STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, July 2026, pp 797–807.

Dynamic and Persistence of the Heteroclinic orbit for the model of the cantilever beam

Andrés Felipe Camelo Muñoz, Diego Alexander Castro Guevara, Guillermo Villa Martínez*

afkamelo@utp.edu.co, xandercastro@utp.edu.co, gvilla@utp.edu.co.
Department of Mathematics , Universidad Tecnológica de Pererira , Colombia

Abstract We study the dynamics for a nonlinear model of a cantilever beam, the dynamics of the system are governed by a quasilinear equation. We analyze the existence of a pitchfork bifurcation on the conservative case with the presence of heteroclinic orbit, then we study the persistence of heteroclinic orbits when the system is subjected to a periodic perturbation of the form $f(t) = \gamma cos(wt)$.

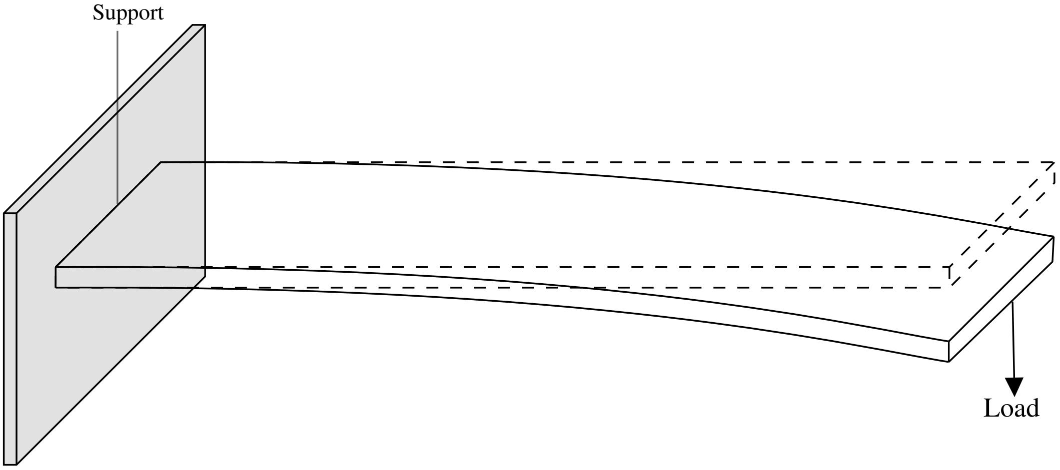
Keywords Nonlinear analysis, Melnikov function, spectral analysis, dynamical systems, beam dynamic

AMS 2010 subject classifications 65P30

DOI: 10.19139/soic-2310-5070-3162

1. Introduction

A cantilever beam is a type of beam that is supported or fixed at only one end, while the other end remains free. Due to this configuration, cantilever beams are subjected to significant bending moments and shear forces at the fixed end.



Cantilever beams are commonly employed in a wide range of structures and infrastructures, including:

1. **Bridges and walkways:** To avoid the need for intermediate supports that may obstruct traffic or water flow.

---

*Correspondence to: Guillermo Villa Martínez (Email: gvilla@utp.edu.co). Department of Department of Mathematics, Universidad Tecnológica de Pererira, Carrera 27 #10-02 Barrio Alamos - Pereira - Risaralda - Colombia (660003).

ISSN 2310-5070 (online) ISSN 2311-004X (print)

2. **Building balconies:** To extend without the need for additional support columns.
3. **Cranes and mechanical arms:** To enable movement in open spaces without support restrictions.

The design process must account for key factors, including material strength, applied loads, and stiffness, to ensure the prevention of excessive deformations or potential structural failure. In the case of free vibrations of a thin and inextensible cantilever beam carrying an intermediate concentrated mass with rotational inertia, The system exhibits dynamics governed by a nonlinear equation.

$$\ddot{u} + u + \alpha u^2 \ddot{u} + \alpha u \dot{u}^2 + \beta u^3 = 0 \tag{1}$$

where α is the geometric nonlinearity coefficient and β is the inertial nonlinearity coefficient. The third and fourth terms in equation (1) represent the cubic inertial nonlinearity resulting from the inextensibility assumption. The last term corresponds to a static nonlinearity associated with the potential energy stored in bending.
A cantilever beam without mass and with a lumped mass attached to its free end, while being harmonically excited at its base, has received considerable attention due to its relevance in applications such as mast antennas, towers, flexible robotic manipulators, and space structures. When the beam's support moves, it may be subjected to external or parametric excitation. In the parametric case, resonance is considered, and nonlinearities begin to influence the motion and therefore cannot be neglected, requiring the use of models of this type.

$$\ddot{u} + u + \alpha u^2 \ddot{u} + \alpha u \dot{u}^2 + \beta u^3 = f(t) \tag{2}$$

In the presence of a periodic forcing term satisfying $f(t+T) = f(t)$ [1, 2], the analysis of the system becomes increasingly difficult due to its pronounced nonlinearity and parameter dependence. In [3], Herişanu and Marinca conducted an analytical investigation of the nonlinear oscillations involving free planar and large-amplitude flexural vibrations of a thin and inextensible cantilever beam carrying a concentrated mass with rotational inertia positioned at an intermediate location along its span. An approximate analytical technique, specifically the optimal homotopy asymptotic method, was applied [4]. Further studies of interest related to the model and its analysis are available in [10, 8]. Whereas previous studies [3, 9] have concentrated on approximate analytical solutions for free vibrations, this study establishes rigorously the existence of heteroclinic dynamics in the forced system.

2. Spectral Analysis of the System

This section focuses on recasting equation (1) as a system with the aim of studying of its solutions in the phase plane. By defining $\xi = \dot{u}$, the equation can thus be reformulated as follows:

$$\begin{cases} \dot{u} = \xi \\ \dot{\xi} = \frac{-u-\beta u^3-\alpha u \xi^2}{1+\alpha u^2} \end{cases} \tag{3}$$

Where $\alpha > 0$ and $\beta \in \mathbb{R}$. The equilibrium states of the system are characterized by:

$$P_0 = (0,0) \qquad\qquad P_{1,2} = (\pm\frac{1}{\sqrt{-\beta}}, 0)$$

For $\beta < 0$. When $\beta \geq 0$, the only equilibrium solution is $P_0 = (0,0)$. Let us consider the function given by:

$$H(u,\xi) = \frac{-u-\beta u^3-\alpha u \xi^2}{1+\alpha u^2},$$

in this case, the Jacobian matrix associated with system (3) can be expressed as:

$$J(u,\xi) = \begin{bmatrix} 0 & 1 \\ H_u & H_\xi \end{bmatrix}$$

where;

$$H_u = \frac{-1 - 3\beta u^2 - \alpha\xi^2 + \alpha u^2 - \alpha\beta u^4 + \alpha^2 u^2 \xi^2}{(1+\alpha u^2)^2}$$

$$H_\xi = \frac{-2\alpha u \xi}{1+\alpha u^2}$$

If $\beta < 0$, the equilibrium solutions $P_{1,2}$ arise. It is important to note that $H_\xi(P_{1,2}) = 0$. Furthermore, owing to the symmetry of the function H, it follows that $H_u(u,\xi) = H_u(-u,\xi)$. Thus:

$$H_u(P_1) = H_u(P_2)$$

Calculating $H_u\left(\frac{1}{\sqrt{\beta}}, 0\right)$, under the change of variable $\beta \to -\beta$, we obtain:

$$H_u(\frac{1}{\sqrt{\beta}}, 0) = \frac{-1 + \frac{3\beta}{\beta} - \frac{\alpha\beta}{\beta^2}}{(1+\frac{\alpha}{\beta})^2} = \frac{2}{(1+\alpha/\beta)^2}$$

which leads to:

$$J(u,\xi) \mid_{P_{1,2}} = \begin{bmatrix} 0 & 1 \\ \frac{2}{(1+\alpha/\beta)^2} & 0 \end{bmatrix}$$

Therefore, the eigenvalues of the Jacobian matrix can be expressed as:

$$\lambda^2 - \frac{2}{(1+\alpha/\beta)^2} = 0, \;\; \lambda = \pm\sqrt{\frac{2}{(1+\alpha/\beta)^2}}$$

Thus, the equilibrium points $P_{1,2}$ will correspond to hyperbolic saddles when $-\beta > 0$.
Now, by analyzing the equilibrium point $P_0 = (0,0)$, it can be seen that the Jacobian matrix takes the following form:

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Consequently, the eigenvalues are $\lambda = \pm i$, implying the existence of centers. The solution trajectories of the system are depicted in the following figure.

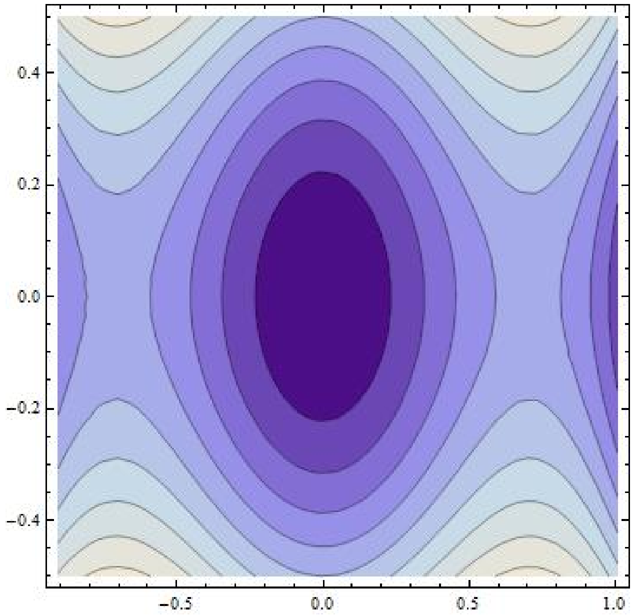


Figure 1. Contour lines for energy

The system (1) can be written as:

$$\underbrace{(1+\alpha u^2)\ddot{u} + u(1+\beta u^2)}_{\text{Hamiltonian part}} + \underbrace{\alpha u \dot{u}^2}_{\text{No Hamiltonian}} = 0$$

Considering the energy of the system associated with the Hamiltonian part, we see that:

$$\ddot{u} + \frac{u(1+\beta u^2)}{1+\alpha u^2} = 0$$

and the associated energy is expressed as:

$$\frac{\dot{u}^2}{2} + \frac{\beta}{2\alpha}u^2 + \frac{\alpha-\beta}{2\alpha^2}\ln(1+\alpha u^2) = E(u,\dot{u})$$

which has potential energy given by:

$$\varphi(u) = \frac{\beta}{2\alpha}u^2 + \frac{\alpha-\beta}{2\alpha^2}\ln(1+\alpha u^2).$$

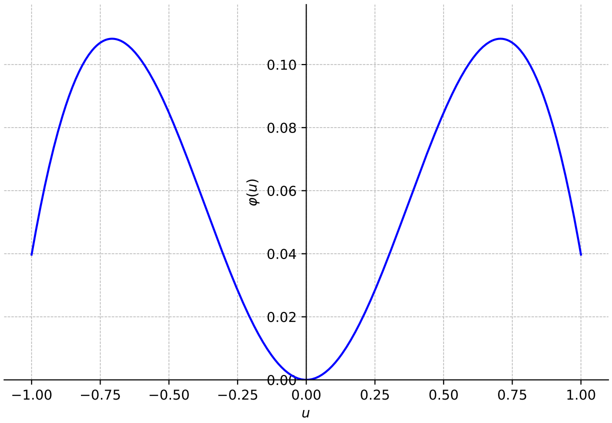


Figure 2. Potential energy $\varphi(u)$, for $-\beta > 0$.

Thus, at P_0 there will be local nonlinear centers. Based on the previous results, we state the following theorem:

Theorem 1
The equilibrium solutions for system (3) are classified as follows:

- For $\beta > 0$, there exists a unique equilibrium point, $P_0 = (0,0)$, which represents a nonlinear center of the corresponding Hamiltonian system.
- For $\beta < 0$, there are three equilibrium points: $P_0 = (0,0)$, which is a nonlinear center, and $P_\pm = (\pm 1/\sqrt{-\beta}, 0)$, which represent hyperbolic saddles.

The bifurcation diagram below provides a summary of the equilibrium solutions as a function of the parameter β.

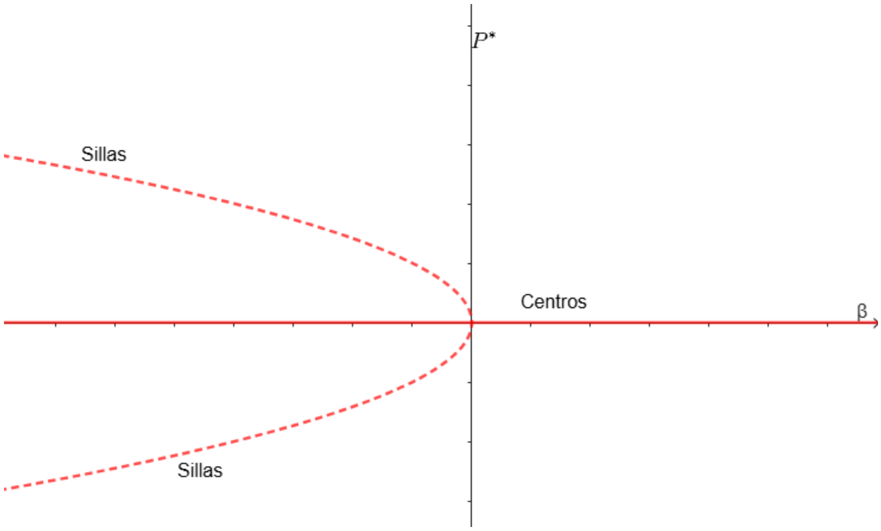


Figure 3. Pitchfork bifurcation diagram associated with the model.

3. Melnikov Function

In this section, we analyze the persistence of heteroclinic orbits when the cantilever model is subjected to a periodic perturbation of the form $f(t) = \gamma\cos(wt)$, the use of a cosine-type forcing in the mathematical model of a cantilever beam is particularly relevant, as it provides a realistic representation of periodic external excitations commonly encountered in mechanical systems, such as actuator-induced vibrations, harmonic dynamic loads, or environmental perturbations. From a dynamical perspective, this type of forcing enables the investigation of fundamental phenomena including resonance and interactions with the natural frequencies of the system. Moreover, being a smooth, bounded, and periodic function, cosine forcing offers a mathematically convenient framework for the rigorous analysis of nonlinear dynamics, facilitating the application of qualitative and analytical tools. In the presence of nonlinearities, such excitation may give rise to complex dynamical behaviors, including bifurcations and heteroclinic connections, making it a canonical choice in the study of forced structural vibrations. Investigating how these orbits behave under small perturbations is of significant practical relevance, as even minor vibrations can destabilize the cantilever's oscillations and potentially result in fractures.
The system will be considered in the following form:

$$x' = f(x) + \epsilon g(x,t) \quad x \in \mathbb{R}^2 \tag{4}$$

where f denotes a Hamiltonian vector field in $\mathbb{R}^2$, $g \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2/(T\mathbb{Z}))$, and $\epsilon \geq 0$. For the unperturbed case of problem (4) (i.e., when $\epsilon = 0$), we assume the existence of a family of periodic orbits described by:

$$\Gamma_e := \{(u,\xi) : H(u,\xi) = e\}, \quad e \in (a,b) \tag{5}$$

where Γ_e tends to a center as $e \to a$ and to an invariant curve, denoted by Γ_β, as $e \to b$. When Γ_β is bounded, it constitutes a heteroclinic orbit connecting two saddle points and enclosing a family of dense periodic orbits. The leading-order approximation of $\Gamma_e(t,\epsilon)$ is characterized by the zeros of the Melnikov function $M_e(t)$, defined as follows:

$$M_e(t) = \int\limits_{H(u,\xi)=e} g \wedge d(u,\xi) = \int\limits_{H(u,\xi)=e} g_2 du - g_1 d\xi \tag{6}$$

where $g(x,t) = (g_1(x,t), g_2(x,t))$, with $x = (u,\xi)$. Thus, identifying the zeros of (6) becomes essential.

Theorem 2
(**[5], Theorem 6.4**) Suppose $e_0 \in (\alpha,\beta)$ and $t_0 \in \mathbb{R}$.

1. If $M_{e_0}(t_0) \neq 0$, then there are no limit cycles near Γ_{e_0} for $\epsilon + |t_0 + t|$ sufficiently small.
2. If $M_{e_0}(t_0) = 0$ is a simple zero, then there exists exactly one limit cycle $\Gamma_{e_0}(t_0,\epsilon)$ for $\epsilon + |t_0 + t|$ sufficiently small, which approaches Γ_{e_0} as $(t,\epsilon) \to (t_0, 0)$.

The Melnikov function can be understood as the first approximation in ϵ of the distance between the stable and unstable manifolds of the saddle points, measured in the direction perpendicular to the unperturbed orbit, and is specifically given by:

$$d(\epsilon) = \epsilon\left(\frac{M_{\beta(t_0)}}{f(\Gamma_\beta)}\right) + O(\epsilon^2)$$

For further details, see [6, 7].
We will now consider the following system:

$$\begin{cases} u' = \xi, \\ \xi' = -\dfrac{u\left(1+\beta u^2\right)}{1+\alpha u^2} + \varepsilon\left(\gamma\cos\big(\omega(t+t_0)\big) - \dfrac{\alpha u \xi^2}{1+\alpha u^2}\right). \end{cases} \tag{7}$$

where the functions $f(u,\xi)$ and $g(u,\xi,t)$ are defined as follows:

$$f(u,\xi) = \begin{pmatrix} \xi \\ -\frac{u(1+\beta u^2)}{1+\alpha u^2} \end{pmatrix}; \; g(u,\xi,t) = \begin{pmatrix} 0 \\ \gamma cos(w(t+t_0) - \frac{\alpha u \xi^2}{1+\alpha u^2} \end{pmatrix}$$

If $\epsilon = 0$, we study the energy associated with the system,

$$\frac{(\dot{u})^2}{2} + \int \frac{u(1+\beta u^2)}{(1+\alpha u^2)} \dot{u} dt = cte$$

$$\int \frac{u(1+\beta u^2)}{(1+\alpha u^2)} = \int \frac{\beta}{\alpha} u + \int \frac{\eta u}{(\alpha u^2 + 1)} = \frac{\beta}{2\alpha} u^2 + \frac{\eta}{2\alpha} ln(1+\alpha u^2)$$

The energy system is derived as follows:

$$\frac{\dot{u}^2}{2} + \frac{\beta}{2\alpha} u^2 + \frac{\eta}{2\alpha} ln(1+\alpha u^2) = E(u,\dot{u}) = cte$$

Where $\eta = 1 + \beta/\alpha$. The heteroclinic orbit is characterized by the following curve:

$$H(1/\sqrt{-\beta}, 0) = \frac{\beta}{2\alpha}\frac{1}{\beta} + \frac{\eta}{2\alpha} ln|1 + \frac{\alpha}{|\beta|}| = H_B$$

The solution for the heteroclinic curve is:

$$\frac{1}{2}(\frac{du}{dt})^2 = H_B - \frac{\beta}{2\alpha}u^2 - \frac{\eta}{2\alpha} ln|1+\alpha u^2|$$

Which leads to:

$$\frac{du}{dt} = \sqrt{2H_B - \frac{\beta}{\alpha}u^2 - \frac{\eta}{2\alpha} ln|1+\alpha u^2|}$$

Separating variables and integrating:

$$t = \int dt = \int \frac{du}{\sqrt{2H_B - \frac{\beta}{\alpha}u^2 - \frac{\gamma}{2\alpha} ln|1+\alpha u^2|}}$$

In this way:

$$u(t) = \gamma_0(t)$$

thereby defining the heteroclinic curve.

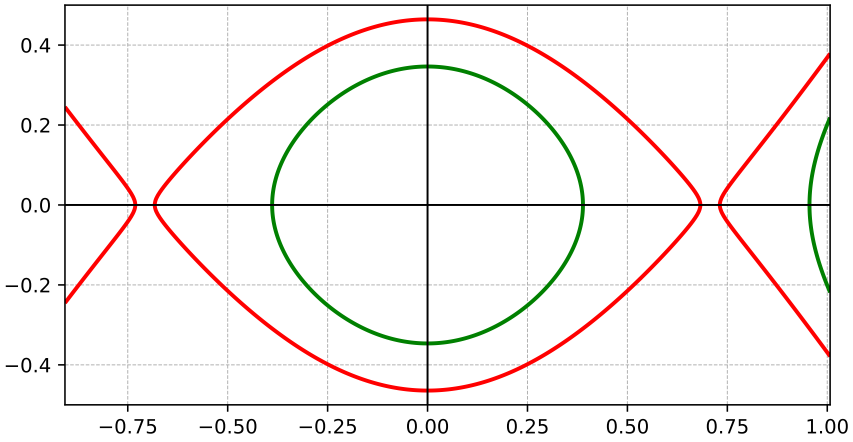


Figure 4. Red: Heteroclinic curve; Green: Periodic solution

Let us define the Melnikov function for the heteroclinic orbit γ_0.

$$M(t_0,\phi_0)=\int_{-\infty}^{\infty} f(\gamma_0(t)\wedge(\gamma_0(t),t+t_0)dt$$

Where $\gamma_0(t)=(u_h(t),\xi_h(t))$ represents the solution of (7) corresponding to the heteroclinic orbit for $\epsilon=0$, and $f\wedge g=f_1g_2-f_2g_1$, with $f=(f_1,f_2)$ and $g=(g_1,g_2)$.

$$f(\gamma_0(t)=\begin{pmatrix}\xi_h(t)\\ \frac{-u_h(1+\beta u_h^2)}{(1+\alpha u_h^2)}\end{pmatrix}$$

$$g((\gamma_0(t),t_0+t)=\begin{pmatrix}0\\ \varsigma cos(w(t+t_0)-\frac{\alpha u_h\xi_h}{(1+\alpha u_h^2)}\end{pmatrix}$$

In this case, the Melnikov function is expressed as:

$$M_\beta(t_0)=\int_{-\infty}^{\infty}\xi_h(t)[\varsigma cos(wt)cos(wt_0)-\eta sen(wt)sen(wt_0)-\frac{-\alpha u_h\xi_h^3}{(1+\alpha u_h^2)}]dt.$$

Note that an important property of this system is that it is reversible, which implies that if $(u(t),\xi(t))$ is a solution for

$$\begin{cases} u'=\xi\\ \xi'=\frac{-u(1+\beta u^2)}{(1+\alpha u^2)}\end{cases}$$

then $(u(-t),-\xi(-t))$ is also a solution, indicating that the solutions are invariant under $t\to -t$ and $\xi\to-\xi$. Furthermore, this suggests that $u_h(t)$ is an even function while $\xi_h(t)$ is an odd function. Thus:

$$\int_{-\infty}^{\infty}cos(wt)\xi_h(t)dt=0$$

then:

$$M_\beta(t_0)=-\gamma sin(wt_o)\int_{-\infty}^{\infty}\xi_h(t)sin(wt)dt-\int_{-\infty}^{\infty}\frac{\alpha u_h(t)\xi_h^3(t)}{(1+\alpha u_h^2(t))}dt$$

let us consider the following function:

$$\varphi(t) = \frac{u_h(t)\xi_h^3(t)}{1+\alpha u_h^2(t)}$$

it follows directly from the properties of $u_h(t)$ and $\xi_h(t)$ that

$$\varphi(-t) = \frac{u_h(-t)\xi_h^3(-t)}{1+\alpha u_h^2(-t)} = \frac{-u_h(t)\xi_h^3(t)}{1+\alpha u_h^2(t)} = -\varphi(t)$$

it is an odd function, and therefore

$$\int_{-\infty}^{\infty} \frac{u_h(t)\xi_h^3(t)}{1+\alpha u_h^2(t)} dt = 0$$

with this result, the Melnikov function can be expressed as:

$$M_\beta(t_0) = -\gamma sin(wt_0) \int_{-\infty}^{\infty} \xi_h(t) sin(wt) dt = M_\beta(t_0) = -\gamma Z sin(wt_0)$$

for

$$Z = \int_{-\infty}^{\infty} \xi_h(t) sin(\omega t) dt$$

Theorem 3
If Z is finite, then under the assumptions stated in item 2 of Theorem 2, the heteroclinic orbit of (7) persists provided that ϵ is sufficiently small.

Proof
Let us consider the change in the limits of integration, derived from the system $du/dt = \xi$, which leads to $dt = du/\xi$, where $0 < t < +\infty$, implying that $-1/\sqrt{-\beta} < u < 1/\sqrt{-\beta}$. Based on this, it can be seen that

$$Z = 2\int_{0}^{+\infty} \xi_h(t)\sin(\omega t) dt$$

$$|Z| \leq 2\int_{0}^{+\infty} |\sin(\omega t)\xi_h(t)\frac{1}{\xi_h(t)}| du = 2\int_{-1/\sqrt{-\beta}}^{1/\sqrt{-\beta}} du = \frac{4}{\sqrt{-\beta}} < \infty$$

This establishes that the Melnikov function;

$$M_\beta(t_0) = \gamma Z \sin(\omega t_0)$$

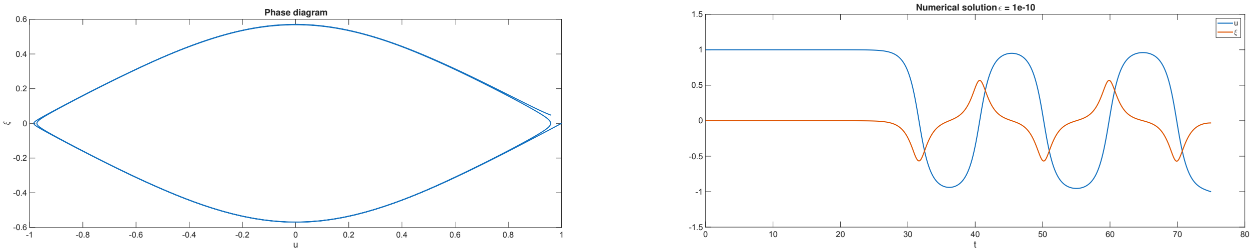
has simple zeros. □

4. Conclusions and numerical analysis.

In this work, the dynamical behavior of a cantilever beam model under periodic forcing was investigated. By means of Melnikov theory, it was shown that for sufficiently small perturbations, the associated Melnikov function admits simple zeros. This result ensures the transversal intersection of the stable and unstable manifolds of the perturbed system, implying the persistence of heteroclinic connections. Consequently, the analysis provides a rigorous theoretical justification for the emergence of complex global dynamics in the forced cantilever beam model.

Table 1. Parameters and initial conditions used in the numerical simulations

Simulation	u_0	ξ_0	ε	α	β	γ	$\omega,\ t_0$
1	1	0	10^{-6}	2	-1	1	$2\pi,\ 2\pi$
2	1	0	10^{-8}	2	-1	1	$2\pi,\ 2\pi$
3	1	0	10^{-10}	2	-1	1	$2\pi,\ 2\pi$
4	1	0	10^{-12}	2	-1	1	$2\pi,\ 2\pi$



(a) Intersection of stable and unstable manifolds.

(b) In blue the solution u and orange ξ.

Figure 5. Phase portraits obtained from numerical simulations with parameters given by case 2 in table 1, where we can note the intersection in the stables and unstable manifolds.

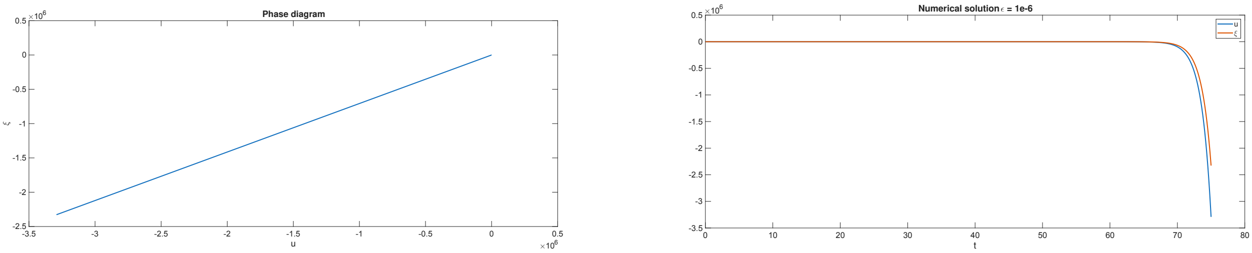


Figure 6. Numerical simulations of the system dynamics. Left: no intersection between the invariant manifolds. Right: transversal intersections of the solutions u and ξ for the values in simulation 1 in the table 1.

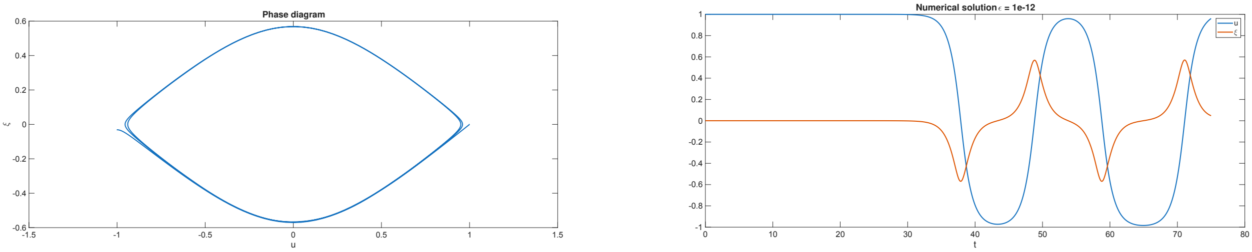


Figure 7. Numerical simulations of the system dynamics. Left: no intersection between the invariant manifolds. Right: transversal intersections of the solutions u and ξ for the values in simulation 3 in the table 1.

For a larger value of the perturbation parameter, the stable and unstable manifolds remain close to each other but no transversal intersections are observed. This behavior illustrates the loss of validity of the Melnikov approximation for moderate values of ε, While transversal intersections occur for sufficiently small ε, Figure **??** shows that for larger perturbations the invariant manifolds approach each other without intersecting.

Acknowledgement

This work was supported by Universidad Tecnológica de Pereira.

REFERENCES

1. S.P. Timoshenko, *Vibration Problems in Engineering* (McGraw-Hill, New York, 1954)
2. Y. Chen, *On the vibration of beams or rods carrying a concentrated mass.* J. Appl. Mech. 30, 310–311 (1963)
3. Herişanu, N., Marinca *Explicit analytical approximation to large-amplitude non-linear oscillations of a uniform cantilever beam carrying an intermediate lumped mass and rotary inertia*. Meccanica 45, 847–855 (2010). https://doi.org/10.1007/s11012-010-9293-0.
4. Herişanu, N., Marinca *Nonlinear Dynamical Systems in Engineering*, Some Approximate Approaches; Springer Berlin, Heidelberg; 978-3-642-22734-9 Published: 05 January 2012.
5. M. Han and P. Yu, *Normal Forms, Melnikov Functions and Bifurcations of limit Cycles*; Springer, Berlin, Heidelberg, 3rd edition, 2004.
6. L. Perko, *Differential equation and Dynamical Systems*, Springer, Berlin, Heidelberg, 3rd edition, 2000.
7. J. Guckenheimer and P. Holmes, *Nonlinear oscillations, Dynamical systems and bifurcations of vector fields*, Springer, Berlin, Heidelberg, 1986.
8. Lei. Xiangsheng, Wang. Yanfeng , Wang. Xinghua, Lin. Gang and Shi. Shihong. (2021). *Revisited on the Free Vibration of a Cantilever Beam with an Asymmetrically Attached Tip Mass.* Mathematical Problems in Engineering. 2021. 1-13. 10.1155/2021/8024879.
9. Maiti. Soumyabrata, Kumar. Anish and Kanchwala. Husain. (2024). Dynamics of a cantilever beam with solitary open crack: nonlinear response to external and base excitation. Nonlinear Dynamics. 113. 7901-7936. 10.1007/s11071-024-10553-7.
10. Kolman. Radek, Kylar. Jaromír, Kotek. Cimrman, Robert, Musil. L.. (2023). *Dynamics of a cantilever beam with piezoelectric sensor: Experimental study*. 119-122. 10.21495/em2023-119.
11. M. Ashhab, V. Salapaka, M. Dahleh et al., *“Melnikov-based dynamical analysis of microcantilevers in scanning probe micrsoscopy,”* Nonlinear Dynamics, vol. 20, pp. 197–229, 1999.
12. M. Han and P. Yu, Normal Forms, Melnikov Functions, and Bifurcations of Limit Cycles, Springer, Berlin, Heidelberg, 2012.
13. Gutiérrez. Alexánder, Zapata. Daniel, Castro G. Diego A (2019). *Saddle-Node Bifurcation and Homoclinic Persistence in AFMs with Periodic Forcing*. Mathematical Problems in Engineering. 2019. 1-6. 10.1155/2019/8925687.