

Joint Type I Censoring in Mixture Weibull Models: Estimation via Maximum Likelihood, Bayesian Methods and Application in Engineering Data

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Abstract In this study, we propose a comprehensive statistical approach to the estimation of the unknown parameters of a mixture of Weibull distribution made of two components using joint Type I censored samples. Bayesian estimation and Maximum Likelihood Estimation (MLE) methods were considered, with a focus on SE and LINEX loss functions. For the purpose of approximating the posterior distribution, we made use of the probabilistic Monte Carlo methods, namely the Metropolis-Hastings algorithm. The accuracy of the estimation was evaluated by constructing asymptotic confidence intervals as well as highest posterior density (HPD) intervals. In addition to this, we conducted a simulation study in order to evaluate the behavior of the suggested estimators, and we used the methodology to a real dataset in order to highlight the illustrations.

Keywords Joint Type I censored scheme, Mixture Weibull distribution, Maximum likelihood estimation, Bayesian estimators and Markov Chain Monte Carlo.

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1. Introduction

Life testing is essential for analyzing time-to-event data, especially in assessing the longevity or dependability of components, systems, creatures, or economic organizations. It includes a wide range of statistical methods employed to evaluate the duration of a subject's functionality or survival under settings. Life testing is essential for understanding failure behavior, refining design, augmenting performance, mitigating risk, and guiding decision-making in practical fields like as engineering, medicine, and economics. Conducting exhaustive life testing on all units until failure is frequently unfeasible due to temporal restrictions, financial limits, and ethical dilemmas. To tackle these issues, numerous censoring methodologies have been devised, enabling the collection of limited information while yet producing accurate statistical judgments. Censoring transpires when the precise failure time of certain units remains unclear, either because the study concludes prior to their failure or they are removed from the study owing to external influences. The joint Type I Censoring Scheme has proven to be an effective and pragmatic design, particularly when managing several samples or product lines under uniform or comparative testing settings.

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Joint Type I censoring schemes assess two or more samples simultaneously and end at a set time. The joint variety allows simultaneous study of multiple populations, unlike Type I censoring. It is especially useful when comparing similar systems or items or when budget constraints require concurrent testing to minimize resources. After the observation period, only failing units are noted, and the rest are redacted. This architecture produces a more complete dataset with fewer failures or accelerated life testing. Joint Type I censored data statistically has pros and cons. They reduce experimental costs and simplify logistics. In contrast, they require advanced inferential methods to estimate distributional parameters, reliability functions, and hazard rates. Selecting statistical models like exponential, Weibull, gamma, or generalized distributions within these frameworks depends on data properties and study goals. Maximum likelihood estimates, Bayesian methods, and nonparametric methods are used to get insights from censored datasets.

Engineering has commonly used joint Type I censoring life testing to evaluate product reliability and component longevity. Under normal or accelerated stress circumstances, manufacturers test electrical circuits, mechanical assembly, and structural materials for failure times. Joint censoring techniques improve quality control and warranty analysis by comparing batches or product designs simultaneously. While obtaining precise life parameter estimations, reliability engineers save time and money on tests. Clinical trials and epidemiological research require life testing and censoring. Survival analysis of patients undergoing different therapies sometimes includes censored data due to loss to follow-up or trial discontinuation before all events occur. The combined Type I censoring model is ideal for comparing two medications or therapy across a fixed observation window. Parallel testing and synchronized study completion allow researchers to draw statistically solid conclusions without the ethical and logistical burden of waiting for all events. In economics, life testing techniques are used to model duration data including loan default, employee job leave, and competitive market company longevity. Real-world datasets right-censor "failure" events like bankruptcy, unemployment, and market departure for economic players including individuals and institutions. Macroeconomic panel studies, which investigate numerous entities over a defined time horizon, benefit from combined Type I censoring. Economists can compare industries, locations, or policies under comparable exposure periods to draw fairer and more interpretable statistical findings. Researchers have recently focused on censored systems and employed them in many disciplines refer to [1] to [18].

Several combined censoring schemes have been proposed in prior statistical research, including joint progressive Type I and type II, joint Type II, and joint progressive hyper Type I censoring. For instance, [19] introduced the Joint Progressive Censoring Type I (JPC-I) scheme, which encompasses the joint Type I censoring scheme as a special case. Their research concentrated on statistically deducing two exponential populations considering combined Type I and JPC-I censoring.

However, a careful study of the literature shows a significant research gap: joint censoring has been looked into a lot for single distributions, but not so much for mixture Weibull models under a Joint Type-I scheme. Mixture models present unique challenges in estimation due to the complexity of the likelihood function and potential identifiability issues. Most existing studies have not adequately addressed the comparative performance of Bayesian estimation (under different loss functions) versus classical maximum likelihood for this specific mixture architecture. Therefore, this study aims to bridge this gap. The following is a summary of this study's main goals:

1. To calculate the Maximum Likelihood Estimators (MLEs) under a Joint Type I censoring for the parameters of the mixture Weibull distribution.
2. To calculate Bayesian estimating processes using both (squared error) and (LINEX) loss functions.
3. To perform Monte Carlo simulation of the suggested estimators.
4. Using the created methodology to show usefulness in practice on an engineering dataset.

In the current study, we enhance this paradigm by implementing joint Type I censoring to mixed distributions. The relevant likelihood function is constructed according to the methodology proposed by [19].

$$L = \frac{m!n!}{(m-m_1)!(n-n_1)!} \prod_{i=1}^r \{ [f(w_i)]^{z_i} [g(w_i)]^{1-z_i} \} (\bar{F}(T))^{m-m_1} (\bar{G}(T))^{n-n_1}. \quad (1)$$

Statistical literature has proposed various types of joint censoring techniques. Notably, references [20] and [21] examined the characteristics of JPC-II censored samples. Also, [22] to [25] looked into Type-II censoring in

different situations. [26] built on the preceding work of [27] by looking at Type-II censored data from k -independent exponential populations together. [28] expanded the approach established by [29] by examining JPC-II censoring in the context of several exponential populations. In a more recent work, [30] tackled the estimation issue for two Weibull populations sharing a common shape parameter as per JPC-I. The researchers calculated maximum likelihood estimates (MLEs) and their related confidence intervals, as well as Bayesian estimates utilizing both squared error and LINEX loss functions, under the assumption of independent gamma priors.

Mixture distributions have found extensive applications across diverse fields, such as reliability, medicine, and engineering. The two-parameter Weibull distribution is specifically chosen due to its flexibility in modeling various hazard rate patterns, making it highly versatile in reliability analysis. The concept of finite mixture models was first introduced in [31], addressing heterogeneous populations consisting of two or more latent subpopulations $s \geq 2$, each characterized by distinct features and represented by an unknown mixing proportion p_j where $0 \leq p_j \leq 1, j = 1, 2, \dots, s$, and $\sum_{j=1}^s p_j = 1$. The PDF for such a mixture model is expressed as

$$f(x) = \sum_{j=1}^s p_j f_j(x),$$

where $f_j(x)$ denotes the PDF of the j -th subpopulation.

In lifetime models, subpopulations frequently correlate to different failure mechanisms. [32] indicated that each unit should preferably be labeled to identify its corresponding subpopulation, which thereafter dictates the form of failure. Nevertheless, in numerous actual situations, this criterion remains unfulfilled, and the classification of individual units is indeterminate. The absence of information affects both the configuration of the likelihood function, and the accuracy of parameter estimate. For further information regarding the application of lifespan models in scientific contexts, refer to [33] - [54].

Numerous studies have examined Type I censoring in relation to mixture distributions. For example, [32] derived maximum likelihood estimates for the parameters of a mixed exponential distribution under JPC-I censoring. [55] expanded this research to the mixed Weibull model, whereas [56] introduced a Weibull-exponential hybrid model and investigated both maximum likelihood estimation and Bayesian inference. [57] employed Bayesian predictive methodologies to mixed exponential models subjected to Type I censoring. Subsequent contributions encompass [58] regarding the mixed generalized exponential distribution and [59], which examined both Type I and Type-II censored samples. [60] and [61] investigated mixed Rayleigh models utilizing JPC-I censoring and addressed a simplified likelihood function when censoring is complete.

When estimating parameters for a mixture model under Type I censoring, the likelihood function takes the form

$$l \propto \prod_{j=1}^s \prod_{i=1}^{r_j} p_j f_j(x_i) \bar{F}(T)^{n-r} \quad (2)$$

where $\bar{F}(T) = 1 - F(T)$ and if all data points are observed ($r = n$), the likelihood reduces to that of a complete sample from the mixture model by [61]. The Weibull distribution is recognized for its versatility and practical utility in diverse fields, including survival analysis, reliability engineering, and biological research. Its capacity to handle both ascending and descending failure rates renders it an appropriate substitute for the exponential model in lifetime data analysis.

The subsequent portions of this work are organized as follows: Section 2 presents the model formulation and notations. Section 3 discusses the parameter estimation methodologies employing both Maximum Likelihood and Bayesian methods. Section 4 presents a quantitative analysis to illustrate the efficacy of the proposed estimators, accompanied by a practical application of real data. In conclusion, Section 5 concludes the paper and encapsulates the findings and potential extensions.

2. Model Description and Notation

Let $X_1, X_2, \dots, X_m$ denote the lifetimes of m units from product A , assumed to be independent and identically distributed (i.i.d.) random variables following a two-component mixture distribution. Each subpopulation is characterized by an unknown proportion p_j cumulative distribution function $F_j(x_{ji})$, and corresponding probability density function $f_j(x_{ji})$, where $j = 1, 2$ and $i = 1, \dots, r_j$. Similarly, let $Y_1, Y_2, \dots, Y_n$, be (i.i.d.) lifetimes of n units from product B , modeled using another two-component mixture distribution with proportions ϕ_j , distribution functions $G_j(y_{ji})$, and densities $g_j(y_{ji})$. Define $W_{(1)} < W_{(2)} < \dots < W_{(N)}$ as the ordered failure times from the combined $N = m + n$ sample. Under JPC-I, the test is terminated at a fixed time T , and only the first r failures are observed, with r_j failures from each subpopulation, such that $\sum_{j=1}^s r_j = r$.

The censored likelihood function is constructed by incorporating the subpopulation indicators $z_i \in \{0, 1\}$, indicating the latent class membership of each observed failure. The structure follows the formulation introduced by [61] and [23] to [25], and is adapted in the current model setting to include the mixed Weibull distributions. The likelihood function in general is represented as follows

$$L(\mathcal{G}|\mathbf{w}) \propto \prod_{j=1}^s \prod_{i=1}^{r_j} [p_j f_j(\mathbf{w}_{ji})]^{z_i} \cdot [\phi_j g_j(\mathbf{w}_{ji})]^{(1-z_i)} \cdot [\bar{F}^*(T) \cdot \bar{G}^*(T)]^{N-r} \quad (3)$$

where

$$\mathcal{G} = (\alpha_j, \beta_j, p_j, \phi_j), \quad j = 1, 2 \text{ is the parameter vector.}$$

$$\bar{F}^*(T) = \sum_{j=1}^s p_j \bar{F}_j(T) \quad \text{and} \quad \bar{G}^*(T) = \sum_{j=1}^s \phi_j \bar{G}_j(T).$$

Assuming $s = 2$ the likelihood simplifies to:

$$L(\vartheta | \mathbf{w}) \propto \prod_{i=1}^{r_1} [p_1 f_1(w_{1i}) p_2 f_2(w_{2i})]^{z_i} \prod_{i=1}^{r_2} [\phi_1 g_1(w_{1i}) \phi_2 g_2(w_{2i})]^{(1-z_i)} \times [\bar{F}^*(T)]^{m-r_1} [\bar{G}^*(T)]^{n-r_2} \quad (4)$$

Subject to the constraints: $p_1 + p_2 = 1$ and $\phi_1 + \phi_2 = 1$. If any of the mixing proportions p_1, ϕ_1 or p_2, ϕ_2 are set to zero, the model reduces to the standard joint Type I censoring likelihood described in Equation (1).

3. Estimation Methods

The maximum likelihood (ML) and Bayesian estimation have been discussed to estimate the unknown parameters of the mixture Weibull model based on joint type I censoring by point estimation and asymptotic confidence intervals in this section.

3.1. Maximum Likelihood Estimation

Assume the mixed distributions for both products A and B follow a two-component Weibull distribution with parameters $\alpha_j > 0$ (scale) and $\beta_j > 0$ (shape), for $j = 1, 2$. The likelihood function is constructed following the principles of parameter estimation for mixture models as discussed by [55] and [56] under joint Type I censoring. The likelihood function becomes:

$$\begin{aligned} L(g|\mathbf{w}) &\propto [p_1(1-p_1)]^n \prod_{i=1}^n [f_1(w_{1i}; \alpha_1, \beta_1) f_2(w_{2i}; \alpha_2, \beta_2)]^{z_i} \\ &\times [\phi_1(1-\phi_1)]^n \prod_{i=1}^n [g_1(w_{1i}; \alpha_1, \beta_1) g_2(w_{2i}; \alpha_2, \beta_2)]^{(1-z_i)} \\ &\times [\bar{F}^*(T)]^{n-r_1} [\bar{G}^*(T)]^{n-r_2} \end{aligned} \quad (5)$$

in which

$$f_j(w_{ji}; \alpha_j, \beta_j) = \frac{\beta_j}{\alpha_j} \left(\frac{w_{ji}}{\alpha_j} \right)^{\beta_j-1} \exp \left[- \left(\frac{w_{ji}}{\alpha_j} \right)^{\beta_j} \right], \quad i = 1, 2, \dots, r_1,$$

$$g_j(w_{ji}; \alpha_j, \beta_j) = \frac{\beta_j}{\alpha_j} \left(\frac{w_{ji}}{\alpha_j} \right)^{\beta_j-1} \exp \left[- \left(\frac{w_{ji}}{\alpha_j} \right)^{\beta_j} \right], \quad i = 1, 2, \dots, r_2.$$

The mixture survival terms are:

$$\bar{F}^*(T) = p_1 \bar{F}_1(T) + p_2 \bar{F}_2(T) \quad \text{and} \quad \bar{G}^*(T) = p_1 \bar{G}_1(T) + p_2 \bar{G}_2(T).$$

The corresponding log likelihood function will be

$$\begin{aligned} \ell(\vartheta \mid \mathbf{w}) &\propto r_1 \log [p_1(1-p_1)] + \sum_{i=1}^{r_1} z_i \log (f_1(w_{1i}; \alpha_1, \beta_1)) + \sum_{i=1}^{r_1} z_i \log (f_2(w_{1i}; \alpha_2, \beta_1)) \\ &+ r_2 \log [\varphi_1(1-\varphi_1)] + \sum_{i=1}^{r_2} (1-z_i) \log (g_1(w_{1i}; \alpha_1, \beta_1)) + \sum_{i=1}^{r_2} (1-z_i) \log (g_2(w_{1i}; \alpha_2, \beta_1)) \\ &+ (m-r_1) \log \left[p_1 \exp \left(- \left(\frac{T}{\alpha_1} \right)^{\beta_1} \right) + (1-p_1) \exp \left(- \left(\frac{T}{\alpha_2} \right)^{\beta_2} \right) \right] \\ &+ (n-r_2) \log \left[\varphi_1 \exp \left(- \left(\frac{T}{\alpha_1} \right)^{\beta_1} \right) + (1-\varphi_1) \exp \left(- \left(\frac{T}{\alpha_2} \right)^{\beta_2} \right) \right]. \end{aligned} \quad (6)$$

By differentiating the log-likelihood function in Equation (6) in relation to parameters $\alpha_1, \beta_1, \alpha_2, \beta_2, p_1$ and φ_1 , and setting the resulting first-order derivatives equal to zero, we obtain the maximum likelihood estimators (MLEs): $\hat{p}_1, \hat{\varphi}_1, \hat{\beta}_1, \hat{\beta}_2, \hat{\alpha}_1$ and $\hat{\alpha}_2$. The corresponding likelihood equations for each parameter are derived as follows:

$$\begin{aligned} \frac{\partial \ln L}{\partial \hat{p}_1} &= \frac{r_1}{\hat{p}_1} - \frac{r_1}{1-\hat{p}_1} + \frac{m-r_1}{\hat{p}_1} - \frac{m-r_1}{1-\hat{p}_1} = 0 \\ \frac{\partial \ln L}{\partial \hat{\varphi}_1} &= \frac{r_2}{\hat{\varphi}_1} - \frac{r_2}{1-\hat{\varphi}_1} + \frac{n-r_2}{\hat{\varphi}_1} - \frac{n-r_2}{1-\hat{\varphi}_1} = 0 \\ \frac{\partial \ln L}{\partial \hat{\alpha}_1} &= \frac{\sum_{i=1}^{r_1} z_i}{\hat{\alpha}_1} - \sum_{i=1}^{r_1} z_i (\hat{\beta}_1 - 1) \left(\frac{w_{1i}}{\hat{\alpha}_1^{\hat{\beta}_1}} \right) + \hat{\beta}_1 (\hat{\alpha}_1)^{-\hat{\beta}_1-1} \sum_{i=1}^{r_1} z_i (w_{1i})^{\hat{\beta}_1} \\ &- \frac{\sum_{i=1}^{r_2} (1-z_i)}{\hat{\alpha}_1} - \sum_{i=1}^{r_2} (1-z_i) (\hat{\beta}_1 - 1) \left(\frac{w_{1i}}{\hat{\alpha}_1^{\hat{\beta}_1}} \right) + \hat{\beta}_1 (\hat{\alpha}_1)^{-\hat{\beta}_1-1} \sum_{i=1}^{r_2} (1-z_i) (w_{1i})^{\hat{\beta}_1} \\ &+ (m-r_1) \hat{\beta}_1 (\hat{\alpha}_1)^{-\hat{\beta}_1-1} (T)^{\hat{\beta}_1} + (n-r_2) \hat{\beta}_1 (\hat{\alpha}_1)^{-\hat{\beta}_1-1} (T)^{\hat{\beta}_1} = 0 \\ \frac{\partial \ln L}{\partial \hat{\beta}_1} &= \frac{\sum_{i=1}^{r_1} z_i}{\hat{\beta}_1} + \sum_{i=1}^{r_1} z_i \ln \left(\frac{w_{1i}}{\hat{\alpha}_1} \right) - \sum_{i=1}^{r_1} z_i \left(\frac{w_{1i}}{\hat{\alpha}_1} \right)^{\hat{\beta}_1} \ln \left(\frac{w_{1i}}{\hat{\alpha}_1} \right) + \frac{\sum_{i=1}^{r_2} (1-z_i)}{\hat{\beta}_1} \\ &+ \sum_{i=1}^{r_2} (1-z_i) \ln \left(\frac{w_{1i}}{\hat{\alpha}_1} \right) - \sum_{i=1}^{r_2} (1-z_i) \left(\frac{w_{1i}}{\hat{\alpha}_1} \right)^{\hat{\beta}_1} \ln \left(\frac{w_{1i}}{\hat{\alpha}_1} \right) \\ &- (m-r_1) \left(\frac{T}{\hat{\alpha}_1} \right)^{\hat{\beta}_1} \ln \left(\frac{T}{\hat{\alpha}_1} \right) - (n-r_2) \left(\frac{T}{\hat{\alpha}_1} \right)^{\hat{\beta}_1} \ln \left(\frac{T}{\hat{\alpha}_1} \right) = 0 \end{aligned}$$

$$\begin{aligned}
\frac{\partial \ln L}{\partial \hat{\alpha}_2} &= -\frac{\sum_{i=1}^{r_1} z_i}{\hat{\alpha}_2} - \sum_{i=1}^{r_1} z_i (\hat{\beta}_2 - 1) \left(\frac{w_{2i}}{\hat{\alpha}_2^2} \right) + \hat{\beta}_2 (\hat{\alpha}_2)^{-\hat{\beta}_2-1} \sum_{i=1}^{r_1} z_i (w_{2i})^{\hat{\beta}_2} \\
&\quad - \frac{\sum_{i=1}^{r_2} (1 - z_i)}{\hat{\alpha}_2} - \sum_{i=1}^{r_2} (1 - z_i) (\hat{\beta}_2 - 1) \left(\frac{w_{2i}}{\hat{\alpha}_2^2} \right) + (n - r_2) \hat{\beta}_2 (\hat{\alpha}_2)^{-\hat{\beta}_2-1} (T)^{\hat{\beta}_2} \\
&\quad + \hat{\beta}_2 (\hat{\alpha}_2)^{-\hat{\beta}_2-1} \sum_{i=1}^{r_2} (1 - z_i) (w_{2i})^{\hat{\beta}_2} + (m - r_1) \hat{\beta}_2 (\hat{\alpha}_2)^{-\hat{\beta}_2-1} (T)^{\hat{\beta}_2} = 0 \\
\\
\frac{\partial \ln L}{\partial \hat{\beta}_2} &= \frac{\sum_{i=1}^{r_1} z_i}{\hat{\beta}_2} + \sum_{i=1}^{r_1} z_i \ln \left(\frac{w_{2i}}{\hat{\alpha}_2} \right) - \sum_{i=1}^{r_1} z_i \left(\frac{w_{2i}}{\hat{\alpha}_2} \right)^{\hat{\beta}_2} \ln \left(\frac{w_{2i}}{\hat{\alpha}_2} \right) \\
&\quad + \frac{\sum_{i=1}^{r_2} (1 - z_i)}{\hat{\beta}_2} + \sum_{i=1}^{r_2} (1 - z_i) \ln \left(\frac{w_{2i}}{\hat{\alpha}_2} \right) - \sum_{i=1}^{r_2} (1 - z_i) \left(\frac{w_{2i}}{\hat{\alpha}_2} \right)^{\hat{\beta}_2} \ln \left(\frac{w_{2i}}{\hat{\alpha}_2} \right) \\
&\quad - (m - r_1) \left(\frac{T}{\hat{\alpha}_2} \right)^{\hat{\beta}_2} \ln \left(\frac{T}{\hat{\alpha}_2} \right) - (n - r_2) \left(\frac{T}{\hat{\alpha}_2} \right)^{\hat{\beta}_2} \ln \left(\frac{T}{\hat{\alpha}_2} \right) = 0
\end{aligned} \tag{7}$$

It is evident from Equation (7) that MLEs of parameters $\alpha_1, \beta_1, \alpha_2, \beta_2, p_1$ and ϕ_1 involve solving a system of nonlinear equations, which are analytically intractable in most cases. Therefore, numerical optimization techniques implemented in R software are employed to estimate the parameters $\hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2, \hat{p}_1$ and $\hat{\phi}_1$. To ensure the reliability of the numerical results for the MLEs, we implemented a robust optimization procedure in R. Given the potential for local maxima in mixture models, multiple sets of initial parameter values were tested. This approach ensures that the numerical algorithm converges to the global maximum of the log-likelihood function, thereby providing consistent and stable estimates.

In order to calculate the variance-covariance matrix of the estimators, the observed Fisher information matrix is attained through evaluating the negative of the log-likelihood function's second-order partial derivatives in relation to parameters. This approach for constructing asymptotic confidence intervals is a standard procedure in life-testing analysis (see [1] and [4]). The inverse of this matrix provides the estimated variance-covariance matrix for these parameter vector. Due to the length and complexity of these second-order partial derivatives, their detailed expressions are provided in the Appendix (A).

Suppose that $\hat{\theta}$ is MLE for parameter vector $\theta = (\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1)$. Under standard regularity conditions, the MLE $\hat{\theta}$ is asymptotically normally distributed with mean θ and covariance matrix I_{θ}^{-1} , in which I_{θ} denotes the Fisher information matrix. In practice, this matrix is estimated by $I_{\hat{\theta}}^{-1}$. Accordingly, the approximate $100(1 - \alpha)\%$ confidence intervals with respect to parameters α_j and $\beta_j, j = 1, 2$, are given by:

$$\hat{\alpha}_j \pm z_{(1-\alpha/2)} \sqrt{\text{var}(\hat{\alpha}_j)} \quad \text{and} \quad \hat{\beta}_j \pm z_{(1-\alpha/2)} \sqrt{\text{var}(\hat{\beta}_j)}.$$

where $z_{(1-\alpha/2)}$ represents the upper $(1 - \alpha/2)$ quantile of the standard normal distribution.

3.2. Bayesian Estimation

Bayesian estimators are obtained for the unknown parameters $\alpha_1, \beta_1, \alpha_2$ and β_2 under two different loss functions: the squared error (SE) loss and LINEX loss. The LINEX loss function displayed by [20]. The Bayesian inference is performed under the assumption that the prior distributions for the parameters $\alpha_1, \beta_1, \alpha_2$ and β_2 follow independent gamma distributions. Specifically, we assume the following priors:

For α_j (scale parameters), the prior distributions are:

$$\pi(\alpha_j) \propto \alpha_j^{a_j-1} e^{-(a_j b_j)}, \quad a_j, b_j, \alpha_j > 0, \quad j = 1, 2$$

For β_j (shape parameters), the prior distributions are:

$$\pi(\beta_j) \propto \beta_j^{c_j-1} e^{-(\beta_j d_j)}, \quad c_j, d_j, \beta_j > 0, \quad j = 1, 2$$

The mixing proportions p_1 and ϕ_1 are assumed to follow independent beta distributions as follows: $p_1 \sim \text{Beta}(C_1, D_1)$ and $\phi_1 \sim \text{Beta}(C_2, D_2)$. Accordingly, the prior densities for p_1 and ϕ_1 are given by:

$$\begin{aligned}\pi(p_1) &\propto p_1^{C_1-1}(1-p_1)^{D_1-1}, & 0 < p_1 < 1, & C_1, D_1 > 0 \\ \pi(\phi_1) &\propto \phi_1^{C_2-1}(1-\phi_1)^{D_2-1}, & 0 < \phi_1 < 1, & C_2, D_2 > 0\end{aligned}$$

Here, all of the hyper parameters ($a_j, b_j, c_j, d_j, C_1, D_1, C_2$ and D_2) are supposedly non-negative and known. Accordingly, the joint prior distribution of the parameters $\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1$ can be expressed (up to a constant of proportionality) as:

$$\pi(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1) \propto \pi(\alpha_j)\pi(\beta_j)\pi(p_1)\pi(\phi_1), \quad a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, C_1, D_1, C_2, D_2 > 0 \quad (8)$$

Combining the likelihood function in Equation (5) with the joint prior density described in Equations (8), the joint posterior density function of the parameter vector $\theta = (\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1)$ is, up to a proportionality constant, given by

$$\pi(\theta|\mathbf{w}) = \frac{1}{\psi} \pi(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1) L(\theta|\mathbf{w}), \quad (9)$$

where $\psi = \int \int \int \int \int \pi(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1) L(\theta|\mathbf{w}) d\alpha_j d\beta_j dp_1 d\phi_1, j = 1, 2$.

Consequently, the Bayes estimator of any function of parameter vector $\theta = \{\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1\}$ under SE loss function can be written as follows

$$\theta_{SE} = E(\theta(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1|\mathbf{w})) \quad (10)$$

In light of the loss function of LINEX, the corresponding Bayes estimate is given in the following form

$$\theta_{LINEX} = -\frac{1}{\tau} \ln[E(e^{-\tau\theta(\alpha_1, \beta_1, \alpha_2, \beta_2, p_1, \phi_1)}|\mathbf{w})], \quad \tau \neq 0 \quad (11)$$

Where $\tau \neq 0$ is a parameter that reflects the asymmetry of the LINEX loss function. Since the expectations in Equations (10) and (11), as well as the posterior distribution in Equation (9), do not admit closed-form solutions, even under heavy censoring, the informative priors ensure component identifiability and robust estimation.

Therefore, we suggest using the MCMC methods for approximating the Bayes estimates. Specifically, the algorithm of Metropolis-Hastings (MH) is employed to generate posterior samples and approximate the required integrals, which often represent analytical challenges in mixture models. Unlike the EM algorithm, which is primarily designed for point estimation and may lead to convergence to local maxima, our MCMC-based approach, supported by the construction of HPD intervals, provides more comprehensive inferential results by giving the full posterior surface and more reliable interval estimates in complex joint exploration scenarios. To guarantee the positivity of the shape and scale parameters α_i, β_i and to improve the convergence of the MCMC algorithm, the parameters were log-transformed during the estimation process.

4. Simulation Study and Real Data Analysis

Here, we aim to evaluate the performance of these proposed estimation methods given above in Sections 2 and 3. A real dataset is used to serve goals of illustrations, and a comprehensive MCMC simulation study was conducted with the aim of assessing the statistical behavior of the estimators under a joint Type I censoring scheme. All computations are carried out through the R language, with the `bbmle` and `HDInterval` packages utilized to calculate HPD and MLEs see [62].

4.1. Simulation Study

In order to perform the assessment of the Bayesian estimation and MLE methods, an MCMC simulation is carried out under JPC-I censoring from a mixture of two components of Weibull distributions. For each scenario, 1000 random samples are generated. The chosen parameter combinations were selected to represent various real-world scenarios, including cases where the sub-populations are well-separated and cases where the failure rates are closely clustered, thus testing the robustness of the proposed estimation methods. The simulation is carried out according to the following parameter settings:

Case	p_1	ϕ_1	α_1	β_1	α_2	β_2
I	0.4	0.6	0.5	1.5	0.75	1.75
II	0.6	0.4	0.5	1.5	0.75	1.75
III	0.4	0.6	0.5	1	0.75	1
IV	0.6	0.4	0.5	1	0.75	1
V	0.4	0.6	0.5	1.5	0.5	1.75
VI	0.6	0.4	0.5	1.5	0.5	1.75

Three varying sample sizes were considered for the two populations:

$$(n, m) = (15, 15), (25, 25), (50, 50)$$

Censoring times T are specified for each case as follows:

- Case I & II: $T = 0.5, 0.75$
- Case III & IV: $T = 0.4, 1.00$
- Case V & VI: $T = 0.3, 0.60$

Note: When $\beta_1 = \beta_2 = 1$, Case III and Case IV reduce to exponential mixture distributions. When $\beta_1 = \beta_2 = 2$ the model simplifies to the Rayleigh mixture distribution.

MLEs and their associated 95% asymptotic confidence intervals (Asy-CIs) are computed using the generated samples. The initial values used in the numerical optimization are set equal to the true parameter values. The corresponding results are reported in Appendix B (Tables 3 to 8).

Bayesian estimates are obtained using informative gamma priors. Three prior settings are considered:

$$\text{prior 0} = (0.5, 0.5, 0.5, 0.5), \quad \text{prior 1} = (1.2, 1.5, 1.5, 2), \quad \text{prior 2} = (1.5, 1.2, 1.5, 1.8)$$

Sensitivity analysis was performed by testing alternative non-informative priors, confirming the stability of the Bayesian estimates. MLEs are employed as initial values for the Metropolis-Hastings (MH) algorithm, in addition to the comparable variance-covariance matrix of $\ln(\hat{\rho}_1), \ln(\hat{\phi}_1), \ln(\hat{\alpha}_1), \ln(\hat{\beta}_1), \ln(\hat{\alpha}_2), \ln(\hat{\beta}_2)$. Out of 6,000 generated posterior samples, the first 1,500 are discarded as burn-in. Bayes estimates are then calculated in light of SE and LINEX (with $\nu = 2$) loss functions. HPD intervals are also constructed following the method developed by [62].

Summary of Results:

- Table 1: This is a summary of the main comparative indicators (MSE) of the estimators across some critical simulation scenarios, where the numerical convergence rate of MLE exceeded 99% in all simulation cases. Also shown is the number of failure units for these scenarios.
- Tables 3 to 8 in Appendix B show the average estimates (Avg.) and the mean squared errors (MSEs) for the MLE and Bayes methods, corresponding to sample sizes $(n, m) = (15, 15), (25, 25), (50, 50)$.
- Tables 9 to 14 in Appendix B report the average interval lengths (AILs) and the coverage probabilities (CPs) for corresponding methods.

Drawing from the results shown in Appendix B the following conclusion can be made:

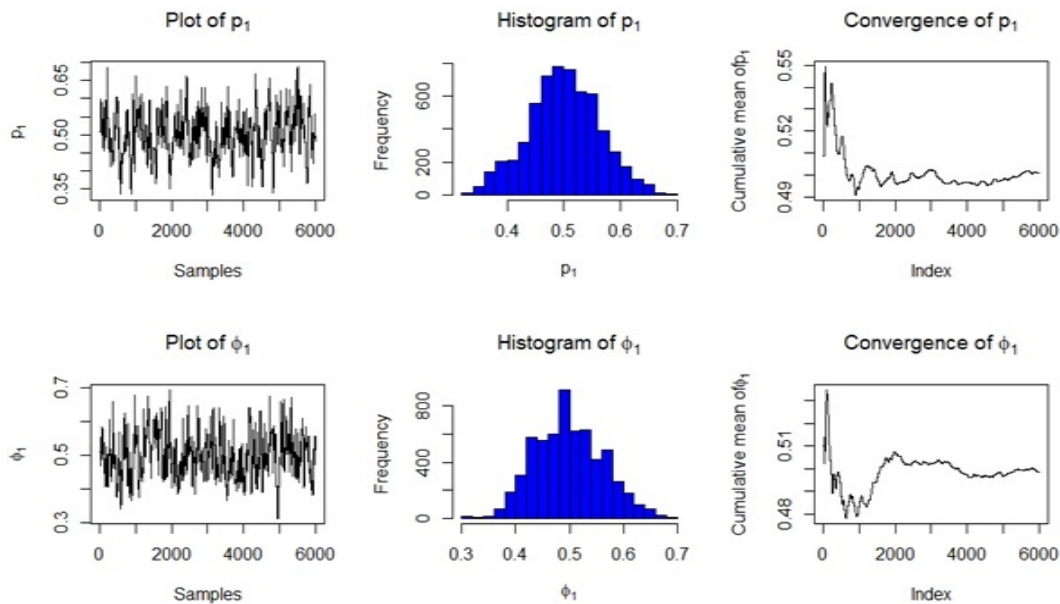
Table 1. Summary of the main comparative indicators (MSE).

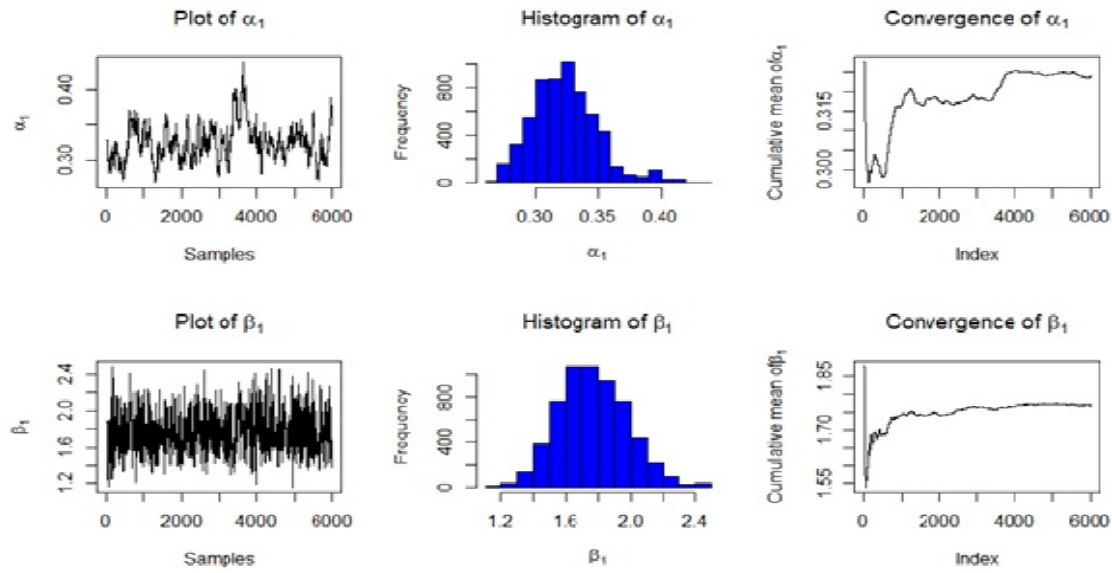
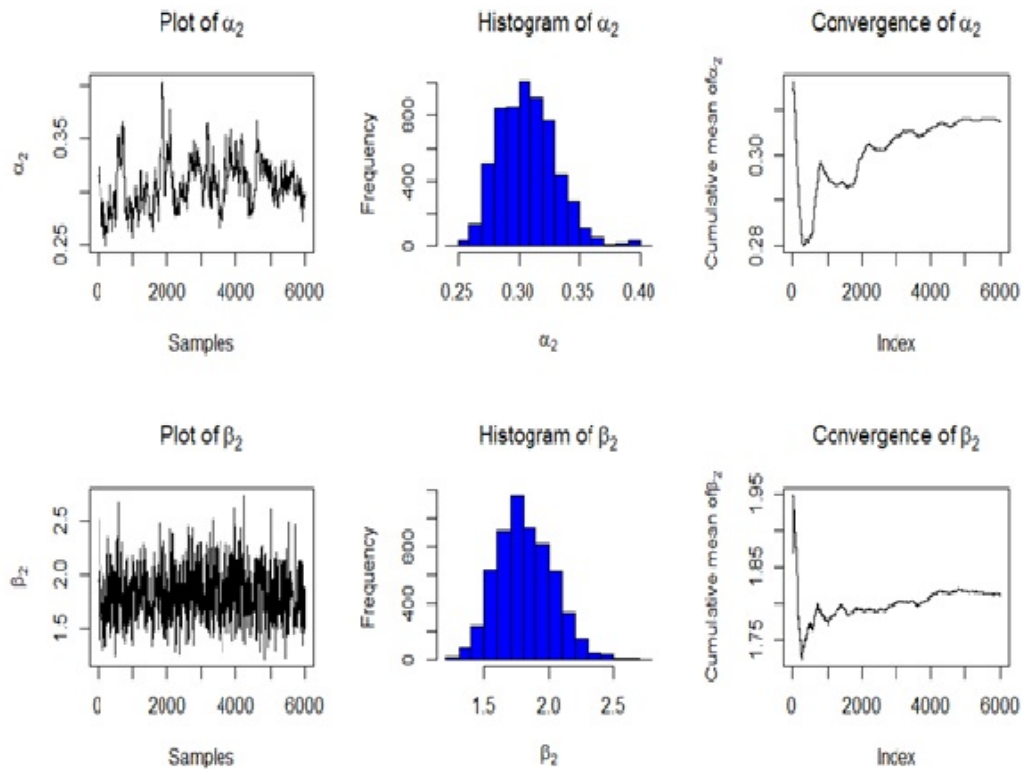
Comparison Effect	(m,n)	T	Avg. Failures $r_1 + r_1$	parameter	MLE (MSE)	Bayesian SE(MSE)	Bayesian INEX(MSE)
Sample Size Effect	(15,15)	0.5	15	p_1	0.0170	1.3219	0.5399
	(50,50)		53		0.0119 ↓	0.0100 ↓	0.0092 ↓
Censoring Time Effect	(15,15)	0.5	15	α_1	0.0228	0.8305	0.0835
		0.75	22		0.0026 ↓	0.1629 ↓	0.0718 ↓
LINEX Superiority	(15,15)	0.5	15	β_1	0.6416	9.8621	0.8031
	(50,50)	0.3	35	β_2	0.7131	0.3750	0.2640

1. The Mean Squared Error decreases for both Maximum Likelihood Estimation and Bayesian estimations with an increase in sample size.
2. Larger censoring time T leads to lower MSEs.
3. Bayes estimates considering the LINEX loss function yield fewer mean squared errors than those derived from the squared error loss function.
4. The asymptotic information limits for maximum likelihood estimates are generally shorter than those for Bayesian estimates.
5. Confidence intervals for maximum likelihood estimates are larger and closer to 100% compared to Bayesian estimates.

Figures 1 to 3 illustrate the performance of the MCMC estimates under the MH algorithm for the following parameter setting: $p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$ with censoring time: $T = 0.5$ (Case I) and Sample size $(n, m) = (50, 50)$.

These plots confirm that the posterior samples for all parameters are approximately normally distributed and exhibit good convergence behavior. Specifically, the trace plots show rapid convergence and excellent mixing around the mean, confirming that the Markov chains have quickly reached their stationary distribution. Furthermore, the histograms confirm that the posterior distributions are generally unimodal and near-symmetric.

Figure 1. Plot, Histogram, and Convergence of p_1 and φ_1 .

Figure 2. Plot, Histogram, and Convergence of α_1 and β_1 .Figure 3. Plot, Histogram, and Convergence of α_2 and β_2 .

4.2. Real Data

This section examines the actual dataset employed to demonstrate the applicability of the proposed methodology. These data, sourced from [64], reflect the breaking strength of jute fibers at two distinct gauge lengths: 10 mm and 20 mm. Each dataset has 30 measurements of fiber strength, as detailed below:

Dataset 1: (Gauge length 10 mm.):

43.93, 50.16, 101.15, 108.94, 123.06, 141.38, 151.48, 163.40, 177.25, 183.16, 212.13, 257.44, 262.90, 291.27, 303.90, 323.83, 353.24, 376.42, 383.43, 422.11, 506.60, 530.55, 590.48, 637.66, 671.49, 693.73, 700.74, 704.66, 727.23, 778.17.

Dataset 2: (Gauge length 20mm.):

36.75, 45.58, 48.01, 71.46, 83.55, 99.72, 113.85, 116.99, 119.86, 145.96, 166.49, 187.13, 187.85, 200.16, 244.53, 284.64, 350.70, 375.81, 419.02, 456.60, 547.44, 578.62, 581.60, 585.57, 594.29, 662.66, 688.16, 707.36, 756.70, 765.14.

The K–S goodness-of-fit test was employed to evaluate the appropriateness of the Weibull distribution for modeling these datasets.

- For Dataset 1, the calculated K–S statistic was 0.10601 with a p-value of 0.8539, based on MLE estimates $\hat{\alpha}_1 = 408.6115$ and $\hat{\beta}_1 = 1.6250$, indicating a good fit.
- For Dataset 2, the K–S statistic was 0.1490 with a p-value of 0.4733, with parameter estimates $\hat{\alpha}_2 = 372.1153$, $\hat{\beta}_2 = 1.3608$, also suggesting an acceptable fit to the Weibull model.

To proceed with the joint Type I censoring analysis, several censoring times were considered: $T = 75, 100, 200, 500, 600$. Assuming a mixture structure with proportions $p_1 = 0.4$ and $\varphi_1 = 0.4$, MLEs were attained concerning the parameters $p_1, \varphi_1, \alpha_1, \beta_1, \alpha_2, \beta_2$, along with their associated 95% asymptotic confidence intervals. For Bayesian analysis, estimation was performed using the Metropolis-Hastings (MH) algorithm under non-informative priors. Specifically, the prior settings were:

$$\text{prior } 0 = (0, 1, 0, 1), \quad \text{and} \quad \text{prior } 1 = \text{prior } 2 = (0, 0, 0, 0).$$

Initial values for the MH algorithm were defined for the MLEs of the parameters

$(p_1^{(0)}, \phi_1^{(0)}, \alpha_1^{(0)}, \beta_1^{(0)}, \alpha_2^{(0)}, \beta_2^{(0)}) = (\hat{p}_1, \hat{\phi}_1, \hat{\alpha}_1, \hat{\beta}_1, \hat{\alpha}_2, \hat{\beta}_2)$. The variance covariance matrix used corresponds to the log-transformed parameters:

$$\{\ln(\hat{p}_1), \ln(\hat{\phi}_1), \ln(\hat{\alpha}_1), \ln(\hat{\beta}_1), \ln(\hat{\alpha}_2), \ln(\hat{\beta}_2)\}$$

Totalling 10,000, the posterior samples were generated, with the first 2,000 discarded as burn-in. Bayes estimates were then computed in view of LINEX and SE loss functions (with $\nu = 0.5$). In addition, the intervals of HPD were obtained using the technique developed in 1999 by Chen and Shao. All MLE estimates, their asymptotic confidence intervals, and SE are presented in Table 2. Similarly, Bayesian estimates derived via MCMC under the two loss functions, along with corresponding HPD intervals and standard errors, are also reported for comparison. Simpler distributions can be used due to their fewer parameters; however, the mixed Weibull model was chosen to accommodate data variability. Many reliability datasets include units from multiple production batches or failure types, making the mixture model more representative of the actual failure process. This extended form ensures that the estimation process encompasses the characteristics of subcategories that a single-component distribution might miss.

Table 2. MLE's and BEs with associated St.Er and CIs based on different joint Type I censoring schemes for the given real data set and assume: $p_1 = 0.4, \phi_1 = 0.4$.

t	Param.	MLE			Bayesian MCMC			
		Estimate	St.Er	Asy-CI	Estimate		St.Er	HPD
					SE	LINEX		
75	p_1	0.5559	0.1656	(0.231, 0.88)	0.6095	0.6086	0.0588	(0.495, 0.728)
	ϕ_1	0.6001	0.2191	(0.171, 1.03)	0.3420	0.3400	0.0890	(0.224, 0.547)
	α_1	144.4020	14.5153	(15.953, 72.852)	60.5533	37.3235	15.2558	(29.759, 85.511)
	β_1	1.2514	0.6741	(-0.07, 2.573)	2.3081	2.1314	0.8295	(0.761, 3.585)
	α_2	61.5760	39.4368	(-15.718, 138.872)	50.1973	43.5442	6.6428	(38.144, 62.264)
	β_2	0.8243	0.3879	(0.131, 1.651)	2.1228	1.9278	0.9567	(0.158, 0.574)
100	p_1	0.5386	0.1383	(0.268, 0.81)	0.5382	0.5354	0.1052	(0.98, 4.021)
	ϕ_1	0.6000	0.2191	(0.171, 1.029)	0.4183	0.4157	0.1015	(0.367, 0.759)
	α_1	81.1787	11.5261	(58.588, 103.77)	82.8951	66.4982	12.9144	(60.458, 111.346)
	β_1	2.4984	0.7275	(1.072, 3.924)	2.2970	2.1956	0.6683	(1.269, 3.539)
	α_2	67.2083	8.2634	(51.012, 83.404)	67.3176	57.6319	6.5474	(54.447, 79.366)
	β_2	3.0560	0.8142	(1.46, 4.652)	2.9146	2.8607	0.4648	(2.024, 3.941)
200	p_1	0.5187	0.0962	(0.33, 0.707)	0.4193	0.4175	0.0871	(0.264, 0.593)
	ϕ_1	0.5238	0.1090	(0.31, 0.737)	0.5273	0.5261	0.0702	(0.391, 0.664)
	α_1	141.4614	12.7375	(116.496, 166.426)	147.0975	128.2391	13.3189	(120.178, 171.877)
	β_1	2.3702	0.4158	(1.555, 3.185)	2.4973	2.4675	0.3489	(1.877, 3.146)
	α_2	130.8472	10.9138	(109.457, 152.238)	131.4989	118.4924	9.7794	(112.539, 150.34)
	β_2	2.6250	0.4546	(1.734, 3.516)	2.3219	2.2914	0.3575	(1.647, 3.096)
500	p_1	0.5123	0.0781	(0.359, 0.665)	0.4507	0.4494	0.0704	(0.32, 0.566)
	ϕ_1	0.5121	0.0781	(0.359, 0.665)	0.5087	0.5074	0.0740	(0.366, 0.643)
	α_1	249.8774	25.3491	(200.194, 299.561)	251.1118	211.3942	23.6903	(204.829, 291.506)
	β_1	1.6160	0.2053	(1.214, 2.018)	1.5603	1.5472	0.2296	(1.144, 1.976)
	α_2	230.5810	21.5956	(188.255, 272.908)	224.1843	181.8581	20.7792	(178.776, 260.973)
	β_2	1.7803	0.2248	(1.34, 2.221)	1.7776	1.7510	0.3244	(1.131, 2.252)
600	p_1	0.5097	0.0700	(0.372, 0.647)	0.4782	0.4765	0.0823	(0.339, 0.638)
	ϕ_1	0.5106	0.0729	(0.368, 0.654)	0.5202	0.5194	0.0570	(0.422, 0.631)
	α_1	314.7314	31.7504	(252.502, 376.961)	304.0040	238.8596	42.8563	(232.221, 388.158)
	β_1	1.4878	0.1734	(1.148, 1.828)	1.3699	1.3621	0.1773	(1.075, 1.734)
	α_2	295.0771	28.7812	(238.667, 351.487)	302.1432	224.6807	31.0429	(241.172, 365.294)
	β_2	1.5612	0.1796	(1.209, 1.913)	1.5488	1.5439	0.1407	(1.308, 1.819)

5. Conclusions and Future Research Directions

In the current investigation, the MLE and Bayesian estimating methods were utilized in order to arrive at estimates of the unknown parameters of a mixed Weibull distribution that was subjected to a joint Type I censoring scheme. Moreover, asymmetric confidence intervals and variance-covariance matrices were constructed from the data presented. The primary objective of this comprehensive simulation study is to thoroughly investigate, how the proposed estimators performed statistically. Additionally, a dataset from the actual world was reanalyzed in order to validate the practical utility of the model. The results demonstrate that the estimators are not only accurate but also extremely dependable, which lends credence to the possibility that they could be utilized in dependability analysis and lifetime data modeling. Based on these findings, it appears that the model that was proposed has the potential to be applied as a trustworthy instrument in applications involving survival analysis and reliability engineering in the real world. Based on our simulation and real data results, we demonstrate that the Bayesian estimation method, particularly with the LINEX loss function, outperforms other estimation methods, especially when sample sizes are small. In the future, this model can be expanded to include different age distributions within the same censoring framework, such as mixed Lindley distributions and mixed generalized exponential distributions and other distributions, such as Kavya–Manoharan cosine inverse Rayleigh distribution and the Half-logistic Gaussian distribution. Additionally, using alternative censoring methods, such as progressive joint censoring, will provide a broader perspective on the model's performance under different levels of data knowledge. Furthermore, it is

useful to explore integrating shrinkage estimation methods and regression modeling within the Weibull mixture framework and to make a comparison between maximum likelihood estimation (MLE) and shrinkage estimation, as well as regression analysis across different types of data, including binary, beta, and standardized gamma data. refer to [64] to [68].

Appendix (A)

The components of the sample information matrix are computed in the following:

$$\begin{aligned}
-\frac{\partial^2 \ln L}{\partial p_1^2} &= \frac{-r_1}{p_1^2} - \frac{r_1}{(1-p_1)^2} - \frac{m-r_1}{p_1^2} - \frac{m-r_1}{(1-p_1)^2} \\
-\frac{\partial^2 \ln L}{\partial \varphi_1^2} &= \frac{-r_2}{\varphi_1^2} - \frac{r_2}{(1-\varphi_1)^2} - \frac{n-r_2}{\varphi_1^2} - \frac{n-r_2}{(1-\varphi_1)^2} \\
-\frac{\partial^2 \ln L}{\partial \alpha_1^2} &= \frac{\sum_{i=1}^{r_1} z_i}{\hat{\alpha}_1^2} + 2 \sum_{i=1}^{r_1} z_i (\beta_1 - 1) \left(\frac{w_{1i}}{\hat{\alpha}_1^3} \right) - \beta_1 (\beta_1 - 1) (\hat{\alpha}_1)^{-\beta_1-2} \sum_{i=1}^{r_1} z_i (w_{1i})^{\beta_1} \\
&+ \frac{\sum_{i=1}^{r_2} (1-z_i)}{\hat{\alpha}_1^2} + 2 \sum_{i=1}^{r_2} (1-z_i) (\beta_1 - 1) \left(\frac{w_{1i}}{\hat{\alpha}_1^3} \right) + \beta_1 (\beta_1 - 1) (\hat{\alpha}_1)^{-\beta_1-2} \sum_{i=1}^{r_2} (1-z_i) (w_{1i})^{\beta_1} \\
&+ (m-r_1) \beta_1 (\beta_1 - 1) (\hat{\alpha}_1)^{-\beta_1-2} (T)^{\beta_1} + (n-r_2) \beta_1 (\beta_1 - 1) (\hat{\alpha}_1)^{-\beta_1-2} (T)^{\beta_1} \\
-\frac{\partial^2 \ln L}{\partial \beta_1^2} &= \frac{-\sum_{i=1}^{r_1} z_i}{\beta_1^2} + \sum_{i=1}^{r_1} z_i \left(\frac{w_{1i}}{\alpha_1} \right)^{\beta_1} \ln \left(\frac{w_{1i}}{\alpha_1} \right) \ln \left(\frac{w_{1i}}{\alpha_1} \right) - \frac{\sum_{i=1}^{r_2} (1-z_i)}{\beta_1^2} \\
&+ \sum_{i=1}^{r_2} (1-z_i) \left(\frac{w_{1i}}{\alpha_1} \right)^{\beta_1} \ln \left(\frac{w_{1i}}{\alpha_1} \right) \ln \left(\frac{w_{1i}}{\alpha_1} \right) \\
&- \left(\frac{T}{\alpha_1} \right)^{\beta_1} \ln \left(\frac{T}{\alpha_1} \right) \ln \left(\frac{T}{\alpha_1} \right) [(m-r_1) + (n-r_2)] \\
-\frac{\partial^2 \ln L}{\partial \alpha_2^2} &= \frac{\sum_{i=1}^{r_1} z_i}{\hat{\alpha}_2^2} + 2 \sum_{i=1}^{r_1} z_i (\beta_2 - 1) \left(\frac{w_{2i}}{\hat{\alpha}_2^3} \right) - \beta_2 (\beta_2 - 1) (\hat{\alpha}_2)^{-\beta_2-2} \sum_{i=1}^{r_1} z_i (w_{2i})^{\beta_2} \\
&+ \frac{\sum_{i=1}^{r_2} (1-z_i)}{\hat{\alpha}_2^2} + 2 \sum_{i=1}^{r_2} (1-z_i) (\beta_2 - 1) \left(\frac{w_{2i}}{\hat{\alpha}_2^3} \right) + \beta_2 (\beta_2 - 1) (\hat{\alpha}_2)^{-\beta_2-2} \sum_{i=1}^{r_2} (1-z_i) (w_{2i})^{\beta_2} \\
&+ (m-r_1) \beta_2 (\beta_2 - 1) (\hat{\alpha}_2)^{-\beta_2-2} (T)^{\beta_2} + (n-r_2) \beta_2 (\beta_2 - 1) (\hat{\alpha}_2)^{-\beta_2-2} (T)^{\beta_2} \\
-\frac{\partial^2 \ln L}{\partial \beta_2^2} &= \frac{-\sum_{i=1}^{r_1} z_i}{\beta_2^2} + \sum_{i=1}^{r_1} z_i \left(\frac{w_{2i}}{\alpha_2} \right)^{\beta_2} \ln \left(\frac{w_{2i}}{\alpha_2} \right) \ln \left(\frac{w_{2i}}{\alpha_2} \right) - \frac{\sum_{i=1}^{r_2} (1-z_i)}{\beta_2^2} \\
&+ \sum_{i=1}^{r_2} (1-z_i) \left(\frac{w_{2i}}{\alpha_2} \right)^{\beta_2} \ln \left(\frac{w_{2i}}{\alpha_2} \right) \ln \left(\frac{w_{2i}}{\alpha_2} \right) \\
&- \left(\frac{T}{\alpha_2} \right)^{\beta_2} \ln \left(\frac{T}{\alpha_2} \right) \ln \left(\frac{T}{\alpha_2} \right) [(m-r_1) + (n-r_2)]
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \beta_1 \partial \alpha_1} &= -\sum_{i=1}^{r_1} z_i \left(\frac{w_{1i}}{\alpha_1^2} \right) - \beta_1 \sum_{i=1}^{r_1} z_i (w_{1i})^{\beta_1} \alpha_1^{-\beta_1-1} - \left(\frac{w_{1i}}{\alpha_1^2} \right) + \sum_{i=1}^{r_1} (1-z_i) \left(\frac{w_{1i}}{\alpha_1^2} \right) \\
&+ \beta_1 \sum_{i=1}^{r_1} (1-z_i) (w_{1i})^{\beta_1} \alpha_1^{-\beta_1-1} - \left(\frac{w_{1i}}{\alpha_1^2} \right) + (m-r_1) \beta_1 (\alpha_1)^{-\beta_1-1} (T)^{\beta_1} - \left(\frac{w_{1i}}{\alpha_1^2} \right) \\
&+ (n-r_2) \beta_1 (\alpha_1)^{-\beta_1-1} (T)^{\beta_1} - \left(\frac{w_{1i}}{\alpha_1^2} \right)
\end{aligned}$$

$$\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \beta_2 \partial \alpha_2} &= -\sum_{i=1}^{r_1} z_i \left(\frac{w_{2i}}{\alpha_2^2} \right) - \beta_2 \sum_{i=1}^{r_1} z_i (w_{2i})^{\beta_2} \alpha_2^{-\beta_2-1} - \left(\frac{w_{2i}}{\alpha_2^2} \right) + \sum_{i=1}^{r_1} (1-z_i) \left(\frac{w_{2i}}{\alpha_2^2} \right) \\
&+ \beta_2 \sum_{i=1}^{r_1} (1-z_i) (w_{2i})^{\beta_2} \alpha_2^{-\beta_2-1} - \left(\frac{w_{2i}}{\alpha_2^2} \right) + (m-r_1) \beta_2 (\alpha_2)^{-\beta_2-1} (T)^{\beta_2} - \left(\frac{w_{2i}}{\alpha_2^2} \right) \\
&+ (n-r_2) \beta_2 (\alpha_2)^{-\beta_2-1} (T)^{\beta_2} - \left(\frac{w_{2i}}{\alpha_2^2} \right)
\end{aligned}$$

$$\frac{\partial^2 \ln l}{\partial p_1 \partial \varphi_1} = \frac{\partial^2 \ln l}{\partial p_1 \partial \beta_1} = \frac{\partial^2 \ln l}{\partial p_1 \partial \beta_2} = \frac{\partial^2 \ln l}{\partial p_1 \partial \alpha_1} = \frac{\partial^2 \ln l}{\partial p_1 \partial \alpha_2} = 0$$

$$\frac{\partial^2 \ln l}{\partial \varphi_1 \partial \beta_1} = \frac{\partial^2 \ln l}{\partial \varphi_1 \partial \beta_2} = \frac{\partial^2 \ln l}{\partial \varphi_1 \partial \alpha_1} = \frac{\partial^2 \ln l}{\partial \varphi_1 \partial \alpha_2} = 0$$

$$\frac{\partial^2 \ln l}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 \ln l}{\partial \beta_1 \partial \alpha_2} = \frac{\partial^2 \ln l}{\partial \beta_2 \partial \alpha_1} = \frac{\partial^2 \ln l}{\partial \alpha_1 \partial \alpha_2} = 0$$

Appendix (B)

Table 3. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$) under joint Type I censoring scheme.

(m,n)	t	Parm.	MLE		Bayesian MCMC-SE		Bayesian MCMC-LINEX	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(15,15)	0.50	p_1	0.5301	0.0170	0.8931	1.3219	0.6973	0.5399
		ϕ_1	0.5335	0.0045	1.1911	3.0097	0.7543	0.8188
		α_1	0.3530	0.0228	0.6134	0.8305	0.4004	0.0835
		β_1	2.0708	0.6416	2.2792	9.8621	1.1429	0.8031
		α_2	0.3086	0.1963	0.4537	0.4187	0.3551	0.2085
		β_2	2.4055	0.9141	2.7848	13.4758	1.2505	1.1645
	0.75	p_1	0.5154	0.0134	0.5608	0.1813	0.5324	0.1146
		ϕ_1	0.5187	0.0067	0.5597	0.1633	0.5244	0.0718
		α_1	0.4738	0.0026	0.5124	0.1629	0.4748	0.0179
		β_1	1.8768	0.2709	1.7367	1.5924	1.4889	0.2674
		α_2	0.4272	1.1728	0.4656	0.2730	0.4299	0.1323
		β_2	2.1116	0.3351	2.0358	3.4960	1.6682	0.5295
(25,25)	0.50	p_1	0.5189	0.0142	0.5503	0.1683	0.5089	0.0461
		ϕ_1	0.5205	0.0063	0.5726	0.2510	0.5117	0.0314
		α_1	0.3374	0.0271	0.3433	0.0338	0.3374	0.0307
		β_1	2.1068	0.5184	1.9840	2.3693	1.6790	0.1889
		α_2	0.3109	0.1936	0.3245	0.2277	0.3146	0.1969
		β_2	2.3166	0.5209	2.1564	1.9130	1.8723	0.5493
	0.75	p_1	0.5121	0.0126	0.4996	0.0100	0.4933	0.0088
		ϕ_1	0.5135	0.0075	0.5000	0.0101	0.4934	0.0115
		α_1	0.4570	0.0032	0.4548	0.0034	0.4528	0.0036
		β_1	1.9261	0.2639	1.7682	0.1326	1.7087	0.0968
		α_2	0.4282	0.1050	0.4262	0.1063	0.4246	0.1073
		β_2	2.0798	0.2121	1.9164	0.1045	1.8477	0.0764
(50,50)	0.50	p_1	0.5092	0.0119	0.5000	0.0100	0.4955	0.0092
		ϕ_1	0.5119	0.0077	0.4997	0.0101	0.4946	0.0111
		α_1	0.3449	0.0241	0.3232	0.0316	0.3226	0.0318
		β_1	2.3909	0.8426	2.0047	0.3231	1.9481	0.2623
		α_2	0.3322	0.1746	0.3100	0.1939	0.3096	0.1943
		β_2	2.5111	0.6378	2.1170	0.2154	2.0549	0.1649
	0.75	p_1	0.5065	0.0114	0.4998	0.0100	0.4966	0.0094
		ϕ_1	0.5069	0.0087	0.4994	0.0102	0.4960	0.0109
		α_1	0.4427	0.0039	0.4410	0.0041	0.4401	0.0042
		β_1	1.7596	0.2398	1.8642	0.1671	1.8315	0.1421
		α_2	1.9468	0.1042	0.4273	0.1048	0.4266	0.1053
		β_2	0.4282	0.1220	1.9473	0.0783	1.9120	0.0629

Table 4. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$) under joint Type I censoring scheme.

(m,n)	t	Parm.	MLE		Bayesian MCMC-SE		Bayesian MCMC-LINEX	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(15,15)	0.50	p_1	0.4289	0.0306	1.0434	2.4239	0.6584	0.5839
		ϕ_1	0.4419	0.0028	0.8240	1.1794	0.5791	0.2926
		α_1	0.4085	0.0109	0.7704	1.0845	0.5117	0.1687
		β_1	1.9559	0.5149	2.2792	8.2624	1.1658	0.8495
		α_2	0.5457	0.0546	0.9242	2.1520	0.5273	0.2170
	0.75	β_2	1.7287	0.1892	2.4298	12.9256	1.1358	1.0741
		p_1	0.4601	0.0201	0.5509	0.2438	0.4873	0.0700
		ϕ_1	0.4662	0.0048	0.5391	0.2109	0.4927	0.0847
		α_1	0.4885	0.0033	0.5854	0.2635	0.5299	0.0422
		β_1	1.8573	0.2726	1.6804	0.9312	1.4217	0.1147
(25,25)	0.50	α_2	0.5632	0.0409	0.6075	0.4018	0.5276	0.0693
		β_2	1.6918	0.1067	1.6663	0.9876	1.4416	0.3943
		p_1	0.4322	0.0289	0.5567	0.3490	0.4986	0.1461
		ϕ_1	0.4450	0.0025	0.5358	0.1772	0.4933	0.0887
		α_1	0.4138	0.0088	0.5525	0.2714	0.4768	0.0375
		β_1	1.8685	0.2820	1.6962	0.9478	1.4712	0.3036
		α_2	0.5326	0.0530	0.5833	0.5212	0.4981	0.1114
		β_2	1.6921	0.1128	1.8400	3.5467	1.4970	1.1652
	0.75	p_1	0.4603	0.0198	0.4841	0.0137	0.4758	0.0157
		ϕ_1	0.4685	0.0049	0.4854	0.0075	0.4783	0.0063
		α_1	0.4947	0.0019	0.5262	0.0029	0.5196	0.0024
		β_1	1.8074	0.1674	1.6298	0.0670	1.5653	0.0467
		α_2	0.5564	0.0409	0.5328	0.0497	0.5263	0.0524
		β_2	1.6791	0.0643	1.6208	0.0672	1.5578	0.0786
(50,50)	0.50	p_1	0.4348	0.0276	0.4777	0.0159	0.4681	0.0184
		ϕ_1	0.4460	0.0023	0.4801	0.0071	0.4723	0.0059
		α_1	0.4187	0.0071	0.4667	0.0028	0.4593	0.0031
		β_1	1.7904	0.1371	1.6509	0.0650	1.5975	0.0468
		α_2	0.5247	0.0531	0.4950	0.0676	0.4871	0.0714
		β_2	1.6477	0.0560	1.6206	0.0576	1.5709	0.0685
	0.75	p_1	0.5069	0.0087	0.5000	0.0101	0.4966	0.0108
		ϕ_1	0.5066	0.0114	0.5001	0.0101	0.4970	0.0094
		α_1	0.4412	0.0041	0.4396	0.0043	0.4388	0.0044
		β_1	1.9529	0.2480	1.8700	0.1736	1.8371	0.1479
		α_2	0.4267	0.1052	0.4258	0.1058	0.4251	0.1063
		β_2	2.0348	0.1301	1.9548	0.0841	1.9191	0.0678

Table 5. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1, \alpha_2 = 0.75, \beta_2 = 1$) under joint Type I censoring scheme.

(m,n)	t	Parm.	MLE		Bayesian MCMC-SE		Bayesian MCMC-LINEX	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(15,15)	0.40	p_1	0.4609	0.0043	0.9832	1.7100	0.6885	0.5300
		ϕ_1	0.4562	0.0214	1.2637	3.5916	0.7907	1.0231
		α_1	0.3437	0.0291	0.8407	1.4597	0.5313	0.3929
		β_1	1.2098	0.1612	1.9264	7.2761	0.9287	1.1048
		α_2	0.4694	0.1030	0.8348	2.2000	0.4937	0.6772
		β_2	1.1190	0.0970	1.7421	6.3728	0.9012	0.8420
	1.00	p_1	0.4808	0.0067	0.4929	0.0295	0.4804	0.0284
		ϕ_1	0.4767	0.0154	0.5341	0.2832	0.4787	0.0374
		α_1	0.4711	0.0080	0.5382	0.0738	0.4970	0.0393
		β_1	1.1601	0.0721	1.1084	0.6071	1.0044	0.0506
		α_2	0.5377	0.0568	0.5345	0.1952	0.5045	0.1663
		β_2	1.0916	0.0494	1.0689	0.2346	1.0111	0.1434
(25,25)	0.40	p_1	0.4638	0.0043	0.5841	0.3513	0.5340	0.1760
		ϕ_1	0.4595	0.0200	0.6474	0.6816	0.5262	0.1639
		α_1	0.3474	0.0257	0.5018	0.3130	0.4045	0.0870
		β_1	1.1636	0.0790	1.2583	1.4618	1.0107	0.2717
		α_2	0.4459	0.1027	0.5417	0.5679	0.4174	0.1688
		β_2	1.1032	0.0573	1.2222	1.4256	0.9973	0.1648
	1.00	p_1	0.4808	0.0066	0.4915	0.0189	0.4817	0.0071
		ϕ_1	0.4773	0.0152	0.4891	0.0140	0.4808	0.0145
		α_1	0.4750	0.0046	0.5068	0.0044	0.4955	0.0040
		β_1	1.1290	0.0416	1.0610	0.0237	1.0349	0.0197
		α_2	0.5283	0.0553	0.5079	0.0636	0.4977	0.0699
		β_2	1.0763	0.0280	1.0600	0.0314	1.0323	0.0190
(50,50)	0.40	p_1	0.4660	0.0045	0.4796	0.0080	0.4744	0.0233
		ϕ_1	0.4607	0.0195	0.4963	0.1415	0.4694	0.0214
		α_1	0.3487	0.0242	0.3912	0.0160	0.3836	0.0614
		β_1	1.1301	0.0395	1.1151	0.3036	1.0595	0.0279
		α_2	0.4357	0.1031	0.4286	0.1443	0.4089	0.1242
		β_2	1.0774	0.0265	1.0696	0.0388	1.0498	0.0505
	1.00	p_1	0.4814	0.0067	0.4941	0.0092	0.4901	0.0082
		ϕ_1	0.4779	0.0150	0.4953	0.0131	0.4907	0.0126
		α_1	0.4806	0.0021	0.5034	0.0031	0.4961	0.0021
		β_1	1.1035	0.0217	1.0641	0.0159	1.0518	0.0143
		α_2	0.5196	0.0558	0.5026	0.0634	0.5041	0.0914
		β_2	1.0668	0.0148	1.0644	0.0142	1.0506	0.0129

Table 6. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1, \alpha_2 = 0.75, \beta_2 = 1$) under joint Type I censoring scheme.

(m,n)	t	Parm.	MLE		Bayesian MCMC-SE		Bayesian MCMC-LINEX	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(15,15)	0.40	p_1	0.4570	0.0212	1.1653	3.0411	0.7127	0.6487
		ϕ_1	0.4631	0.0045	0.9260	1.4978	0.6982	0.6045
		α_1	0.3403	0.0300	0.7724	1.3833	0.5225	0.4086
		β_1	1.2089	0.1465	1.7185	5.4376	0.8661	0.4973
		α_2	0.4555	0.1099	0.8699	2.6919	0.4389	0.3122
		β_2	1.1311	0.1018	1.8020	7.0747	0.9383	1.0285
	1.00	p_1	0.4777	0.0152	0.5034	0.0808	0.5006	0.1377
		ϕ_1	0.4798	0.0066	0.5150	0.0920	0.4756	0.0132
		α_1	0.4685	0.0078	0.5560	0.2062	0.4943	0.0129
		β_1	1.1610	0.0706	1.0850	0.3451	1.0042	0.0661
		α_2	0.5347	0.0580	0.5465	0.3269	0.4944	0.0890
		β_2	1.0920	0.0472	1.0755	0.1965	1.0016	0.0416
(25,25)	0.40	p_1	0.4603	0.0198	0.6767	0.7830	0.5296	0.1661
		ϕ_1	0.4638	0.0043	0.5919	0.3455	0.5211	0.1462
		α_1	0.3462	0.0262	0.5243	0.4295	0.4114	0.0971
		β_1	1.1606	0.0783	1.2574	1.5530	1.0368	0.5231
		α_2	0.4442	0.1037	0.5593	0.7281	0.4233	0.2605
		β_2	1.1005	0.0568	1.3183	1.8727	0.9956	0.2128
	1.00	p_1	0.4775	0.0151	0.4930	0.0591	0.4796	0.0151
		ϕ_1	0.4808	0.0066	0.4896	0.0146	0.4809	0.0069
		α_1	0.4747	0.0046	0.5065	0.0056	0.4939	0.0042
		β_1	1.1215	0.0400	1.0656	0.1442	1.0404	0.1068
		α_2	0.5263	0.0565	0.5051	0.0652	0.5019	0.1331
		β_2	1.0707	0.0261	1.0592	0.1185	1.0260	0.0192
(50,50)	0.40	p_1	0.4614	0.0193	0.4841	0.0317	0.4787	0.0426
		ϕ_1	0.4654	0.0044	0.4831	0.0280	0.4761	0.0259
		α_1	0.3497	0.0238	0.4012	0.0644	0.3867	0.0307
		β_1	1.1228	0.0376	1.0798	0.0311	1.0583	0.0403
		α_2	0.4373	0.1023	0.4253	0.1172	0.4118	0.1278
		β_2	1.0714	0.0258	1.0735	0.1168	1.0431	0.0510
	1.00	p_1	0.4785	0.0148	0.4949	0.0147	0.4895	0.0127
		ϕ_1	0.4815	0.0067	0.4955	0.0136	0.4909	0.0093
		α_1	0.4780	0.0024	0.4992	0.0024	0.4927	0.0024
		β_1	1.1024	0.0210	1.0688	0.0334	1.0515	0.0130
		α_2	0.5160	0.0578	0.5010	0.0690	0.4931	0.0684
		β_2	1.0668	0.0146	1.0660	0.0200	1.0562	0.0353

Table 7. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.5, \beta_2 = 1.75$) under joint Type I censoring scheme.

(m,n)	t	Parm.	MLE		Bayesian MCMC-SE		Bayesian MCMC-LINEX	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(15,15)	0.30	p_1	0.4840	0.0078	1.0158	1.8637	0.7135	0.7624
		ϕ_1	0.4836	0.0137	1.3966	4.2894	0.8381	1.5007
		α_1	0.3616	0.0251	0.8643	1.6846	0.5368	0.4680
		β_1	1.9366	0.6682	2.6281	15.8194	1.0997	2.9062
		α_2	0.3866	0.0233	0.8333	1.5087	0.5039	0.3093
		β_2	1.9093	0.4867	3.2152	25.0402	1.1181	4.2993
	0.60	p_1	0.5033	0.0109	0.5365	0.1740	0.4933	0.0445
		ϕ_1	0.5054	0.0096	0.6288	0.7353	0.5156	0.1243
		α_1	0.4316	0.0070	0.4651	0.1002	0.4355	0.0463
		β_1	1.8429	0.2437	1.7886	1.9851	1.4978	0.1703
		α_2	0.4216	0.0082	0.4813	0.1706	0.4385	0.0379
		β_2	1.8757	0.1481	1.8307	2.6555	1.4942	0.3841
(25,25)	0.30	p_1	0.4859	0.0074	0.7758	0.9582	0.6082	0.3697
		ϕ_1	0.4850	0.0133	1.0194	2.3761	0.6094	0.4187
		α_1	0.3572	0.0233	0.7394	1.1959	0.4663	0.2070
		β_1	1.8664	0.3662	2.5350	11.8253	1.1667	0.5478
		α_2	0.3774	0.0197	0.5990	0.7542	0.4150	0.0912
		β_2	1.8427	0.2342	2.8228	19.2128	1.2319	1.6296
	0.60	p_1	0.5029	0.0106	0.4898	0.0272	0.4782	0.0065
		ϕ_1	0.5028	0.0094	0.4853	0.0134	0.4784	0.0153
		α_1	0.4289	0.0064	0.4279	0.0066	0.4237	0.0072
		β_1	1.7977	0.1585	1.6913	0.0927	1.6229	0.0638
		α_2	0.4234	0.0072	0.4322	0.0063	0.4282	0.0068
		β_2	1.8154	0.0785	1.6849	0.0591	1.6174	0.0668
(50,50)	0.30	p_1	0.5141	0.0130	0.5064	0.0253	0.4960	0.0111
		ϕ_1	0.5144	0.0073	0.5032	0.0117	0.4942	0.0118
		α_1	0.2079	0.0855	0.2076	0.0858	0.2072	0.0860
		β_1	2.3160	0.8215	2.0816	0.4588	1.9880	0.3509
		α_2	0.1971	0.0919	0.1961	0.0926	0.1958	0.0927
		β_2	2.4724	0.7131	2.2346	0.3750	2.1265	0.2640
	0.60	p_1	0.5028	0.0106	0.4922	0.0086	0.4882	0.0079
		ϕ_1	0.5027	0.0095	0.4921	0.0117	0.4881	0.0126
		α_1	0.4279	0.0058	0.4259	0.0061	0.4240	0.0063
		β_1	1.7831	0.1100	1.7347	0.0820	1.6980	0.0641
		α_2	0.4244	0.0063	0.4293	0.0057	0.4273	0.0059
		β_2	1.7928	0.0323	1.7296	0.0277	1.6934	0.0284

Table 8. Avg. estimated values and MSEs of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.5, \beta_2 = 1.75$) under joint Type I censoring scheme.

(m,n)	t	Parm.	MLE		Bayesian MCMC-SE		Bayesian MCMC-LINEX	
			Avg.	MSE	Avg.	MSE	Avg.	MSE
(15,15)	0.30	p_1	0.4837	0.0137	1.4466	4.6815	0.8253	1.2840
		ϕ_1	0.4837	0.0071	1.0606	2.0398	0.7699	0.8901
		α_1	0.3623	0.0241	0.9327	2.0947	0.5626	0.4736
		β_1	1.9313	0.6869	2.7546	17.5759	1.0382	2.3465
		α_2	0.3881	0.0214	0.8390	1.6725	0.4992	0.3077
		β_2	1.8990	0.4927	2.8973	22.0151	1.1235	4.4278
	0.60	p_1	0.5030	0.0094	0.5506	0.2201	0.4981	0.1146
		ϕ_1	0.5031	0.0106	0.5481	0.2306	0.5005	0.0910
		α_1	0.4362	0.0062	0.4879	0.1654	0.4429	0.0467
		β_1	1.8433	0.2475	1.7613	1.3030	1.5021	0.4600
		α_2	0.4262	0.0076	0.4782	0.1084	0.4403	0.0256
		β_2	1.8840	0.1823	1.7133	0.7347	1.4771	0.2138
(25,25)	0.30	p_1	0.4850	0.0133	0.9358	1.9557	0.6518	0.6765
		ϕ_1	0.4858	0.0074	0.8172	1.2304	0.5984	0.3510
		α_1	0.3555	0.0236	0.7342	1.3264	0.4566	0.1899
		β_1	1.8498	0.3511	2.2724	9.1178	1.2745	1.6870
		α_2	0.3762	0.0201	0.6108	0.7087	0.4256	0.1216
		β_2	1.8259	0.2283	2.3259	10.4124	1.2496	1.6407
	0.60	p_1	0.5028	0.0092	0.4854	0.0138	0.4773	0.0153
		ϕ_1	0.5029	0.0106	0.4866	0.0119	0.4772	0.0062
		α_1	0.4280	0.0064	0.4270	0.0073	0.4222	0.0072
		β_1	1.7919	0.1570	1.6937	0.1379	1.6196	0.0630
		α_2	0.4235	0.0070	0.4324	0.0062	0.4281	0.0066
		β_2	1.8045	0.0771	1.6866	0.1283	1.6122	0.0682
(50,50)	0.30	p_1	0.4863	0.0129	0.5411	0.3965	0.4870	0.1876
		ϕ_1	0.4867	0.0075	0.5086	0.1653	0.4702	0.0691
		α_1	0.3548	0.0223	0.3920	0.1227	0.3543	0.0350
		β_1	1.7638	0.1722	1.7887	1.5371	1.5477	0.1300
		α_2	0.3728	0.0180	0.4239	0.0893	0.4016	0.0487
		β_2	1.7430	0.1003	1.7463	2.1495	1.5150	0.4293
	0.60	p_1	0.5028	0.0090	0.4915	0.0119	0.4875	0.0127
		ϕ_1	0.5028	0.0102	0.4914	0.0085	0.4874	0.0077
		α_1	0.4291	0.0056	0.4267	0.0060	0.4248	0.0063
		β_1	1.7824	0.1126	1.7348	0.0847	1.6979	0.0663
		α_2	0.4256	0.0061	0.4309	0.0055	0.4290	0.0057
		β_2	1.7919	0.0353	1.7272	0.0300	1.6912	0.0307

Table 9. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$) under joint Type I censoring scheme.

(m,n)	t	Parm.	Asy-CI		HPD	
			AIL	CP	AIL	CP
(15,15)	0.50	p_1	0.0399	96.1	3.5413	95.1
		ϕ_1	0.0429	94.5	5.2347	95.1
		α_1	0.1333	97.7	2.0148	95.1
		β_1	2.2039	95.7	7.3343	95.0
		α_2	0.1531	97.5	1.2005	95.0
		β_2	2.7296	95.5	10.8372	95.0
	0.75	p_1	0.0405	99.9	0.4749	96.6
		ϕ_1	0.0377	98.6	0.4187	95.6
		α_1	0.1698	97.9	0.2968	96.7
		β_1	1.4079	96.9	1.4091	96.2
		α_2	0.1842	97.9	0.3243	96.8
		β_2	1.6891	96.6	1.8496	97.4
(25,25)	0.50	p_1	0.0155	96.4	0.1308	95.9
		ϕ_1	0.0172	93.3	0.2397	96.1
		α_1	0.1051	97.9	0.1584	97.1
		β_1	1.5199	96.1	1.3921	96.3
		α_2	0.1128	98.5	0.1468	97.1
		β_2	1.7535	95.9	1.5374	96.2
	0.75	p_1	0.0129	99.9	0.0398	97.9
		ϕ_1	0.0098	96.9	0.0405	97.9
		α_1	0.1424	97.8	0.1426	96.5
		β_1	1.1256	97.0	0.9354	96.2
		α_2	0.1488	97.8	0.1471	96.8
		β_2	1.2608	97.2	1.0507	97.0
(50,50)	0.50	p_1	0.0074	100.0	0.0343	98.1
		ϕ_1	0.0061	100.0	0.0349	97.6
		α_1	0.0347	100.0	0.0761	98.1
		β_1	0.9697	100.0	0.9766	96.4
		α_2	0.0368	100.0	0.0759	96.8
		β_2	1.0609	100.0	1.0557	96.5
	0.75	p_1	0.0021	96.3	0.0281	96.5
		ϕ_1	0.0024	96.1	0.0300	97.9
		α_1	0.0999	97.4	0.1005	98.0
		β_1	0.7856	97.2	0.7036	96.7
		α_2	0.1022	97.3	0.1028	97.9
		β_2	0.8368	97.1	0.7524	97.2

Table 10. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.75, \beta_2 = 1.75$) under joint Type I censoring scheme.

(m,n)	t	Parm.	Asy-CI		HPD	
			AIL	CP	AIL	CP
(15,15)	0.50	p_1	0.1452	100.0	4.3932	95.1
		ϕ_1	0.1258	100.0	3.2228	95.0
		α_1	0.1985	98.6	2.8391	95.1
		β_1	2.1732	96.0	6.9158	95.1
		α_2	0.4441	96.5	3.5804	95.1
		β_2	1.7038	95.6	8.5531	95.1
	0.75	p_1	0.0924	100.0	0.4179	95.8
		ϕ_1	0.0765	100.0	0.3488	96.5
		α_1	0.2195	97.8	0.4581	96.3
		β_1	1.4932	95.5	1.3415	96.4
		α_2	0.3054	96.7	0.4445	96.3
		β_2	1.2604	95.8	1.1980	96.5
(25,25)	0.50	p_1	0.1070	99.9	0.6015	95.1
		ϕ_1	0.0877	100.0	0.4752	95.7
		α_1	0.1465	97.8	0.6461	95.9
		β_1	1.4996	96.0	1.5104	97.3
		α_2	0.2973	95.8	0.5566	96.9
		β_2	1.2975	95.9	1.5669	95.9
	0.75	p_1	0.0691	99.6	0.0581	98.6
		ϕ_1	0.0564	99.9	0.0518	97.9
		α_1	0.1686	97.6	0.1735	97.2
		β_1	1.0589	96.2	0.8664	95.5
		α_2	0.2281	95.9	0.1943	95.8
		β_2	0.9550	96.3	0.8581	95.7
(50,50)	0.50	p_1	0.0701	99.3	0.1158	98.1
		ϕ_1	0.0578	99.6	0.1006	97.8
		α_1	0.0897	97.4	0.1610	97.3
		β_1	0.9012	96.2	0.7983	96.3
		α_2	0.1902	96.1	0.1944	96.6
		β_2	0.8373	96.3	0.7829	96.2
	0.75	p_1	0.0026	96.1	0.0289	97.5
		ϕ_1	0.0022	95.9	0.0263	97.5
		α_1	0.1000	97.5	0.0991	96.3
		β_1	0.8120	96.9	0.7135	96.3
		α_2	0.1023	97.6	0.1021	96.9
		β_2	0.8680	97.1	0.7544	96.7

Table 11. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1, \alpha_2 = 0.75, \beta_2 = 1$) under joint Type I censoring scheme.

(m,n)	t	Parm.	Asy-CI		HPD	
			AIL	CP	AIL	CP
(15,15)	0.40	p_1	0.0949	100.0	3.9037	95.1
		ϕ_1	0.1042	100.0	5.9966	95.1
		α_1	0.2694	97.5	3.5171	95.1
		β_1	1.3424	95.6	8.4992	95.0
		α_2	0.6113	96.5	3.5889	95.1
		β_2	1.1284	95.4	7.9681	95.0
	1.00	p_1	0.0520	100.0	0.0795	97.3
		ϕ_1	0.0573	100.0	0.0829	96.1
		α_1	0.3321	96.9	0.3940	96.7
		β_1	0.8454	96.9	0.7287	96.5
		α_2	0.4254	96.4	0.3860	96.6
		β_2	0.7723	96.3	0.7251	97.3
(25,25)	0.40	p_1	0.0621	100.0	0.9928	95.3
		ϕ_1	0.0685	100.0	1.3451	95.6
		α_1	0.1947	96.4	1.0156	95.2
		β_1	0.8963	96.3	1.7258	95.0
		α_2	0.3963	95.4	0.9049	95.0
		β_2	0.8722	96.2	0.9233	96.3
	1.00	p_1	0.0383	99.5	0.0354	98.5
		ϕ_1	0.0424	99.8	0.0357	97.7
		α_1	0.2479	97.4	0.2505	97.6
		β_1	0.6196	96.2	0.5395	96.6
		α_2	0.3065	96.2	0.2709	96.5
		β_2	0.5712	96.1	0.5214	95.8
(50,50)	0.40	p_1	0.0379	99.5	0.1046	97.6
		ϕ_1	0.0420	99.6	0.1180	97.2
		α_1	0.1407	96.6	0.2191	96.4
		β_1	0.5889	95.8	0.5697	98.0
		α_2	0.2585	96.1	0.2544	95.9
		β_2	0.5232	96.7	0.4901	96.7
	1.00	p_1	0.0262	99.0	0.0264	98.5
		ϕ_1	0.0292	99.6	0.0261	97.8
		α_1	0.1645	96.9	0.1790	98.1
		β_1	0.4102	96.7	0.3827	97.6
		α_2	0.2054	96.6	0.1886	96.5
		β_2	0.3978	96.7	0.3848	97.3

Table 12. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1, \alpha_2 = 0.75, \beta_2 = 1$) under joint Type I censoring scheme.

(m,n)	t	Parm.	Asy-CI		HPD	
			AIL	CP	AIL	CP
(15,15)	0.40	p_1	0.1081	100.0	5.1381	95.0
		ϕ_1	0.0875	100.0	3.6833	95.0
		α_1	0.2632	96.6	3.0474	95.1
		β_1	1.2579	95.9	6.4732	95.1
		α_2	0.5973	96.9	3.9126	95.0
		β_2	1.1408	96.0	8.2214	95.0
	1.00	p_1	0.0566	100.0	0.0740	96.7
		ϕ_1	0.0538	100.0	0.0815	96.3
		α_1	0.3241	96.9	0.3855	97.3
		β_1	0.8286	96.5	0.7171	97.5
		α_2	0.4233	96.6	0.3704	97.3
		β_2	0.7723	96.3	0.7251	97.3
(25,25)	0.40	p_1	0.0665	100.0	1.7350	95.2
		ϕ_1	0.0616	100.0	1.2001	95.1
		α_1	0.1967	95.8	1.0699	95.1
		β_1	0.8986	95.8	1.8756	95.1
		α_2	0.3967	95.4	0.8585	95.1
		β_2	0.8473	95.7	2.5094	95.0
	1.00	p_1	0.0434	99.8	0.0398	97.8
		ϕ_1	0.0387	100.0	0.0358	97.0
		α_1	0.2460	97.4	0.2618	97.0
		β_1	0.6229	96.0	0.5276	96.3
		α_2	0.3150	96.7	0.2731	96.4
		β_2	0.5693	96.7	0.5407	95.7
(50,50)	0.40	p_1	0.0435	99.6	0.1235	97.8
		ϕ_1	0.0393	99.8	0.1136	98.6
		α_1	0.1381	96.6	0.2193	96.4
		β_1	0.5888	96.4	0.5436	97.2
		α_2	0.2634	96.3	0.2773	96.2
		β_2	0.5637	96.4	0.5456	97.0
	1.00	p_1	0.0316	99.3	0.0261	97.1
		ϕ_1	0.0266	98.9	0.0242	97.9
		α_1	0.1701	98.1	0.1784	97.7
		β_1	0.4019	96.5	0.3859	96.6
		α_2	0.2144	97.1	0.1822	97.2
		β_2	0.3954	96.6	0.3835	96.2

Table 13. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.4, \phi_1 = 0.6, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.5, \beta_2 = 1.75$) under joint Type I censoring scheme.

(m,n)	t	Parm.	Asy-CI		HPD	
			AIL	CP	AIL	CP
(15,15)	0.30	p_1	0.0483	100.0	3.7961	95.1
		ϕ_1	0.0500	100.0	5.9833	95.1
		α_1	0.3027	95.6	3.7544	95.0
		β_1	2.7102	96.5	9.8327	95.1
		α_2	0.4012	95.6	3.5791	95.1
		β_2	2.6639	96.5	15.1234	95.0
	0.60	p_1	0.0069	93.7	0.5154	96.5
		ϕ_1	0.0072	93.2	0.7259	95.3
		α_1	0.1869	97.3	0.5159	96.4
		β_1	1.3929	96.1	1.4466	96.6
		α_2	0.1793	97.6	0.4056	96.5
		β_2	1.4265	95.6	1.5292	96.8
(25,25)	0.30	p_1	0.0264	100.0	2.9549	95.1
		ϕ_1	0.0302	100.0	4.5493	95.2
		α_1	0.2108	97.0	3.0100	95.0
		β_1	1.8888	96.4	10.8100	95.1
		α_2	0.2668	97.0	1.7842	95.0
		β_2	1.8629	96.4	13.0524	95.1
	0.60	p_1	0.0055	95.6	0.0549	97.6
		ϕ_1	0.0051	96.2	0.0516	98.0
		α_1	0.1451	97.9	0.1435	97.7
		β_1	1.0368	96.5	0.8864	97.1
		α_2	0.1441	98.3	0.1575	97.0
		β_2	1.0688	96.2	0.8793	97.3
(50,50)	0.30	p_1	0.0166	100.0	0.3332	95.8
		ϕ_1	0.0171	99.9	0.5745	95.0
		α_1	0.1327	95.6	0.3345	95.4
		β_1	1.1639	96.3	1.2424	96.3
		α_2	0.1581	95.5	0.3675	97.4
		β_2	1.1497	96.3	1.1520	96.3
	0.60	p_1	0.0037	95.7	0.0377	97.0
		ϕ_1	0.0036	96.4	0.0370	98.2
		α_1	0.0929	97.2	0.0927	97.3
		β_1	0.6780	96.8	0.6244	96.7
		α_2	0.0913	97.3	0.1002	97.6
		β_2	0.6848	96.6	0.6203	96.4

Table 14. AILs and CP(%) values of the ML and BE for mixture Weibull distributions ($p_1 = 0.6, \phi_1 = 0.4, \alpha_1 = 0.5, \beta_1 = 1.5, \alpha_2 = 0.5, \beta_2 = 1.75$) under joint Type I censoring scheme.

(m,n)	t	Parm.	Asy-CI		HPD	
			AIL	CP	AIL	CP
(15,15)	0.30	p_1	0.0466	100.0	6.3944	95.1
		ϕ_1	0.0483	100.0	4.1733	95.2
		α_1	0.2815	95.3	4.4233	95.1
		β_1	2.7755	95.4	12.1250	95.2
		α_2	0.3686	95.8	3.6939	95.1
		β_2	2.6902	95.4	14.3614	95.3
	0.60	p_1	0.0073	93.5	0.5181	95.9
		ϕ_1	0.0076	97.1	0.5214	95.8
		α_1	0.1795	97.3	0.4856	95.5
		β_1	1.4120	96.3	1.3620	96.5
		α_2	0.1806	97.6	0.4463	95.5
		β_2	1.5898	95.6	1.3945	97.2
(25,25)	0.30	p_1	0.0399	100.0	3.9920	95.5
		ϕ_1	0.0269	100.0	3.2552	95.0
		α_1	0.2059	95.4	3.1492	95.1
		β_1	1.8758	95.5	6.3454	95.1
		α_2	0.2702	96.2	1.8798	95.1
		β_2	1.8501	95.7	8.6698	95.1
	0.60	p_1	0.0052	96.6	0.0540	97.9
		ϕ_1	0.0050	97.2	0.0514	98.2
		α_1	0.1348	97.6	0.1330	97.1
		β_1	1.0509	95.5	0.8992	96.4
		α_2	0.1305	97.8	0.1440	97.3
		β_2	1.0677	95.7	0.9008	96.1
(50,50)	0.30	p_1	0.0172	100.0	0.4082	95.5
		ϕ_1	0.0162	99.8	0.3643	95.4
		α_1	0.1371	95.5	0.3133	96.5
		β_1	1.2566	95.7	1.4192	96.6
		α_2	0.1655	95.6	0.3558	97.2
		β_2	1.2418	95.7	1.3292	96.9
	0.60	p_1	0.0035	95.9	0.0392	99.2
		ϕ_1	0.0036	96.8	0.0369	97.1
		α_1	0.0971	97.1	0.0972	97.1
		β_1	0.7112	96.8	0.6611	97.5
		α_2	0.0952	97.4	0.1001	97.7
		β_2	0.7185	96.7	0.6697	96.4

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