiple bivariate type GWP-G classes are introduced in this text and made necessitate by the discussed copulas creates opportunities in the work of further research that can be performed, as discussed in Section 2. The section 3 of the manuscript takes the readers into the explanation and analysis of pertinent statistical and mathematical characteristics related to the new type bivariate classes of type G. Section 4 is in the so-called most popular maximum probabilistic approach which extensively examines its uses. In Section 5, the discussion is continued with reference to practical applications on the presented classes. The final observations at the sixth section provide a summary of main findings and contributions made by the study and the importance of the new bivariate type GWP-G classes and their possible role in statistical modeling and analysis.

2. Simple Type Copulas

In this section, we introduce novel versions of bivariate type GWP-G (Bv-GWP-G) models, employing the FAGM copula, which was developed by Gumbel[15] [16] (1958 and 1960), Morgenstern[26] (1956), Farlie[13] (1960), Johnson and Kotz[22] [23](1975, 1977), Balakreshnan and Lai[10] (2009). Additionally, we consider the ”modified FAGM (MFAGM)” copula, the ”Clayton” copula, the ”Archimedean-Ali-Haq-Mikhail” (AR-AMH) copula, and the ”Renyi (the entropy)” copula, as outlined in works by Ali et al[5]., (1978), and Pougaza and Djafari[28] (2011). Furthermore, we present the Multivariate GWP-G (Mv GWP-G) type, with more detailed information available in Balakreshnan and Lai[10] (2009). these models have been introduced in this section but at the same time, we recognize that in the future, there is still an opportunity to conduct further studies and analysis of such new models. This provides avenue to further research and development of the suggested Bv-GWP-G versions, which may lead to the discovery of new knowledge and use in statistical modeling and analysis.

2.1. BGWP-G type via MFAGM copula

By constructing the copula on the basis of the fundamental structure of the FAGM, it is feasible to differentiate the MFAGM copula, which is a versatile instrument that can be utilized to model a variety of complex dependent systems. The fact that it is simple to use and possesses a variety of advantages, such as the capacity to imitate mixed dependence, distinguishes it from other kinds of copulas. Additional material regarding the same topic can be found in the works of Rodriguez-Lallena and Ubeda-Flores[30] (2004), Abdullah and Masmoudi[2] (2023), and Abdullah et al[1]., (2023). It is for this reason that the FAGM copula has remained to be an effective solution in a variety of applications. In this sub-part, we are going to investigate the possibility of taking into account one of the variants of the MFAGM copula.

$$C(\Delta, l, \varepsilon) = \varepsilon(\tilde{G}(\Delta))\tilde{E}(l) + \Delta l,$$

where $\tilde{G}(\Delta) = \bar{\Delta}G(\Delta)$, $\tilde{E}(l) = \bar{l}E(l)$ and $G(\Delta)$ and $E(l)$ are two functions on (0,1) where

$$0 = G(0|\Delta) = E(0|l) = G(1|\Delta) = E(1|l),$$

and

$$\alpha|\Delta = \alpha(\Delta) = \inf\left\{\frac{\partial}{\partial\Delta}\tilde{G}(\Delta) : h_1(\Delta)\right\} < 0, \quad c|\Delta = c(\Delta) = \sup\left\{\frac{\partial}{\partial\Delta}\tilde{G}(\Delta) : h_1(\Delta)\right\} < 0,$$

$$\xi|l = \xi(l) = \inf\left\{\frac{\partial}{\partial l}\tilde{E}(l) : h_2(l)\right\} > 0, \quad \eta|l = \eta(l) = \sup\left\{\frac{\partial}{\partial l}\tilde{E}(l) : h_2(l)\right\} > 0.$$

Then,

$$\min(\alpha(\Delta)c(\Delta), \xi(l)\eta(l)) \geq 1,$$

where

$$\frac{\partial}{\partial\Delta}\tilde{G}(\Delta) = G(\Delta) + \Delta\frac{\partial}{\partial\Delta}G(\Delta)$$

$$h_1(\Delta) = \{\Delta : \Delta \in (0,1)|\frac{\partial}{\partial\Delta}\tilde{G}(\Delta) \text{ exists}\},$$

and

$$h_2(l) = \{l : l \in (0,1)|\frac{\partial}{\partial l}\tilde{E}(l) \text{ exists}\}.$$

The MFAGM copula is an easy to use model that can be used to model the interrelationships among prices of assets. The risk of an asset portfolio can be analyzed using it and hedging strategies created. It can also be used to model the interconnected nature of the environmental variables which allows the estimation of the probabilistic of environmental disaster and develop mitigation strategies. The MFAGM copula is also necessary in demonstrating the interdependence of insurance claims which makes it simpler to ascertain the risk profile of an insurance company and establish pricing and reserve rules. In spite of the intrinsic constraints of the current research, we will theorize about the four variants of the MFAGM copula illustrated below.

2.1.1. Type I model The BGWP-G of type I, as per the utilization of the MFAGM copula, can be mathematically expressed as:

$$C_\varepsilon(\Delta, l) = \varepsilon\left[\tilde{G}(\Delta)\tilde{E}(l)\right] + \left(\left\{C_{a_1}^{-1}\left[1 - \exp(-a_1 Q_{b_1,c_1,\xi}(\Delta))\right]\right\} \times \left\{C_{a_2}^{-1}\left[1 - \exp(-a_2 Q_{b_2,c_2,\xi}(l))\right]\right\}\right),$$

where

$$\tilde{G}(\Delta) = \Delta\left\{1 - C_{a_1}^{-1}\left[1 - \exp(-a_1 Q_{b_1,c_1,\xi}(\Delta))\right]\right\}\Big|_{\xi U_1 > 0},$$

and

$$\tilde{E}(l) = l\left\{1 - C_{a_2}^{-1}\left[1 - \exp(-a_2 Q_{b_2,c_2,\xi}(l))\right]\right\}\Big|_{\xi U_2 > 0}.$$

2.1.2. Type II model The BGWP-G of type II, as per the utilization of the MFAGM copula, can be mathematically expressed as:

$G(\Delta) \mid (\varepsilon_1 > 0) = \Delta^{\varepsilon_1}(1-\Delta)^{1-\varepsilon_1}$ and $E(l) \mid (\varepsilon_2 > 0) = l^{\varepsilon_2}(1-l)^{1-\varepsilon_2}$.

Then,

$$C_{\varepsilon,\varepsilon_1,\varepsilon_2}(\Delta, l) = \left[1 + \varepsilon\Delta^{\varepsilon_1}l^{\varepsilon_2}(1-l)^{1-\varepsilon_1}(1-l)^{1-\varepsilon_2}\right]\Delta l.$$

2.1.3. Type III model The BGWP-G of type III, as per the utilization of the MFAGM copula, can be mathematically expressed as:

$$C_\varepsilon(\Delta, l) = \Delta l\left[1 + \varepsilon\tilde{C}(\Delta)\tilde{D}(l)\right],$$

where

$$\tilde{C}(\Delta) = \Delta\left[\log(1 + \bar{\Delta})\right] \text{ and } \tilde{D}(l) = l\left[\log(1 + \bar{l})\right].$$

2.2. AR-AMH copula

The AR-AMH copula greatly helps in the mathematical and statistical characterization of bi-variate data since it enables the development of bi-variate Statistical models in the absence of lengthy statistical derivations. Bivariate version of AR-AMH copula as recorded by Ali et al[5]., (1978) is easily derivable. Notably, the introduction of a new parameter is restricted, in case it is used, by the nature of the AR-AMH copula. Ali et al[5]., (1978) are the major records that are most commonly mentioned whenever meetings address the AR-AMH copula. Later, a more restrictive Lipschitz condition can be used to produce the unique joint CDF of the AR-AMH copula type.

$$C(\Delta, l) = \Delta l \frac{1}{(1-\vartheta\bar{\Delta}\bar{l})}\Big|_{\varepsilon\in(-1,1)}$$

Therefore, the corresponding joint-PDF of the AR-AMH copula can then be derived as

$$c(\Delta, l) = \frac{1-\vartheta+2\vartheta\Delta l/(1-\vartheta\bar{\Delta}\bar{l})}{(1-\vartheta\bar{\Delta}\bar{l})^2}\Big|_{\varepsilon\in(-1,1)}$$

Then,

$$C(\Delta, l) = \frac{C_{a_2}^{-1}\left[1-\exp(-a_2 Q_{b_2,c_2,\xi}(x))\right] C_{a_1}^{-1}\left[1-\exp(-a_1 Q_{b_1,c_1,\xi}(t))\right]}{1-\vartheta\left(\left\{1-C_{a_2}^{-1}\left[1-\exp(-a_2 Q_{b_2,c_2,\xi}(x))\right]\right\}\times\left\{1-C_{a_1}^{-1}\left[1-\exp(-a_1 Q_{b_1,c_1,\xi}(t))\right]\right\}\right)}\Big|_{\varepsilon\in(-1,1)}$$

2.3. Multivariate type copula

Clayton type of copulas has been known to be the simplest in its mathematical formulation and easy to use, thus it is one of the most readily available copulas to derive multivariate models or classes. Its simplicity also makes the Clayton copula popular in other practical uses especially in modeling multivariate data in the engineering, medical journals, validity tests, insurance, reinsurance, and other areas. Using the idea of the Clayton type, a multivariate type model can be obtained based on this copula, which has additional flexibility in its use in other areas.

$$H(l_o) = \left(\sum_{o=1}^{\varpi}\left\{C_{a_o}^{-1}\left[1-\exp\left(-a_o Q_{b_o,c_o,\xi}(t)\right)\right]\right\}^{-\varepsilon}+1-\varpi\right)^{-1/\varepsilon}.$$

3. Mathematical Characteristics

In exploring new types of continuous models it is very important to undertake a careful analysis of the properties that the new G class possesses [21]. The investigation is all-inclusive and is necessary as a preliminary step to parameter estimation or use of the model in statistical and mathematical modeling especially in real-world data applications where a variety of factual data are involved. The examination of the mathematical properties of the new family can be invaluable, simplifying the incorporation of the new family into different mathematical and statistical modelling processes. The symbolic computing software platforms, including Mathematica and Maple, are very good at working with complex expressions, allowing formulas developed in research to be easily worked with. This could be more effective than the straightforward numeric integration by using explicit formulas that are that have been created over the years to do the indirect calculation of statistical measures. This part is detailed in the mathematical representation, estimation and inference of the class of compounds, and is deeply covered. Within the context of Statistical models, the derivation of explicit mathematical formulas is normally the first stage in calculating mathematical and statistical properties. In this case, we give a simplified form of the GWP-G density. These properties are important in determining future behaviour of a random variable. The ability to predict a certain occurrence with the aid of the probabilistic mass function (PMF) or probability density function (PDF) allows one to decide on the choice grounded on that prediction. Overall, the design and testing of statistical models requires the deep knowledge of the following mathematical concepts, which open the door to the creation of statistical models that accurately describe the behavior of real-life events based on Statistical models that have clear mathematical properties.

Theorem 3.1
For the PDF in (5), let $X \sim GWP\text{-}G(U)$, then

$$f_U(x) = \sum_{\Delta,i=0}^{n} \upsilon_{\Delta,i} w_{c^*}(x)\Big|_{c^*=[\Delta+1]c+i},$$

where $w_{c^*}(x) = \frac{\varpi W_{c^*}(x)}{\varpi x} = c^* w(x) W_\xi(x)^{c^*-1}$ is the exp-G PDF with power parameter c^* and

$$\upsilon_{\Delta,i} = \sum_{\varpi,o=0}^{n} \left[a^{1+\varpi} bcC\right]_a^{-1} \frac{1}{(\varpi!o!\Delta!i!c^*)} (-1)^{\varpi+\Delta+o}(o+1)^{\Delta} V_\Gamma,$$

where

$$V_\Gamma = \frac{1}{\Gamma(b(\varpi+1)-o)\Gamma([\Delta+1]c+1)} \Gamma(b(\varpi+1))\Gamma(c^*+1).$$

proof:
First, we expand $A_o(x)$ and replace the result in 3, the we get

$$f_\xi(x) = cbC_a^{-1} \sum_{\varpi=0}^{n} \left((-1)^{\varpi} a^{1+\varpi} \exp\left[-\Omega_{c,\xi}(x)\right] w_\xi(x)\right) / \left(\varpi! \bar{W}_\xi(x)^{(c+1)} W_\xi(x)^{(-c+1)}\right) \underbrace{\left(1-\exp\left[-\Omega_{c,\xi}(x)\right]\right)^{b(\varpi+1)-1}} (B_o(x)). \tag{4}$$

Let us consider

$$\left(1 - \frac{1}{m_2} m_1\right)^{(m_3)} = \sum_{m_4=0}^{n} \left(\frac{(-1)^{m_4}\Gamma(1+m_3)}{m_4!\Gamma(1+m_3-m_4)}\right)\left(\frac{m_1}{m_2}\right)^{(m_4)} |_{\left(\left|\frac{m_1}{m_2}\right|<1 \text{ and } m_3>0\right)}. \tag{5}$$

Apply 5 to $B_o(x)$ we arrive at

$$f_\xi(x) = C_a^{-1} cbw_\xi(x) \frac{\left(W_\xi(x)^{(c-1)}\right)}{\left(\bar{W}_\xi(x)^{(c+1)}\right)} \sum_{\varpi,o=0}^{n} \left\{ a^{1+\varpi} \frac{\left((-1)^{(\varpi+o)}\Gamma(b(\varpi+1))\right)}{o!\varpi!\Gamma(b(\varpi+1)-o)} \right\} \underbrace{\left(\exp\left[-(o+1)\Omega_{c,\xi}(x)\right]\right)} (C_o(x)). \tag{6}$$

Then, Expand the quantity $C_o(x)$ in power series again, we can write

$$C_o(x) = \sum_{\Delta=0}^{n} \left\{(-1)^{\Delta}(o+1)^{\Delta} \frac{1}{\Delta!}\right\} \frac{\left(W_\xi(x)^{\Delta c}\right)}{\left(\bar{W}_\xi(x)^{\Delta c}\right)}. \tag{7}$$

Inserting the 7 in 6, we get

$$f_\xi(x) = bcC_a^{-1} \sum_{\varpi,o,\Delta=0}^{n} a^{1+\varpi} \frac{(-1)^{(\varpi+\Delta+o)}\left(\Gamma(b(\varpi+1))(o+1)^{\Delta}\right)}{\varpi!o!\Delta!\Gamma(b(\varpi+1)-o)} w_\xi(x) \frac{\left(W_\xi(x)^{(\Delta+1)c-1}\right)}{\left(\bar{W}_\xi(x)^{(\Delta+1)c+1}\right)}. \tag{8}$$

Then, using the generalized binomial theory for $[1-W_\xi(x)]^{-[(\Delta+1)c+1]}$, we have

$$[1-W_\xi(x)]^{-[(\Delta+1)c+1]} = \sum_{i=0}^{n} \left\{\frac{1}{i!} \frac{\Gamma([\Delta+1]c+i+1)}{\Gamma([\Delta+1]c+1)}\right\} W_\xi(x)^i. \tag{9}$$

Putting 9 into 8, the new PDF of the GWP-G class can be re-expressed by

$$f_\xi(x) = \sum_{\Delta,i=0}^{n} \upsilon_{\Delta,i} w_{(c^*)}(x) \,|_{(c^*=[\Delta+1]c+i)}, \tag{10}$$

where $w_{(c^*)}(x) = \frac{\varpi W_{(c^*)}(x)}{\varpi x} = c^* w(x) W_\xi(x)^{(c^*-1)}$ is the PDF of the exp-G models with power parameter c^* and

$$\sum_{\varpi,o=0}^{n} \left\{ a^{1+\varpi} bc C_a^{-1} \frac{1}{\varpi! o! \Delta! i! c^*} (-1)^{(\varpi+\Delta+o)} (o+1)^{\Delta} V_\Gamma \right\}.$$

According to equation 10, the PDF of the GWP-G family can be expressed as a linear combination of probabilistic density functions from the exp-G model. Consequently, gaining an understanding of the characteristics of the exp-G model enables the deduction of several mathematical characteristics of the new GWP-G family. Similarly, mirroring the exp-G CDFs, the CDF of the GWP-G family can also be represented as a linear combination of CDFs from:

$$F_\xi(x) = \sum_{\Delta,i=0}^{n} \upsilon_{\Delta,i} W_{(c^*)}(x), \tag{11}$$

where $W_{(c^*)}(x)$ is considered as the CDF of common exp-G, and with the c^* refer to its corresponding power-parameter.

Result 1:

Following Theorem 1, the rth moment of X, say $\mu'_{r,X}$, follows from equation (12) as

$$\mu'_{r,X} = E(X^r) = \sum_{\Delta,i=0}^{n} \upsilon_{\Delta,i} E(Y_{c^*}^r),$$

where Y_{c^*} is our new RV which is considered as the exp-G RV. The nth central moment of X, say M_n, is given by

$$M_{n,X} = E(X-\mu'_1)^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu'_1)^{n-r} E(X^r) = \sum_{r=0}^{n} \sum_{\Delta,i=0}^{n} \upsilon_{\Delta,i} \binom{n}{r} (-\mu'_1)^{n-r} E(Y_{c^*}^r).$$

Result 2:

Following Theorem 1 and result 1, the moment generating function (MGF) $M_X(t) = E(\exp(tX))$ of X can be derived from

$$M_X(t) = \sum_{\Delta,i=0}^{n} \upsilon_{\Delta,i} M_{c^*}(t),$$

The function $M_{c^*}(t)$ is the MGF of Y_{c^*} ($\forall \Delta, i \geq 0$).

Result 3:

Incomplete moments are valuable tools in reliability analysis, offering a means to understand and model the distribution of failure times in situations where not all data points are observed. This is particularly useful in assessing the reliability and durability of systems in various fields such as engineering, finance, and healthcare. The pth incomplete moment, say $\psi_p(t)$, of X can be expressed from (12) as

$$\psi_{p,X}(t) = \int_{-\infty}^{t} x^p f(x) dx = \sum_{\Delta,i=0}^{n} \upsilon_{\Delta,i} \int_{-\infty}^{t} x^p w_{c^*}(x) dx. \tag{12}$$

Clearly, the integral $\int_{-\infty}^{t} x^p w_{c^*}(x)dx$ in 12 denotes the pth incomplete moment of our new Y_{c^*} of the exp-G models.

4. Parameter Estimation

A key idea in statistical modeling, especially when discussing maximal likelihood estimation (MLE) [31] [32], is the log-likelihood function. It is a metric for evaluating how well observed data is explained by a probabilistic model. A key tool in statistical inference, hypothesis testing, and model selection is the log-likelihood function [17]. It provides a quantitative measure of how well a given set of parameters explains the observed data, and its maximization leads to parameter estimates that are most consistent with the observed information. The log-likelihood function l for the GWP-G model is given by

$$l_{\text{U}} = n\log a + n\log c + n\log b - n\log C_a + \sum_{o=1}^{n}\log w_{\xi}(x_o)$$

$$+(c-1)\sum_{o=1}^{n}\log W_{\xi}(x_o) - (1+c)\sum_{o=1}^{n}\bar{W}_{\xi}(x_o) - \sum_{o=1}^{n}\Omega_{c,\xi}(x_o)$$

$$-(1-b)\sum_{o=1}^{n}\{1-\exp[-\Omega_{c,\xi}(x_o)]\} - a\sum_{o=1}^{n}\{1-\exp[-\Omega_{c,\xi}(x_o)]\}^{b}.$$

We can maximize l_U using many possible software or by solving the score vector elements if possible[33]. Anyway, the score vector components, are given by [34] [35].

$$U_a = \frac{\partial}{\partial a} l_{|U} = \frac{n}{a} - n\frac{\exp(-a)}{C_a} - \sum_{o=1}^{n}\left\{1-\exp\left[-\Omega_{c,|\xi}(x_o)\right]\right\}^{b},$$

$$U_b = \frac{\partial}{\partial b} l_{|U} = \frac{n}{b} + \sum_{o=1}^{n}\left\{1-\exp\left[-\Omega_{c,|\xi}(x_o)\right]\right\}^{b}$$

$$-a\sum_{o=1}^{n}\left\{1-\exp\left[-O_{c,|\xi}(x_o)\right]\right\}^{b}\log\left\{1-\exp\left[-\Omega_{c,|\xi}(x_o)\right]\right\},$$

$$U_c = \frac{\partial}{\partial c} l_{|U} = \frac{n}{c} + \sum_{o=1}^{n}\log\left[\Pi_{|\xi}(x_o)\right] - \sum_{o=1}^{n}\left[\Omega_{c,|\xi}(x)\log\Omega_{1,|\xi}(x)\right] - \sum_{o=1}^{n}\left[\bar{\Pi}_{|\xi}(x_o)\right]$$

$$-(1-b)\sum_{o=1}^{n}\left\{\exp\left[-\Omega_{c,|\xi}(x_o)\right]\Omega_{c,|\xi}(x)\log\Omega_{1,|\xi}(x)\right\}$$

$$-ab\sum_{o=1}^{n}\left\{1-\exp\left[-\Omega_{c,|\xi}(x_o)\right]\right\}^{(b-1)}\exp\left[-O_{c,|\xi}(x_o)\right]\Omega_{c,|\xi}(x)\log\Omega_{1,|\xi}(x),$$

and

$$U_{\xi_\Delta} = \frac{\partial}{\partial \xi_\Delta} l_U = \sum_{o=1}^{n}\frac{w_{\xi}^{\setminus}(x_o)}{w_{\xi}(x_o)} + (c-1)\sum_{o=1}^{n}\frac{W_{\xi}^{\setminus}(x_o)}{W_{\xi}(x_o)} \pm c\sum_{o=1}^{n}\Omega_{c-1,\xi}(x)p_o$$

$$+(c+1)\sum_{o=1}^{n}W_{\xi}(x_o) - (1-b)c\sum_{o=1}^{n}\exp\left[-\Omega_{c,\xi}(x_o)\right]\Omega_{c-1,\xi}(x)p_o$$

$$-abc\sum_{o=1}^{n}\{1-\exp[-\Omega_{c,\xi}(x_o)]\}^{b-1}\exp\left[-\Omega_{c,\xi}(x_o)\right]\Omega_{c-1,\xi}(x)p_o,$$

where $p_o = W_{\|\xi}^{\setminus}(x_o)(\overline{W}_{\|\xi}(x_o))^{-2}$, $w_{\|\xi}^{\setminus}(x_o) = \varpi/(\varpi_{\|\xi\Delta})w_{\|\xi}(x)$ and $W_{\|\xi}^{\setminus}(x_o) = \varpi/(\varpi_{\|\xi\Delta})W_{\|\xi}(x)$. The MLE method is often used with the numerical optimization method, the Newton-Raphson method in statistics. The MLE involves defining the amount of the parameters that decrease the negative log-likelihood function or increase the likelihood function. The Newton-Raphson method is an iterative optimization method that successfully approaches to a maximum or a minimum of a given function. Newton-Raphson is a suitable method in the optimization of the log-likelihood function in MLE since the method is effective with smooth and well-behaved curves. The Newton-Raphson method in comparison with other optimization techniques frequently tends to converge to the optimum within a short time. This may be critical particularly in cases where large data sets are involved or elaborate models. The Newton-Raphson algorithm takes use of the first and second derivatives of the objective function (the log-likelihood function in this case). Applying second-order information may result in quicker convergence than approaches, which use first-order information only (e.g., gradient-based methods). In the case where the likelihood function is well-behaved, the Newton-Raphson method gives more accurate estimates of the parameters. It takes into account curvature of the function hence making the estimates more accurate. The MLE when used with the Newton-Raphson procedure can easily estimate parameters, confidence interval, and hypothesis testing. The validity of these statistical deductions is very important in making useful conclusions to data. The Newton-Raphson method of obtaining the maximum likelihood is implemented in many statistical software packages such as R, Python (SciPy library) and others. These tools are commonly used in the statistical analyses by researchers and practitioners. The Newton-Raphson algorithm is generalized to a variety of models and can be applicable to a large variety of statistical estimation tasks. It is very versatile and thus is a preferred choice in the usage in different statistics. In spite of these benefits, it should be mentioned that Newton-Raphson method can be complicated in situations where the likelihood function is characterized by areas of poor curvature or when the starting point is far away. Other optimization techniques or algorithm alterations can be discussed in this case.

5. Applications

The focus on the significance of actual data modeling is essential in this respect in order to compare different Statistical models. Real data modeling is one of the essential instruments of a researcher, which offers a possibility to evaluate the validity of different Statistical models specific to a particular set of data. In practice, it may not be adequate to use one probabilistic model in order to provide a full explanation to a variety of datasets. The inherent complex patterns in datasets can require a more complex or versatile model. It is due to the systematic analysis of many Statistical models based on real data modeling that researchers can identify the one that best describes the complexities of the underlying process of generating data. The strategy does not only guarantee a subtle interpretation of data but also increases the ability to capture the complexities of the data. Moreover, the real data modelling allows the scientists to make conclusions about the population features based on the fitted model. After identifying the most appropriate model, researchers can be able to estimate the model parameters and make conclusions regarding the characteristics of the population. The analysis of data, testing hypotheses, and future predictions can be carried out using such conclusions.

The real data modeling task is to compare the goodness of fit of various Statistical models to real data. Using this method, a scientist can determine what model fits best with a specific data set and draw conclusions regarding the characteristics of the population. There are various reasons as to why real data modeling is important. To start with, the real data modeling enables the researchers to determine the viability of Statistical models in the case of a specific dataset. As a matter of fact, multiple datasets cannot be adequately described through one probabilistic model. Alternatively the data may show us complicated trends which would require a more sophisticated or versatile model.

Researchers can evaluate many Statistical models using real data modelling to find the one that most accurately captures the underlying data generation process. Consider the base line one parameter Pareto (Pa) model where

$$\Omega_{(c,\beta)}(x) = [(x+1)^{\beta} - 1]^{c}.$$

Then, based 2, the CDF of the GWPP model can be written as

$$F_U(x) = C_a^{-1}\left[1 - \exp\left(-a\{1 - \exp[-\Omega_{(c,\beta)}(x)]\}^{b}\right)\right]\Big|_{x>0}.$$

In Table 2, we present many classes which are considered in the below comparison.

Table 2. The competitive model

Model;Author	Abbreviation
Special generalized mixture -Pareto;Altun et al., (2018a)	SLGMPa
Gamma-Pareto;Cordeiro et al., (2015)	GamPa
Transmuted Topp-Leone -Pareto;Yousof et al., (2017b)	TTLPa
Expentiated-Pareto;Gupta et al., (1998)	ExpPa
RBH-Pareto;Yousof et al., (2018a)	RBHPa
Log-logistic -Pareto;Chesneau and Yousof (2020)	OLLPa
Reduced TTL-Pareto;Yousof et al., (2017b)	RTTLPa
Kumaraswamy-Pareto;Lemonte et al., (2013)	KUMYPa
Reduced log-logistic -Pareto;Chesneau and Yousof (2020)	ROLLPa
Pareto;Lomax (1954)	Pa
Proportional reversed hazard rate -Pareto;-	PRHRPa

6. The 1st Data set (84 Aircraft-Windshield): Failure/ Survival times

The data of survival plays a vital role in actuarial science, which is one of the keystones in determining and measuring risks related to insurance policies. It provides the important information on the time to event (TTE) of particular events, e.g. death, failure and disability, or the expiry of a policy term. Survival data help actuaries to model a variety of risks, such as mortality, morbidity and longevity, and predict the value of future losses.

Reinsurance and actuarial science In insurance, survival data has been very important in actuarial table development. These tabular products are based on strong survival data that are vital in computing insurance premiums, reserves and structuring insurance products. Also, survival data is used to determine the impact of different variables on the possibility of an event occurring, which also includes such aspects as age, gender, health conditions, and lifestyle aspects. This priceless information is used in setting up the underwriting rules and in shaping the decisions regarding policy approvals and price process.

Basically, survival/reliability data forms a significant component of the actuarial research and is important in finding risk in insurance plans. The value and quality of this data will be the key to the reliability of actuarial models and projections that will enable insurance companies to successfully manage their financial needs and risks. The first data set provided here is a real life example of failure times of 84 aircraft -windshields, which were reported (Murthy et al[27]., 2004). This dataset is a viable example of how survival data can be used when determining the risks in a particular scenario.

7. The 2nd Data set (63 Aircraft-Windshield): Service times

The queuing theory can be useful in modeling a system with a waiting line like call centers, traffic and manufacturing processes. Here, Statistical models are followed to model both the service time and the arrival time of customers. The stochastic properties are extremely important in the determination of important measures such as the average waiting time and queue length, which give an insight into the performance of the system. The second real life dataset obtained by Murthy et al[27]. (2004) deals with the service times of 63 aircraft-windshields. Moreover, different sources, such as Altun et al[8] [7]. (2018a, b), Ibrahim[18] (2019), Ibrahim and Yousof[20] (2020), Ibrahim[19] (2020), Yadav et al[36]. (2020), Mansour et al[25]. (2020), and Goual et al[14]. (2020), provide an abundance of real-life examples, broadening the scope of applications and analyses of the queuing theory. These datasets contribute to the study and analysis of various domains, enabling one to gain a better insight into the behavior of the system, its optimization, and decision making. Such datasets can help the researchers create and test queuing models, which will result in the creation of more efficient approaches to the handling of waiting lines, the increased efficiency of the service system, and the overall system performance.

Some of the tools are applied in this application like the Quantile-Quantile (QL-QL), Kernel density estimation (KRDE) and the total time under test (TTUT) plots. A QL-QL plot also commonly known as a QL-QL plot is a tool used to compare the model of a dataset to a theoretical model. This is done by making a plot of the quantiles of the dataset versus the quantiles of the theoretical model which basically represents observed values in terms of expected values. Arrangement in a straight line suggests that there is a similarity between the two groups of quantiles. The QL-QL plot plays a crucial role in the visual evaluation of the assumption of a particular model, including the normality distribution. A straight line of points on the plot is an indication that the data is modeled well, whereas the large variance indicates that it may not fit the model adopted. Compared to that, QL-QL plots are also used in the comparison of the models of two datasets and in the interpretation of the plot of quantiles of a dataset versus the quantiles of another dataset, the movement of a straight-line would indicate that the models associated with the two datasets are similar.

The KRDE is a non-parametric procedure used to estimate the initial PDF of some RV as a result of the data set. This is a KRDE plot, which plots the density of the data as estimated. The KRDE can be constructed using many functions in the kernel. The KRDE plot is made smooth depending on the bandwidth of the kernel with a bandwidth of less giving a finer plot and bandwidth of more giving a smooth plot. The choice of the best bandwidth is one of the most significant stages in the estimation procedure because it affects the quality and precision of the KRDE plot. The KRDE plot is used in data analysis, and it is especially useful to visualize the model of a dataset and in cases where the underlying model is unknown or where the dataset is too small to parametrically model. In addition, KRDE plots help in comparing the models to two or more datasets as well as identify patterns and anomalies in the data. The TTUT plot is related to the time or a time span that a particular treatment (or a set of interventions) was applied to the test plot in question. This variable is important because it may affect the final results of the experiment and the accuracy of the results obtained. TTUT plot can include the duration of treatment as well as the time taken in the monitoring and data collection of the experiment.

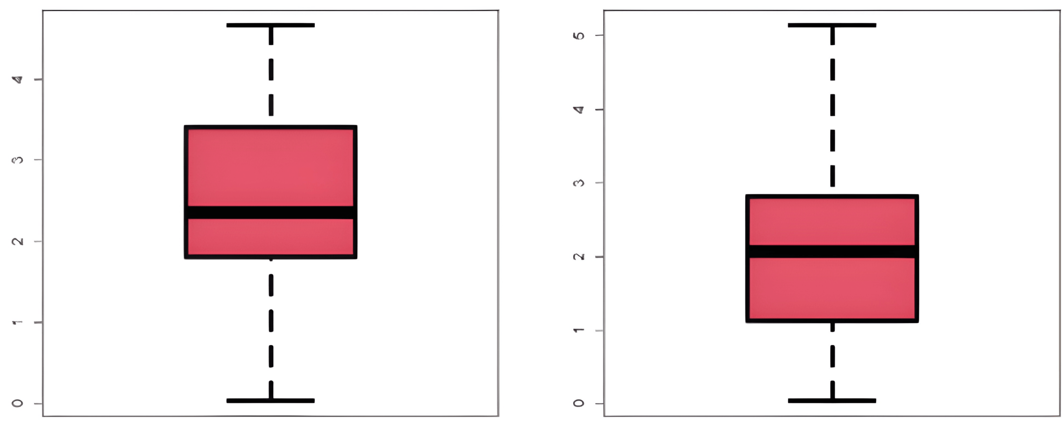


Figure 1. Box plotting for data set I (left one) and for data set II (right panel)

Figure 1 illustrates a ”box plot” designed to investigate the presence of outlier observations, revealing that no outliers were identified. Moving to Figure 2, a ”QL-QL plot” is employed to assess the ”normality” of the data, with the observation that normality is nearly present. The exploration of the empirical Hazard Rate Function’s (HRF) shape is conducted through the ”total time test (TTUT)” plot, as shown in Figure 3. The HRF for both data sets is observed to be ”asymmetric monotonically increasing.” To delve into the nonparametric initial shape of real data, Figure 4 provides the Kernel Density Estimation (KRDE). Additionally, Figure 5 and Figure 6 present the estimated Probabilistic Density Function (EPDF) and estimated HRF (EHRF) for the two datasets, respectively.

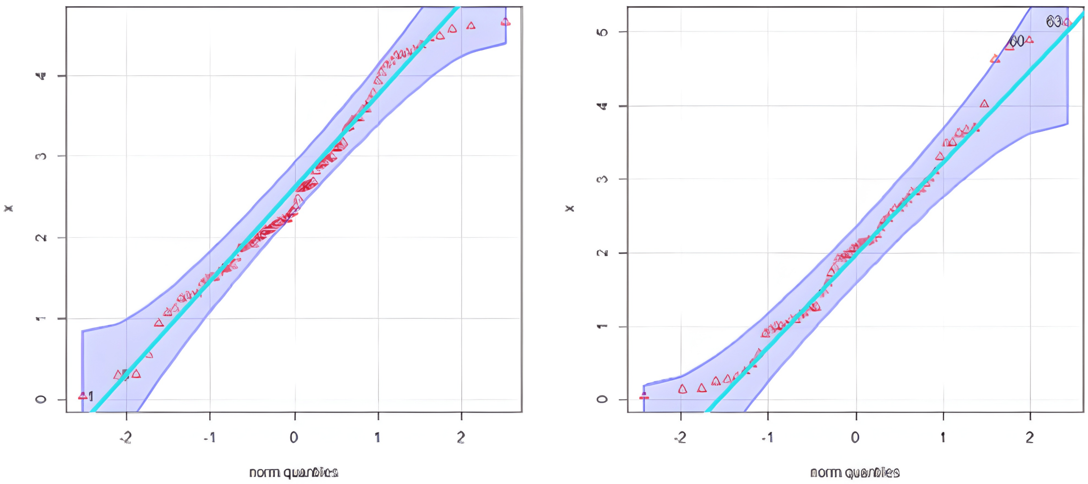


Figure 2. QL-QL plotting for the first life time dataset (left) and for second life time dataset (right panel)

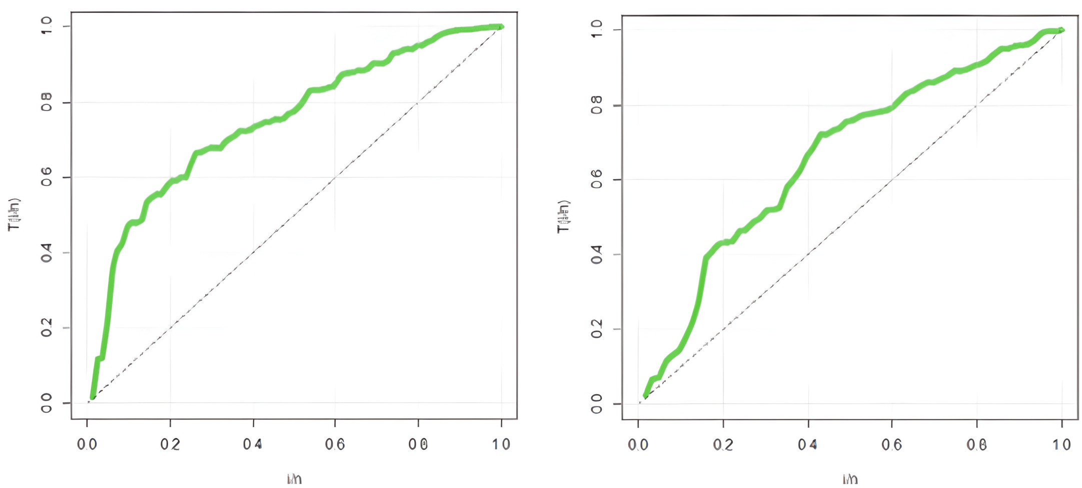


Figure 3. TTUT plot for first life time dataset (left) and for second life time dataset (right).

For achieving our main aims in comparing some competing models, the following statistics are used as main goodness-of-fit (GOF) tests:

1. Akaike-Information-Criterion (AKICR),
2. Bayes-IC (BYIRC),
3. Consisting-AKICR (CAKICR),
4. HannanQuinn-IC (HQIC)

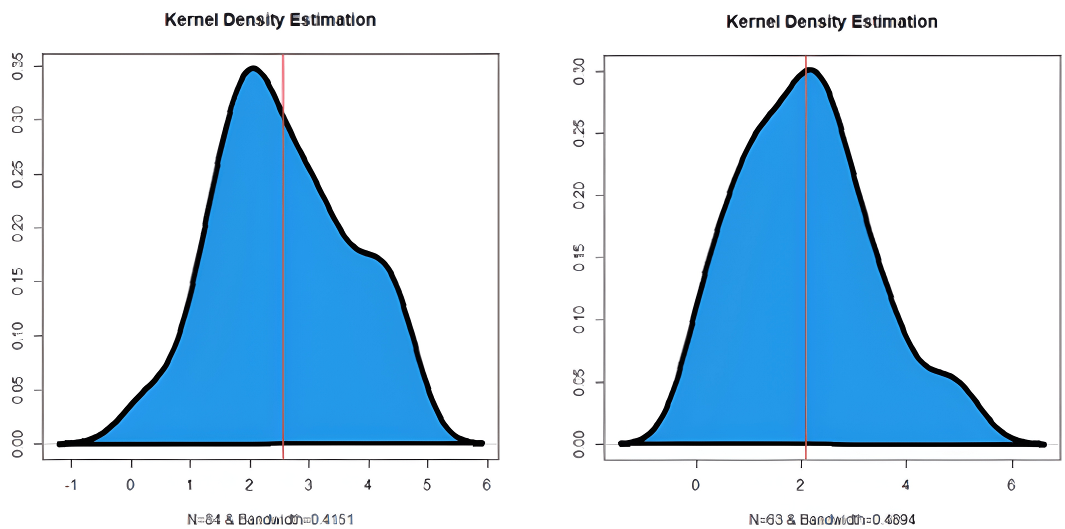


Figure 4. KRDE for first lifetime dataset (left) and for second life time dataset (right)

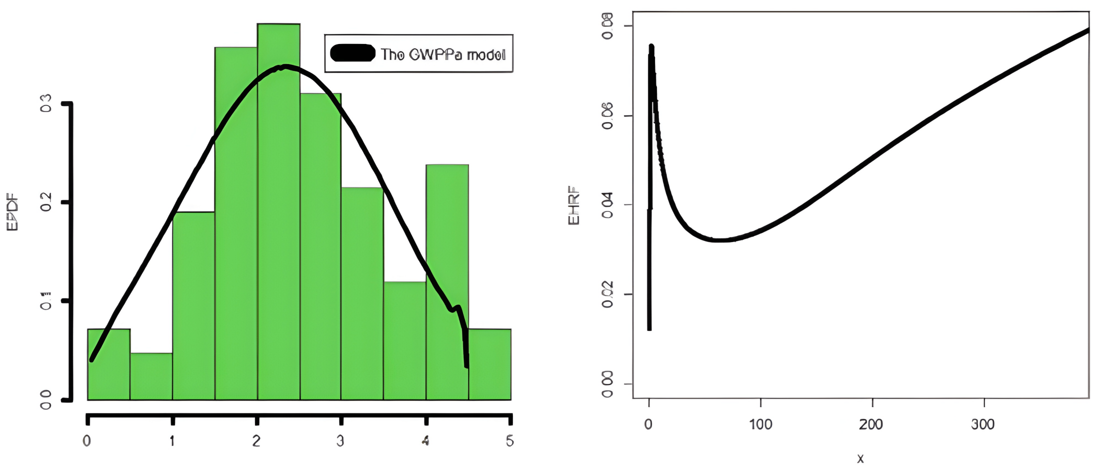


Figure 5. EPDF of the GWPPa model (left panel) and EHRF of the GWPPa (right panel) for the first lifetime data.

For the first data, comprehensive results are provided in Table 3 and Table 4. First, Table 3 presents the MLEs along with their standard errors (SE), while Table 4 displays the results of $-\ell$, AKICR, CAKICR, BYIRC and HQIC for the first lifetime data. Similarly, for times of service data, the pertinent findings are outlined in Table 5 and Table 6. Firstly, Table 5 includes the estimates and their corresponding SE, while Table 6 provides the results of $-\ell$, AKICR, CAKICR, BYIRC and HQIC for the second lifetime data.

Tables 5 and 6 indicate that, among all the fitted probability models, the GWPPa model yields the lowest values for AKICR, CAKICR, BYIRC, and HQIC. Given these considerations, it may be considered the most suitable model.

Table 3. The estimating for the first life time dataset.

Model	Estimates			
GWPPa(a,b,c,β)	1.904434 (0.88155)	0.254385 (0.03362)	6.657875 (0.01019)	0.156315 (0.006326)
KPa(a,b,c,β)	2.615015 (0.38324)	100.2762 (120.448)	5.27715 (9.8136)	78.67746 (186.1931)
TTLPa(a,b,c,β)	-0.807754 (0.13972)	2.476653 (0.54878)	(156082) (1602.44)	(3862823) (123.9443)
TTLPa(a,b,c,β)	3.603675 (0.618731)	33.63870 (63.7154)	4.83070 (9.23826)	118.8377 (428.9323)
PRHRPa(a,b,c)	3.724310^6 1.115310^6	4.706310^{-1} (0.001117)	4.50310^6 37.14654	
SLGMPa(a,b,c)	-1.05310^{-1} (0.12245)	9.86610^6 (48481.5)	1.22010^7 (511.243)	
RTTLP(a,b,c)	-0.847344 (0.10012)	5.520454 (1.18482)	1.1567586 (0.09582)	
OLLPa(a,b,c)	2.326362 (2.1410^{-1})	(7.210^5) (1.1910^6)	(2.3410^6) (2.61×10)	
ExpPa(a,b,c)	3.626104 (0.623364)	20074.53 (2041.86)	26257.73 (99.7454)	
GamPa(a,b,c)	3.587603 (0.513338)	52001.43 (795586.21)	37029.754 (81.16544)	
ROLLPa(a,b)	3.890564 (0.365254)	0.573165 (0.01946)		
RBHPa(a,b)	10801733 (983309.23)	51367189 (232312.99)		
Pa(a,b)	51425.476 (5933.532)	131790.32 (296.1222)		

Table 4. The results of l, AKICR, CAKICR, BYIRC and HQIC for the first life time data.

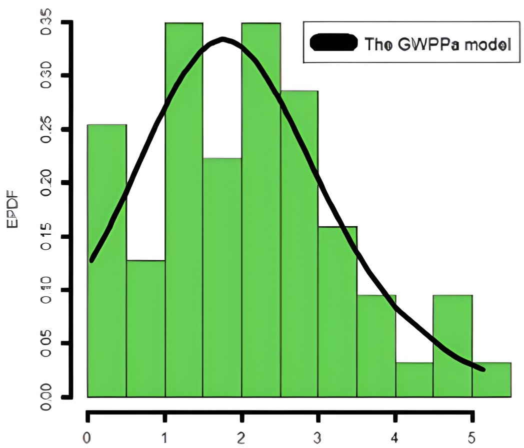
Model	l	AKICR	CAKICR	BYIRC	HQIC
GWPPa	-126.733	261.473	261.977	271.178	265.381
OLLPa	-135.428	274.846	275.148	285.131	277.773
PRHRPa	-163.855	332.743	335.051	339.045	335.645
TTLP	-134.575	279.145	279.646	288.867	284.048
GamPa	-139.405	283.810	286.115	290.135	285.755
TTLP	-139.719	285.425	285.922	295.210	289.344
ExpPa	-140.422	288.792	289.091	296.122	293.741
RBHPa	-165.694	347.212	342.355	346.073	343.155
ROLLPa	-142.843	289.693	289.851	294.554	294.644
RTTLPa	-156.777	313.966	314.269	325.259	316.895
SLGMPa	-144.082	293.171	294.471	299.460	296.136
Pa	-165.958	337.974	335.125	339.869	334.947

Table 5. Estimating results for the second life time dataset.

Model	Estimates			
GWPPa(a,b,c,b)	-1.08434 (1.82432)	0.856694 (0.52233)	1.129143 (0.55832)	0.313234 (0.130431)
TTLPa(a,b,c,β)	1.921843 (0.32185)	31.25974 (316.834)	4.968544 (50.25528)	169.57352 (339.2551)
KUMYPa(a,b,c,β)	1.663993 (0.256674)	60.56745 (86.0165)	2.526494 (4.75891)	65.0644 (177.52964)
TTLPa(a,b,c,β)	-0.604743 (0.213033)	1.785708 (0.4155522)	2123.394 (163.9215)	4822.7692 (200.016)
RTTLPa(a,b,c)	-0.671453 (0.187463)	2.744914 (0.663963)	1.0123284 (0.11414)	
PRHRPa(a,b,c)	1.59310^6 2.01410^3	3.93310^{-1} 0.004310^{-1}	1.30310^6 0.95310^6	
SLGMPa(a,b,c)	-1.04510^{-1} (4.14310^{-10})	6.45410^6 (3.21310^6)	6.33410^6 (3.85734)	
GamPa(a,b,c)	1.9027324 (0.321352)	35842.435 (6945.0745)	39197.575 (151.6533)	
OLLPa(a,b,c)	1.6641943 (1.85410^{-1})	6.34010^5 (1.69810^4)	2.014510^6 7.22510^6	
ExpPa(a,b,c)	1.9145543 (0.348434)	22971.155 (3209.535)	32882.875 (162.2538)	
RBHPa(a,b)	14055522.4 (422.0154)	53203423.3 (28.52254)		
ROLLPa(a,b)	2.3723533 (0.268332)	0.6910943 (0.044945)		
Pa(a,b)	99269.832 (11864.54)	207019.454 (301.2373)		

Table 6. The results of l, AKICR, CAKICR, BYIRC and HQIC for the second life time data.

Model	l	AKICR	CAKICR	BYIRC	HQIC
GWPPa	-98.1405	204.287	204.977	212.859	207.6584
RBHPa	-112.605	229.209	229.441	233.485	230.857
KUMYPa	-101.865	209.738	212.424	218.318	213.197
RTTLPa	-114.186	230.375	232.768	236.844	232.944
TTLPa	-103.450	213.932	213.582	222.462	216.277
ExpPa	-103.553	213.096	214.546	219.549	217.658
GamPa	-110.835	214.667	215.076	217.087	214.195
PRHRPa	-119.300	224.593	225.014	231.427	226.326
SLGMPa	-103.895	213.786	214.194	216.248	214.347
TTLPa	-103.960	214.924	214.692	222.494	217.295
OLLPa	-104.906	215.803	217.225	222.258	218.357
ROLLPa	-111.725	226.452	225.647	229.746	226.343
Pa	-108.294	223.593	223.788	228.885	225.083



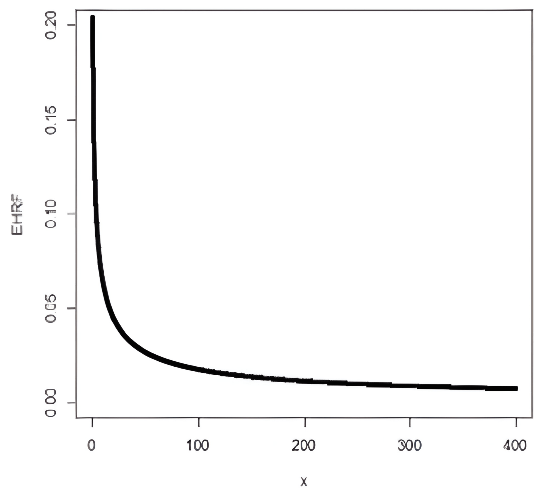


Figure 6. EPDF of the GWPPa model (left panel) and EHRF of the GWPPa (right panel) for the second lifetime data.

Conclusion

In this study, a novel compound G-family based on the zero-truncated Poisson and generated Weibull framework was successfully introduced and analyzed. The proposed GWP-G family demonstrated high flexibility in modeling complex lifetime and reliability data. Several important statistical properties, including moments, generating functions, and incomplete moments, were derived in closed forms. The model was further extended to bivariate and multivariate cases using various copulas, enhancing its applicability in dependent data analysis. Parameter estimation was effectively performed using the maximum likelihood method combined with the Newton–Raphson algorithm. The practical performance of the model was validated through two real datasets involving aircraft windshield failure and service times. Goodness-of-fit measures confirmed that the proposed model outperforms several well-known competing models. Graphical tools such as QL–QL, KRDE, and TTUT plots supported the adequacy of the model in capturing data behavior. The results highlight the importance of compound distributions in handling over-dispersion and complex hazard rate shapes. Overall, the proposed GWP-G family provides a powerful and versatile tool for statistical modeling and future research in reliability and survival analysis.

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