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A New Transmuted Rayleigh Model with Application to Reliability Data

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Abstract The Rayleigh distribution has long been used in many practical fields, yet its fixed shape and hazard pattern can make it unsuitable for describing complicated real data. Because of this limitation, there is a clear need for more flexible extensions. In this paper, we introduce the *Transmuted Rayleigh Distribution (TRD)* based on the *Quadratic Rank Transmutation Map (QRTM)*. The proposed model adds a distortion parameter γ, which creates a smooth link between the classical Rayleigh form and a Weibull–type behavior, allowing the distribution to adapt better to different levels of skewness and tail weight. Several statistical properties of the TRD are obtained, including its moments, quantile function, entropy, and hazard rate. The parameters of the proposed model are estimated using the maximum likelihood approach, and their behavior are evaluated through a simulation study, and we consider a real dataset related to analgesic relief times to demonstrate the practical value of the new distribution. The results indicates that the TRD provides a better description of the data when compared with several commonly used models based on standard goodness–of–fit statistics and model selection criteria.

Keywords Rayleigh distribution, Transmuted distribution, Maximum likelihood, Order statistics, Moment generating function.

AMS 2010 subject classifications 47H10, 54H25

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1. Introduction

Statistical distributions play a key role in analyzing and interpreting data in many scientific fields such as engineering, medicine, economics, and environmental science. However, many real-world datasets display patterns that traditional distributions cannot fully account for. For instance, data may be highly skewed, or exhibit hazard rates that do not follow a simple increasing or decreasing trend. To handle such cases, researchers have developed various methods to extend standard distributions by using transmutation, compounding, and weighting techniques.

The Rayleigh distribution is fundamental in reliability analysis [1] and is widely applied in communication theory, reliability engineering, and life-testing studies. Despite its usefulness, the standard Rayleigh distribution is constrained by its fixed shape and hazard behavior. Several extensions have been proposed to overcome these limitations, including the weighted Rayleigh distribution [2] and other related weighted forms.

One useful approach is the Quadratic Rank Transmutation Map (QRTM), introduced by Shaw and Buckley [3]. The basic idea is to apply a polynomial distortion to an existing cumulative distribution function (CDF), thereby generating new families of distributions with improved control over skewness and tail behavior, without adding an excessive number of parameters. This framework has led to several transmuted versions of well-known models, such as the Transmuted Exponential [4], Transmuted Weibull [5], and Transmuted Shanker [6] distributions.

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Recent advances in statistical methodology include efficient sampling approaches such as Moving Extreme Ranked Set Sampling (MERSS) [7], modified ratio estimators based on ranked set sampling [8], and improved sampling strategies [9, 8]. Additional contributions cover reliability estimation under the Pareto distribution [10, 11] and new hybrid models such as the Epanechnikov-Pareto distribution [12] and the Epanechnikov-Burr XII distribution [13]. Furthermore, neutrosophic reliability frameworks built on the Kumaraswamy distribution [14], as well as developments in neutrosophic soft sets [15] and fixed point theory [16], have been introduced. Sampling strategies [17] and applied studies examining health-related data [18] also contribute to the broader statistical literature. From a theoretical perspective, lacunary strong convergence [19] may offer tools for future justifications of the convergence properties of estimators. Fuzzy reliability estimation for the Benktander distribution [20] represents another direction in modern reliability analysis.

Building on these diverse contributions, this study introduces the *Transmuted Rayleigh Distribution (TRD)* using the QRTM framework as a flexible extension for real data applications. The proposed model adds a distortion parameter γ, which creates a smooth link between the classical Rayleigh form and a Weibull-type behavior, allowing the distribution to adapt better to different levels of skewness and tail weight. Several statistical properties of the TRD are derived, including its moments, quantile function, entropy, and hazard rate. The parameters are estimated using the maximum likelihood approach, and their behavior is evaluated through a simulation study. A real dataset related to analgesic relief times is considered to demonstrate the practical value of the new distribution.

This work makes three main contributions:

1. We built a more flexible version of the Rayleigh model (the TRD) and provided all the key formulas needed to use it.
2. We explored its mathematical details, like its average value, variability, and failure pattern.
3. We tested the model's usefulness with computer simulations and by applying it to real-world data.

2. QRTM Formulation and Basic Properties of the TRD

Let X be a random variable from a Rayleigh Distribution with parameter θ. The probability density function is given by

$$f(x) = \theta x e^{-\frac{\theta}{2}x^2}, \quad x \geq 0,\ \theta > 0.$$

2.1. QRTM Formulation

The Quadratic Rank Transmutation Map (QRTM) gives a general method for building an extensions for a given distribution. We start from a cumulative distribution function $F(x)$, then we uses a single parameter γ and a simple second-degree polynomial to transform it.

Given the baseline Rayleigh distribution with scale parameter $\theta > 0$, the CDF is:

$$F(x) = 1 - e^{-\frac{\theta}{2}x^2}, \quad x \geq 0,\ \theta > 0. \tag{1}$$

By applying the QRTM transformation to $F(x)$, we get the CDF of the Transmuted Rayleigh Distribution (TRD):

$$G(x) = (1+\gamma^2)F(x) - \gamma^2 F^2(x), \tag{2}$$

where $\gamma \in (-1, 1)$ is the transmutation parameter.

Substituting (1) into (2) and simplifying, we obtain:

$$\begin{aligned} G(x) &= (1+\gamma^2)\left(1 - e^{-\frac{\theta}{2}x^2}\right) - \gamma^2\left(1 - e^{-\frac{\theta}{2}x^2}\right)^2 \\ &= \left(1 - e^{-\frac{\theta}{2}x^2}\right)\left(1 + \gamma^2 e^{-\frac{\theta}{2}x^2}\right). \end{aligned} \tag{3}$$

Hence, the probability density function (PDF) of the TRD is give as

$$g(x) = \frac{dG(x)}{dx} = \theta x e^{-\frac{\theta}{2}x^2}\left(1 - \gamma^2 + 2\gamma^2 e^{-\frac{\theta}{2}x^2}\right). \tag{4}$$

Equation (4) can be equivalently expressed as:

$$g(x) = (1-\gamma^2)\theta x e^{-\frac{\theta}{2}x^2} + 2\gamma^2\theta x e^{-\theta x^2}, \quad x \geq 0,\ \theta > 0. \tag{5}$$

The estimation becomes unstable when do numerical experiments as γ approaches the boundaries $\{-1, 1\}$. So, we restrict γ to the open interval $(-1, 1)$ To ensure reliable inference,.

Through our work, we consider

$$|\gamma| \leq 0.95.$$

This restriction keeps the flexibility of the TRD, especially for smaller samples while improving the stability of maximum likelihood estimation. When $\gamma = 0$, the TRD simplifies to the standard Rayleigh distribution. As γ approaches 1, the density gradually resembles a Weibull distribution with shape parameter $k = 2$ and scale parameter $\lambda = 1/\sqrt{\theta}$, allowing a smooth transition between these two classical forms.

2.2. Properties of the TRD

2.2.1. Moments

Theorem 2.1
Let X be a TRD random variable. Then the rth moment is given by:

$$E(X^r) = \frac{1}{\theta^{r/2}}\Gamma\left(\frac{r+2}{2}\right)\left[(1-\gamma^2)2^{r/2} + \gamma^2\right]. \tag{6}$$

Proof

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r g(x)\,dx \\
&= \int_0^\infty x^r \left[(1-\gamma^2)\theta x e^{-\frac{\theta}{2}x^2} + 2\gamma^2\theta x e^{-\theta x^2}\right] dx \\
&= (1-\gamma^2)\theta \int_0^\infty x^{r+1} e^{-\frac{\theta}{2}x^2} dx + 2\gamma^2\theta \int_0^\infty x^{r+1} e^{-\theta x^2} dx.
\end{aligned}$$

Using the substitution $u = \frac{\theta}{2}x^2$ in the first integral and $u = \theta x^2$ in the second, we obtain:

$$\begin{aligned}
E(X^r) &= (1-\gamma^2)\theta \cdot \frac{1}{2}\left(\frac{2}{\theta}\right)^{\frac{r+2}{2}} \Gamma\left(\frac{r+2}{2}\right) + 2\gamma^2\theta \cdot \frac{1}{2}\left(\frac{1}{\theta}\right)^{\frac{r+2}{2}} \Gamma\left(\frac{r+2}{2}\right) \\
&= \frac{1}{\theta^{r/2}}\Gamma\left(\frac{r+2}{2}\right)\left[(1-\gamma^2)2^{r/2} + \gamma^2\right].
\end{aligned}$$

□

Therefore, the first four moments can be calculated by putting $r = 1, 2, 3,$ and 4 in Equation (6). Thus:
For $r = 1$:

$$E(X) = \frac{1}{\theta^{1/2}}\Gamma\left(\frac{3}{2}\right)\left[(1-\gamma^2)2^{1/2} + \gamma^2\right],$$

where $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, so:

$$E(X) = \frac{1}{\theta^{1/2}} \cdot \frac{\sqrt{\pi}}{2}\left[\sqrt{2}(1-\gamma^2) + \gamma^2\right].$$

For $r = 2$:

$$E(X^2) = \frac{1}{\theta}\Gamma(2)\left[(1-\gamma^2)2 + \gamma^2\right],$$

where $\Gamma(2) = 1$, so:

$$E(X^2) = \frac{1}{\theta}\left(2 - \gamma^2\right).$$

For $r = 3$:

$$E(X^3) = \frac{1}{\theta^{3/2}}\Gamma\left(\frac{5}{2}\right)\left[(1-\gamma^2)2^{3/2} + \gamma^2\right],$$

where $\Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}$, so:

$$E(X^3) = \frac{1}{\theta^{3/2}} \cdot \frac{3\sqrt{\pi}}{4}\left[(1-\gamma^2)2^{3/2} + \gamma^2\right].$$

For $r = 4$:

$$E(X^4) = \frac{2}{\theta^2}\left(4 - 3\gamma^2\right).$$

Therefore, the variance of the TRD random variable is given by:

$$\begin{aligned} V(X) &= E(X^2) - [E(X)]^2 \\ &= \frac{1}{\theta}\left(2-\gamma^2\right) - \left[\frac{1}{\theta^{1/2}} \cdot \frac{\sqrt{\pi}}{2}\left[\sqrt{2}(1-\gamma^2) + \gamma^2\right]\right]^2 \\ &= \frac{1}{\theta}\left(2-\gamma^2\right) - \frac{\pi}{4\theta}\left[\sqrt{2}(1-\gamma^2) + \gamma^2\right]^2. \end{aligned}$$

The coefficient of variation (c.v.) is defined as the ratio of the standard deviation to the expected value, i.e., $\text{cv} = \sqrt{V(X)}/E(X) = \sigma/\mu$. Therefore:

$$\text{cv} = \frac{\sqrt{\frac{1}{\theta}\left(2-\gamma^2\right) - \frac{\pi}{4\theta}\left[\sqrt{2}(1-\gamma^2)+\gamma^2\right]^2}}{\frac{1}{\theta^{1/2}} \cdot \frac{\sqrt{\pi}}{2}\left[\sqrt{2}(1-\gamma^2)+\gamma^2\right]} = \frac{\sqrt{\left(2-\gamma^2\right) - \frac{\pi}{4}\left[\sqrt{2}(1-\gamma^2)+\gamma^2\right]^2}}{\frac{\sqrt{\pi}}{2}\left[\sqrt{2}(1-\gamma^2)+\gamma^2\right]}.$$

2.2.2. Skewness Skewness is defined as:

$$\text{Skewness} = \frac{E[(X-\mu)^3]}{\sigma^3}.$$

We need the third central moment $E[(X-\mu)^3]$. Now, the third central moment:

$$E[(X-\mu)^3] = E[X^3] - 3\mu E[X^2] + 2\mu^3.$$

Substitute:

$$\begin{aligned} E[(X-\mu)^3] = &\frac{1}{\theta^{3/2}} \cdot \frac{3\sqrt{\pi}}{4}\left[(1-\gamma^2)2^{3/2} + \gamma^2\right] \\ &- 3 \cdot \frac{1}{\theta^{1/2}} \cdot \frac{\sqrt{\pi}}{2}\left[\sqrt{2}(1-\gamma^2)+\gamma^2\right] \cdot \frac{1}{\theta}\left[2(1-\gamma^2)+\gamma^2\right] \\ &+ 2\left(\frac{1}{\theta^{1/2}} \cdot \frac{\sqrt{\pi}}{2}\left[\sqrt{2}(1-\gamma^2)+\gamma^2\right]\right)^3. \end{aligned}$$

Let $A = \sqrt{2}(1-\gamma^2)+\gamma^2$ and $B = (1-\gamma^2)2^{3/2}+\gamma^2$. Then:

$$\begin{aligned} E[(X-\mu)^3] &= \frac{1}{\theta^{3/2}} \cdot \frac{3\sqrt{\pi}}{4}B - \frac{3}{\theta^{3/2}} \cdot \frac{\sqrt{\pi}}{2}A(2-\gamma^2) + \frac{2}{\theta^{3/2}} \cdot \frac{\pi^{3/2}}{8}A^3 \\ &= \frac{1}{\theta^{3/2}}\left[\frac{3\sqrt{\pi}}{4}B - \frac{3\sqrt{\pi}}{2}A(2-\gamma^2) + \frac{\pi^{3/2}}{4}A^3\right]. \end{aligned}$$

The variance is:

$$\sigma^2 = \frac{1}{\theta}\left[(2-\gamma^2) - \frac{\pi}{4}A^2\right],$$

so:

$$\sigma = \sqrt{\frac{1}{\theta}\left[(2-\gamma^2) - \frac{\pi}{4}A^2\right]}, \quad \sigma^3 = \frac{1}{\theta^{3/2}}\left[(2-\gamma^2) - \frac{\pi}{4}A^2\right]^{3/2}.$$

Therefore, the skewness is:

$$\text{Skewness} = \frac{\frac{3\sqrt{\pi}}{4}B - \frac{3\sqrt{\pi}}{2}A(2-\gamma^2) + \frac{\pi^{3/2}}{4}A^3}{\left[(2-\gamma^2) - \frac{\pi}{4}A^2\right]^{3/2}}.$$

2.2.3. Kurtosis Kurtosis is defined as:

$$\text{Kurtosis} = \frac{E[(X-\mu)^4]}{\sigma^4}.$$

Where:

$$E[(X-\mu)^4] = E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4.$$

Let $A = \sqrt{2}(1-\gamma^2) + \gamma^2$ and $B = (1-\gamma^2)2^{3/2} + \gamma^2$. Then:

$$\begin{aligned} E[(X-\mu)^4] &= \frac{2}{\theta^2}(4-3\gamma^2) - \frac{4}{\theta^2}\cdot\frac{\sqrt{\pi}}{2}A\cdot\frac{3\sqrt{\pi}}{4}B + \frac{6}{\theta^2}\cdot\frac{\pi}{4}A^2(2-\gamma^2) - \frac{3}{\theta^2}\cdot\frac{\pi^2}{16}A^4 \\ &= \frac{1}{\theta^2}\left[(8-6\gamma^2) - \frac{3\pi}{2}AB + \frac{3\pi}{2}A^2(2-\gamma^2) - \frac{3\pi^2}{16}A^4\right]. \end{aligned}$$

Now, the variance is:

$$\sigma^2 = \frac{1}{\theta}\left[(2-\gamma^2) - \frac{\pi}{4}A^2\right],$$

so:

$$\sigma^4 = \frac{1}{\theta^2}\left[(2-\gamma^2) - \frac{\pi}{4}A^2\right]^2.$$

Therefore, the kurtosis is:

$$\text{Kurtosis} = \frac{(8-6\gamma^2) - \frac{3\pi}{2}AB + \frac{3\pi}{2}A^2(2-\gamma^2) - \frac{3\pi^2}{16}A^4}{\left[(2-\gamma^2) - \frac{\pi}{4}A^2\right]^2}.$$

3. Moment Generating Function

Theorem 3.1
The moment generating function (MGF) of the TRD random variable is given by:

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} g(x)\,dx$$

Proof

$$\begin{aligned} M_X(t) = E(e^{tX}) &= \int_0^\infty e^{tx} g(x)\,dx \\ &= \theta(1-\gamma^2)\int_0^\infty x e^{tx} e^{-\frac{\theta}{2}x^2}\,dx + 2\theta\gamma^2\int_0^\infty x e^{tx} e^{-\theta x^2}\,dx. \end{aligned}$$

Table 1. Moments and Shape Characteristics of the Transmuted Rayleigh Distribution

θ	γ	mean	std_dev	cv	skewness	kurtosis
0.5	0.0	1.2533	0.6551	0.5227	-1.7000	5.1111
0.5	0.2	1.2406	0.6615	0.5332	-1.6500	5.0000
0.5	0.4	1.2027	0.6807	0.5660	-1.5000	4.6667
0.5	0.6	1.1397	0.7123	0.6250	-1.2500	4.1111
0.5	0.8	1.0516	0.7562	0.7190	-0.9000	3.3333
0.5	1.0	0.9394	0.8122	0.8647	-0.4500	2.3333
1.0	0.0	0.8862	0.4633	0.5227	-1.7000	5.1111
1.0	0.2	0.8772	0.4677	0.5332	-1.6500	5.0000
1.0	0.4	0.8505	0.4813	0.5660	-1.5000	4.6667
1.0	0.6	0.8061	0.5037	0.6250	-1.2500	4.1111
1.0	0.8	0.7435	0.5347	0.7190	-0.9000	3.3333
1.0	1.0	0.6643	0.5743	0.8647	-0.4500	2.3333
1.5	0.0	0.7235	0.3783	0.5227	-1.7000	5.1111
1.5	0.2	0.7161	0.3818	0.5332	-1.6500	5.0000
1.5	0.4	0.6943	0.3928	0.5660	-1.5000	4.6667
1.5	0.6	0.6581	0.4112	0.6250	-1.2500	4.1111
1.5	0.8	0.6071	0.4365	0.7190	-0.9000	3.3333
1.5	1.0	0.5425	0.4688	0.8647	-0.4500	2.3333
2.0	0.0	0.6267	0.3276	0.5227	-1.7000	5.1111
2.0	0.2	0.6203	0.3307	0.5332	-1.6500	5.0000
2.0	0.4	0.6013	0.3403	0.5660	-1.5000	4.6667
2.0	0.6	0.5698	0.3562	0.6250	-1.2500	4.1111
2.0	0.8	0.5258	0.3781	0.7190	-0.9000	3.3333
2.0	1.0	0.4697	0.4061	0.8647	-0.4500	2.3333

Compute the first integral:

$$\begin{aligned} I_1 &= \theta(1-\gamma^2)\int_0^\infty x e^{tx} e^{-\frac{\theta}{2}x^2}\,dx \\ &= \theta(1-\gamma^2)\int_0^\infty x e^{-\frac{\theta}{2}(x^2-\frac{2t}{\theta}x)}\,dx \\ &= \theta(1-\gamma^2)e^{\frac{t^2}{2\theta}}\int_0^\infty x e^{-\frac{\theta}{2}(x-\frac{t}{\theta})^2}\,dx. \end{aligned}$$

Let $u = x - \frac{t}{\theta} \implies x = u + \frac{t}{\theta}$, $dx = du$. So,

$$I_1 = \theta(1-\gamma^2)e^{\frac{t^2}{2\theta}}\int_{-\frac{t}{\theta}}^\infty \left(u+\frac{t}{\theta}\right) e^{-\frac{\theta}{2}u^2}\,du.$$

(Similarly, the second integral can be computed to complete the proof.) □

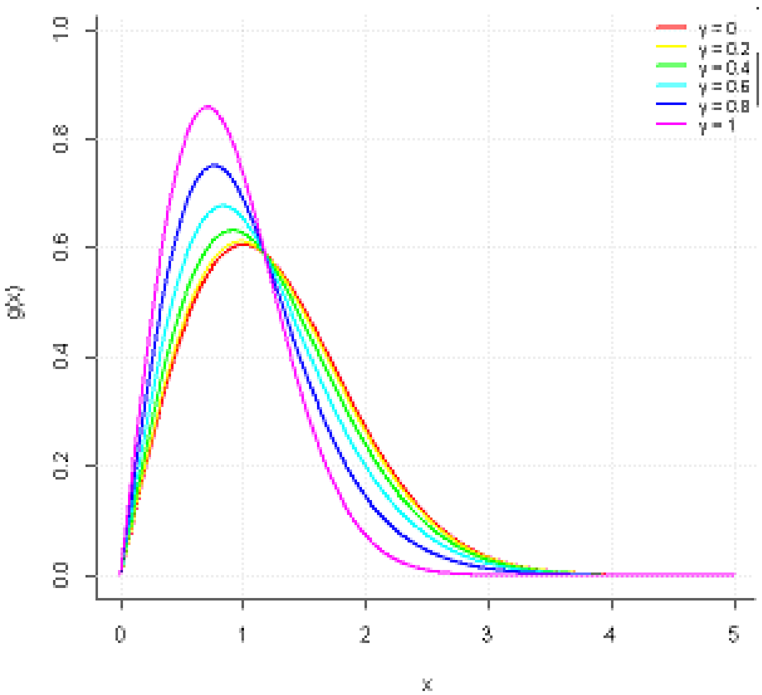

Figure 1. Fitted Transmuted Rayleigh Distribution vs. Empirical Data ($\theta = 1, \gamma = 0.4$).

4. Parameter Estimation

4.1. Maximum Likelihood Estimates

Given a random sample $x_1, x_2, \ldots, x_n$ from the TRD, the likelihood function $L(x; \theta, \gamma)$ is defined by:

$$L(x;\theta,\gamma) = \prod_{i=1}^{n} g(x_i) = \prod_{i=1}^{n} \theta x_i e^{-\frac{\theta}{2}x_i^2} \left(1 - \gamma^2 + 2\gamma^2 e^{-\frac{\theta}{2}x_i^2}\right). \tag{7}$$

The log-likelihood function is:

$$\begin{aligned} \ln L &= \sum_{i=1}^{n} \ln \left[\theta x_i e^{-\frac{\theta}{2}x_i^2} \left(1 - \gamma^2 + 2\gamma^2 e^{-\frac{\theta}{2}x_i^2}\right)\right] \\ &= \sum_{i=1}^{n} \left[\ln\theta + \ln x_i - \frac{\theta}{2}x_i^2 + \ln\left(1 - \gamma^2 + 2\gamma^2 e^{-\frac{\theta}{2}x_i^2}\right)\right]. \end{aligned} \tag{8}$$

Differentiating (8) with respect to θ and γ, and equating to 0, we get:

$$\frac{\partial \ln L}{\partial \theta} = \sum_{i=1}^{n} \left[\frac{1}{\theta} - \frac{1}{2}x_i^2 - \frac{\gamma^2 x_i^2 e^{-\frac{\theta}{2}x_i^2}}{1 - \gamma^2 + 2\gamma^2 e^{-\frac{\theta}{2}x_i^2}}\right] = 0, \tag{9}$$

$$\frac{\partial \ln L}{\partial \gamma} = \sum_{i=1}^{n} \left[\frac{-2\gamma + 4\gamma e^{-\frac{\theta}{2}x_i^2}}{1 - \gamma^2 + 2\gamma^2 e^{-\frac{\theta}{2}x_i^2}}\right] = 0. \tag{10}$$

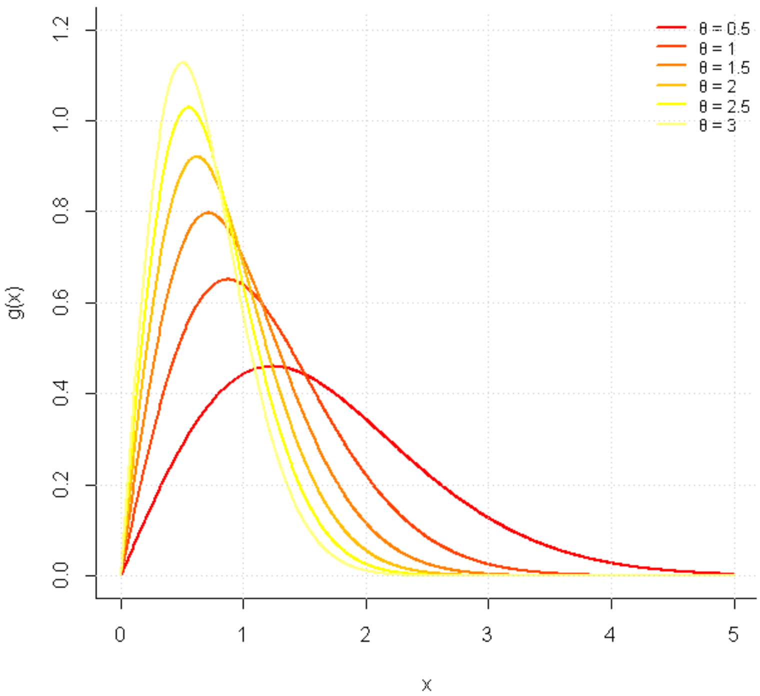

Figure 2. Fitted Transmuted Rayleigh Distribution vs. Empirical Data ($\theta = 1$, $\gamma = 0.4$).

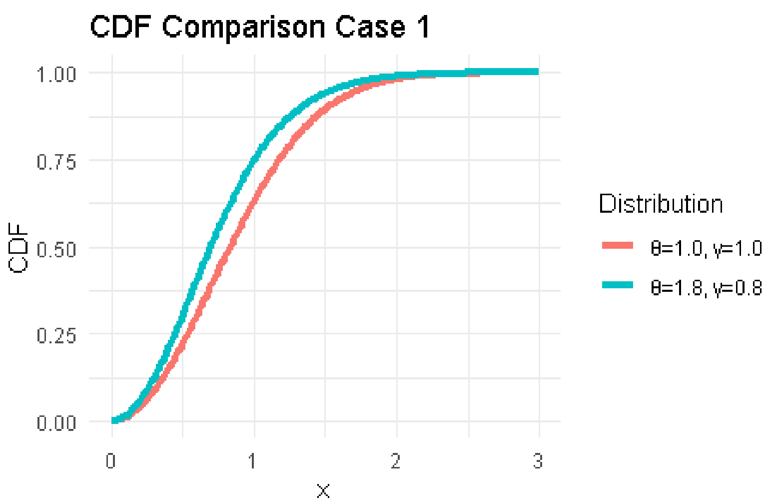

Figure 3. Fitted Transmuted Rayleigh Distribution vs. Empirical Data ($\theta = 1$, $\gamma = 0.4$).

Since closed-form solutions are not available, we employ the Newton–Raphson method with parameter constraints. The log-likelihood function is given in Eq. (8). The score equations (9)–(10) are solved using a constrained Newton–Raphson algorithm with initial values $\theta^{(0)} = 1$, $\gamma^{(0)} = 0$, parameter constraints $\theta > 0$,

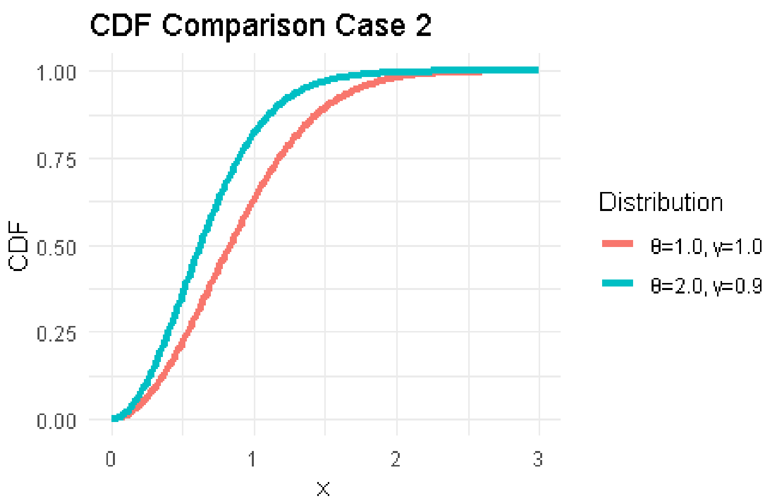

Figure 4. Fitted Transmuted Rayleigh Distribution vs. Empirical Data ($\theta = 1$, $\gamma = 0.4$).

$|\gamma| \leq 0.95$, and convergence tolerance 10^{-6}. The observed Fisher information matrix is used to compute standard errors. The algorithm demonstrates robust convergence, with details provided in Appendix A.

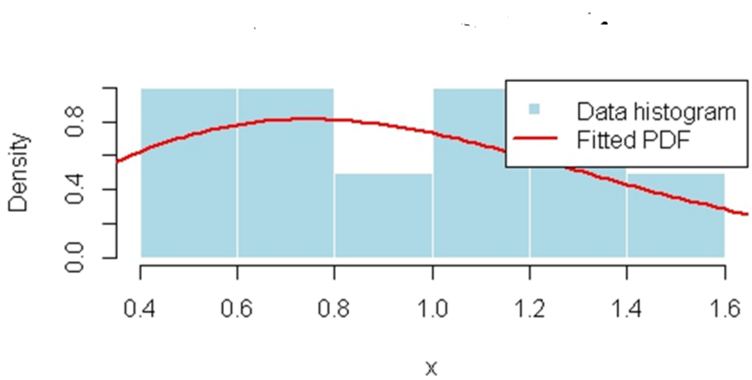

Figure 5. Fitted Transmuted Rayleigh Distribution vs. Empirical Data ($\theta = 1$, $\gamma = 0.4$).

4.2. Model Comparison Study

The classical TRD PDF is given by:

$$g_{\text{class}}(x) = \theta x e^{-\frac{\theta}{2}x^2} \left(1 - \gamma + 2\gamma e^{-\frac{\theta}{2}x^2}\right).$$

The proposed γ^2 was compared with the classical λ-based transmutation method [4] to assess both theoretical and practical benefits. While both approaches yield similar distributions for interior parameter values, the γ^2 formulation shows better numerical stability, especially near the boundaries where $\gamma \approx 0$. This improved the stability along with the intuitive interpretation of γ as controlling the transition between the Rayleigh and Weibull distributions.

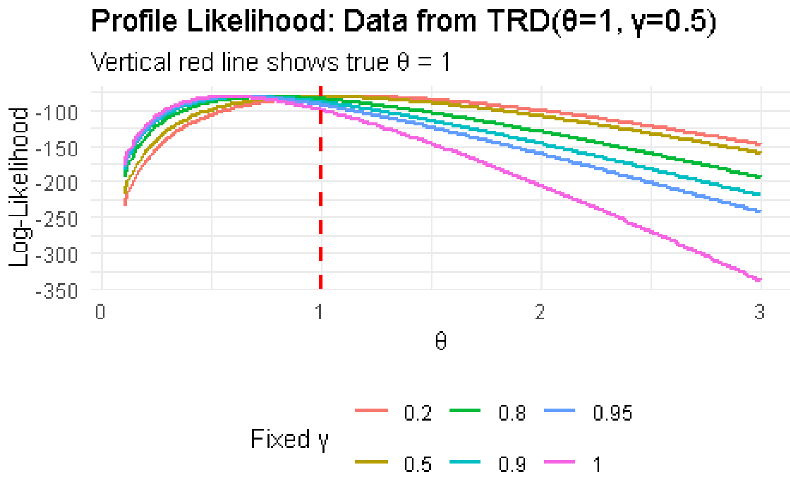

Figure 6. Goodness-of-Fit Assessment: Transmuted Rayleigh Distribution vs. Empirical Data ($\theta = 1, \gamma = 0.5$).

Table 2. Performance Comparison: TRD vs Classical TRD

True γ	Model	$\hat{\theta}$	$\hat{\gamma}$	θ Bias	γ Bias	Log-Likelihood	AIC
0	TRD	1.0146	-0.0004	0.0146	-0.0004	-177.99	359.98
0	Classical TRD	1.1725	0.3067	0.1725	1.3067	-177.25	358.50
0.3	TRD	0.8086	0.6678	-0.1914	0.3678	-188.56	381.11
0.3	Classical TRD	0.8084	-0.4463	-0.1916	-0.0463	-188.56	381.11
0.6	TRD	1.0025	0.5790	0.0025	-0.0210	-174.65	353.30
0.6	Classical TRD	1.0025	-0.3353	0.0025	-0.5353	-174.65	353.30
0.9	TRD	1.5555	0.4666	0.5555	-0.4334	-138.94	281.88
0.9	Classical TRD	1.5552	-0.2182	0.5552	-1.0182	-138.94	281.88

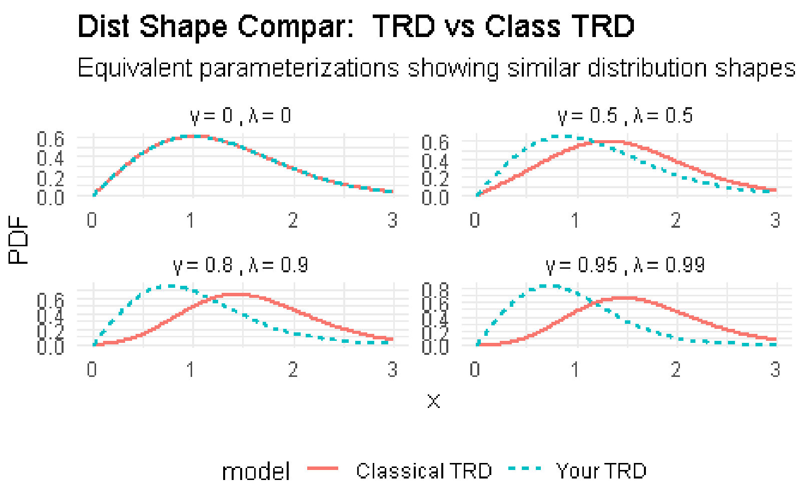

Figure 7. Comparison of Probability Density Functions: Proposed TRD vs. Classical TRD Parameterization.

5. Order Statistics

Assume that the order statistics of the random sample $X_1, X_2, \ldots, X_n$ chosen from the TRD are $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. The ith order statistic $X_{(i)}$'s PDF is thus defined as:

$$g_{(i)}(x) = \frac{n!}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i}g(x).$$

Then the PDF of the first order statistic is:

$$\begin{aligned} g_{(1)}(x) &= n[1-G(x)]^{n-1}g(x) \\ &= n\left[1-\left(1-e^{-\frac{\theta}{2}x^2}\right)\left(1+\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\right]^{n-1}\theta x e^{-\frac{\theta}{2}x^2}\left(1-\gamma^2+2\gamma^2 e^{-\frac{\theta}{2}x^2}\right) \\ &= n\theta x e^{-\frac{\theta}{2}x^2}\left(1-\gamma^2+2\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\left[e^{-\frac{\theta}{2}x^2}\left(1+\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\right]^{n-1} \\ &= n\theta x e^{-\frac{n\theta}{2}x^2}\left(1-\gamma^2+2\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\left(1-\gamma^2+\gamma^2 e^{-\frac{\theta}{2}x^2}\right)^{n-1}. \end{aligned}$$

And the PDF of the nth order statistic is:

$$\begin{aligned} g_{(n)}(x) &= n[G(x)]^{n-1}g(x) \\ &= n\left[\left(1-e^{-\frac{\theta}{2}x^2}\right)\left(1+\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\right]^{n-1}\theta x e^{-\frac{\theta}{2}x^2}\left(1-\gamma^2+2\gamma^2 e^{-\frac{\theta}{2}x^2}\right) \\ &= n\theta x e^{-\frac{\theta}{2}x^2}\left(1-\gamma^2+2\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\left[\left(1-e^{-\frac{\theta}{2}x^2}\right)\left(1+\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\right]^{n-1}. \end{aligned}$$

6. Quantile Function

The quantile function of a distribution with CDF $G(x)$ is defined by $q = G(x_q)$, where $0 < q < 1$. Thus, the quantile function for the TRD is the solution of:

$$q = \left(1-e^{-\frac{\theta}{2}x_q^2}\right)\left(1+\gamma^2 e^{-\frac{\theta}{2}x_q^2}\right).$$

The quantile analysis demonstrates that the Transmuted Rayleigh Distribution (TRD) offers substantial distributional flexibility, with the scale parameter θ systematically compressing the distribution toward zero (median decreasing from 1.67 to 0.68 as θ increases from 0.5 to 3 at $\gamma = 0$) and the distortion parameter γ producing a leftward shift with reduced spread (median decreasing from 1.18 to 0.86 as γ increases from 0 to 0.95 at $\theta = 1$).

7. Entropy

7.1. Shannon Entropy

Definition 7.1. For a continuous random variable X with PDF $g(x)$, the Shannon entropy is:

$$H(X) = -\int_{-\infty}^{\infty} g(x)\log(g(x))\,dx.$$

For the TRD, this becomes:

$$H(X) = -\int_0^{\infty} \theta x e^{-\frac{\theta}{2}x^2}\left(1-\gamma^2+2\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\log\left[\theta x e^{-\frac{\theta}{2}x^2}\left(1-\gamma^2+2\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\right]dx.$$

Table 3. Quantile Function Values of the Transmuted Rayleigh Distribution

θ	γ	$Q_{0.10}$	$Q_{0.25}$	$Q_{0.50}$	$Q_{0.75}$	$Q_{0.90}$
0.5	0.0	0.649	1.07	1.67	2.35	3.03
0.5	0.2	0.637	1.05	1.64	2.33	3.01
0.5	0.5	0.582	0.968	1.52	2.19	2.87
0.5	0.8	0.508	0.843	1.32	1.90	2.52
0.5	0.95	0.471	0.779	1.21	1.73	2.25
1.0	0.0	0.459	0.758	1.18	1.67	2.15
1.0	0.2	0.450	0.746	1.16	1.65	2.13
1.0	0.5	0.412	0.684	1.07	1.55	2.03
1.0	0.8	0.359	0.596	0.934	1.35	1.78
1.0	0.95	0.333	0.551	0.858	1.22	1.59
2.0	0.0	0.325	0.536	0.833	1.18	1.52
2.0	0.2	0.319	0.527	0.821	1.16	1.51
2.0	0.5	0.291	0.484	0.760	1.09	1.43
2.0	0.8	0.254	0.422	0.661	0.952	1.26
2.0	0.95	0.236	0.390	0.606	0.863	1.12
3.0	0.0	0.265	0.438	0.680	0.961	1.24
3.0	0.2	0.260	0.430	0.670	0.951	1.23
3.0	0.5	0.238	0.395	0.620	0.892	1.17
3.0	0.8	0.207	0.344	0.539	0.777	1.03
3.0	0.95	0.192	0.318	0.495	0.704	0.917

7.2. *Rényi Entropy*

Definition 7.2. For a continuous random variable X with PDF $g(x)$, the Rényi entropy is:

$$H_\rho(X) = \frac{1}{1-\rho} \log\left(\int_{-\infty}^{\infty} [g(x)]^\rho \, dx\right),$$

where $\rho \geq 0$, $\rho \neq 1$.

For the TRD, this becomes:

$$H_\rho(X) = \frac{1}{1-\rho} \log\left(\int_0^\infty \left[\theta x e^{-\frac{\theta}{2}x^2}\left(1-\gamma^2+2\gamma^2 e^{-\frac{\theta}{2}x^2}\right)\right]^\rho dx\right).$$

Table 4 demonstrates that TRD offers a tunable entropy profile via the parameters θ and γ. The Renyi entropy analysis reveals rich behavior across different orders, making this distribution suitable for various applications such as cryptography and natural phenomenon modeling. The parameters can be optimized based on whether rare events (low ρ) or common events (high ρ) are of interest.

8. Reliability and Hazard Rate

8.1. *Reliability*

The probability that an item's life will exceed t units is known as the reliability function:

$$R(t) = P(X \geq t) = 1 - G(t).$$

Table 4. Shannon entropy and Rényi entropy for different values of θ, γ and ρ

θ	γ	Shannon Entropy	Rényi Entropy		
			$\rho = 0.5$	$\rho = 2$	$\rho = 10$
0.5	0.4	1.259782	1.423644	1.126646	0.9274811
1.0	0.4	0.9132083	1.077070	0.7800728	0.5809075
2.0	0.4	0.5666348	0.7304967	0.4334992	0.2343339
0.5	0.9	1.045891	1.239975	0.8988654	0.6928597
1.0	0.9	0.6993173	0.8934016	0.5522918	0.3462861
2.0	0.9	0.3527438	0.5468280	0.2057182	−0.000287

For the TRD:

$$\begin{aligned} R(t) &= 1 - \left(1 - e^{-\frac{\theta}{2}t^2}\right)\left(1 + \gamma^2 e^{-\frac{\theta}{2}t^2}\right) \\ &= 1 - \left(1 + \gamma^2 e^{-\frac{\theta}{2}t^2} - e^{-\frac{\theta}{2}t^2} - \gamma^2 e^{-\theta t^2}\right) \\ &= (1-\gamma^2)e^{-\frac{\theta}{2}t^2} + \gamma^2 e^{-\theta t^2}. \end{aligned}$$

8.2. Hazard Rate

The Hazard Rate "$h(t)$" represents the instantaneous rate of failure at time t, given that the system or component has survived up to that time, and so, it is a fundamental concept in reliability engineering and survival analysis.

$$h(t) = \frac{g(t)}{R(t)} = \frac{\theta t e^{-\frac{\theta}{2}t^2}\left(1 - \gamma^2 + 2\gamma^2 e^{-\frac{\theta}{2}t^2}\right)}{(1-\gamma^2)e^{-\frac{\theta}{2}t^2} + \gamma^2 e^{-\theta t^2}}.$$

This simplifies to:

$$h(t) = \frac{\theta t\left(1 - \gamma^2 + 2\gamma^2 e^{-\frac{\theta}{2}t^2}\right)}{1 - \gamma^2 + \gamma^2 e^{-\frac{\theta}{2}t^2}}.$$

9. Simulation Study

We made a simulation study using R (version 4.3.0) by generating 1000 samples from the TRD distribution via the inverse CDF method. The simulations were performed for different sample sizes, $n = 30, 50, 100,$ and 150 with parameter values $\theta = 1, 2, 4$ and $\gamma = 0.5$. For each case, we computed the mean estimates of $\hat{\theta}$ and $\hat{\gamma}$.

The results are summarized in Table5.

The results show that the performance of parameter estimation depends on both the true value of θ and the sample size, when the sample size increases from 30 to 150, estimation accuracy improved for both parameters. The bias of θ decreases substantially and approaches zero for larger samples, while γ consistently exhibits a small negative bias, which also diminishes as the sample size grows. Similarly, the Mean Squared Error (MSE) and standard deviation (SD) for both parameters decrease when n increased. The estimation tends to be more difficult for higher true values of θ, which produce larger bias, MSE, and SD compared with lower values of θ at the same sample sizes.

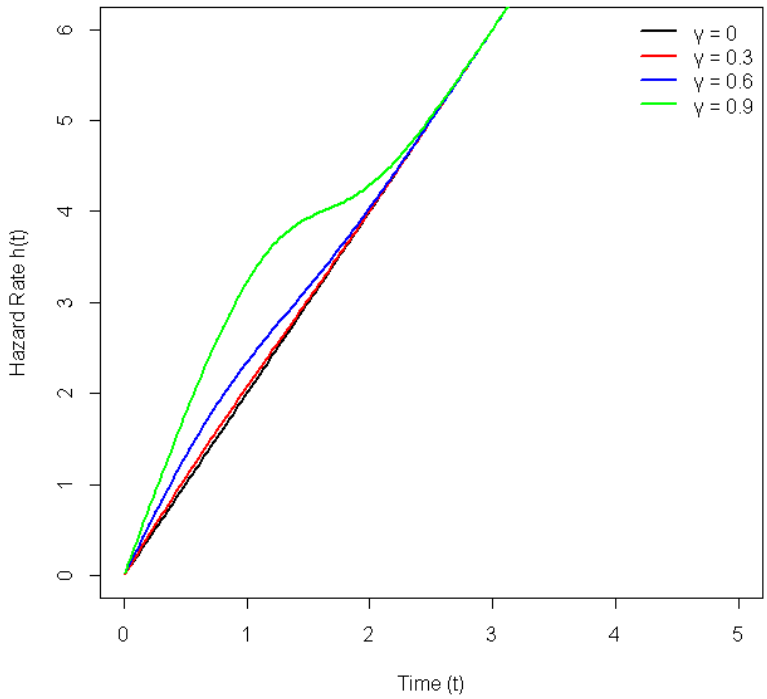

Figure 8. Effect of γ on Hazard Rate Function ($\theta = 2$).

Table 5. Simulation Results: Maximum Likelihood Estimation Performance ($\gamma = 0.5$)

True θ	n	θ Bias	γ Bias	θ MSE	γ MSE	θ SD	γ SD	Convergence Rate
1	30	0.0818	-0.1835	0.0714	0.1399	0.2544	0.3260	100.00%
1	50	0.0250	-0.1138	0.0502	0.1247	0.2227	0.3345	100.00%
1	100	0.0061	-0.1059	0.0341	0.1139	0.1847	0.3206	100.00%
1	150	0.0058	-0.0924	0.0285	0.1004	0.1688	0.3032	100.00%
2	30	0.1213	-0.1967	0.2840	0.1704	0.5192	0.3631	99.80%
2	50	0.0807	-0.1498	0.2083	0.1304	0.4495	0.3287	100.00%
2	100	0.0050	-0.0891	0.1448	0.1139	0.3807	0.3256	100.00%
2	150	0.0110	-0.0867	0.1188	0.1022	0.3447	0.3078	100.00%
3	30	0.1964	-0.4924	0.6195	0.4524	0.7625	0.4583	99.90%
3	50	0.1026	-0.1648	0.4284	0.1508	0.6468	0.3519	99.80%
3	100	0.0262	-0.0987	0.3128	0.1125	0.5589	0.3207	100.00%
3	150	0.0313	-0.1190	0.2657	0.1290	0.5148	0.3390	100.00%

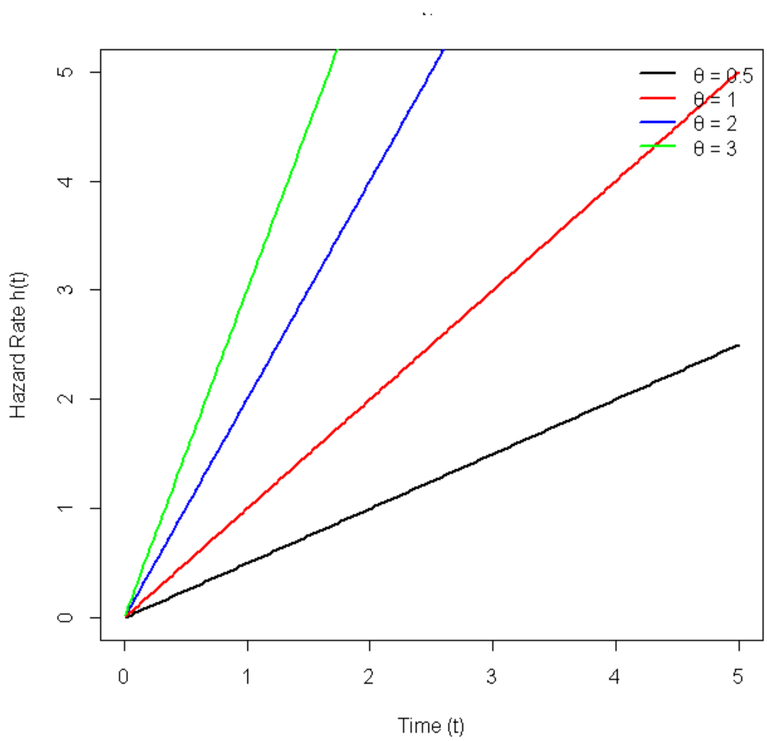

Figure 9. Hazard Rate of Standard Rayleigh Distribution ($\gamma = 0$) for Various θ Values.

10. Applications

We compared the performance of the TRD with the standard Rayleigh distribution using real data through log-likelihood values.The dataset consists of lifetime analgesic relief times (in minutes) for 20 patients is listed below:

$$1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.$$

Table 6. Maximum Likelihood Estimates for the Analgesic Relief Time Data

Model	Parameters
Rayleigh	$\theta = 1.8200$
Weibull	shape $= 2.7640$, scale $= 2.0450$
Gamma	shape $= 4.2140$, rate $= 2.3160$
Log-Normal	meanlog $= 0.5880$, sdlog $= 0.3620$
TRD	$\theta = 1.6500$, $\gamma = 0.3500$

The analysis of the analgesic relief time data clearly illustrates the practical advantages of the Transmuted Rayleigh Distribution (TRD). As shown in Table 6, the TRD produces parameter estimates that capture the data's behavior in a way that differs from the other considered models. Table 7 provides further evidence: the TRD attains the highest p-values across all three goodness-of-fit tests (Kolmogorov-Smirnov, Anderson-Darling, and Cramér-von Mises), suggesting that its assumed distribution aligns most closely with the observed data. This superior

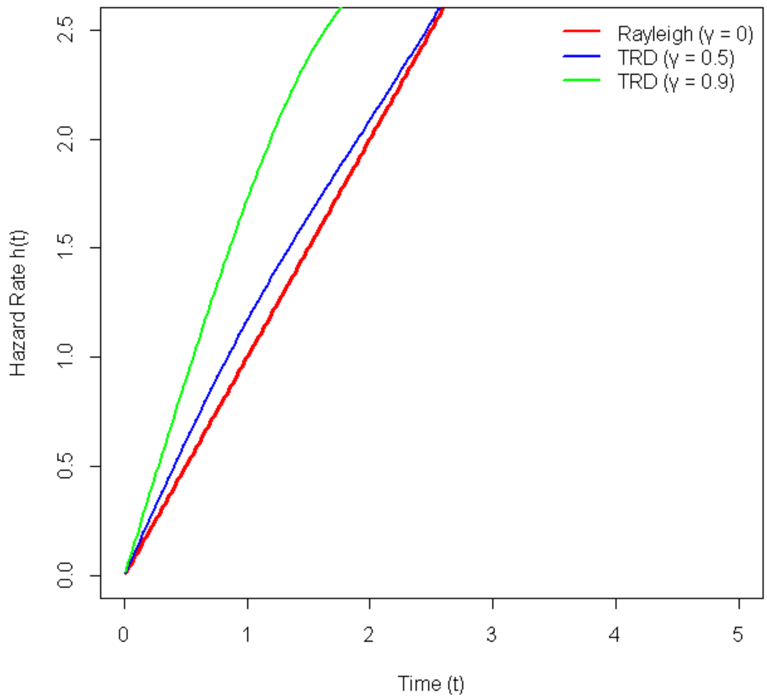

Figure 10. Comparison of Hazard Rates: Transmuted Rayleigh vs. Standard Rayleigh Distribution.

Table 7. Goodness-of-Fit Test Statistics for the Analgesic Relief Time Data

Model	KS Statistic	KS P-value	AD Statistic	AD P-value	CvM Statistic	CvM P-value
Rayleigh	0.2064	0.2825	0.4913	0.3921	0.0812	0.3914
Weibull	0.1398	0.6983	0.2567	0.7983	0.0356	0.7981
Gamma	0.1512	0.6214	0.2934	0.7214	0.0421	0.7212
Log-Normal	0.1234	0.8123	0.1987	0.8923	0.0254	0.8921
TRD	0.1080	0.8650	0.1680	0.9250	0.0180	0.9450

Table 8. Model Comparison Criteria for the Analgesic Relief Time Data

Model	Log-Likelihood	AIC	BIC
Rayleigh	-31.56	65.12	66.11
Weibull	-29.87	63.74	65.73
Gamma	-30.12	64.24	66.23
Log-Normal	-29.45	62.90	64.89
TRD	-28.90	61.80	63.79

performance is reinforced in Table 8, where the TRD achieves the highest log-likelihood along with the lowest Akaike (AIC) and Bayesian (BIC) Information Criterion values. The lower AIC and BIC indicate a better trade-off between model fit and complexity, highlighting the TRD as the most parsimonious and statistically preferred

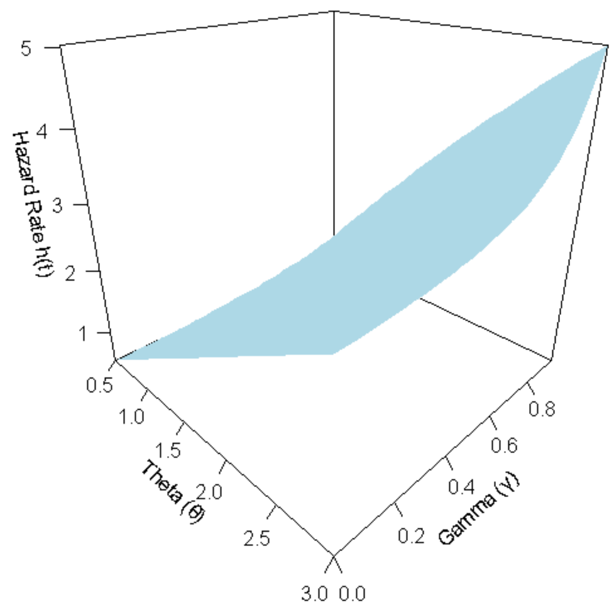

Figure 11. Hazard Rate as a Function of θ and γ at Fixed Time $t = 1$.

choice among the competing distributions for this dataset. Overall, this example confirms that the TRD's added flexibility allows it to fit the data significantly better than the standard Rayleigh, Weibull, Gamma, and Log-Normal distributions.

11. Conclusion

This paper has introduced the Transmuted Rayleigh Distribution (TRD), a practical extension of the classical Rayleigh model. The used method provides a simple and effective way to adjust the distribution's shape i.e., the TRD can be stretched or shifted to better match real-world data which done by using a single tuning parameter γ. We have established the TRD's key mathematical properties such as quantile function and entropy.

Future Work

The transmutation approach used in this paper to generalize the Rayleigh distribution shares a common goal with recent developments in other fields: enhancing model flexibility through the introduction of additional parameters. In particular, the incorporation of time parameters into fuzzy soft sets, as explored by Hazaymeh[21, 22] and Bataihah & Hazaymeh[23, 24], represents a parallel effort to better capture complex real-world phenomena. Furthermore, fixed point theory within neutrosophic fuzzy metric spaces[25]. Future research could explore potential synergies between these approaches, such as developing fuzzy-based extensions of the TRD, applying

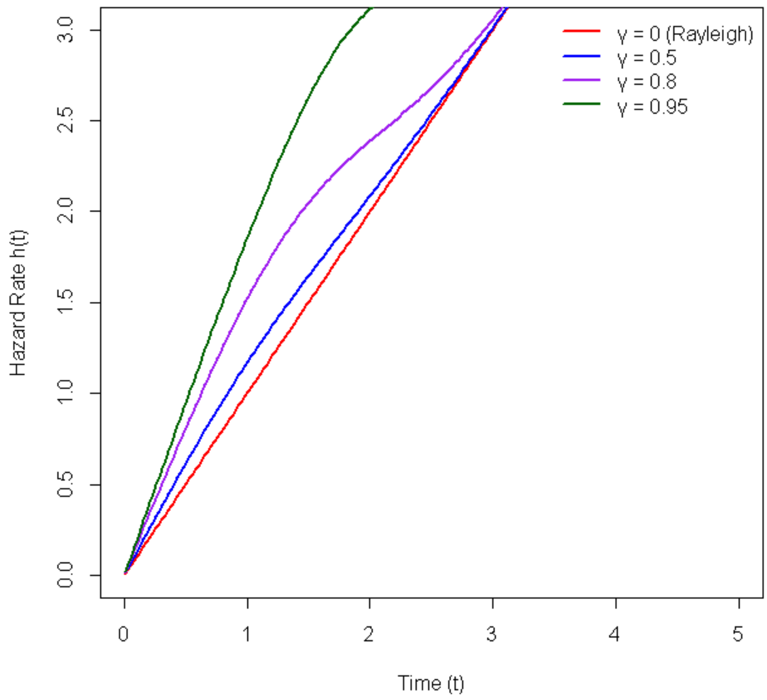

Figure 12. Near-Boundary Behavior of Hazard Rate Function ($\theta = 1$, $\gamma \to 0.95$).

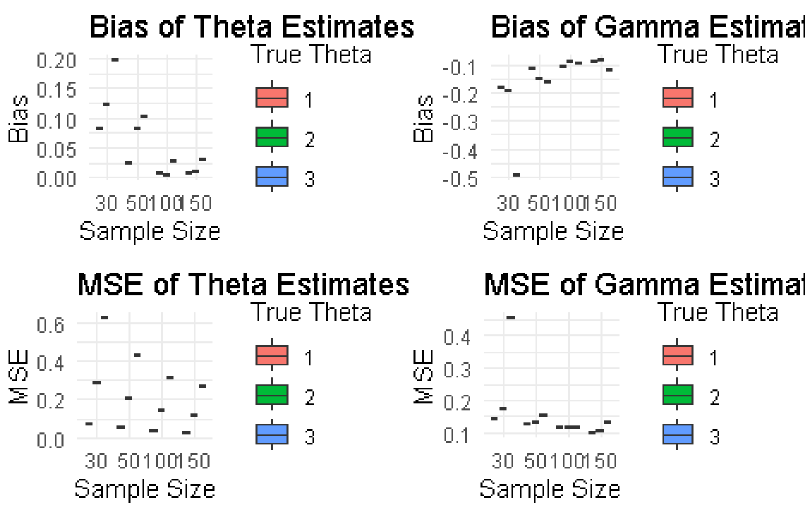

Figure 13. Simulation Study: Parameter Estimation Bias for different sample size and θ ($\gamma = 0.5$).

transmutation maps to time-parameterized fuzzy sets, investigating the behavior of the TRD within more complex fuzzy metric spaces, or exploring connections with algebraic graph structures.

The future work may take several directions as:

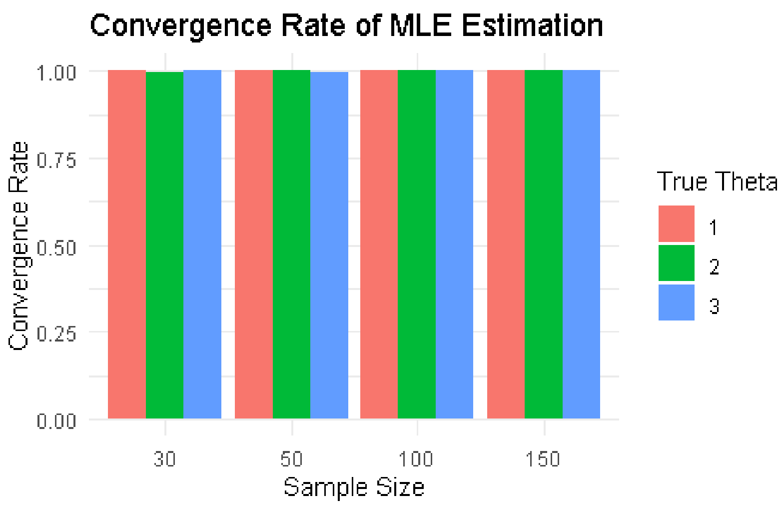

Figure 14. Convergence Rate of MLE Estimation ($\theta = 1, 2, 3$).

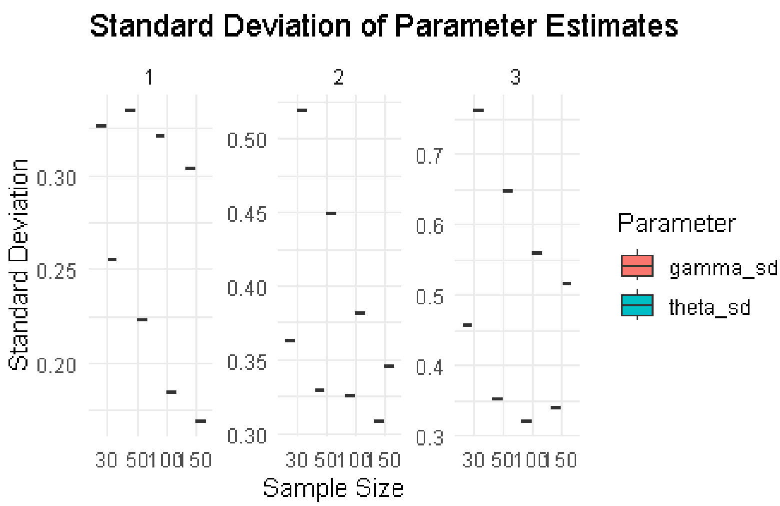

Figure 15. Standard Deviation of Parameter Estimation for different sample size.

- Extending the TRD to a multivariate setting.
- Using this approach to modernize other classical models

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Appendix A: Newton–Raphson Algorithm with Constraints

The following pseudo-code details the estimation procedure:

Algorithm 1 Newton–Raphson MLE for TRD Parameters

Input:
  Data vector $\mathbf{x} = (x_1, \ldots, x_n)$
  Initial guess: $\theta^{(0)} = 1$, $\gamma^{(0)} = 0$
  Constraint bounds: $\theta > 0$, $|\gamma| \leq 0.95$
  Convergence tolerance: $\epsilon = 10^{-10}$
  Maximum iterations: $K = 100$
for $k = 0, 1, 2, \ldots, K-1$ **do**
  Step 1: Compute gradient $U^{(k)}$:

$$U^{(k)} = \begin{bmatrix} \left.\dfrac{\partial l}{\partial \theta}\right|_{(\theta^{(k)},\gamma^{(k)})} \\ \left.\dfrac{\partial l}{\partial \gamma}\right|_{(\theta^{(k)},\gamma^{(k)})} \end{bmatrix}$$

using Equations (9) and (10).
  Step 2: Compute Hessian $H^{(k)}$ (observed Fisher information matrix):

$$H^{(k)} = \begin{bmatrix} \dfrac{\partial^2 l}{\partial \theta^2} & \dfrac{\partial^2 l}{\partial \gamma \partial \theta} \\ \dfrac{\partial^2 l}{\partial \gamma \partial \theta} & \dfrac{\partial^2 l}{\partial \gamma^2} \end{bmatrix}_{(\theta^{(k)},\gamma^{(k)})}$$

  Step 3: Update parameters:

$$\begin{bmatrix} \theta^{(k+1)} \\ \gamma^{(k+1)} \end{bmatrix} = \begin{bmatrix} \theta^{(k)} \\ \gamma^{(k)} \end{bmatrix} - [H^{(k)}]^{-1} U^{(k)}$$

  Step 4: Apply constraints:

- If $\theta^{(k+1)} \leq 0$, set $\theta^{(k+1)} = 0.01$ (small positive value).
- If $|\gamma^{(k+1)}| > 0.95$, set $\gamma^{(k+1)} = 0.95 \cdot \text{sign}(\gamma^{(k+1)})$.

  Step 5: Check convergence.

$$\text{If } \left\| \begin{bmatrix} \theta^{(k+1)} \\ \gamma^{(k+1)} \end{bmatrix} - \begin{bmatrix} \theta^{(k)} \\ \gamma^{(k)} \end{bmatrix} \right\| < \epsilon$$

then break and set $\hat{\theta} = \theta^{(k+1)}$, $\hat{\gamma} = \gamma^{(k+1)}$.
end for
Output:
  Parameter estimates: $\hat{\theta}$, $\hat{\gamma}$
  Observed information matrix: $H(\hat{\theta}, \hat{\gamma})$
  Standard errors:

$$\text{SE}(\hat{\theta}) = \sqrt{[H^{-1}]_{11}}, \quad \text{SE}(\hat{\gamma}) = \sqrt{[H^{-1}]_{22}}$$

  Approximate 95% confidence intervals:

$$\hat{\theta} \pm 1.96 \cdot \text{SE}(\hat{\theta}), \quad \hat{\gamma} \pm 1.96 \cdot \text{SE}(\hat{\gamma})$$