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# Joint Type I Censoring in Mixture Weibull Models: Estimation via Maximum Likelihood, Bayesian Methods and Application in Engineering Data

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**Abstract** In this study, we propose a comprehensive statistical approach to the estimation of the unknown parameters of a mixture of Weibull distribution made of two components using joint Type I censored samples. Bayesian estimation and Maximum Likelihood Estimation (MLE) methods were considered, with a focus on SE and LINEX loss functions. For the purpose of approximating the posterior distribution, we made use of the probabilistic Monte Carlo methods, namely the Metropolis-Hastings algorithm. The accuracy of the estimation was evaluated by constructing asymptotic confidence intervals as well as highest posterior density (HPD) intervals. In addition to this, we conducted a simulation study in order to evaluate the behavior of the suggested estimators, and we used the methodology to a real dataset in order to highlight the illustrations.

**Keywords** Joint Type I censored scheme, Mixture Weibull distribution, Maximum likelihood estimation, Bayesian estimators and Markov Chain Monte Carlo.

**AMS 2010 subject classifications:** 62F10, 62F15, 62N01, 62N02, 62N05

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## 1. Introduction

Life testing is essential for analyzing time-to-event data, especially in assessing the longevity or dependability of components, systems, creatures, or economic organizations. It includes a wide range of statistical methods employed to evaluate the duration of a subject's functionality or survival under settings. Life testing is essential for understanding failure behavior, refining design, augmenting performance, mitigating risk, and guiding decision-making in practical fields like as engineering, medicine, and economics. Conducting exhaustive life testing on all units until failure is frequently unfeasible due to temporal restrictions, financial limits, and ethical dilemmas. To tackle these issues, numerous censoring methodologies have been devised, enabling the collection of limited information while yet producing accurate statistical judgments. Censoring transpires when the precise failure time of certain units remains unclear, either because the study concludes prior to their failure or they are removed from the study owing to external influences. The joint Type I Censoring Scheme has proven to be an effective and pragmatic design, particularly when managing several samples or product lines under uniform or comparative testing settings.

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Joint Type I censoring schemes assess two or more samples simultaneously and end at a set time. The joint variety allows simultaneous study of multiple populations, unlike Type I censoring. It is especially useful when comparing similar systems or items or when budget constraints require concurrent testing to minimize resources. After the observation period, only failing units are noted, and the rest are redacted. This architecture produces a more complete dataset with fewer failures or accelerated life testing. Joint Type I censored data statistically has pros and cons. They reduce experimental costs and simplify logistics. In contrast, they require advanced inferential methods to estimate distributional parameters, reliability functions, and hazard rates. Selecting statistical models like exponential, Weibull, gamma, or generalized distributions within these frameworks depends on data properties and study goals. Maximum likelihood estimates, Bayesian methods, and nonparametric methods are used to get insights from censored datasets.

Engineering has commonly used joint Type I censoring life testing to evaluate product reliability and component longevity. Under normal or accelerated stress circumstances, manufacturers test electrical circuits, mechanical assembly, and structural materials for failure times. Joint censoring techniques improve quality control and warranty analysis by comparing batches or product designs simultaneously. While obtaining precise life parameter estimations, reliability engineers save time and money on tests. Clinical trials and epidemiological research require life testing and censoring. Survival analysis of patients undergoing different therapies sometimes includes censored data due to loss to follow-up or trial discontinuation before all events occur. The combined Type I censoring model is ideal for comparing two medications or therapy across a fixed observation window. Parallel testing and synchronized study completion allow researchers to draw statistically solid conclusions without the ethical and logistical burden of waiting for all events. In economics, life testing techniques are used to model duration data including loan default, employee job leave, and competitive market company longevity. Real-world datasets right-censor "failure" events like bankruptcy, unemployment, and market departure for economic players including individuals and institutions. Macroeconomic panel studies, which investigate numerous entities over a defined time horizon, benefit from combined Type I censoring. Economists can compare industries, locations, or policies under comparable exposure periods to draw fairer and more interpretable statistical findings. Researchers have recently focused on censored systems and employed them in many disciplines refer to [1] to [18].

Several combined censoring schemes have been proposed in prior statistical research, including joint progressive Type I and type II, joint Type II, and joint progressive hyper Type I censoring. For instance, [19] introduced the Joint Progressive Censoring Type I (JPC-I) scheme, which encompasses the joint Type I censoring scheme as a special case. Their research concentrated on statistically deducing two exponential populations considering combined Type I and JPC-I censoring.

However, a careful study of the literature shows a significant research gap: joint censoring has been looked into a lot for single distributions, but not so much for mixture Weibull models under a Joint Type-I scheme. Mixture models present unique challenges in estimation due to the complexity of the likelihood function and potential identifiability issues. Most existing studies have not adequately addressed the comparative performance of Bayesian estimation (under different loss functions) versus classical maximum likelihood for this specific mixture architecture. Therefore, this study aims to bridge this gap. The following is a summary of this study's main goals:

1. To calculate the Maximum Likelihood Estimators (MLEs) under a Joint Type I censoring for the parameters of the mixture Weibull distribution.
2. To calculate Bayesian estimating processes using both (squared error) and (LINEX) loss functions.
3. To perform Monte Carlo simulation of the suggested estimators.
4. Using the created methodology to show usefulness in practice on an engineering dataset.

In the current study, we enhance this paradigm by implementing joint Type I censoring to mixed distributions. The relevant likelihood function is constructed according to the methodology proposed by [19].

$$L = \frac{m!n!}{(m-m_1)!(n-n_1)!}\prod_{i=1}^{r}\{[f(w_i)]^{z_i}[g(w_i)]^{1-z_i}\}(\bar{F}(T))^{m-m_1}(\bar{G}(T))^{n-n_1}. \tag{1}$$

Statistical literature has proposed various types of joint censoring techniques. Notably, references [20] and [21] examined the characteristics of JPC-II censored samples. Also, [22] to [25] looked into Type-II censoring in

different situations. [26] built on the preceding work of [27] by looking at Type-II censored data from k-independent exponential populations together. [28] expanded the approach established by [29] by examining JPC-II censoring in the context of several exponential populations. In a more recent work, [30] tackled the estimation issue for two Weibull populations sharing a common shape parameter as per JPC-I. The researchers calculated maximum likelihood estimates (MLEs) and their related confidence intervals, as well as Bayesian estimates utilizing both squared error and LINEX loss functions, under the assumption of independent gamma priors.

Mixture distributions have found extensive applications across diverse fields, such as reliability, medicine, and engineering. The two-parameter Weibull distribution is specifically chosen due to its flexibility in modeling various hazard rate patterns, making it highly versatile in reliability analysis. The concept of finite mixture models was first introduced in [31], addressing heterogeneous populations consisting of two or more latent subpopulations $s \geq 2$, each characterized by distinct features and represented by an unknown mixing proportion $p_j$ where $0 \leq p_j \leq 1, j = 1, 2, \ldots, s$, and $\sum_{j=1}^{s} p_j = 1$. The PDF for such a mixture model is expressed as

$$f(x) = \sum_{j=1}^{s} p_j f_j(x),$$

where $f_j(x)$ denotes the PDF of the $j$-th subpopulation.

In lifetime models, subpopulations frequently correlate to different failure mechanisms. [32] indicated that each unit should preferably be labeled to identify its corresponding subpopulation, which thereafter dictates the form of failure. Nevertheless, in numerous actual situations, this criterion remains unfulfilled, and the classification of individual units is indeterminate. The absence of information affects both the configuration of the likelihood function, and the accuracy of parameter estimate. For further information regarding the application of lifespan models in scientific contexts, refer to [33] - [54].

Numerous studies have examined Type I censoring in relation to mixture distributions. For example, [32] derived maximum likelihood estimates for the parameters of a mixed exponential distribution under JPC-I censoring. [55] expanded this research to the mixed Weibull model, whereas [56] introduced a Weibull-exponential hybrid model and investigated both maximum likelihood estimation and Bayesian inference. [57] employed Bayesian predictive methodologies to mixed exponential models subjected to Type I censoring. Subsequent contributions encompass [58] regarding the mixed generalized exponential distribution and [59], which examined both Type I and Type-II censored samples. [60] and [61] investigated mixed Rayleigh models utilizing JPC-I censoring and addressed a simplified likelihood function when censoring is complete.

When estimating parameters for a mixture model under Type I censoring, the likelihood function takes the form

$$l \propto \prod_{j=1}^{s} \prod_{i=1}^{r_j} p_j f_j(x_i) \bar{F}(T)^{n-r} \tag{2}$$

where $\bar{F}(T) = 1 - F(T)$ and if all data points are observed $(r = n)$, the likelihood reduces to that of a complete sample from the mixture model by [61]. The Weibull distribution is recognized for its versatility and practical utility in diverse fields, including survival analysis, reliability engineering, and biological research. Its capacity to handle both ascending and descending failure rates renders it an appropriate substitute for the exponential model in lifetime data analysis.

The subsequent portions of this work are organized as follows: Section 2 presents the model formulation and notations. Section 3 discusses the parameter estimation methodologies employing both Maximum Likelihood and Bayesian methods. Section 4 presents a quantitative analysis to illustrate the efficacy of the proposed estimators, accompanied by a practical application of real data. In conclusion, Section 5 concludes the paper and encapsulates the findings and potential extensions.

## 2. Model Description and Notation

Let $X_1, X_2, \ldots, X_m$ denote the lifetimes of $m$ units from product $A$, assumed to be independent and identically distributed (i.i.d.) random variables following a two-component mixture distribution. Each subpopulation is characterized by an unknown proportion $p_j$ cumulative distribution function $F_j(x_{ji})$, and corresponding probability density function $f_j(x_{ji})$, where $j = 1, 2$ and $i = 1, \ldots, r_j$. Similarly, let $Y_1, Y_2, \ldots, Y_n$, be (i.i.d.) lifetimes of $n$ units from product $B$, modeled using another two-component mixture distribution with proportions $\phi_j$, distribution functions $G_j(y_{ji})$, and densities $g_j(y_{ji})$. Define $W_{(1)} < W_{(2)} < \cdots < W_{(N)}$ as the ordered failure times from the combined $N = m + n$ sample. Under JPC-I, the test is terminated at a fixed time $T$, and only the first $r$ failures are observed, with $r_j$ failures from each subpopulation, such that $\sum_{j=1}^{s} r_j = r$.

The censored likelihood function is constructed by incorporating the subpopulation indicators $z_i \in \{0, 1\}$, indicating the latent class membership of each observed failure. The structure follows the formulation introduced by [61] and [23] to [25], and is adapted in the current model setting to include the mixed Weibull distributions. The likelihood function in general is represented as follows

$$L(\mathcal{G}|\mathbf{w}) \propto \prod_{j=1}^{s}\prod_{i=1}^{r_j} [p_j f_j(\mathbf{w}_{ji})]^{z_i} \cdot [\phi_j g_j(\mathbf{w}_{ji})]^{(1-z_i)} \cdot \left[\bar{F}^*(T) \cdot \bar{G}^*(T)\right]^{N-r} \tag{3}$$

where

$$\mathcal{G} = (\alpha_j, \beta_j, p_j, \phi_j), \quad j = 1, 2 \text{ is the parameter vector.}$$
$$\bar{F}^*(T) = \sum_{j=1}^{s} p_j \bar{F}_j(T) \quad \text{and} \quad \bar{G}^*(T) = \sum_{j=1}^{s} p_j \bar{G}_j(T).$$

Assuming $s = 2$ the likelihood simplifies to:

$$L(\vartheta \mid \mathbf{w}) \propto \prod_{i=1}^{r_1} [p_1 f_1(w_{1i})\, p_2 f_2(w_{2i})]^{z_i} \prod_{i=1}^{r_2} [\varphi_1 g_1(w_{1i})\, \varphi_2 g_2(w_{2i})]^{(1-z_i)} \times \left[\overline{F}^*(T)\right]^{m-r_1} \left[\overline{G}^*(T)\right]^{n-r_2} \tag{4}$$

Subject to the constraints: $p_1 + p_2 = 1$ and $\phi_1 + \phi_2 = 1$. If any of the mixing proportions $p_1, \phi_1$ or $p_2, \phi_2$ are set to zero, the model reduces to the standard joint Type I censoring likelihood described in Equation (1).

## 3. Estimation Methods

The maximum likelihood (ML) and Bayesian estimation have been discussed to estimate the unknown parameters of the mixture Weibull model based on joint type I censoring by point estimation and asymptotic confidence intervals in this section.

### *3.1. Maximum Likelihood Estimation*

Assume the mixed distributions for both products A and B follow a two-component Weibull distribution with parameters $\alpha_j > 0$ (scale) and $\beta_j > 0$ (shape), for $j = 1, 2$. The likelihood function is constructed following the principles of parameter estimation for mixture models as discussed by [55] and [56] under joint Type I censoring. The likelihood function becomes:

$$\begin{aligned} L(g|\mathbf{w}) \quad &\propto \quad [p_1(1-p_1)]^n \prod_{i=1}^{n} [f_1(w_{1i}; \alpha_1, \beta_1) f_2(w_{2i}; \alpha_2, \beta_2)]^{z_i} \\ &\times \quad [\phi_1(1-\phi_1)]^n \prod_{i=1}^{n} [g_1(w_{1i}; \alpha_1, \beta_1) g_2(w_{2i}; \alpha_2, \beta_2)]^{(1-z_i)} \\ &\times \quad \left[\bar{F}^*(T)\right]^{n-r_1} \left[\bar{G}^*(T)\right]^{n-r_2} \end{aligned} \tag{5}$$

in which

$$f_j(w_{ji};\alpha_j,\beta_j)=\frac{\beta_j}{\alpha_j}\left(\frac{w_{ji}}{\alpha_j}\right)^{\beta_j-1}\exp\left[-\left(\frac{w_{ji}}{\alpha_j}\right)^{\beta_j}\right],\quad i=1,2,\ldots,r_1,$$

$$g_j(w_{ji};\alpha_j,\beta_j)=\frac{\beta_j}{\alpha_j}\left(\frac{w_{ji}}{\alpha_j}\right)^{\beta_j-1}\exp\left[-\left(\frac{w_{ji}}{\alpha_j}\right)^{\beta_j}\right],\quad i=1,2,\ldots,r_2.$$

The mixture survival terms are:

$$\bar{F}^*(T)=p_1\bar{F}_1(T)+p_2\bar{F}_2(T)\quad\text{and}\quad\bar{G}^*(T)=p_1\bar{G}_1(T)+p_2\bar{G}_2(T).$$

The corresponding log likelihood function will be

$$\begin{aligned}\ell(\vartheta\mid\mathbf{w}) \;\propto\;& r_1\log\left[p_1(1-p_1)\right]+\sum_{i=1}^{r_1}z_i\log\left(f_1(w_{1i};\alpha_1,\beta_1)\right)+\sum_{i=1}^{r_1}z_i\log\left(f_2(w_{1i};\alpha_2,\beta_1)\right)\\
+\;& r_2\log\left[\varphi_1(1-\varphi_1)\right]+\sum_{i=1}^{r_2}(1-z_i)\log\left(g_1(w_{1i};\alpha_1,\beta_1)\right)+\sum_{i=1}^{r_2}(1-z_i)\log\left(g_2(w_{1i};\alpha_2,\beta_1)\right)\\
+\;& (m-r_1)\log\left[p_1\exp\left(-\left(\frac{T}{\alpha_1}\right)^{\beta_1}\right)+(1-p_1)\exp\left(-\left(\frac{T}{\alpha_2}\right)^{\beta_2}\right)\right]\\
+\;& (n-r_2)\log\left[\varphi_1\exp\left(-\left(\frac{T}{\alpha_1}\right)^{\beta_1}\right)+(1-\varphi_1)\exp\left(-\left(\frac{T}{\alpha_2}\right)^{\beta_2}\right)\right].\end{aligned}\tag{6}$$

By differentiating the log-likelihood function in Equation (6) in relation to parameters $\alpha_1,\beta_1,\alpha_2,\beta_2,p_1$ and $\varphi_1$, and setting the resulting first-order derivatives equal to zero, we obtain the maximum likelihood estimators (MLEs): $\hat{p}_1,\hat{\varphi}_1,\hat{\beta}_1,\hat{\beta}_2,\hat{\alpha}_1$ and $\hat{\alpha}_2$. The corresponding likelihood equations for each parameter are derived as follows:

$$\frac{\partial\ln L}{\partial\hat{p}_1}=\frac{r_1}{\hat{p}_1}-\frac{r_1}{1-\hat{p}_1}+\frac{m-r_1}{\hat{p}_1}-\frac{m-r_1}{1-\hat{p}_1}=0$$

$$\frac{\partial\ln L}{\partial\hat{\varphi}_1}=\frac{r_2}{\hat{\varphi}_1}-\frac{r_2}{1-\hat{\varphi}_1}+\frac{n-r_2}{\hat{\varphi}_1}-\frac{n-r_2}{1-\hat{\varphi}_1}=0$$

$$\begin{aligned}\frac{\partial\ln L}{\partial\hat{\alpha}_1}=\;& \frac{\sum_{i=1}^{r_1}z_i}{\hat{\alpha}_1}-\sum_{i=1}^{r_1}z_i(\hat{\beta}_1-1)\left(\frac{w_{1i}}{\hat{\alpha}_1^2}\right)+\hat{\beta}_1(\hat{\alpha}_1)^{-\hat{\beta}_1-1}\sum_{i=1}^{r_1}z_i(w_{1i})^{\hat{\beta}_1}\\
-\;& \frac{\sum_{i=1}^{r_2}(1-z_i)}{\hat{\alpha}_1}-\sum_{i=1}^{r_2}(1-z_i)(\hat{\beta}_1-1)\left(\frac{w_{1i}}{\hat{\alpha}_1^2}\right)+\hat{\beta}_1(\hat{\alpha}_1)^{-\hat{\beta}_1-1}\sum_{i=1}^{r_2}(1-z_i)(w_{1i})^{\hat{\beta}_1}\\
+\;& (m-r_1)\hat{\beta}_1(\hat{\alpha}_1)^{-\hat{\beta}_1-1}(T)^{\hat{\beta}_1}+(n-r_2)\hat{\beta}_1(\hat{\alpha}_1)^{-\hat{\beta}_1-1}(T)^{\hat{\beta}_1}=0\end{aligned}$$

$$\begin{aligned}\frac{\partial\ln L}{\partial\hat{\beta}_1}=\;& \frac{\sum_{i=1}^{r_1}z_i}{\hat{\beta}_1}+\sum_{i=1}^{r_1}z_i\ln\left(\frac{w_{1i}}{\hat{\alpha}_1}\right)-\sum_{i=1}^{r_1}z_i\left(\frac{w_{1i}}{\hat{\alpha}_1}\right)^{\hat{\beta}_1}\ln\left(\frac{w_{1i}}{\hat{\alpha}_1}\right)+\frac{\sum_{i=1}^{r_2}(1-z_i)}{\hat{\beta}_1}\\
+\;& \sum_{i=1}^{r_2}(1-z_i)\ln\left(\frac{w_{1i}}{\hat{\alpha}_1}\right)-\sum_{i=1}^{r_2}(1-z_i)\left(\frac{w_{1i}}{\hat{\alpha}_1}\right)^{\hat{\beta}_1}\ln\left(\frac{w_{1i}}{\hat{\alpha}_1}\right)\\
-\;& (m-r_1)\left(\frac{T}{\hat{\alpha}_1}\right)^{\hat{\beta}_1}\ln\left(\frac{T}{\hat{\alpha}_1}\right)-(n-r_2)\left(\frac{T}{\hat{\alpha}_1}\right)^{\hat{\beta}_1}\ln\left(\frac{T}{\hat{\alpha}_1}\right)=0\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \hat{\alpha}_2} &= -\frac{\sum_{i=1}^{r_1} z_i}{\hat{\alpha}_2} - \sum_{i=1}^{r_1} z_i(\hat{\beta}_2 - 1)\left(\frac{w_{2i}}{\hat{\alpha}_2^2}\right) + \hat{\beta}_2(\hat{\alpha}_2)^{-\hat{\beta}_2-1}\sum_{i=1}^{r_1} z_i(w_{2i})^{\hat{\beta}_2} \\ &- \frac{\sum_{i=1}^{r_2}(1-z_i)}{\hat{\alpha}_2} - \sum_{i=1}^{r_2}(1-z_i)(\hat{\beta}_2 - 1)\left(\frac{w_{2i}}{\hat{\alpha}_2^2}\right) + (n-r_2)\hat{\beta}_2(\hat{\alpha}_2)^{-\hat{\beta}_2-1}(T)^{\hat{\beta}_2} \\ &+ \hat{\beta}_2(\hat{\alpha}_2)^{-\hat{\beta}_2-1}\sum_{i=1}^{r_2}(1-z_i)(w_{2i})^{\hat{\beta}_2} + (m-r_1)\hat{\beta}_2(\hat{\alpha}_2)^{-\hat{\beta}_2-1}(T)^{\hat{\beta}_2} = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \hat{\beta}_2} &= \frac{\sum_{i=1}^{r_1} z_i}{\hat{\beta}_2} + \sum_{i=1}^{r_1} z_i \ln\left(\frac{w_{2i}}{\hat{\alpha}_2}\right) - \sum_{i=1}^{r_1} z_i\left(\frac{w_{2i}}{\hat{\alpha}_2}\right)^{\hat{\beta}_2}\ln\left(\frac{w_{2i}}{\hat{\alpha}_2}\right) \\ &+ \frac{\sum_{i=1}^{r_2}(1-z_i)}{\hat{\beta}_2} + \sum_{i=1}^{r_2}(1-z_i)\ln\left(\frac{w_{2i}}{\hat{\alpha}_2}\right) - \sum_{i=1}^{r_2}(1-z_i)\left(\frac{w_{2i}}{\hat{\alpha}_2}\right)^{\hat{\beta}_2}\ln\left(\frac{w_{2i}}{\hat{\alpha}_2}\right) \\ &- (m-r_1)\left(\frac{T}{\hat{\alpha}_2}\right)^{\hat{\beta}_2}\ln\left(\frac{T}{\hat{\alpha}_2}\right) - (n-r_2)\left(\frac{T}{\hat{\alpha}_2}\right)^{\hat{\beta}_2}\ln\left(\frac{T}{\hat{\alpha}_2}\right) = 0 \qquad (7)\end{aligned}$$

It is evident from Equation (7) that MLEs of parameters $\alpha_1, \beta_1, \alpha_2, \beta_2, p_1$ and $\phi_1$ involve solving a system of nonlinear equations, which are analytically intractable in most cases. Therefore, numerical optimization techniques implemented in R software are employed to estimate the parameters $\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{p}_1$ and $\hat{\phi}_1$. To ensure the reliability of the numerical results for the MLEs, we implemented a robust optimization procedure in R. Given the potential for local maxima in mixture models, multiple sets of initial parameter values were tested. This approach ensures that the numerical algorithm converges to the global maximum of the log-likelihood function, thereby providing consistent and stable estimates.

In order to calculate the variance-covariance matrix of the estimators, the observed Fisher information matrix is attained through evaluating the negative of the log-likelihood function's second-order partial derivatives in relation to parameters. This approach for constructing asymptotic confidence intervals is a standard procedure in life-testing analysis (see [1] and [4]). The inverse of this matrix provides the estimated variance-covariance matrix for these parameter vector. Due to the length and complexity of these second-order partial derivatives, their detailed expressions are provided in the Appendix (A).

Suppose that $\hat{\theta}$ is MLE for parameter vector $\theta = (\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1)$. Under standard regularity conditions, the MLE $\hat{\theta}$ is asymptotically normally distributed with mean $\theta$ and covariance matrix $I_\theta^{-1}$, in which $I_\theta$ denotes the Fisher information matrix. In practice, this matrix is estimated by $I_\theta^{-1}$. Accordingly, the approximate $100(1-\alpha)\%$ confidence intervals with respect to parameters $\alpha_j$ and $\beta_j, j = 1, 2$, are given by:

$$\hat{\alpha}_j \pm z_{(1-\alpha/2)}\sqrt{\mathrm{var}(\hat{\alpha}_j)} \quad \text{and} \quad \hat{\beta}_j \pm z_{(1-\alpha/2)}\sqrt{\mathrm{var}(\hat{\beta}_j)}.$$

where $z_{(1-\alpha/2)}$ represents the upper $(1-\alpha/2)$ quantile of the standard normal distribution.

### *3.2. Bayesian Estimation*

Bayesian estimators are obtained for the unknown parameters $\alpha_1, \beta_1, \alpha_2$ and $\beta_2$ under two different loss functions: the squared error (SE) loss and LINEX loss. The LINEX loss function displayed by [20]. The Bayesian inference is performed under the assumption that the prior distributions for the parameters $\alpha_1, \beta_1, \alpha_2$ and $\beta_2$ follow independent gamma distributions. Specifically, we assume the following priors:

For $\alpha_j$ (scale parameters), the prior distributions are:

$$\pi(\alpha_j) \propto \alpha_j^{a_j-1}e^{-(a_j b_j)}, \quad a_j, b_j, \alpha_j > 0, \quad j = 1, 2$$

For $\beta_j$ (shape parameters), the prior distributions are:

$$\pi(\beta_j) \propto \beta_j^{c_j-1}e^{-(\beta_j d_j)}, \quad c_j, d_j, \beta_j > 0, \quad j = 1, 2$$

The mixing proportions $p_1$ and $\phi_1$ are assumed to follow independent beta distributions as follows: $p_1 \sim \text{Beta}(C_1, D_1)$ and $\phi_1 \sim \text{Beta}(C_2, D_2)$. Accordingly, the prior densities for $p_1$ and $\phi_1$ are given by:

$$\begin{aligned} \pi(p_1) &\propto p_1^{C_1-1}(1-p_1)^{D_1-1}, \qquad 0 < p_1 < 1, \qquad C_1, D_1 > 0 \\ \pi(\phi_1) &\propto \phi_1^{C_2-1}(1-\phi_1)^{D_2-1}, \qquad 0 < \phi_1 < 1, \qquad C_2, D_2 > 0 \end{aligned}$$

Here, all of the hyper parameters $(a_j, b_j, c_j, d_j, C_1, D_1, C_2$ and $D_2)$ are supposedly non-negative and known. Accordingly, the joint prior distribution of the parameters $\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1$ can be expressed (up to a constant of proportionality) as:

$$\pi(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1) \propto \pi(\alpha_j)\pi(\beta_j)\pi(p_1)\pi(\phi_1), \quad a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, C_1, D_1, C_2, D_2 > 0 \tag{8}$$

Combining the likelihood function in Equation (5) with the joint prior density described in Equations (8), the joint posterior density function of the parameter vector $\theta = (\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1)$ is, up to a proportionality constant, given by

$$\pi(\theta|\mathbf{w}) = \frac{1}{\psi}\pi(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1)L(\theta|\mathbf{w}), \tag{9}$$

where $\psi = \int\int\int\int\int\int \pi(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1)L(\theta|\mathbf{w})\, d\alpha_j\, d\beta_j\, dp_1\, d\phi_1$, $j = 1, 2$.

Consequently, the Bayes estimator of any function of parameter vector $\theta = \{\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1\}$ under SE loss function can be written as follows

$$\theta_{SE} = E(\theta(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1|\mathbf{w})) \tag{10}$$

In light of the loss function of LINEX, the corresponding Bayes estimate is given in the following form

$$\theta_{LINEX} = -\frac{1}{\tau}\ln[E(e^{-\tau\theta(\alpha_1,\beta_1,\alpha_2,\beta_2,p_1,\phi_1)}/\mathbf{w})], \quad \tau \neq 0 \tag{11}$$

Where $\tau \neq 0$ is a parameter that reflects the asymmetry of the LINEX loss function. Since the expectations in Equations (10) and (11), as well as the posterior distribution in Equation (9), do not admit closed-form solutions, even under heavy censoring, the informative priors ensure component identifiability and robust estimation.
Therefore, we suggest using the MCMC methods for approximating the Bayes estimates. Specifically, the algorithm of Metropolis-Hastings (MH) is employed to generate posterior samples and approximate the required integrals, which often represent analytical challenges in mixture models. Unlike the EM algorithm, which is primarily designed for point estimation and may lead to convergence to local maxima, our MCMC-based approach, supported by the construction of HPD intervals, provides more comprehensive inferential results by giving the full posterior surface and more reliable interval estimates in complex joint exploration scenarios. To guarantee the positivity of the shape and scale parameters $\alpha_i, \beta_i$ and to improve the convergence of the MCMC algorithm, the parameters were log-transformed during the estimation process.

## 4. Simulation Study and Real Data Analysis

Here, we aim to evaluate the performance of these proposed estimation methods given above in Sections 2 and 3. A real dataset is used to serve goals of illustrations, and a comprehensive MCMC simulation study was conducted with the aim of assessing the statistical behavior of the estimators under a joint Type I censoring scheme. All computations are carried out through the R language, with the bbmlle and HDInterval packages utilized to calculate HPD and MLEs see [62].

### *4.1. Simulation Study*

In order to perform the assessment of the Bayesian estimation and MLE methods, an MCMC simulation is carried out under JPC-I censoring from a mixture of two components of Weibull distributions. For each scenario, 1000 random samples are generated. The chosen parameter combinations were selected to represent various real-world scenarios, including cases where the sub-populations are well-separated and cases where the failure rates are closely clustered, thus testing the robustness of the proposed estimation methods. The simulation is carried out according to the following parameter settings:

| Case | $p_1$ | $\phi_1$ | $\alpha_1$ | $\beta_1$ | $\alpha_2$ | $\beta_2$ |
|---|---|---|---|---|---|---|
| I | 0.4 | 0.6 | 0.5 | 1.5 | 0.75 | 1.75 |
| II | 0.6 | 0.4 | 0.5 | 1.5 | 0.75 | 1.75 |
| III | 0.4 | 0.6 | 0.5 | 1 | 0.75 | 1 |
| IV | 0.6 | 0.4 | 0.5 | 1 | 0.75 | 1 |
| V | 0.4 | 0.6 | 0.5 | 1.5 | 0.5 | 1.75 |
| VI | 0.6 | 0.4 | 0.5 | 1.5 | 0.5 | 1.75 |

Three varying sample sizes were considered for the two populations:

$$(n, m) = (15, 15), (25, 25), (50, 50)$$

Censoring times $T$ are specified for each case as follows:

- Case I & II: $T = 0.5, 0.75$
- Case III & IV: $T = 0.4, 1.00$
- Case V & VI: $T = 0.3, 0.60$

Note: When $\beta_1 = \beta_2 = 1$, Case III and Case IV reduce to exponential mixture distributions. When $\beta_1 = \beta_2 = 2$ the model simplifies to the Rayleigh mixture distribution.

MLEs and their associated $95\%$ asymptotic confidence intervals (Asy-CIs) are computed using the generated samples. The initial values used in the numerical optimization are set equal to the true parameter values. The corresponding results are reported in Appendix B (Tables 3 to 8).

Bayesian estimates are obtained using informative gamma priors. Three prior settings are considered:

$$\text{prior } 0 = (0.5, 0.5, 0.5, 0.5), \quad \text{prior } 1 = (1.2, 1.5, 1.5, 2), \quad \text{prior } 2 = (1.5, 1.2, 1.5, 1.8)$$

Sensitivity analysis was performed by testing alternative non-informative priors, confirming the stability of the Bayesian estimates. MLEs are employed as initial values for the Metropolis-Hastings (MH) algorithm, in addition to the comparable variance-covariance matrix of $\ln(\hat{\rho}_1), \ln(\hat{\phi}_1), \ln(\hat{\alpha}_1), \ln(\hat{\beta}_1), \ln(\hat{\alpha}_2), \ln(\hat{\beta}_2)$. Out of 6,000 generated posterior samples, the first 1,500 are discarded as burn-in. Bayes estimates are then calculated in light of SE and LINEX (with $\nu = 2$) loss functions. HPD intervals are also constructed following the method developed by [62].

Summary of Results:

- Table 1: This is a summary of the main comparative indicators (MSE) of the estimators across some critical simulation scenarios, where the numerical convergence rate of MLE exceeded $99\%$ in all simulation cases. Also shown is the number of failure units for these scenarios.
- Tables 3 to 8 in Appendix B show the average estimates (Avg.) and the mean squared errors (MSEs) for the MLE and Bayes methods, corresponding to sample sizes $(n, m) = (15, 15), (25, 25), (50, 50)$.
- Tables 9 to 14 in Appendix B report the average interval lengths (AILs) and the coverage probabilities (CPs) for corresponding methods.

Drawing from the results shown in Appendix B the following conclusion can be made:

Table 1. Summary of the main comparative indicators (MSE).

| Comparison Effect | (m,n) | T | Avg. Failures $r_1 + r_1$ | parameter | MLE (MSE) | Bayesian SE(MSE) | Bayesian INEX(MSE) |
|---|---|---|---|---|---|---|---|
| **Sample Size Effect** | (15,15) | 0.5 | 15 | $p_1$ | 0.0170 | 1.3219 | 0.5399 |
| | (50,50) | | 53 | | 0.0119 ⇓ | 0.0100 ⇓ | 0.0092 ⇓ |
| **Censoring Time Effect** | (15,15) | 0.5 | 15 | $\alpha_1$ | 0.0228 | 0.8305 | 0.0835 |
| | | 0.75 | 22 | | 0.0026 ⇓ | 0.1629 ⇓ | 0.0718 ⇓ |
| **LINEX Superiority** | (15,15) | 0.5 | 15 | $\beta_1$ | 0.6416 | 9.8621 | 0.8031 |
| | (50,50) | 0.3 | 35 | $\beta_2$ | 0.7131 | 0.3750 | 0.2640 |

1. The Mean Squared Error decreases for both Maximum Likelihood Estimation and Bayesian estimations with an increase in sample size.
2. Larger censoring time $T$ leads to lower MSEs.
3. Bayes estimates considering the LINEX loss function yield fewer mean squared errors than those derived from the squared error loss function.
4. The asymptotic information limits for maximum likelihood estimates are generally shorter than those for Bayesian estimates.
5. Confidence intervals for maximum likelihood estimates are larger and closer to $100\%$ compared to Bayesian estimates.

Figures 1 to 3 illustrate the performance of the MCMC estimates under the MH algorithm for the following parameter setting: $p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$ with censoring time: $T = 0.5$ (Case I) and Sample size $(n, m) = (50, 50)$.
These plots confirm that the posterior samples for all parameters are approximately normally distributed and exhibit good convergence behavior. Specifically, the trace plots show rapid convergence and excellent mixing around the mean, confirming that the Markov chains have quickly reached their stationary distribution. Furthermore, the histograms confirm that the posterior distributions are generally unimodal and near-symmetric.

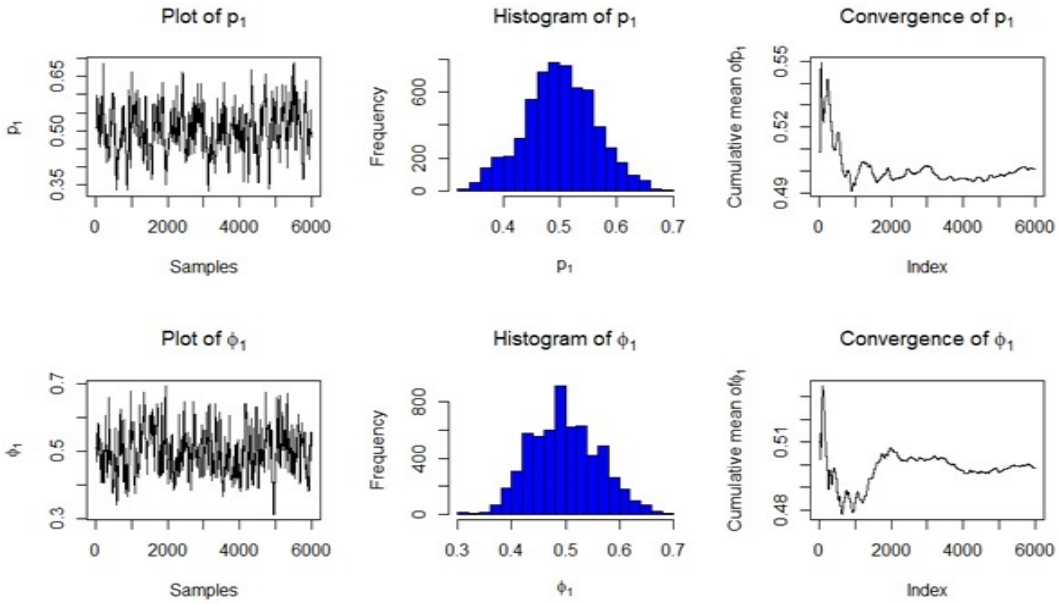



Figure 1. Plot, Histogram, and Convergence of $p_1$ and $\varphi_1$.

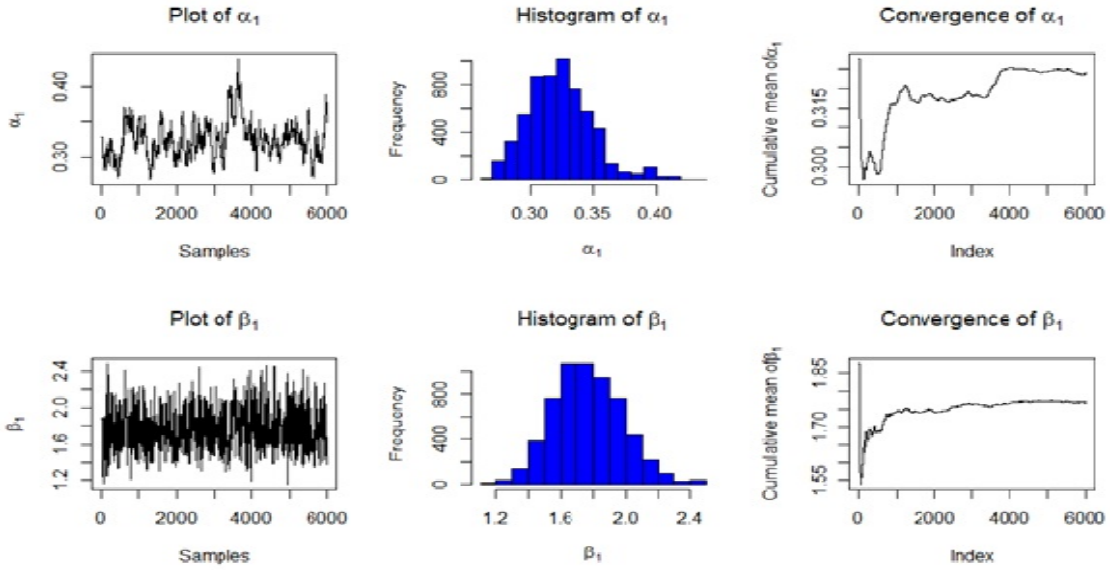



Figure 2. Plot, Histogram, and Convergence of $\alpha_1$ and $\beta_1$.

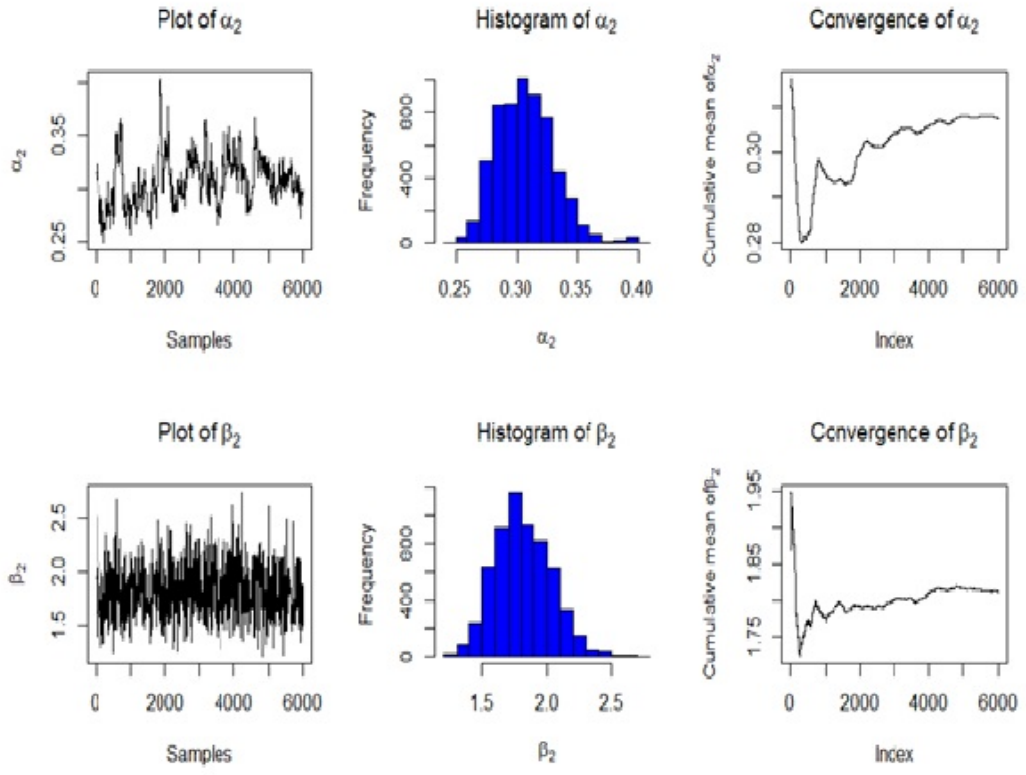



Figure 3. Plot, Histogram, and Convergence of $\alpha_2$ and $\beta_2$.

### *4.2. Real Data*

This section examines the actual dataset employed to demonstrate the applicability of the proposed methodology. These data, sourced from [64], reflect the breaking strength of jute fibers at two distinct gauge lengths: 10 mm and 20 mm. Each dataset has 30 measurements of fiber strength, as detailed below:

Dataset 1: (Gauge length 10 mm.):
43.93, 50.16, 101.15, 108.94, 123.06, 141.38, 151.48, 163.40, 177.25, 183.16, 212.13, 257.44, 262.90, 291.27, 303.90, 323.83, 353.24, 376.42, 383.43, 422.11, 506.60, 530.55, 590.48, 637.66, 671.49, 693.73, 700.74, 704.66, 727.23, 778.17.

Dataset 2: (Gauge length 20mm.):
36.75, 45.58, 48.01, 71.46, 83.55, 99.72, 113.85, 116.99, 119.86, 145.96, 166.49, 187.13, 187.85, 200.16, 244.53, 284.64, 350.70, 375.81, 419.02, 456.60, 547.44, 578.62, 581.60, 585.57, 594.29, 662.66, 688.16, 707.36, 756.70, 765.14.

The K–S goodness-of-fit test was employed to evaluate the appropriateness of the Weibull distribution for modeling these datasets.

- For Dataset 1, the calculated K–S statistic was 0.10601 with a p-value of 0.8539, based on MLE estimates $\hat{\alpha}_1 = 408.6115$ and $\hat{\beta}_1 = 1.6250$, indicating a good fit.
- For Dataset 2, the K–S statistic was 0.1490 with a p-value of 0.4733, with parameter estimates $\hat{\alpha}_2 = 372.1153$ , $\hat{\beta}_2 = 1.3608$, also suggesting an acceptable fit to the Weibull model.

To proceed with the joint Type I censoring analysis, several censoring times were considered: $T = 75, 100, 200, 500, 600$. Assuming a mixture structure with proportions $p_1 = 0.4$ and $\varphi_1 = 0.4$, MLEs were attained concerning the parameters $p_1, \varphi_1, \alpha_1, \beta 1, \alpha_2, \beta 2$, along with their associated 95% asymptotic confidence intervals. For Bayesian analysis, estimation was performed using the Metropolis-Hastings (MH) algorithm under non-informative priors. Specifically, the prior settings were:

$$\text{prior}\,0 = (0, 1, 0, 1), \quad \text{and} \quad \text{prior}\,1 = \text{prior}\,2 = (0, 0, 0, 0).$$

Initial values for the MH algorithm were defined for the MLEs of the parameters $(p_1^{(0)}, \phi_1^{(0)}, \alpha_1^{(0)}, \beta_1^{(0)}, \alpha_2^{(0)}, \beta_2^{(0)}) = (\hat{p}_1, \hat{\phi}_1, \hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)$. The variance covariance matrix used corresponds to the log-transformed parameters:

$$\{\ln(\hat{p}_1), \ln(\hat{\phi}_1), \ln(\hat{\alpha}_1), \ln(\hat{\beta}_1), \ln(\hat{\alpha}_2), \ln(\hat{\beta}_2)\}$$

Totaling 10,000, the posterior samples were generated, with the first 2,000 discarded as burn-in. Bayes estimates were then computed in view of LINEX and SE loss functions (with $\nu = 0.5$). In addition, the intervals of HPD were obtained using the technique developed in 1999 by Chen and Shao. All MLE estimates, their asymptotic confidence intervals, and SE are presented in Table 2. Similarly, Bayesian estimates derived via MCMC under the two loss functions, along with corresponding HPD intervals and standard errors, are also reported for comparison. Simpler distributions can be used due to their fewer parameters; however, the mixed Weibull model was chosen to accommodate data variability. Many reliability datasets include units from multiple production batches or failure types, making the mixture model more representative of the actual failure process. This extended form ensures that the estimation process encompasses the characteristics of subcategories that a single-component distribution might miss.

Table 2. MLE's and BEs with associated St.Er and CIs based on different joint Type I censoring schemes for the given real data set and assume: $p_1 = 0.4, \phi_1 = 0.4$.

| t | Param. | MLE | | | Bayesian MCMC | | | |
|---|---|---|---|---|---|---|---|---|
| | | Estimate | St.Er | Asy-CI | Estimate | | St.Er | HPD |
| | | | | | SE | LINEX | | |
| 75 | $p_1$ | 0.5559 | 0.1656 | (0.231, 0.88) | 0.6095 | 0.6086 | 0.0588 | (0.495,0.728) |
| | $\phi_1$ | 0.6001 | 0.2191 | (0.171, 1.03) | 0.3420 | 0.3400 | 0.0890 | (0.224,0.547) |
| | $\alpha_1$ | 144.4020 | 14.5153 | (15.953, 72.852) | 60.5533 | 37.3235 | 15.2558 | (29.759,85.511) |
| | $\beta_1$ | 1.2514 | 0.6741 | (-0.07, 2.573) | 2.3081 | 2.1314 | 0.8295 | (0.761,3.585) |
| | $\alpha_2$ | 61.5760 | 39.4368 | (-15.718, 138.872) | 50.1973 | 43.5442 | 6.6428 | (38.144,62.264) |
| | $\beta_2$ | 0.8243 | 0.3879 | (0.131, 1.651) | 2.1228 | 1.9278 | 0.9567 | (0.158,0.574) |
| 100 | $p_1$ | 0.5386 | 0.1383 | (0.268, 0.81) | 0.5382 | 0.5354 | 0.1052 | (0.98,4.021) |
| | $\phi_1$ | 0.6000 | 0.2191 | (0.171, 1.029) | 0.4183 | 0.4157 | 0.1015 | (0.367,0.759) |
| | $\alpha_1$ | 81.1787 | 11.5261 | (58.588, 103.77) | 82.8951 | 66.4982 | 12.9144 | (60.458,111.346) |
| | $\beta_1$ | 2.4984 | 0.7275 | (1.072, 3.924) | 2.2970 | 2.1956 | 0.6683 | (1.269,3.539) |
| | $\alpha_2$ | 67.2083 | 8.2634 | (51.012, 83.404) | 67.3176 | 57.6319 | 6.5474 | (54.447,79.366) |
| | $\beta_2$ | 3.0560 | 0.8142 | (1.46, 4.652) | 2.9146 | 2.8607 | 0.4648 | (2.024,3.941) |
| 200 | $p_1$ | 0.5187 | 0.0962 | (0.33, 0.707) | 0.4193 | 0.4175 | 0.0871 | (0.264,0.593) |
| | $\phi_1$ | 0.5238 | 0.1090 | (0.31, 0.737) | 0.5273 | 0.5261 | 0.0702 | (0.391,0.664) |
| | $\alpha_1$ | 141.4614 | 12.7375 | (116.496, 166.426) | 147.0975 | 128.2391 | 13.3189 | (120.178,171.877) |
| | $\beta_1$ | 2.3702 | 0.4158 | (1.555, 3.185) | 2.4973 | 2.4675 | 0.3489 | (1.877,3.146) |
| | $\alpha_2$ | 130.8472 | 10.9138 | (109.457, 152.238) | 131.4989 | 118.4924 | 9.7794 | (112.539,150.34) |
| | $\beta_2$ | 2.6250 | 0.4546 | (1.734, 3.516) | 2.3219 | 2.2914 | 0.3575 | (1.647,3.096) |
| 500 | $p_1$ | 0.5123 | 0.0781 | (0.359, 0.665) | 0.4507 | 0.4494 | 0.0704 | (0.32,0.566) |
| | $\phi_1$ | 0.5121 | 0.0781 | (0.359, 0.665) | 0.5087 | 0.5074 | 0.0740 | (0.366,0.643) |
| | $\alpha_1$ | 249.8774 | 25.3491 | (200.194, 299.561) | 251.1118 | 211.3942 | 23.6903 | (204.829,291.506) |
| | $\beta_1$ | 1.6160 | 0.2053 | (1.214, 2.018) | 1.5603 | 1.5472 | 0.2296 | (1.144,1.976) |
| | $\alpha_2$ | 230.5810 | 21.5956 | (188.255, 272.908) | 224.1843 | 181.8581 | 20.7792 | (178.776,260.973) |
| | $\beta_2$ | 1.7803 | 0.2248 | (1.34, 2.221) | 1.7776 | 1.7510 | 0.3244 | (1.131,2.252) |
| 600 | $p_1$ | 0.5097 | 0.0700 | (0.372, 0.647) | 0.4782 | 0.4765 | 0.0823 | (0.339,0.638) |
| | $\phi_1$ | 0.5106 | 0.0729 | (0.368, 0.654) | 0.5202 | 0.5194 | 0.0570 | (0.422,0.631) |
| | $\alpha_1$ | 314.7314 | 31.7504 | (252.502, 376.961) | 304.0040 | 238.8596 | 42.8563 | (232.221,388.158) |
| | $\beta_1$ | 1.4878 | 0.1734 | (1.148, 1.828) | 1.3699 | 1.3621 | 0.1773 | (1.075,1.734) |
| | $\alpha_2$ | 295.0771 | 28.7812 | (238.667, 351.487) | 302.1432 | 224.6807 | 31.0429 | (241.172,365.294) |
| | $\beta_2$ | 1.5612 | 0.1796 | (1.209, 1.913) | 1.5488 | 1.5439 | 0.1407 | (1.308,1.819) |

## 5. Conclusions and Future Research Directions

In the current investigation, the MLE and Bayesian estimating methods were utilized in order to arrive at estimates of the unknown parameters of a mixed Weibull distribution that was subjected to a joint Type I censoring scheme. Moreover, asymmetric confidence intervals and variance-covariance matrices were constructed from the data presented. The primary objective of this comprehensive simulation study is to thoroughly investigate, how the proposed estimators performed statistically. Additionally, a dataset from the actual world was reanalyzed in order to validate the practical utility of the model. The results demonstrate that the estimators are not only accurate but also extremely dependable, which lends credence to the possibility that they could be utilized in dependability analysis and lifetime data modeling. Based on these findings, it appears that the model that was proposed has the potential to be applied as a trustworthy instrument in applications involving survival analysis and reliability engineering in the real world. Based on our simulation and real data results, we demonstrate that the Bayesian estimation method, particularly with the LINEX loss function, outperforms other estimation methods, especially when sample sizes are small.In the future, this model can be expanded to include different age distributions within the same censoring framework, such as mixed Lindley distributions and mixed generalized exponential distributions and other distributions, such as Kavya–Manoharan cosine inverse Rayleigh distribution and the Half-logistic Gaussian distribution. Additionally, using alternative censoring methods, such as progressive joint censoring, will provide a broader perspective on the model's performance under different levels of data knowledge. Furthermore, it is

useful to explore integrating shrinkage estimation methods and regression modeling within the Weibull mixture framework and to make a comparison between maximum likelihood estimation (MLE) and shrinkage estimation, as well as regression analysis across different types of data, including binary, beta, and standardized gamma data. refer to [64] to [68].

## Appendix (A)

The components of the sample information matrix are computed in the following:

$$-\frac{\partial^2 \ln L}{\partial p_1^2} = \frac{-r_1}{p_1^2} - \frac{r_1}{(1-p_1)^2} - \frac{m-r_1}{p_1^2} - \frac{m-r_1}{(1-p_1)^2}$$

$$-\frac{\partial^2 \ln L}{\partial \varphi_1^2} = \frac{-r_2}{\varphi_1^2} - \frac{r_2}{(1-\varphi_1)^2} - \frac{n-r_2}{\varphi_1^2} - \frac{n-r_2}{(1-\varphi_1)^2}$$

$$\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \alpha_1^2} &= \frac{\sum_{i=1}^{r_1} z_i}{\hat{\alpha}_1^2} + 2\sum_{i=1}^{r_1} z_i(\beta_1-1)\left(\frac{w_{1i}}{\hat{\alpha}_1^3}\right) - \beta_1(\beta_1-1)(\hat{\alpha}_1)^{-\beta_1-2}\sum_{i=1}^{r_1} z_i(w_{1i})^{\beta_1} \\
&+ \frac{\sum_{i=1}^{r_2}(1-z_i)}{\hat{\alpha}_1^2} + 2\sum_{i=1}^{r_2}(1-z_i)(\beta_1-1)\left(\frac{w_{1i}}{\hat{\alpha}_1^3}\right) + \beta_1(\beta_1-1)(\hat{\alpha}_1)^{-\beta_1-2}\sum_{i=1}^{r_2}(1-z_i)(w_{1i})^{\beta_1} \\
&+ (m-r_1)\beta_1(\beta_1-1)(\hat{\alpha}_1)^{-\beta_1-2}(T)^{\beta_1} + (n-r_2)\beta_1(\beta_1-1)(\hat{\alpha}_1)^{-\beta_1-2}(T)^{\beta_1}
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \beta_1^2} &= \frac{-\sum_{i=1}^{r_1} z_i}{\beta_1^2} + \sum_{i=1}^{r_1} z_i\left(\frac{w_{1i}}{\alpha_1}\right)^{\beta_1}\ln\left(\frac{w_{1i}}{\alpha_1}\right)\ln\left(\frac{w_{1i}}{\alpha_1}\right) - \frac{\sum_{i=1}^{r_2}(1-z_i)}{\beta_1^2} \\
&+ \sum_{i=1}^{r_2}(1-z_i)\left(\frac{w_{1i}}{\alpha_1}\right)^{\beta_1}\ln\left(\frac{w_{1i}}{\alpha_1}\right)\ln\left(\frac{w_{1i}}{\alpha_1}\right) \\
&- \left(\frac{T}{\alpha_1}\right)^{\beta_1}\ln\left(\frac{T}{\alpha_1}\right)\ln\left(\frac{T}{\alpha_1}\right)[(m-r_1)+(n-r_2)]
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \alpha_2^2} &= \frac{\sum_{i=1}^{r_1} z_i}{\hat{\alpha}_2^2} + 2\sum_{i=1}^{r_1} z_i(\beta_2-1)\left(\frac{w_{2i}}{\hat{\alpha}_2^3}\right) - \beta_2(\beta_2-1)(\hat{\alpha}_2)^{-\beta_2-2}\sum_{i=1}^{r_1} z_i(w_{2i})^{\beta_2} \\
&+ \frac{\sum_{i=1}^{r_2}(1-z_i)}{\hat{\alpha}_2^2} + 2\sum_{i=1}^{r_2}(1-z_i)(\beta_2-1)\left(\frac{w_{2i}}{\hat{\alpha}_2^3}\right) + \beta_2(\beta_2-1)(\hat{\alpha}_2)^{-\beta_2-2}\sum_{i=1}^{r_2}(1-z_i)(w_{2i})^{\beta_2} \\
&+ (m-r_1)\beta_2(\beta_2-1)(\hat{\alpha}_2)^{-\beta_2-2}(T)^{\beta_2} + (n-r_2)\beta_2(\beta_2-1)(\hat{\alpha}_2)^{-\beta_2-2}(T)^{\beta_2}
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \beta_2^2} &= \frac{-\sum_{i=1}^{r_1} z_i}{\beta_2^2} + \sum_{i=1}^{r_1} z_i\left(\frac{w_{2i}}{\alpha_2}\right)^{\beta_2}\ln\left(\frac{w_{2i}}{\alpha_2}\right)\ln\left(\frac{w_{2i}}{\alpha_2}\right) - \frac{\sum_{i=1}^{r_2}(1-z_i)}{\beta_2^2} \\
&+ \sum_{i=1}^{r_2}(1-z_i)\left(\frac{w_{2i}}{\alpha_2}\right)^{\beta_2}\ln\left(\frac{w_{2i}}{\alpha_2}\right)\ln\left(\frac{w_{2i}}{\alpha_2}\right) \\
&- \left(\frac{T}{\alpha_2}\right)^{\beta_2}\ln\left(\frac{T}{\alpha_2}\right)\ln\left(\frac{T}{\alpha_2}\right)[(m-r_1)+(n-r_2)]
\end{aligned}$$

$$
\begin{aligned}
-\frac{\partial^2 \ln L}{\partial\beta_1\,\partial\alpha_1} &= -\sum_{i=1}^{r_1} z_i \left(\frac{w_{1i}}{\alpha_1^2}\right) - \beta_1 \sum_{i=1}^{r_1} z_i (w_{1i})^{\beta_1} \alpha_1^{-\beta_1-1} - \left(\frac{w_{1i}}{\alpha_1^2}\right) + \sum_{i=1}^{r_1} (1-z_i) \left(\frac{w_{1i}}{\alpha_1^2}\right) \\
&+ \beta_1 \sum_{i=1}^{r_1} (1-z_i)(w_{1i})^{\beta_1} \alpha_1^{-\beta_1-1} - \left(\frac{w_{1i}}{\alpha_1^2}\right) + (m-r_1)\beta_1(\alpha_1)^{-\beta_1-1}(T)^{\beta_1} - \left(\frac{w_{1i}}{\alpha_1^2}\right) \\
&+ (n-r_2)\beta_1(\alpha_1)^{-\beta_1-1}(T)^{\beta_1} - \left(\frac{w_{1i}}{\alpha_1^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\partial^2 \ln L}{\partial\beta_2\,\partial\alpha_2} &= -\sum_{i=1}^{r_1} z_i \left(\frac{w_{2i}}{\alpha_2^2}\right) - \beta_2 \sum_{i=1}^{r_1} z_i (w_{2i})^{\beta_2} \alpha_2^{-\beta_2-1} - \left(\frac{w_{2i}}{\alpha_2^2}\right) + \sum_{i=1}^{r_1} (1-z_i) \left(\frac{w_{2i}}{\alpha_2^2}\right) \\
&+ \beta_2 \sum_{i=1}^{r_1} (1-z_i)(w_{2i})^{\beta_2} \alpha_2^{-\beta_2-1} - \left(\frac{w_{2i}}{\alpha_2^2}\right) + (m-r_1)\beta_2(\alpha_2)^{-\beta_2-1}(T)^{\beta_2} - \left(\frac{w_{2i}}{\alpha_2^2}\right) \\
&+ (n-r_2)\beta_2(\alpha_2)^{-\beta_2-1}(T)^{\beta_2} - \left(\frac{w_{2i}}{\alpha_2^2}\right)
\end{aligned}
$$

$$
\frac{\partial^2 \ln l}{\partial p_1\,\partial\varphi_1} = \frac{\partial^2 \ln l}{\partial p_1\,\partial\beta_1} = \frac{\partial^2 \ln l}{\partial p_1\,\partial\beta_2} = \frac{\partial^2 \ln l}{\partial p_1\,\partial_{a1}} = \frac{\partial^2 \ln l}{\partial p_1\,\partial\alpha_2} = 0
$$

$$
\frac{\partial^2 \ln l}{\partial\varphi_1\,\partial\beta_1} = \frac{\partial^2 \ln l}{\partial\varphi_1\,\partial\beta_2} = \frac{\partial^2 \ln l}{\partial\varphi_1\,\partial\alpha_1} = \frac{\partial^2 \ln l}{\partial\varphi_1\,\partial\alpha_2} = 0
$$

$$
\frac{\partial^2 \ln l}{\partial\beta_1\,\partial\beta_2} = \frac{\partial^2 \ln l}{\partial\beta_1\,\partial\alpha_2} = \frac{\partial^2 \ln l}{\partial\beta_2\,\partial\alpha_1} = \frac{\partial^2 \ln l}{\partial\alpha_1\,\partial\alpha_2} = 0
$$

## Appendix (B)

Table 3. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | MLE | | Bayesian MCMC-SE | | Bayesian MCMC-LINEX | |
|---|---|---|---|---|---|---|---|---|
| | | | Avg. | MSE | Avg. | MSE | Avg. | MSE |
| (15,15) | 0.50 | $p_1$ | 0.5301 | 0.0170 | 0.8931 | 1.3219 | 0.6973 | 0.5399 |
| | | $\phi_1$ | 0.5335 | 0.0045 | 1.1911 | 3.0097 | 0.7543 | 0.8188 |
| | | $\alpha_1$ | 0.3530 | 0.0228 | 0.6134 | 0.8305 | 0.4004 | 0.0835 |
| | | $\beta_1$ | 2.0708 | 0.6416 | 2.2792 | 9.8621 | 1.1429 | 0.8031 |
| | | $\alpha_2$ | 0.3086 | 0.1963 | 0.4537 | 0.4187 | 0.3551 | 0.2085 |
| | | $\beta_2$ | 2.4055 | 0.9141 | 2.7848 | 13.4758 | 1.2505 | 1.1645 |
| | 0.75 | $p_1$ | 0.5154 | 0.0134 | 0.5608 | 0.1813 | 0.5324 | 0.1146 |
| | | $\phi_1$ | 0.5187 | 0.0067 | 0.5597 | 0.1633 | 0.5244 | 0.0718 |
| | | $\alpha_1$ | 0.4738 | 0.0026 | 0.5124 | 0.1629 | 0.4748 | 0.0179 |
| | | $\beta_1$ | 1.8768 | 0.2709 | 1.7367 | 1.5924 | 1.4889 | 0.2674 |
| | | $\alpha_2$ | 0.4272 | 1.1728 | 0.4656 | 0.2730 | 0.4299 | 0.1323 |
| | | $\beta_2$ | 2.1116 | 0.3351 | 2.0358 | 3.4960 | 1.6682 | 0.5295 |
| (25,25) | 0.50 | $p_1$ | 0.5189 | 0.0142 | 0.5503 | 0.1683 | 0.5089 | 0.0461 |
| | | $\phi_1$ | 0.5205 | 0.0063 | 0.5726 | 0.2510 | 0.5117 | 0.0314 |
| | | $\alpha_1$ | 0.3374 | 0.0271 | 0.3433 | 0.0338 | 0.3374 | 0.0307 |
| | | $\beta_1$ | 2.1068 | 0.5184 | 1.9840 | 2.3693 | 1.6790 | 0.1889 |
| | | $\alpha_2$ | 0.3109 | 0.1936 | 0.3245 | 0.2277 | 0.3146 | 0.1969 |
| | | $\beta_2$ | 2.3166 | 0.5209 | 2.1564 | 1.9130 | 1.8723 | 0.5493 |
| | 0.75 | $p_1$ | 0.5121 | 0.0126 | 0.4996 | 0.0100 | 0.4933 | 0.0088 |
| | | $\phi_1$ | 0.5135 | 0.0075 | 0.5000 | 0.0101 | 0.4934 | 0.0115 |
| | | $\alpha_1$ | 0.4570 | 0.0032 | 0.4548 | 0.0034 | 0.4528 | 0.0036 |
| | | $\beta_1$ | 1.9261 | 0.2639 | 1.7682 | 0.1326 | 1.7087 | 0.0968 |
| | | $\alpha_2$ | 0.4282 | 0.1050 | 0.4262 | 0.1063 | 0.4246 | 0.1073 |
| | | $\beta_2$ | 2.0798 | 0.2121 | 1.9164 | 0.1045 | 1.8477 | 0.0764 |
| (50,50) | 0.50 | $p_1$ | 0.5092 | 0.0119 | 0.5000 | 0.0100 | 0.4955 | 0.0092 |
| | | $\phi_1$ | 0.5119 | 0.0077 | 0.4997 | 0.0101 | 0.4946 | 0.0111 |
| | | $\alpha_1$ | 0.3449 | 0.0241 | 0.3232 | 0.0316 | 0.3226 | 0.0318 |
| | | $\beta_1$ | 2.3909 | 0.8426 | 2.0047 | 0.3231 | 1.9481 | 0.2623 |
| | | $\alpha_2$ | 0.3322 | 0.1746 | 0.3100 | 0.1939 | 0.3096 | 0.1943 |
| | | $\beta_2$ | 2.5111 | 0.6378 | 2.1170 | 0.2154 | 2.0549 | 0.1649 |
| | 0.75 | $p_1$ | 0.5065 | 0.0114 | 0.4998 | 0.0100 | 0.4966 | 0.0094 |
| | | $\phi_1$ | 0.5069 | 0.0087 | 0.4994 | 0.0102 | 0.4960 | 0.0109 |
| | | $\alpha_1$ | 0.4427 | 0.0039 | 0.4410 | 0.0041 | 0.4401 | 0.0042 |
| | | $\beta_1$ | 1.7596 | 0.2398 | 1.8642 | 0.1671 | 1.8315 | 0.1421 |
| | | $\alpha_2$ | 1.9468 | 0.1042 | 0.4273 | 0.1048 | 0.4266 | 0.1053 |
| | | $\beta_2$ | 0.4282 | 0.1220 | 1.9473 | 0.0783 | 1.9120 | 0.0629 |

Table 4. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | MLE | | Bayesian MCMC-SE | | Bayesian MCMC-LINEX | |
|---|---|---|---|---|---|---|---|---|
| | | | Avg. | MSE | Avg. | MSE | Avg. | MSE |
| (15,15) | 0.50 | $p_1$ | 0.4289 | 0.0306 | 1.0434 | 2.4239 | 0.6584 | 0.5839 |
| | | $\phi_1$ | 0.4419 | 0.0028 | 0.8240 | 1.1794 | 0.5791 | 0.2926 |
| | | $\alpha_1$ | 0.4085 | 0.0109 | 0.7704 | 1.0845 | 0.5117 | 0.1687 |
| | | $\beta_1$ | 1.9559 | 0.5149 | 2.2792 | 8.2624 | 1.1658 | 0.8495 |
| | | $\alpha_2$ | 0.5457 | 0.0546 | 0.9242 | 2.1520 | 0.5273 | 0.2170 |
| | | $\beta_2$ | 1.7287 | 0.1892 | 2.4298 | 12.9256 | 1.1358 | 1.0741 |
| | 0.75 | $p_1$ | 0.4601 | 0.0201 | 0.5509 | 0.2438 | 0.4873 | 0.0700 |
| | | $\phi_1$ | 0.4662 | 0.0048 | 0.5391 | 0.2109 | 0.4927 | 0.0847 |
| | | $\alpha_1$ | 0.4885 | 0.0033 | 0.5854 | 0.2635 | 0.5299 | 0.0422 |
| | | $\beta_1$ | 1.8573 | 0.2726 | 1.6804 | 0.9312 | 1.4217 | 0.1147 |
| | | $\alpha_2$ | 0.5632 | 0.0409 | 0.6075 | 0.4018 | 0.5276 | 0.0693 |
| | | $\beta_2$ | 1.6918 | 0.1067 | 1.6663 | 0.9876 | 1.4416 | 0.3943 |
| (25,25) | 0.50 | $p_1$ | 0.4322 | 0.0289 | 0.5567 | 0.3490 | 0.4986 | 0.1461 |
| | | $\phi_1$ | 0.4450 | 0.0025 | 0.5358 | 0.1772 | 0.4933 | 0.0887 |
| | | $\alpha_1$ | 0.4138 | 0.0088 | 0.5525 | 0.2714 | 0.4768 | 0.0375 |
| | | $\beta_1$ | 1.8685 | 0.2820 | 1.6962 | 0.9478 | 1.4712 | 0.3036 |
| | | $\alpha_2$ | 0.5326 | 0.0530 | 0.5833 | 0.5212 | 0.4981 | 0.1114 |
| | | $\beta_2$ | 1.6921 | 0.1128 | 1.8400 | 3.5467 | 1.4970 | 1.1652 |
| | 0.75 | $p_1$ | 0.4603 | 0.0198 | 0.4841 | 0.0137 | 0.4758 | 0.0157 |
| | | $\phi_1$ | 0.4685 | 0.0049 | 0.4854 | 0.0075 | 0.4783 | 0.0063 |
| | | $\alpha_1$ | 0.4947 | 0.0019 | 0.5262 | 0.0029 | 0.5196 | 0.0024 |
| | | $\beta_1$ | 1.8074 | 0.1674 | 1.6298 | 0.0670 | 1.5653 | 0.0467 |
| | | $\alpha_2$ | 0.5564 | 0.0409 | 0.5328 | 0.0497 | 0.5263 | 0.0524 |
| | | $\beta_2$ | 1.6791 | 0.0643 | 1.6208 | 0.0672 | 1.5578 | 0.0786 |
| (50,50) | 0.50 | $p_1$ | 0.4348 | 0.0276 | 0.4777 | 0.0159 | 0.4681 | 0.0184 |
| | | $\phi_1$ | 0.4460 | 0.0023 | 0.4801 | 0.0071 | 0.4723 | 0.0059 |
| | | $\alpha_1$ | 0.4187 | 0.0071 | 0.4667 | 0.0028 | 0.4593 | 0.0031 |
| | | $\beta_1$ | 1.7904 | 0.1371 | 1.6509 | 0.0650 | 1.5975 | 0.0468 |
| | | $\alpha_2$ | 0.5247 | 0.0531 | 0.4950 | 0.0676 | 0.4871 | 0.0714 |
| | | $\beta_2$ | 1.6477 | 0.0560 | 1.6206 | 0.0576 | 1.5709 | 0.0685 |
| | 0.75 | $p_1$ | 0.5069 | 0.0087 | 0.5000 | 0.0101 | 0.4966 | 0.0108 |
| | | $\phi_1$ | 0.5066 | 0.0114 | 0.5001 | 0.0101 | 0.4970 | 0.0094 |
| | | $\alpha_1$ | 0.4412 | 0.0041 | 0.4396 | 0.0043 | 0.4388 | 0.0044 |
| | | $\beta_1$ | 1.9529 | 0.2480 | 1.8700 | 0.1736 | 1.8371 | 0.1479 |
| | | $\alpha_2$ | 0.4267 | 0.1052 | 0.4258 | 0.1058 | 0.4251 | 0.1063 |
| | | $\beta_2$ | 2.0348 | 0.1301 | 1.9548 | 0.0841 | 1.9191 | 0.0678 |

Table 5. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1, \alpha_2 = 0.75, \beta_2 = 1$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | MLE | | Bayesian MCMC-SE | | Bayesian MCMC-LINEX | |
|---|---|---|---|---|---|---|---|---|
| | | | Avg. | MSE | Avg. | MSE | Avg. | MSE |
| (15,15) | 0.40 | $p_1$ | 0.4609 | 0.0043 | 0.9832 | 1.7100 | 0.6885 | 0.5300 |
| | | $\phi_1$ | 0.4562 | 0.0214 | 1.2637 | 3.5916 | 0.7907 | 1.0231 |
| | | $\alpha_1$ | 0.3437 | 0.0291 | 0.8407 | 1.4597 | 0.5313 | 0.3929 |
| | | $\beta_1$ | 1.2098 | 0.1612 | 1.9264 | 7.2761 | 0.9287 | 1.1048 |
| | | $\alpha_2$ | 0.4694 | 0.1030 | 0.8348 | 2.2000 | 0.4937 | 0.6772 |
| | | $\beta_2$ | 1.1190 | 0.0970 | 1.7421 | 6.3728 | 0.9012 | 0.8420 |
| | 1.00 | $p_1$ | 0.4808 | 0.0067 | 0.4929 | 0.0295 | 0.4804 | 0.0284 |
| | | $\phi_1$ | 0.4767 | 0.0154 | 0.5341 | 0.2832 | 0.4787 | 0.0374 |
| | | $\alpha_1$ | 0.4711 | 0.0080 | 0.5382 | 0.0738 | 0.4970 | 0.0393 |
| | | $\beta_1$ | 1.1601 | 0.0721 | 1.1084 | 0.6071 | 1.0044 | 0.0506 |
| | | $\alpha_2$ | 0.5377 | 0.0568 | 0.5345 | 0.1952 | 0.5045 | 0.1663 |
| | | $\beta_2$ | 1.0916 | 0.0494 | 1.0689 | 0.2346 | 1.0111 | 0.1434 |
| (25,25) | 0.40 | $p_1$ | 0.4638 | 0.0043 | 0.5841 | 0.3513 | 0.5340 | 0.1760 |
| | | $\phi_1$ | 0.4595 | 0.0200 | 0.6474 | 0.6816 | 0.5262 | 0.1639 |
| | | $\alpha_1$ | 0.3474 | 0.0257 | 0.5018 | 0.3130 | 0.4045 | 0.0870 |
| | | $\beta_1$ | 1.1636 | 0.0790 | 1.2583 | 1.4618 | 1.0107 | 0.2717 |
| | | $\alpha_2$ | 0.4459 | 0.1027 | 0.5417 | 0.5679 | 0.4174 | 0.1688 |
| | | $\beta_2$ | 1.1032 | 0.0573 | 1.2222 | 1.4256 | 0.9973 | 0.1648 |
| | 1.00 | $p_1$ | 0.4808 | 0.0066 | 0.4915 | 0.0189 | 0.4817 | 0.0071 |
| | | $\phi_1$ | 0.4773 | 0.0152 | 0.4891 | 0.0140 | 0.4808 | 0.0145 |
| | | $\alpha_1$ | 0.4750 | 0.0046 | 0.5068 | 0.0044 | 0.4955 | 0.0040 |
| | | $\beta_1$ | 1.1290 | 0.0416 | 1.0610 | 0.0237 | 1.0349 | 0.0197 |
| | | $\alpha_2$ | 0.5283 | 0.0553 | 0.5079 | 0.0636 | 0.4977 | 0.0699 |
| | | $\beta_2$ | 1.0763 | 0.0280 | 1.0600 | 0.0314 | 1.0323 | 0.0190 |
| (50,50) | 0.40 | $p_1$ | 0.4660 | 0.0045 | 0.4796 | 0.0080 | 0.4744 | 0.0233 |
| | | $\phi_1$ | 0.4607 | 0.0195 | 0.4963 | 0.1415 | 0.4694 | 0.0214 |
| | | $\alpha_1$ | 0.3487 | 0.0242 | 0.3912 | 0.0160 | 0.3836 | 0.0614 |
| | | $\beta_1$ | 1.1301 | 0.0395 | 1.1151 | 0.3036 | 1.0595 | 0.0279 |
| | | $\alpha_2$ | 0.4357 | 0.1031 | 0.4286 | 0.1443 | 0.4089 | 0.1242 |
| | | $\beta_2$ | 1.0774 | 0.0265 | 1.0696 | 0.0388 | 1.0498 | 0.0505 |
| | 1.00 | $p_1$ | 0.4814 | 0.0067 | 0.4941 | 0.0092 | 0.4901 | 0.0082 |
| | | $\phi_1$ | 0.4779 | 0.0150 | 0.4953 | 0.0131 | 0.4907 | 0.0126 |
| | | $\alpha_1$ | 0.4806 | 0.0021 | 0.5034 | 0.0031 | 0.4961 | 0.0021 |
| | | $\beta_1$ | 1.1035 | 0.0217 | 1.0641 | 0.0159 | 1.0518 | 0.0143 |
| | | $\alpha_2$ | 0.5196 | 0.0558 | 0.5026 | 0.0634 | 0.5041 | 0.0914 |
| | | $\beta_2$ | 1.0668 | 0.0148 | 1.0644 | 0.0142 | 1.0506 | 0.0129 |

Table 6. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1, \alpha_2 = 0.75, \beta_2 = 1$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | MLE | | Bayesian MCMC-SE | | Bayesian MCMC-LINEX | |
|---|---|---|---|---|---|---|---|---|
| | | | Avg. | MSE | Avg. | MSE | Avg. | MSE |
| (15,15) | 0.40 | $p_1$ | 0.4570 | 0.0212 | 1.1653 | 3.0411 | 0.7127 | 0.6487 |
| | | $\phi_1$ | 0.4631 | 0.0045 | 0.9260 | 1.4978 | 0.6982 | 0.6045 |
| | | $\alpha_1$ | 0.3403 | 0.0300 | 0.7724 | 1.3833 | 0.5225 | 0.4086 |
| | | $\beta_1$ | 1.2089 | 0.1465 | 1.7185 | 5.4376 | 0.8661 | 0.4973 |
| | | $\alpha_2$ | 0.4555 | 0.1099 | 0.8699 | 2.6919 | 0.4389 | 0.3122 |
| | | $\beta_2$ | 1.1311 | 0.1018 | 1.8020 | 7.0747 | 0.9383 | 1.0285 |
| | 1.00 | $p_1$ | 0.4777 | 0.0152 | 0.5034 | 0.0808 | 0.5006 | 0.1377 |
| | | $\phi_1$ | 0.4798 | 0.0066 | 0.5150 | 0.0920 | 0.4756 | 0.0132 |
| | | $\alpha_1$ | 0.4685 | 0.0078 | 0.5560 | 0.2062 | 0.4943 | 0.0129 |
| | | $\beta_1$ | 1.1610 | 0.0706 | 1.0850 | 0.3451 | 1.0042 | 0.0661 |
| | | $\alpha_2$ | 0.5347 | 0.0580 | 0.5465 | 0.3269 | 0.4944 | 0.0890 |
| | | $\beta_2$ | 1.0920 | 0.0472 | 1.0755 | 0.1965 | 1.0016 | 0.0416 |
| (25,25) | 0.40 | $p_1$ | 0.4603 | 0.0198 | 0.6767 | 0.7830 | 0.5296 | 0.1661 |
| | | $\phi_1$ | 0.4638 | 0.0043 | 0.5919 | 0.3455 | 0.5211 | 0.1462 |
| | | $\alpha_1$ | 0.3462 | 0.0262 | 0.5243 | 0.4295 | 0.4114 | 0.0971 |
| | | $\beta_1$ | 1.1606 | 0.0783 | 1.2574 | 1.5530 | 1.0368 | 0.5231 |
| | | $\alpha_2$ | 0.4442 | 0.1037 | 0.5593 | 0.7281 | 0.4233 | 0.2605 |
| | | $\beta_2$ | 1.1005 | 0.0568 | 1.3183 | 1.8727 | 0.9956 | 0.2128 |
| | 1.00 | $p_1$ | 0.4775 | 0.0151 | 0.4930 | 0.0591 | 0.4796 | 0.0151 |
| | | $\phi_1$ | 0.4808 | 0.0066 | 0.4896 | 0.0146 | 0.4809 | 0.0069 |
| | | $\alpha_1$ | 0.4747 | 0.0046 | 0.5065 | 0.0056 | 0.4939 | 0.0042 |
| | | $\beta_1$ | 1.1215 | 0.0400 | 1.0656 | 0.1442 | 1.0404 | 0.1068 |
| | | $\alpha_2$ | 0.5263 | 0.0565 | 0.5051 | 0.0652 | 0.5019 | 0.1331 |
| | | $\beta_2$ | 1.0707 | 0.0261 | 1.0592 | 0.1185 | 1.0260 | 0.0192 |
| (50,50) | 0.40 | $p_1$ | 0.4614 | 0.0193 | 0.4841 | 0.0317 | 0.4787 | 0.0426 |
| | | $\phi_1$ | 0.4654 | 0.0044 | 0.4831 | 0.0280 | 0.4761 | 0.0259 |
| | | $\alpha_1$ | 0.3497 | 0.0238 | 0.4012 | 0.0644 | 0.3867 | 0.0307 |
| | | $\beta_1$ | 1.1228 | 0.0376 | 1.0798 | 0.0311 | 1.0583 | 0.0403 |
| | | $\alpha_2$ | 0.4373 | 0.1023 | 0.4253 | 0.1172 | 0.4118 | 0.1278 |
| | | $\beta_2$ | 1.0714 | 0.0258 | 1.0735 | 0.1168 | 1.0431 | 0.0510 |
| | 1.00 | $p_1$ | 0.4785 | 0.0148 | 0.4949 | 0.0147 | 0.4895 | 0.0127 |
| | | $\phi_1$ | 0.4815 | 0.0067 | 0.4955 | 0.0136 | 0.4909 | 0.0093 |
| | | $\alpha_1$ | 0.4780 | 0.0024 | 0.4992 | 0.0024 | 0.4927 | 0.0024 |
| | | $\beta_1$ | 1.1024 | 0.0210 | 1.0688 | 0.0334 | 1.0515 | 0.0130 |
| | | $\alpha_2$ | 0.5160 | 0.0578 | 0.5010 | 0.0690 | 0.4931 | 0.0684 |
| | | $\beta_2$ | 1.0668 | 0.0146 | 1.0660 | 0.0200 | 1.0562 | 0.0353 |

Table 7. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.5, \beta_2 = 1.75$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | MLE | | Bayesian MCMC-SE | | Bayesian MCMC-LINEX | |
|---|---|---|---|---|---|---|---|---|
| | | | Avg. | MSE | Avg. | MSE | Avg. | MSE |
| (15,15) | 0.30 | $p_1$ | 0.4840 | 0.0078 | 1.0158 | 1.8637 | 0.7135 | 0.7624 |
| | | $\phi_1$ | 0.4836 | 0.0137 | 1.3966 | 4.2894 | 0.8381 | 1.5007 |
| | | $\alpha_1$ | 0.3616 | 0.0251 | 0.8643 | 1.6846 | 0.5368 | 0.4680 |
| | | $\beta_1$ | 1.9366 | 0.6682 | 2.6281 | 15.8194 | 1.0997 | 2.9062 |
| | | $\alpha_2$ | 0.3866 | 0.0233 | 0.8333 | 1.5087 | 0.5039 | 0.3093 |
| | | $\beta_2$ | 1.9093 | 0.4867 | 3.2152 | 25.0402 | 1.1181 | 4.2993 |
| | 0.60 | $p_1$ | 0.5033 | 0.0109 | 0.5365 | 0.1740 | 0.4933 | 0.0445 |
| | | $\phi_1$ | 0.5054 | 0.0096 | 0.6288 | 0.7353 | 0.5156 | 0.1243 |
| | | $\alpha_1$ | 0.4316 | 0.0070 | 0.4651 | 0.1002 | 0.4355 | 0.0463 |
| | | $\beta_1$ | 1.8429 | 0.2437 | 1.7886 | 1.9851 | 1.4978 | 0.1703 |
| | | $\alpha_2$ | 0.4216 | 0.0082 | 0.4813 | 0.1706 | 0.4385 | 0.0379 |
| | | $\beta_2$ | 1.8757 | 0.1481 | 1.8307 | 2.6555 | 1.4942 | 0.3841 |
| (25,25) | 0.30 | $p_1$ | 0.4859 | 0.0074 | 0.7758 | 0.9582 | 0.6082 | 0.3697 |
| | | $\phi_1$ | 0.4850 | 0.0133 | 1.0194 | 2.3761 | 0.6094 | 0.4187 |
| | | $\alpha_1$ | 0.3572 | 0.0233 | 0.7394 | 1.1959 | 0.4663 | 0.2070 |
| | | $\beta_1$ | 1.8664 | 0.3662 | 2.5350 | 11.8253 | 1.1667 | 0.5478 |
| | | $\alpha_2$ | 0.3774 | 0.0197 | 0.5990 | 0.7542 | 0.4150 | 0.0912 |
| | | $\beta_2$ | 1.8427 | 0.2342 | 2.8228 | 19.2128 | 1.2319 | 1.6296 |
| | 0.60 | $p_1$ | 0.5029 | 0.0106 | 0.4898 | 0.0272 | 0.4782 | 0.0065 |
| | | $\phi_1$ | 0.5028 | 0.0094 | 0.4853 | 0.0134 | 0.4784 | 0.0153 |
| | | $\alpha_1$ | 0.4289 | 0.0064 | 0.4279 | 0.0066 | 0.4237 | 0.0072 |
| | | $\beta_1$ | 1.7977 | 0.1585 | 1.6913 | 0.0927 | 1.6229 | 0.0638 |
| | | $\alpha_2$ | 0.4234 | 0.0072 | 0.4322 | 0.0063 | 0.4282 | 0.0068 |
| | | $\beta_2$ | 1.8154 | 0.0785 | 1.6849 | 0.0591 | 1.6174 | 0.0668 |
| (50,50) | 0.30 | $p_1$ | 0.5141 | 0.0130 | 0.5064 | 0.0253 | 0.4960 | 0.0111 |
| | | $\phi_1$ | 0.5144 | 0.0073 | 0.5032 | 0.0117 | 0.4942 | 0.0118 |
| | | $\alpha_1$ | 0.2079 | 0.0855 | 0.2076 | 0.0858 | 0.2072 | 0.0860 |
| | | $\beta_1$ | 2.3160 | 0.8215 | 2.0816 | 0.4588 | 1.9880 | 0.3509 |
| | | $\alpha_2$ | 0.1971 | 0.0919 | 0.1961 | 0.0926 | 0.1958 | 0.0927 |
| | | $\beta_2$ | 2.4724 | 0.7131 | 2.2346 | 0.3750 | 2.1265 | 0.2640 |
| | 0.60 | $p_1$ | 0.5028 | 0.0106 | 0.4922 | 0.0086 | 0.4882 | 0.0079 |
| | | $\phi_1$ | 0.5027 | 0.0095 | 0.4921 | 0.0117 | 0.4881 | 0.0126 |
| | | $\alpha_1$ | 0.4279 | 0.0058 | 0.4259 | 0.0061 | 0.4240 | 0.0063 |
| | | $\beta_1$ | 1.7831 | 0.1100 | 1.7347 | 0.0820 | 1.6980 | 0.0641 |
| | | $\alpha_2$ | 0.4244 | 0.0063 | 0.4293 | 0.0057 | 0.4273 | 0.0059 |
| | | $\beta_2$ | 1.7928 | 0.0323 | 1.7296 | 0.0277 | 1.6934 | 0.0284 |

Table 8. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.5, \beta_2 = 1.75$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | MLE | | Bayesian MCMC-SE | | Bayesian MCMC-LINEX | |
|---|---|---|---|---|---|---|---|---|
| | | | Avg. | MSE | Avg. | MSE | Avg. | MSE |
| (15,15) | 0.30 | $p_1$ | 0.4837 | 0.0137 | 1.4466 | 4.6815 | 0.8253 | 1.2840 |
| | | $\phi_1$ | 0.4837 | 0.0071 | 1.0606 | 2.0398 | 0.7699 | 0.8901 |
| | | $\alpha_1$ | 0.3623 | 0.0241 | 0.9327 | 2.0947 | 0.5626 | 0.4736 |
| | | $\beta_1$ | 1.9313 | 0.6869 | 2.7546 | 17.5759 | 1.0382 | 2.3465 |
| | | $\alpha_2$ | 0.3881 | 0.0214 | 0.8390 | 1.6725 | 0.4992 | 0.3077 |
| | | $\beta_2$ | 1.8990 | 0.4927 | 2.8973 | 22.0151 | 1.1235 | 4.4278 |
| | 0.60 | $p_1$ | 0.5030 | 0.0094 | 0.5506 | 0.2201 | 0.4981 | 0.1146 |
| | | $\phi_1$ | 0.5031 | 0.0106 | 0.5481 | 0.2306 | 0.5005 | 0.0910 |
| | | $\alpha_1$ | 0.4362 | 0.0062 | 0.4879 | 0.1654 | 0.4429 | 0.0467 |
| | | $\beta_1$ | 1.8433 | 0.2475 | 1.7613 | 1.3030 | 1.5021 | 0.4600 |
| | | $\alpha_2$ | 0.4262 | 0.0076 | 0.4782 | 0.1084 | 0.4403 | 0.0256 |
| | | $\beta_2$ | 1.8840 | 0.1823 | 1.7133 | 0.7347 | 1.4771 | 0.2138 |
| (25,25) | 0.30 | $p_1$ | 0.4850 | 0.0133 | 0.9358 | 1.9557 | 0.6518 | 0.6765 |
| | | $\phi_1$ | 0.4858 | 0.0074 | 0.8172 | 1.2304 | 0.5984 | 0.3510 |
| | | $\alpha_1$ | 0.3555 | 0.0236 | 0.7342 | 1.3264 | 0.4566 | 0.1899 |
| | | $\beta_1$ | 1.8498 | 0.3511 | 2.2724 | 9.1178 | 1.2745 | 1.6870 |
| | | $\alpha_2$ | 0.3762 | 0.0201 | 0.6108 | 0.7087 | 0.4256 | 0.1216 |
| | | $\beta_2$ | 1.8259 | 0.2283 | 2.3259 | 10.4124 | 1.2496 | 1.6407 |
| | 0.60 | $p_1$ | 0.5028 | 0.0092 | 0.4854 | 0.0138 | 0.4773 | 0.0153 |
| | | $\phi_1$ | 0.5029 | 0.0106 | 0.4866 | 0.0119 | 0.4772 | 0.0062 |
| | | $\alpha_1$ | 0.4280 | 0.0064 | 0.4270 | 0.0073 | 0.4222 | 0.0072 |
| | | $\beta_1$ | 1.7919 | 0.1570 | 1.6937 | 0.1379 | 1.6196 | 0.0630 |
| | | $\alpha_2$ | 0.4235 | 0.0070 | 0.4324 | 0.0062 | 0.4281 | 0.0066 |
| | | $\beta_2$ | 1.8045 | 0.0771 | 1.6866 | 0.1283 | 1.6122 | 0.0682 |
| (50,50) | 0.30 | $p_1$ | 0.4863 | 0.0129 | 0.5411 | 0.3965 | 0.4870 | 0.1876 |
| | | $\phi_1$ | 0.4867 | 0.0075 | 0.5086 | 0.1653 | 0.4702 | 0.0691 |
| | | $\alpha_1$ | 0.3548 | 0.0223 | 0.3920 | 0.1227 | 0.3543 | 0.0350 |
| | | $\beta_1$ | 1.7638 | 0.1722 | 1.7887 | 1.5371 | 1.5477 | 0.1300 |
| | | $\alpha_2$ | 0.3728 | 0.0180 | 0.4239 | 0.0893 | 0.4016 | 0.0487 |
| | | $\beta_2$ | 1.7430 | 0.1003 | 1.7463 | 2.1495 | 1.5150 | 0.4293 |
| | 0.60 | $p_1$ | 0.5028 | 0.0090 | 0.4915 | 0.0119 | 0.4875 | 0.0127 |
| | | $\phi_1$ | 0.5028 | 0.0102 | 0.4914 | 0.0085 | 0.4874 | 0.0077 |
| | | $\alpha_1$ | 0.4291 | 0.0056 | 0.4267 | 0.0060 | 0.4248 | 0.0063 |
| | | $\beta_1$ | 1.7824 | 0.1126 | 1.7348 | 0.0847 | 1.6979 | 0.0663 |
| | | $\alpha_2$ | 0.4256 | 0.0061 | 0.4309 | 0.0055 | 0.4290 | 0.0057 |
| | | $\beta_2$ | 1.7919 | 0.0353 | 1.7272 | 0.0300 | 1.6912 | 0.0307 |

Table 9. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | Asy-CI | | HPD | |
|---|---|---|---|---|---|---|
| | | | AIL | CP | AIL | CP |
| (15,15) | 0.50 | $p_1$ | 0.0399 | 96.1 | 3.5413 | 95.1 |
| | | $\phi_1$ | 0.0429 | 94.5 | 5.2347 | 95.1 |
| | | $\alpha_1$ | 0.1333 | 97.7 | 2.0148 | 95.1 |
| | | $\beta_1$ | 2.2039 | 95.7 | 7.3343 | 95.0 |
| | | $\alpha_2$ | 0.1531 | 97.5 | 1.2005 | 95.0 |
| | | $\beta_2$ | 2.7296 | 95.5 | 10.8372 | 95.0 |
| | 0.75 | $p_1$ | 0.0405 | 99.9 | 0.4749 | 96.6 |
| | | $\phi_1$ | 0.0377 | 98.6 | 0.4187 | 95.6 |
| | | $\alpha_1$ | 0.1698 | 97.9 | 0.2968 | 96.7 |
| | | $\beta_1$ | 1.4079 | 96.9 | 1.4091 | 96.2 |
| | | $\alpha_2$ | 0.1842 | 97.9 | 0.3243 | 96.8 |
| | | $\beta_2$ | 1.6891 | 96.6 | 1.8496 | 97.4 |
| (25,25) | 0.50 | $p_1$ | 0.0155 | 96.4 | 0.1308 | 95.9 |
| | | $\phi_1$ | 0.0172 | 93.3 | 0.2397 | 96.1 |
| | | $\alpha_1$ | 0.1051 | 97.9 | 0.1584 | 97.1 |
| | | $\beta_1$ | 1.5199 | 96.1 | 1.3921 | 96.3 |
| | | $\alpha_2$ | 0.1128 | 98.5 | 0.1468 | 97.1 |
| | | $\beta_2$ | 1.7535 | 95.9 | 1.5374 | 96.2 |
| | 0.75 | $p_1$ | 0.0129 | 99.9 | 0.0398 | 97.9 |
| | | $\phi_1$ | 0.0098 | 96.9 | 0.0405 | 97.9 |
| | | $\alpha_1$ | 0.1424 | 97.8 | 0.1426 | 96.5 |
| | | $\beta_1$ | 1.1256 | 97.0 | 0.9354 | 96.2 |
| | | $\alpha_2$ | 0.1488 | 97.8 | 0.1471 | 96.8 |
| | | $\beta_2$ | 1.2608 | 97.2 | 1.0507 | 97.0 |
| (50,50) | 0.50 | $p_1$ | 0.0074 | 100.0 | 0.0343 | 98.1 |
| | | $\phi_1$ | 0.0061 | 100.0 | 0.0349 | 97.6 |
| | | $\alpha_1$ | 0.0347 | 100.0 | 0.0761 | 98.1 |
| | | $\beta_1$ | 0.9697 | 100.0 | 0.9766 | 96.4 |
| | | $\alpha_2$ | 0.0368 | 100.0 | 0.0759 | 96.8 |
| | | $\beta_2$ | 1.0609 | 100.0 | 1.0557 | 96.5 |
| | 0.75 | $p_1$ | 0.0021 | 96.3 | 0.0281 | 96.5 |
| | | $\phi_1$ | 0.0024 | 96.1 | 0.0300 | 97.9 |
| | | $\alpha_1$ | 0.0999 | 97.4 | 0.1005 | 98.0 |
| | | $\beta_1$ | 0.7856 | 97.2 | 0.7036 | 96.7 |
| | | $\alpha_2$ | 0.1022 | 97.3 | 0.1028 | 97.9 |
| | | $\beta_2$ | 0.8368 | 97.1 | 0.7524 | 97.2 |

Table 10. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | Asy-CI | | HPD | |
|---|---|---|---|---|---|---|
| | | | AIL | CP | AIL | CP |
| (15,15) | 0.50 | $p_1$ | 0.1452 | 100.0 | 4.3932 | 95.1 |
| | | $\phi_1$ | 0.1258 | 100.0 | 3.2228 | 95.0 |
| | | $\alpha_1$ | 0.1985 | 98.6 | 2.8391 | 95.1 |
| | | $\beta_1$ | 2.1732 | 96.0 | 6.9158 | 95.1 |
| | | $\alpha_2$ | 0.4441 | 96.5 | 3.5804 | 95.1 |
| | | $\beta_2$ | 1.7038 | 95.6 | 8.5531 | 95.1 |
| | 0.75 | $p_1$ | 0.0924 | 100.0 | 0.4179 | 95.8 |
| | | $\phi_1$ | 0.0765 | 100.0 | 0.3488 | 96.5 |
| | | $\alpha_1$ | 0.2195 | 97.8 | 0.4581 | 96.3 |
| | | $\beta_1$ | 1.4932 | 95.5 | 1.3415 | 96.4 |
| | | $\alpha_2$ | 0.3054 | 96.7 | 0.4445 | 96.3 |
| | | $\beta_2$ | 1.2604 | 95.8 | 1.1980 | 96.5 |
| (25,25) | 0.50 | $p_1$ | 0.1070 | 99.9 | 0.6015 | 95.1 |
| | | $\phi_1$ | 0.0877 | 100.0 | 0.4752 | 95.7 |
| | | $\alpha_1$ | 0.1465 | 97.8 | 0.6461 | 95.9 |
| | | $\beta_1$ | 1.4996 | 96.0 | 1.5104 | 97.3 |
| | | $\alpha_2$ | 0.2973 | 95.8 | 0.5566 | 96.9 |
| | | $\beta_2$ | 1.2975 | 95.9 | 1.5669 | 95.9 |
| | 0.75 | $p_1$ | 0.0691 | 99.6 | 0.0581 | 98.6 |
| | | $\phi_1$ | 0.0564 | 99.9 | 0.0518 | 97.9 |
| | | $\alpha_1$ | 0.1686 | 97.6 | 0.1735 | 97.2 |
| | | $\beta_1$ | 1.0589 | 96.2 | 0.8664 | 95.5 |
| | | $\alpha_2$ | 0.2281 | 95.9 | 0.1943 | 95.8 |
| | | $\beta_2$ | 0.9550 | 96.3 | 0.8581 | 95.7 |
| (50,50) | 0.50 | $p_1$ | 0.0701 | 99.3 | 0.1158 | 98.1 |
| | | $\phi_1$ | 0.0578 | 99.6 | 0.1006 | 97.8 |
| | | $\alpha_1$ | 0.0897 | 97.4 | 0.1610 | 97.3 |
| | | $\beta_1$ | 0.9012 | 96.2 | 0.7983 | 96.3 |
| | | $\alpha_2$ | 0.1902 | 96.1 | 0.1944 | 96.6 |
| | | $\beta_2$ | 0.8373 | 96.3 | 0.7829 | 96.2 |
| | 0.75 | $p_1$ | 0.0026 | 96.1 | 0.0289 | 97.5 |
| | | $\phi_1$ | 0.0022 | 95.9 | 0.0263 | 97.5 |
| | | $\alpha_1$ | 0.1000 | 97.5 | 0.0991 | 96.3 |
| | | $\beta_1$ | 0.8120 | 96.9 | 0.7135 | 96.3 |
| | | $\alpha_2$ | 0.1023 | 97.6 | 0.1021 | 96.9 |
| | | $\beta_2$ | 0.8680 | 97.1 | 0.7544 | 96.7 |

Table 11. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1, \alpha_2 = 0.75, \beta_2 = 1$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | Asy-CI | | HPD | |
|---|---|---|---|---|---|---|
| | | | AIL | CP | AIL | CP |
| (15,15) | 0.40 | $p_1$ | 0.0949 | 100.0 | 3.9037 | 95.1 |
| | | $\phi_1$ | 0.1042 | 100.0 | 5.9966 | 95.1 |
| | | $\alpha_1$ | 0.2694 | 97.5 | 3.5171 | 95.1 |
| | | $\beta_1$ | 1.3424 | 95.6 | 8.4992 | 95.0 |
| | | $\alpha_2$ | 0.6113 | 96.5 | 3.5889 | 95.1 |
| | | $\beta_2$ | 1.1284 | 95.4 | 7.9681 | 95.0 |
| | 1.00 | $p_1$ | 0.0520 | 100.0 | 0.0795 | 97.3 |
| | | $\phi_1$ | 0.0573 | 100.0 | 0.0829 | 96.1 |
| | | $\alpha_1$ | 0.3321 | 96.9 | 0.3940 | 96.7 |
| | | $\beta_1$ | 0.8454 | 96.9 | 0.7287 | 96.5 |
| | | $\alpha_2$ | 0.4254 | 96.4 | 0.3860 | 96.6 |
| | | $\beta_2$ | 0.7723 | 96.3 | 0.7251 | 97.3 |
| (25,25) | 0.40 | $p_1$ | 0.0621 | 100.0 | 0.9928 | 95.3 |
| | | $\phi_1$ | 0.0685 | 100.0 | 1.3451 | 95.6 |
| | | $\alpha_1$ | 0.1947 | 96.4 | 1.0156 | 95.2 |
| | | $\beta_1$ | 0.8963 | 96.3 | 1.7258 | 95.0 |
| | | $\alpha_2$ | 0.3963 | 95.4 | 0.9049 | 95.0 |
| | | $\beta_2$ | 0.8722 | 96.2 | 0.9233 | 96.3 |
| | 1.00 | $p_1$ | 0.0383 | 99.5 | 0.0354 | 98.5 |
| | | $\phi_1$ | 0.0424 | 99.8 | 0.0357 | 97.7 |
| | | $\alpha_1$ | 0.2479 | 97.4 | 0.2505 | 97.6 |
| | | $\beta_1$ | 0.6196 | 96.2 | 0.5395 | 96.6 |
| | | $\alpha_2$ | 0.3065 | 96.2 | 0.2709 | 96.5 |
| | | $\beta_2$ | 0.5712 | 96.1 | 0.5214 | 95.8 |
| (50,50) | 0.40 | $p_1$ | 0.0379 | 99.5 | 0.1046 | 97.6 |
| | | $\phi_1$ | 0.0420 | 99.6 | 0.1180 | 97.2 |
| | | $\alpha_1$ | 0.1407 | 96.6 | 0.2191 | 96.4 |
| | | $\beta_1$ | 0.5889 | 95.8 | 0.5697 | 98.0 |
| | | $\alpha_2$ | 0.2585 | 96.1 | 0.2544 | 95.9 |
| | | $\beta_2$ | 0.5232 | 96.7 | 0.4901 | 96.7 |
| | 1.00 | $p_1$ | 0.0262 | 99.0 | 0.0264 | 98.5 |
| | | $\phi_1$ | 0.0292 | 99.6 | 0.0261 | 97.8 |
| | | $\alpha_1$ | 0.1645 | 96.9 | 0.1790 | 98.1 |
| | | $\beta_1$ | 0.4102 | 96.7 | 0.3827 | 97.6 |
| | | $\alpha_2$ | 0.2054 | 96.6 | 0.1886 | 96.5 |
| | | $\beta_2$ | 0.3978 | 96.7 | 0.3848 | 97.3 |

Table 12. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1, \alpha_2 = 0.75, \beta_2 = 1$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | Asy-CI | | HPD | |
|---|---|---|---|---|---|---|
| | | | AIL | CP | AIL | CP |
| (15,15) | 0.40 | $p_1$ | 0.1081 | 100.0 | 5.1381 | 95.0 |
| | | $\phi_1$ | 0.0875 | 100.0 | 3.6833 | 95.0 |
| | | $\alpha_1$ | 0.2632 | 96.6 | 3.0474 | 95.1 |
| | | $\beta_1$ | 1.2579 | 95.9 | 6.4732 | 95.1 |
| | | $\alpha_2$ | 0.5973 | 96.9 | 3.9126 | 95.0 |
| | | $\beta_2$ | 1.1408 | 96.0 | 8.2214 | 95.0 |
| | 1.00 | $p_1$ | 0.0566 | 100.0 | 0.0740 | 96.7 |
| | | $\phi_1$ | 0.0538 | 100.0 | 0.0815 | 96.3 |
| | | $\alpha_1$ | 0.3241 | 96.9 | 0.3855 | 97.3 |
| | | $\beta_1$ | 0.8286 | 96.5 | 0.7171 | 97.5 |
| | | $\alpha_2$ | 0.4233 | 96.6 | 0.3704 | 97.3 |
| | | $\beta_2$ | 0.7723 | 96.3 | 0.7251 | 97.3 |
| (25,25) | 0.40 | $p_1$ | 0.0665 | 100.0 | 1.7350 | 95.2 |
| | | $\phi_1$ | 0.0616 | 100.0 | 1.2001 | 95.1 |
| | | $\alpha_1$ | 0.1967 | 95.8 | 1.0699 | 95.1 |
| | | $\beta_1$ | 0.8986 | 95.8 | 1.8756 | 95.1 |
| | | $\alpha_2$ | 0.3967 | 95.4 | 0.8585 | 95.1 |
| | | $\beta_2$ | 0.8473 | 95.7 | 2.5094 | 95.0 |
| | 1.00 | $p_1$ | 0.0434 | 99.8 | 0.0398 | 97.8 |
| | | $\phi_1$ | 0.0387 | 100.0 | 0.0358 | 97.0 |
| | | $\alpha_1$ | 0.2460 | 97.4 | 0.2618 | 97.0 |
| | | $\beta_1$ | 0.6229 | 96.0 | 0.5276 | 96.3 |
| | | $\alpha_2$ | 0.3150 | 96.7 | 0.2731 | 96.4 |
| | | $\beta_2$ | 0.5693 | 96.7 | 0.5407 | 95.7 |
| (50,50) | 0.40 | $p_1$ | 0.0435 | 99.6 | 0.1235 | 97.8 |
| | | $\phi_1$ | 0.0393 | 99.8 | 0.1136 | 98.6 |
| | | $\alpha_1$ | 0.1381 | 96.6 | 0.2193 | 96.4 |
| | | $\beta_1$ | 0.5888 | 96.4 | 0.5436 | 97.2 |
| | | $\alpha_2$ | 0.2634 | 96.3 | 0.2773 | 96.2 |
| | | $\beta_2$ | 0.5637 | 96.4 | 0.5456 | 97.0 |
| | 1.00 | $p_1$ | 0.0316 | 99.3 | 0.0261 | 97.1 |
| | | $\phi_1$ | 0.0266 | 98.9 | 0.0242 | 97.9 |
| | | $\alpha_1$ | 0.1701 | 98.1 | 0.1784 | 97.7 |
| | | $\beta_1$ | 0.4019 | 96.5 | 0.3859 | 96.6 |
| | | $\alpha_2$ | 0.2144 | 97.1 | 0.1822 | 97.2 |
| | | $\beta_2$ | 0.3954 | 96.6 | 0.3835 | 96.2 |

Table 13. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.5, \beta_2 = 1.75$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | Asy-CI | | HPD | |
|---|---|---|---|---|---|---|
| | | | AIL | CP | AIL | CP |
| (15,15) | 0.30 | $p_1$ | 0.0483 | 100.0 | 3.7961 | 95.1 |
| | | $\phi_1$ | 0.0500 | 100.0 | 5.9833 | 95.1 |
| | | $\alpha_1$ | 0.3027 | 95.6 | 3.7544 | 95.0 |
| | | $\beta_1$ | 2.7102 | 96.5 | 9.8327 | 95.1 |
| | | $\alpha_2$ | 0.4012 | 95.6 | 3.5791 | 95.1 |
| | | $\beta_2$ | 2.6639 | 96.5 | 15.1234 | 95.0 |
| | 0.60 | $p_1$ | 0.0069 | 93.7 | 0.5154 | 96.5 |
| | | $\phi_1$ | 0.0072 | 93.2 | 0.7259 | 95.3 |
| | | $\alpha_1$ | 0.1869 | 97.3 | 0.5159 | 96.4 |
| | | $\beta_1$ | 1.3929 | 96.1 | 1.4466 | 96.6 |
| | | $\alpha_2$ | 0.1793 | 97.6 | 0.4056 | 96.5 |
| | | $\beta_2$ | 1.4265 | 95.6 | 1.5292 | 96.8 |
| (25,25) | 0.30 | $p_1$ | 0.0264 | 100.0 | 2.9549 | 95.1 |
| | | $\phi_1$ | 0.0302 | 100.0 | 4.5493 | 95.2 |
| | | $\alpha_1$ | 0.2108 | 97.0 | 3.0100 | 95.0 |
| | | $\beta_1$ | 1.8888 | 96.4 | 10.8100 | 95.1 |
| | | $\alpha_2$ | 0.2668 | 97.0 | 1.7842 | 95.0 |
| | | $\beta_2$ | 1.8629 | 96.4 | 13.0524 | 95.1 |
| | 0.60 | $p_1$ | 0.0055 | 95.6 | 0.0549 | 97.6 |
| | | $\phi_1$ | 0.0051 | 96.2 | 0.0516 | 98.0 |
| | | $\alpha_1$ | 0.1451 | 97.9 | 0.1435 | 97.7 |
| | | $\beta_1$ | 1.0368 | 96.5 | 0.8864 | 97.1 |
| | | $\alpha_2$ | 0.1441 | 98.3 | 0.1575 | 97.0 |
| | | $\beta_2$ | 1.0688 | 96.2 | 0.8793 | 97.3 |
| (50,50) | 0.30 | $p_1$ | 0.0166 | 100.0 | 0.3332 | 95.8 |
| | | $\phi_1$ | 0.0171 | 99.9 | 0.5745 | 95.0 |
| | | $\alpha_1$ | 0.1327 | 95.6 | 0.3345 | 95.4 |
| | | $\beta_1$ | 1.1639 | 96.3 | 1.2424 | 96.3 |
| | | $\alpha_2$ | 0.1581 | 95.5 | 0.3675 | 97.4 |
| | | $\beta_2$ | 1.1497 | 96.3 | 1.1520 | 96.3 |
| | 0.60 | $p_1$ | 0.0037 | 95.7 | 0.0377 | 97.0 |
| | | $\phi_1$ | 0.0036 | 96.4 | 0.0370 | 98.2 |
| | | $\alpha_1$ | 0.0929 | 97.2 | 0.0927 | 97.3 |
| | | $\beta_1$ | 0.6780 | 96.8 | 0.6244 | 96.7 |
| | | $\alpha_2$ | 0.0913 | 97.3 | 0.1002 | 97.6 |
| | | $\beta_2$ | 0.6848 | 96.6 | 0.6203 | 96.4 |

Table 14. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.5, \beta_2 = 1.75$) under joint Type I censoring scheme.

| (m,n) | t | Parm. | Asy-CI | | HPD | |
|---|---|---|---|---|---|---|
| | | | AIL | CP | AIL | CP |
| (15,15) | 0.30 | $p_1$ | 0.0466 | 100.0 | 6.3944 | 95.1 |
| | | $\phi_1$ | 0.0483 | 100.0 | 4.1733 | 95.2 |
| | | $\alpha_1$ | 0.2815 | 95.3 | 4.4233 | 95.1 |
| | | $\beta_1$ | 2.7755 | 95.4 | 12.1250 | 95.2 |
| | | $\alpha_2$ | 0.3686 | 95.8 | 3.6939 | 95.1 |
| | | $\beta_2$ | 2.6902 | 95.4 | 14.3614 | 95.3 |
| | 0.60 | $p_1$ | 0.0073 | 93.5 | 0.5181 | 95.9 |
| | | $\phi_1$ | 0.0076 | 97.1 | 0.5214 | 95.8 |
| | | $\alpha_1$ | 0.1795 | 97.3 | 0.4856 | 95.5 |
| | | $\beta_1$ | 1.4120 | 96.3 | 1.3620 | 96.5 |
| | | $\alpha_2$ | 0.1806 | 97.6 | 0.4463 | 95.5 |
| | | $\beta_2$ | 1.5898 | 95.6 | 1.3945 | 97.2 |
| (25,25) | 0.30 | $p_1$ | 0.0399 | 100.0 | 3.9920 | 95.5 |
| | | $\phi_1$ | 0.0269 | 100.0 | 3.2552 | 95.0 |
| | | $\alpha_1$ | 0.2059 | 95.4 | 3.1492 | 95.1 |
| | | $\beta_1$ | 1.8758 | 95.5 | 6.3454 | 95.1 |
| | | $\alpha_2$ | 0.2702 | 96.2 | 1.8798 | 95.1 |
| | | $\beta_2$ | 1.8501 | 95.7 | 8.6698 | 95.1 |
| | 0.60 | $p_1$ | 0.0052 | 96.6 | 0.0540 | 97.9 |
| | | $\phi_1$ | 0.0050 | 97.2 | 0.0514 | 98.2 |
| | | $\alpha_1$ | 0.1348 | 97.6 | 0.1330 | 97.1 |
| | | $\beta_1$ | 1.0509 | 95.5 | 0.8992 | 96.4 |
| | | $\alpha_2$ | 0.1305 | 97.8 | 0.1440 | 97.3 |
| | | $\beta_2$ | 1.0677 | 95.7 | 0.9008 | 96.1 |
| (50,50) | 0.30 | $p_1$ | 0.0172 | 100.0 | 0.4082 | 95.5 |
| | | $\phi_1$ | 0.0162 | 99.8 | 0.3643 | 95.4 |
| | | $\alpha_1$ | 0.1371 | 95.5 | 0.3133 | 96.5 |
| | | $\beta_1$ | 1.2566 | 95.7 | 1.4192 | 96.6 |
| | | $\alpha_2$ | 0.1655 | 95.6 | 0.3558 | 97.2 |
| | | $\beta_2$ | 1.2418 | 95.7 | 1.3292 | 96.9 |
| | 0.60 | $p_1$ | 0.0035 | 95.9 | 0.0392 | 99.2 |
| | | $\phi_1$ | 0.0036 | 96.8 | 0.0369 | 97.1 |
| | | $\alpha_1$ | 0.0971 | 97.1 | 0.0972 | 97.1 |
| | | $\beta_1$ | 0.7112 | 96.8 | 0.6611 | 97.5 |
| | | $\alpha_2$ | 0.0952 | 97.4 | 0.1001 | 97.7 |
| | | $\beta_2$ | 0.7185 | 96.7 | 0.6697 | 96.4 |

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