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Boundary Control of Systems Governed by Semilinear Elliptic Equation for Infinite Order Operator with Finite Dimension

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Abstract
This paper studies a boundary optimal control problem governed by a semilinear elliptic equation involving an elliptic operator of infinite-order with a finite dimension. The state equation is defined on a bounded domain with a nonlinear boundary condition where the control variable acts on the boundary. The analysis is carried out within the framework of Sobolev spaces of infinite order. We first establish the existence and uniqueness of the solution to the state equation and define the associated control-to-state mapping. Under suitable assumptions on nonlinear boundary term, we give the differentiability properties of this mapping. Furthermore, we derive the first-order necessary optimality conditions for the optimal control problem through the associated adjoint system and a variational inequality. Finally, we analyze the second order derivatives of the reduced cost functional.

Keywords Boundary control, semilinear elliptic equation, infinite order operator, pointwise control constraint, optimality conditions.

AMS 2010 subject classifications 93C10, 35G60, 35J15, 35J25, 35J57, 35J61, 49K20.

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1. Introduction

This research paper focuses on the development of mathematical conditions necessary and sufficient for solving boundary control problems that are governed by semilinear elliptic equations involving an infinite order operator. Such problems are important in many areas of applied mathematics, physics and engineering where one seeks to influence or control the behavior of a system through its boundary rather than it is interior.

The presence of an infinite order operator introduces technical challenges in proving existence and differentiability of the control-to-state mapping.

The main goal of this paper is to derive first-order and second-order optimality conditions for such boundary control problems. These conditions are essential in optimization theory because they help determine when a particular control function is optimal that is, when it minimizes a given cost functional.

In Dubinskii [5, 6] studied the Cauchy Dirichlet problem

$$\mathcal{L}(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma} u) = h(x), \qquad x \in \Omega$$
$$D^{|\omega|} u(x)|_{\partial\Omega} = 0, \qquad |\omega| = 0, 1, 2, \cdots$$

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Infinite order Sobolev spaces

$$W^{\infty}\{a_{\alpha}, p_{\alpha}\}(\Omega) = \{u(x) \in C_0^{\infty}(\Omega) : p(u) \equiv \sum_{|\alpha|=0}^{\infty} \|D^{\alpha}u\|_{p_{\alpha}}^{p_{\alpha}} < \infty\}$$

where $a_{\alpha} \geq 0$ and $p_{\alpha} \geq 1$ are numerical sequences and established of $W^{\infty}\{a_{\alpha}, p_{\alpha}\}$ and boundary value problem above is investigated where $\Omega \subset R^N$.

Gali et al. [19] presented a set of inequalities defining on optimal control of a system governed by self-adjoint elliptic operators with an infinite number of variables.

Subsequently Lions[1] suggested a problem related to this result but in different direction by taking the case of operators of infinite order with finite dimensions.

Gali has solved this problem, the result has been published in [18].

I. M. Gali and S.-A. El-Saify and S. A. El-Zahaby [20, 21, 22] presented some control problems generated by both elliptic and hyperbolic linear operator of infinite order with finite number of variables.

El-Zahaby et al [10, 11] obtained the optimal control of problems governed by variational inequalities of infinite order with bounded domain.

In this paper, we study a non-convex optimal control problem, using the books of Tiba and Tröltzsch [9] and [23], we derive first-order necessary and second-order sufficient optimality conditions for semilinear elliptic equation with infinite order operator.

We refer for instance, to Casas [3] for first-order necessary optimality conditions, Casas, Tröltzsch and Unger [4] for second-order sufficient condition. For the elliptic case for linear equations of infinite order were established in papers by [12, 16], while for semilinear control problem of infinite order obtained by [15, 17, 14]. The paper which a near connection to our work is obtained by [24, 25, 26].

1.1. Contribution and Relation to Previous Work

The study of optimal control problems governed by elliptic equations involving infinite order operators has been considered in several previous works by the authors and other researchers. In particular, earlier papers mainly focused on distributed control problems and linear or semilinear state equations with various types of constraints.

The main contribution of the present paper is the analysis of a boundary optimal control problem governed by a semilinear elliptic equation involving an elliptic operator of infinite with finite dimension. Unlike distributed control problems, the control variable in this work acts on the boundary of the domain which introduce additional analytical difficulties related the boundary conditions and the functional framework.

Furthermore, the nonlinear boundary term $b(x, y)$ requires specific assumptions in order to ensure the well-posedness of the state equation and the differentiability of the control-to-state mapping.

Another contribution of this work is the derivation of first-order necessary optimality conditions and the analysis of second -order derivatives of the reduced cost functional within the framework of infinite order Sobolev spaces.

These results extend the classical theory of semilinear optimal control problems (see, for example Lions[1], Casas[2], Tröltzsch [23] and Barbu[7]) to the case of elliptic operator of infinite order and boundary control constraints.

This manuscript addresses an optimal boundary control problem governed by a semilinear elliptic equation involving an infinite order operator with finite dimensions. Our aim to establish the well-posedness of the state equation and derive first order necessary optimality condition and establish second order derivative of the cost functional. The work builds upon existing literature regarding infinite order Sobolev spaces (Dubinskii, Gali, et al.) and standard optimal control theory for semilinear elliptic equations (Tröltzsch[23], Casas[2]). The remainder of this paper is organized as follows. In section two, we introduce the framework and recall some properties of sobolev spaces of infinite order.

The existence and uniqueness of the solution to the state equation in section three

In section four: we reduced the cost functional and derive the first order necessary optimality conditions for control problem. In section five: we derive the second order of the reduced cost functional in order to provide further insights into the structure of the optimal control.

2. Some Functional Spaces

The object of this section is to give the definition of some function spaces of infinite order and the chains of the constructed spaces which will be used later.

We define the Sobolev space of infinite order which shall denoted by $W^\infty\{a_\alpha, 2\}$ of periodic functions $\phi(x)$ defined on the whole Euclidean space R^n, $n \geq 1$, as follows

$$W^\infty\{a_\alpha, 2\} = \{\phi \in C^\infty(R^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha \phi\|_2^2 < \infty\}$$

where $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is the canonical norm with space $L^2(R^n)$ all functions are assumed to be the real valued on

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots (\partial x_n^{\alpha_n})},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index for differentiation $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$.

The duality paring of the space $W^\infty\{a_\alpha, 2\}$ and $W^{-\infty}\{a_\alpha, 2\}$ is postulated by the formula

$$(\phi, \psi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{R^n} \psi_\alpha(x) D^\alpha \phi(x)\ dx$$

where

$$\phi \in W^\infty\{a_\alpha, 2\}, \qquad \psi \in W^{-\infty}\{a_\alpha, 2\}$$

From above, $W^\infty\{a_\alpha, 2\}$ is everywhere dense in $L^2(R^n)$ with topological inclusions and $W^{-\infty}\{a_\alpha, 2\}$ dense the topological dual space with respect to $L^2(R^n)$, so we have the following chain

$$W^\infty\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W^{-\infty}\{a_\alpha, 2\}$$

Analogous to the above chain we have

$$W_0^\infty\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W_0^{-\infty}\{a_\alpha, 2\}$$

$$W_0^\infty\{a_\alpha, 2\} = \{\phi(x) \in C_0^\infty(R^n) : \|\phi\|^2 = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha \phi\|_2^2 < \infty,$$
$$D^{|w|}\phi|_\Gamma = 0,\ |w| = 0, 1, \cdots\}$$

Let us consider the elliptic operator of infinite order with finite dimension.

$$Ay = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y \quad a_\alpha > 0. \tag{2.1}$$

This operator is bounded self-adjoint elliptic operator mapping $W_0^\infty\{a_\alpha, 2\}$ onto $W_0^{-\infty}\{a_\alpha, 2\}$.

We introduce a continuous bilinear form on $W_0^\infty\{a_\alpha, 2\}$

$$\begin{aligned}
\pi(y, \phi) &= (Ay, \phi) \\
&= \sum_{|\alpha|=0}^{\infty} \Big((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), \phi(x)\Big)_{L^2(R^n)},\ a_\alpha \geq 0 \\
&= \sum_{|\alpha|=1}^{\infty} \Big((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), \phi(x)\Big)_{L^2(R^n)} + q(x)(y(x), \phi(x))_{L^2(R^n)}
\end{aligned}$$

where $q(x)$ is a real valued function from $L_2(R^n)$ such that $q(x) \geq \nu$, $1 \geq \nu > 0$.
The ellipticity of A is sufficient from the coerciveness of $\pi(u,v)$ on $W^\infty\{a_\alpha, 2\}$
In fact,

$$\begin{aligned}\pi(u,u) &= \sum_{|\alpha|=1}^{\infty}(a_\alpha D^\alpha u(x), D^\alpha u(x))_{L^2(R^n)} \\ &\quad + (q(x)u(x), u(x))_{L^2(R^n)} \\ &\geq (\sum_{|\alpha|=1}^{\infty} a_\alpha D^\alpha u(x), D^\alpha u(x))_{L^2(R^n)} + \nu(u(x), u(x))_{L^2(R^n)} \\ &= \sum_{|\alpha|=1}^{\infty} a_\alpha \|D^\alpha u(x)\|^2_{L^2(R^n)} + \nu \sum_{|\alpha|=1}^{\infty} a_\alpha \|D^\alpha u(x)\|^2_{L^2(R^n)} - \nu \sum_{\|\alpha\|=1}^{\infty} a_\alpha \|D^\alpha u(x)\|^2_{L^2(R^n)} + \nu\|u\|^2_{L^2(R^n)} \\ &= \nu\|u\|^2_{W^\infty\{a_\alpha,2\}} + (1-\nu)\sum_{|\alpha|=1}^{\infty} a_\alpha \|D^\alpha u(x)\|^2_{L^2(R^n)}\end{aligned}$$

Then

$$\pi(u,u) \geq \nu\|u\|^2_{W^\infty\{a_\alpha,2\}}. \tag{2.2}$$

3. Semilinear Elliptic Control Problem for Infinite Order Operator

3.1. Problem statement

We consider optimal control problems governed by semilinear elliptic equation for infinite order with finite dimension.

$$(P) \quad \begin{cases} \min J(y,u) := \dfrac{1}{2}\|y - y_\Omega\|^2_{L^2(\Omega)} + \dfrac{\lambda}{2}\|u\|_{L^2(\Gamma)}, & (3.1) \\ \text{subject to} & \\ Ay + y = 0 \qquad \text{in} \quad \Omega & (3.2) \\ y^{|\omega|}|_\Gamma + b(x,y) = u \qquad \text{on} \quad \Gamma, \quad |w| = 0,1,2,\cdots & \\ \text{and} & \\ u_a(x) \leq u(x) \leq u_b(x). & (3.3) \end{cases}$$

where A denotes an elliptic operator of infinite order having the form (2.1) and $b : \Gamma \times R \to R$ is a function.
The function u denotes the control and $y(u)$ is the solution (state of the function) associated to the control u.
Let us consider the set of admissible control by

$$U_{ad} = \{u \in L^2(\Gamma) : u_a(x) \leq u(x) \leq u_b(x) \quad f.a.a \quad x \in \Gamma\}$$

Note that U_{ad} is non-empty, convex and bounded in $\mathcal{U} = L^2(\Gamma)$.

Definition 3.1. A function $y \in W^\infty\{a_\alpha, 2\}$ is said to be a weak solution of the partial differential equation (3.2) if for all $v \in W^\infty\{a_\alpha, 2\}$ the equation

$$\sum_{|\alpha|=0}^{\infty}\int_\Omega a_\alpha (D^\alpha y)(x)(D^\alpha v)(x)dx + \int_\Omega yvdx + \int_\Gamma b(x,y)vds = \int_\Gamma uvds$$

is valid

To make this well defined, we impose the following assumptions:

Assumption A1:

Let $\Omega \subset R^N$ be a bounded Lipschitz domain with boundary Γ and A is an elliptic operator and the coerciveness condition (1.2) of $\pi(y, y)$ on $W^\infty\{a_\alpha, 2\}$ is satisfied if $q(x) \geq \nu$, $1 \geq \nu > 0$.

The function $b : \Gamma \times R \to R$ is bounded and measurable with respect to $x \in \Gamma$ for any fixed $y \in R$ and is continuous and monotone increasing with respect to y for almost all $x \in \Gamma$. It follow from this assumption, in particular, that $b(x, 0)$ is bounded and measurable in Γ. This property of the function b guarantees the existence of unique weak solution of partial differential equation

Assumption A2:

i) Let $\Omega \subset R^N$ be a bounded domain with Lipschitz-Continuous boundary Γ

ii)The function b is measurable with respect to x for every $y \in R$ and twice differentiable with respect y for almost every $x \in \Gamma$. Moreover, it satisfies the boundedness and local lipschitz conditions of order $k = 2$; for b this means, for example, that there are constants $K > 0$ and $L(M) > 0$ such that for almost every $x \in \Gamma$ we have

$$|b(x, 0)| + |b_y(x, 0)| + |b_y y(x, 0)| \leq K,$$

$$|b_{yy}(x, y_1) - b_{yy}(x, y_2)| \leq L(M)|y_1 - y_2| \forall\ y_1, y_2 \in [-M, M]$$

iii) Additionally $b_y(x, y) \geq 0$ for almost every $x \in \Gamma$ and all $y \in R$. Moreover, there is a set $E_d \subset \Gamma$ of positive measure and constant λ_b such that

$$b_y(x, y) \geq \lambda_b \qquad \forall \quad x \in E_b\ \forall\ y \in R.$$

The above set of assumptions is too restrictive. In fact, for the existence of optimal controls the conditions in (ii) including Lipschitz continuity, are needed only for the functions themselves (order k=0). For first-order necessary optimality conditions, (ii) needs to postulated up to order $k = 1$ only, while Assumption A2 is needed in its entirety for second-order conditions.

Theorem 3.1. *Suppose that Assumption A1 and A2 hold. Then the elliptic bondary value problem (3.2) has for any $u \in L^2(\Gamma)$ a unique solution $y \in W^\infty\{a_\alpha, 2\}$. If in addition $b(x, 0) = 0$, then there exist a constant $C > 0$ which is independent of b, u, such that and the estimate*

$$\|y\|_{W^\infty\{a_\alpha,2\}} \leq C\|u\|_{L^2(\Gamma)} \tag{3.4}$$

By apply Schauder fixed point theorem to prove the existence of a solution $y \in W^\infty\{a_\alpha, 2\}$ see [13]

3.2. Standard results

We recall some well-known results on (P)

We consider y in the state space $Y = W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$ and the control $u \in U_{ad} \subset L^2(\Gamma)$. Moreover, we introduce the control-to-state operator $G : L^2(\Gamma) \to Y$ that assigns y to u.

The following result to show that the weak solution $y \in W^\infty\{a_\alpha, 2\}$ is actually even essentially bounded.

Theorem 3.2. *Under the assumption A1 on Ω and b the state equation (3.2) admits for all $u \in L^2(\Gamma)$ exactly on solution $y = G(u) \in Y$ and the estimate*

$$\|y\|_{W^\infty\{a_\alpha,2\}} + \|y\|_{C(\Omega)} \leq C_\infty\|u\|_{L^2(\Gamma)} \tag{3.5}$$

holds true with a constant C_∞ .

Remark 3.1. In [23] the boundedness of b is not essential for the previous existence results. The important property is the monotonicity. This fact is utilized to prove the existence solution and a weakly assumptions are presented so that the existence of a unique solution $y \in H^1(\Omega) \cap C(\bar{\Omega})$ to

$$-\Delta y + y = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial y}{\partial \nu} + y^3|y| = u \quad \text{on} \quad \Gamma$$

Summarizing, the first-order necessary optimality condition (3.2) is a non-convex optimal control problem. Although J is convex, the nonlinear term in (3.2) makes the problem (P) non-convex.

Therefore, the first-order necessary optimality condition are not sufficient. So that the second-order conditions come into play.

4. First-order Necessary Optimality conditions

First-order Necessary Optimality conditions describe the relationship between the optimal control, the state and an associated adjoint state. These form a system of equations known as the optimality system, which must be satisfied by the optimal control.

4.1. Existence of optimal control

Definition 4.1. A control $\bar{u} \in U_{ad}$ is said to be optimal if it satisfies, together with the associated optimal state $\bar{y} = y(\bar{u})$, the inequality

$$J(y(\bar{u}), \bar{u}) \leq J(y(u), u) \qquad \forall \quad u \in U_{ad}.$$

A control is said to be locally optimal in the sense of $L^2(\Gamma)$ if there exists some $\epsilon > 0$ such that the above inequality holds for all $u \in U_{ad}$ such that $\|u - \bar{u}\|_{L^2(\Gamma)} < \epsilon$.

The solution mapping $G := u \mapsto y$, the solution $y = G(u)$. Obviously G is a nonlinear mapping also continuous

We transformed the control problems (3.1) under inverting into reduced quadratic optimization problems in term of u, namely

$$\begin{aligned} f(u) &:= J(y, u) \\ &= J(G(u), u) \\ &= \frac{1}{2}\|G(u) - y_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Gamma)} \\ &= \frac{1}{2}\int_\Omega (G(u) - y_d)^2 \, dx + \frac{\lambda}{2}\int_\Gamma u^2 \, ds \end{aligned}$$

the problem is to find $\min f(u)$. where

$$U_{ad} = \{u \in L^2(\Gamma) : u_a(x) \leq u(x) \leq u_b(x)\}$$

Theorem 4.1. *Suppose that Assumptions A1 and A2 hold. If the admissible set U_{ad} is not empty, the problem (P) has at least one optimal control.*

Proof

The proof is more or less standard. In all what follows, we denote the optimal solution by $(\bar{y}, \bar{u})$ where $\bar{y} = G(\bar{u})$ and $\bar{u}$ is said to be an optimal control.

By $\lambda > 0$, and Assumption A1, we find a bounded minimizing sequence $\{u_n\} \subset L^2(\Gamma)$, $y_n = G(u_n)$ and we can assume without loss of generality $u_n \to \bar{u}$, $n \to \infty$.

By Theorem (3.1) the associated sequence $\{y_n\}$ is bounded in $W^\infty\{a_\alpha, 2\}$. Moreover, since $W^\infty\{a_\alpha, 2\}$ is by [23][Th 7.4, page356] is completely embedded in $L^2(\Omega)$, we also have the strong convergence

$$y_n \to \bar{y} \qquad \text{in} \quad L^2(\Omega)$$

We know that $y \in C(\Omega)$, it follows from the boundedness of $y_n \in L^\infty(\Omega)$ that

$$\|b(x, y_n) - b(x, \bar{y})\|_{L^2(\Gamma)} \leq L\|y_n - \bar{y}\|_{L^2(\Omega)}$$

and thus

$$b(x, y_n) \to b(x, y) \qquad \text{in} \quad L^2(\Gamma)$$

$$\bar{y} = G(\bar{u})$$

The optimality of $\bar{u}$ is a standard conclusion, therefore

$$j = \lim_{n\to\infty} J(y_n, u_n) = J(\bar{y}, \bar{u})$$

□

Suppose that Assumptions A1 and A2 hold for Γ and b. Then G is a Lipschitz continuous mapping from $L^2(\Gamma)$ into $W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$, that, there is a constant $L > 0$ such that

$$\|G(u_1) - G(u_2)\| \leq L\|u_1 - u_2\|_{L^2(\Gamma)}$$

i.e.

$$\|y_1 - y_2\|_{W^\infty\{a_\alpha,2\}} + \|y_1 - y_2\|_{C(\bar{\Omega})} \leq L\|u_1 - u_2\|_{L^2(\Gamma)}$$

wherever $u_i \in L^2(\Gamma)$ and $y_i = G(u_i)$, $i = 1, 2$.

Lemma 4.2. *Suppose that Assumptions A1 and A2 are satisfied. Then the operator G is twice continuously Frechet differentiable from $L^2(\Gamma) \to Y$. Its first derivative, denoted by $w = G'(u)h$, $h \in L^2(\Gamma)$ is given by the solution of the linearlized equation*

$$\begin{aligned} &Aw + w = 0, \\ &w^{|\alpha|}|_\Gamma + b_y(x, y)w = h, \quad |\alpha| = 0, 1, 2, \cdots \end{aligned} \tag{4.1}$$

with $y = G(u)$.

Moreover the second derivative $z = G''(u)[u_1, u_2]$ solves equation

$$\begin{aligned} &Az + z = 0 && \textit{in} \quad \Omega \\ &z^{|w|}|_\Gamma + b_y(x, y)z = -b_{yy}(x, y)y_1y_2, && |w| = 0, 1, 2, \cdots \end{aligned} \tag{4.2}$$

with $y = G(u)$ and $y_i = G'(u)u_i \in W^\infty\{a_\alpha, 2\}$ for $i = 1, 2$,.

The detailed proof of this Lemma can be found in [23] [Th 4.24 Page 239].

5. Second-order derivative

Since b is twice continuously differentiable together with the differentiability of G (of Lemma (4.2)) this yields the following Theorem.

Theorem 5.1. *Under the Assumptions of lemma (4.2), f is twice continuously Frechet differentiable from $L^2(\Gamma)$ to R its first derivative is given by*

$$\begin{aligned} f'(u)h &= f'(u)(u - \bar{u}) \quad \forall\ u \in U_{ad} \\ &= \int_\Gamma (p(x) + \lambda\bar{u}(x))h(x)\ ds \end{aligned} \tag{5.1}$$

where p solves the adjoint equation

$$\begin{aligned} &Ap + p = y - y_d && \textit{in} \quad \Omega \\ &p^{|w|}|_\Gamma + b_y(x, y)p = 0, && |w| = 0, 1, 2, \cdots \end{aligned} \tag{5.2}$$

with $y = G(u)$. For the second derivative, we obtain

$$f''(u)[u_1, u_2] = (y_1, y_2)_{L^2(\Omega)} + \lambda(u_1, u_2)_{L^2(\Gamma)} - \int_\Gamma b_{yy}(x, y)y_1y_2\ p\ ds \tag{5.3}$$

where y and p are as defined above and

$$y_i = G'(u)u_i \qquad i = 1, 2, \cdots .$$

Proof

From

$$f(u) = J(G(u), u) = \frac{1}{2}\|G(u) - y_d\|^2 + \frac{\lambda}{2}\|u\|^2_{L^2(\Gamma)},$$

we get

$$f'(u)h = (y - y_d, w)_{L^2(\Omega)} + \lambda(u, h)_{L^2(\Gamma)}$$

where $y = G(u)$ and $w = G'(u)h$ denotes the weak solution of the linearized equation (4.1) with to right hand side h.

Now, inserting p as test function in the weak formulate of (4.1), we obtain

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}\int_\Omega a_\alpha D^{2\alpha}wp\ dx + \int_\Omega wp\ dx + \int_\Gamma b_y(x,y)wp\ ds = \int_\Gamma (u-\bar{u})p\ ds$$

$$\sum_{|\alpha|=0}^{\infty}\int_\Omega a_\alpha D^{\alpha}wD^{\alpha}p\ dx + \int_\Omega wp\ dx + \int_\Gamma b_y(x,y)wp\ ds = \int_\Gamma (u-\bar{u})p\ ds$$

On the other hand, we insert w in the weak formulation of equation (5.2) yields

$$\sum_{|\alpha|=0}^{\infty}\int_\Omega a_\alpha D^{\alpha}D^{\alpha}pw\ dx + \int_\Omega pw\ dx + \int_\Gamma b_y(x,y)pw\ ds = \int_\Omega (y-y_d)w\ dx$$

subtracting one equation from the other finally yields

$$(y - y_d, w)_{L^2(\Omega)} = (h, p)_{L^2(\Gamma)}.$$

Applying again the chain rule, we arrive at

$$f''(u)[u_1, u_2] = (G'(u)u_1, G'(u)u_2)_{L^2(\Omega)} + (G(u) - y_d, G''(u)[u_1, u_2])_{L^2(\Omega)} + \lambda(u_1, u_2)_{L^2(\Gamma)}$$

A similar discussion as above, where

$$z = G''(u)[u_1, u_2]$$

denote the weak solution of (4.2), then gives

$$(y - y_d, z)_{L^2(\Omega)} = -(b_{yy}(x,y)y_1y_2, p)_{L^2(\Omega)}$$

□

For a given right hand side in $L^2(\Omega)$, equation (5.2) admits a solution p in $Y = W^\infty\{a_\alpha, 2\} \cap C(\bar{\Omega})$, since the operator A has the form (2.1).

Theorem 5.2. *Let $U = L^2(\Gamma)$ be a Banach space, let $C \subset U$ be convex, and suppose that the functional $f : U \to R$ is twice continuously Frechet differentiable in an open neighborhood of $\bar{u} \in C$. Let the control $\bar{u}$ satisfy the first-order necessary condition*

$$f'(\bar{u})(u - \bar{u}) \geq 0$$

and assume there is some $\delta > 0$ such that

$$f''(\bar{u})(u - \bar{u})^2 \geq \delta\|u - \bar{u}\|^2_{L^2(\Gamma)} \qquad \forall\ (u - \bar{u}) \in U \tag{5.4}$$

Then there exist $\epsilon > 0$ such that we have the quadratic growth condition holds:

$$\frac{\delta}{4}\|u - \bar{u}\|^2_{L^2(\Gamma)} + f(\bar{u}) \leq f(u)\ if\ \|u - \bar{u}\|_{L^2(\Gamma)} < \epsilon\ \forall u \in U \tag{5.5}$$

This theorem will be applicable to optimal control problems involving semilinear partial differential equations if the control-to-state operator G is twice continuously differentiable as a mapping from L^2 into the state space and if the control appears only linearly or quadratically in the cost functional and only linearly in the differential equation.

6. Conclusion

In this paper, we have investigated a boundary optimal control problem governed by a semilinear elliptic equations involving an elliptic operator of infinite-order with finite dimension. The control variable acts on the boundary of the domain and is subject to box constraints, while the state system is characterized by a nonlinear boundary condition. The analysis has been carried out within the framework of sobolev spaces of infinite order. Under suitable assumptions on the nonlinear boundary term, we established the existence and uniqueness of the solution to the state equation and analyzed the associated control-to-state mapping. In particular we discussed its continuity and differentiability properties, which are essential for the formulation of the optimal control problem. Furthermore, we introduced the reduced cost functional and derived the first-order necessary optimality conditions using the adjoint system and variational inequality. In addition, we investigated the second order derivatives of the reduced cost functional providing further insight into the structure of the control problem. The result presented in this work extend classical optimal control theory for semilinear elliptic equations to the setting of infinite-order elliptic operators with boundary control. This extension is not straightforward due to the specific analytical difficulties associated with infinite-order boundary conditions which require a careful adaptation of standard techniques.

Finally, we emphasize that the framework developed in this paper opens the way for several possible extensions. Future work may include the study of more general nonlinearities, derivation of sufficient second order optimality conditions, Lagrange method, numerical approximation schemes and the analysis of more complex control structures in the context of infinite order operators.

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