STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **15**, June 2026, pp 4577–4601.
Published online in International Academic Press (www.IAPress.org)

A New Two-parameter Flexible Extension of the Lindley Distribution with Variable Shapes for the Hazard Rate Function

Gorgees Shaheed Mohammad *

University of AL-Qadisiyah, College of Education, Department of Mathematics, IRAQ

Abstract Statistical distributions are important to describe phenomena in the real world. A new two-parameter extension of the Lindley distribution, called new Lindley (NLi) is introduced and studied in detail. This model is right-skewed and left-skewed in PDF. It is a unimodal PDF. The hazard rate function of this model is very flexible. Statistically important properties, including the quantile function, asymptotics for CDF, PDF, and HRF, extreme value, and moments of the new model, are obtained. Parameter estimates process are conducted by the well-known methods of maximum likelihood, weighted least squares method, Cramér–von Mises method, and Anderson-Darling method. The tables of simulation show that the Anderson-Darling method and Cramér–von Mises methods are better for estimating the parameters of the NLi model. We fit our new model to five real data sets and compare it with some Lindley extensions and some well-known two-parameter distributions like Gamma, Weibull, and Generalized Exponential distribution. The results of tables 12-16 verified that this model is more consistent than other competitive models for real data sets.

Keywords Lindley distribution, estimation, moments; quantile function, Simulation.

AMS 2010 subject classifications 60Exx, 60E05

DOI: 10.19139/soic-2310-5070-3154

1. Introduction

Statistical distributions are important tools for modelling real data sets in various fields such as industry, social sciences, sports, and health science. One of the popular distributions is the Lindley distribution. The Lindley distribution (Li) was introduced by Ghitany et al. (2008). The cdf of the Lindley distribution with parameter ϵ is given by

$$F_{Li(\epsilon)}(w) = 1 - \mathrm{e}^{-\epsilon\,\mathrm{w}}\left(\frac{\epsilon(\mathrm{w}+1)+1}{1+\epsilon}\right) = 1 - \mathrm{H}_\epsilon(\mathrm{w})\mathrm{e}^{-\epsilon\,\mathrm{w}}, \qquad \mathrm{w} > 0,\ \epsilon > 0. \tag{1}$$

where $H_\epsilon(w) = \frac{\epsilon\, w+\epsilon+1}{\epsilon+1}$. The Lindley distribution is a combination of the Gamma and Exponential distributions. The hazard rate function (HRF) ofthe Lindley distribution is incraesing and it is right skew pdf. Many extensions of the Lindley distribution were introduced by several authors in recent years.

The Generalized Lindley distribution (GL) was introduced by Nadarajah et al. (2013). The cdf of GLϵ, δ is given by

$$F_{GL(\epsilon,\delta)}(w) = (1 - H_\epsilon(w)\mathrm{e}^{-\epsilon\,\mathrm{w}})^\delta, \quad \mathrm{w} > 0, \epsilon > 0, \delta > 0, \gamma > 0.$$

The Kumaraswamy Lindley distribution (KwLi) was introduced by Merovci and Sharma (2014). The cdf of KwLi$(\epsilon, \delta, \gamma)$ distribution is given by

$$F_{KwLi(\epsilon,\delta,\gamma)}(w) = 1 - \left[1 - F_{Li(\epsilon)}(w)^\delta\right]^\gamma, \quad w > 0, \epsilon > 0, \delta > 0, \gamma > 0.$$

*Correspondence to: Gorgees Shaheed Mohammad (Email: gorgees.alsalamy@qu.edu.iq). University of AL-Qadisiyah, College of Education, Department of Mathematics, IRAQ

ISSN 2310-5070 (online) ISSN 2311-004X (print)

The Beta Lindley distribution (BLi) was introduced by Cakmakyapan and Kadilar (2014). The cdf of BLi$(\epsilon, \delta, \gamma)$ distribution is given by

$$F_{BLi(\epsilon,\delta,\gamma)}(w) = \frac{1}{B(\delta,\gamma)} \int_0^{F_{Li}(w;\epsilon)} s^{\delta-1}(1-s)^{\gamma-1} ds$$

where $B(\delta,\gamma) = \int_0^1 s^{\delta-1}(1-s)^{\gamma-1} ds$ denote the beta function with parameters δ and γ.

The Power Lindley (PL) distribution was proposed by Ghitany and others in 2013. The cdf of PL(ϵ, δ) is given by

$$F_{PL(\epsilon,\delta)}(w) = 1 - H_\epsilon(w^\delta)\mathrm{e}^{-\epsilon\,\mathrm{w}^\delta}, \quad \mathrm{w} > 0, \epsilon > 0, \delta > 0.$$

Gleaton and Lynch are credited with the introduction of the generalized log-Logistic (Odd log-logistic family. The Lindley case of this family (OLL-Li) was distributed in 2006. The cdf of OLL-Li $(\epsilon, \delta)$is given byThe

$$F_{OLL-Li(\epsilon,\delta)}(w) = F_{Li(\epsilon)}(w)^\delta \left[F_{Li(\epsilon)}(w)^\delta + (1 - F_{Li(\epsilon)}(w))^\delta\right]^{-1}, \quad w > 0, \epsilon > 0, \delta > 0.$$

Alizadeh et al. (2017) are credited with the development of the odd log-logistic Power Lindley (OLL-PL) distribution. The cdf of OLL-PL$(\epsilon, \delta, \gamma)$ is given by

$$F_{OLL-PL(\epsilon,\delta,\gamma)}(w;\epsilon,\delta) = F_{PL(\epsilon,\delta)}(w)^\gamma \left[F_{PL(\epsilon,\delta)}(w)^\gamma + (1 - F_{PL(\epsilon,\delta)}(w))^\gamma\right]^{-1}, \quad w > 0, \epsilon > 0, \delta > 0, \gamma > 0.$$

The odd Burr-Lindley (OBu-Li) distribution was proposed by Altun and colleagues in 2017. The cdf of OBu-Li$(\epsilon, \delta, \gamma)$ is given by

$$F_{OBu-Li(\epsilon,\delta,\gamma)}(w) = 1 - \left[1 - F_{OLL-Li(\epsilon,\delta)}(w)\right]^\gamma, \quad w > 0, \epsilon > 0, \delta > 0, \gamma > 0.$$

The HRF of the GL distribution can exhibit decreasing, increasing, and bathtub-shaped behaviours, depending on the parameter values. The HRF of the PL distribution can exhibit decreasing, increasing, and bathtub-shaped behaviours, depending on the parameter values. The HRF of the OLLLi distribution can exhibit decreasing, increasing, upside-down, and bathtub-shaped behaviours, depending on the parameter values.

In the present work, we introduced a new extension of the Lindley distribution. The cdf of the new Lindley extension is given by

$$\begin{aligned} F_{(\epsilon,\delta)}(w) &= \int_0^{\frac{F_{Li(\epsilon)}(w)}{1-F_{Li(\delta)}(w)}} \frac{dt}{(1+t)^2} \\ &= \left[1 - \frac{\mathrm{e}^{-\epsilon\,\mathrm{w}}(\epsilon\,\mathrm{w}+\epsilon+1)}{\epsilon+1}\right]\left[1 - \frac{\mathrm{e}^{-\epsilon\,\mathrm{w}}(\epsilon\,\mathrm{w}+\epsilon+1)}{\epsilon+1} + \frac{\mathrm{e}^{-\delta\,\mathrm{w}}(\delta\,\mathrm{w}+\delta+1)}{\delta+1}\right]^{-1} \\ &= \left[1 - H_\epsilon(w)\,\mathrm{e}^{-\epsilon\,\mathrm{w}}\right]\left[1 - H_\epsilon(w)\,\mathrm{e}^{-\epsilon\,\mathrm{w}} + \mathrm{H}_\delta(\mathrm{w})\,\mathrm{e}^{-\delta\,\mathrm{w}}\right]^{-1}. \end{aligned} \quad (2)$$

where $\epsilon > 0$ and $\delta > 0$ are two shape parameters and $H_\epsilon(w) = \frac{\epsilon\,w+\epsilon+1}{\epsilon+1}$. We denote it by $NLi(\epsilon, \delta)$. The pdf of $NLi(\epsilon, \delta)$ is given by

$$\begin{aligned} f_{(\epsilon,\delta)}(w) &= \mathrm{e}^{-\delta\,\mathrm{w}}\left[\mathrm{h}_\epsilon(\mathrm{w};)\mathrm{H}_\delta(\mathrm{w})\mathrm{e}^{-\epsilon\,\mathrm{w}} + \mathrm{h}_\delta(\mathrm{w})(1 - \mathrm{H}_\epsilon(\mathrm{w})\mathrm{e}^{-\epsilon\,\mathrm{w}})\right] \\ &\times \left[1 - H_\epsilon(w)\mathrm{e}^{-\epsilon\,\mathrm{w}} + \mathrm{H}_\delta(\mathrm{w};\delta)\mathrm{e}^{-\delta\,\mathrm{w}}\right]^{-2} \end{aligned} \quad (3)$$

where $h_\epsilon(w) = (\epsilon^2 + \epsilon^2\,w)(\epsilon+1)^{-1}$. The related hazard rate function (HRF) is given by

$$\begin{aligned} K_{(\epsilon,\delta)}(w) &= \frac{f_{(\epsilon,\delta)}(w)}{1 - F_{(\epsilon,\delta)}(w)} = \left[h_\epsilon(w)H_\delta(w)\mathrm{e}^{-\epsilon\,\mathrm{w}} + \mathrm{h}_\delta(\mathrm{w})(1 - \mathrm{H}_\epsilon(\mathrm{w})\mathrm{e}^{-\epsilon\,\mathrm{w}})\right] \\ &\times \left[1 - H_\epsilon(w)\mathrm{e}^{-\epsilon\,\mathrm{w}} + \mathrm{H}_\delta(\mathrm{w})\mathrm{e}^{-\delta\,\mathrm{w}}\right]^{-1} H_\delta(w)^{-1} \end{aligned} \quad (4)$$

and the related reversed hazard rate function (RHRF) is given by

$$\begin{aligned} L_{(\epsilon,\delta)}(w) &= \frac{f_{(\epsilon,\delta)}(w)}{F_{(\epsilon,\delta)}(w)} = \mathrm{e}^{-\delta\,\mathrm{w}}\left[\mathrm{h}_\epsilon(\mathrm{w};)\mathrm{H}_\delta(\mathrm{w})\mathrm{e}^{-\epsilon\,\mathrm{w}} + \mathrm{h}_\delta(\mathrm{w})(1-\mathrm{H}_\epsilon(\mathrm{w})\mathrm{e}^{-\epsilon\,\mathrm{w}})\right] \\ &\times \left[1 - H_\epsilon(w)\mathrm{e}^{-\epsilon\,\mathrm{w}} + \mathrm{H}_\delta(\mathrm{w};\delta)\mathrm{e}^{-\delta\,\mathrm{w}}\right]^{-1} \end{aligned} \tag{5}$$

For $\epsilon = \delta$, $NLi(\epsilon, \delta)$ reduces to Lindley distribution. Figures 1 and 2 represent the plots of the density function and hazard rate function for some subset of (ϵ, δ) parameters.

Figure 1. **pdf of NLI(ϵ, δ).**

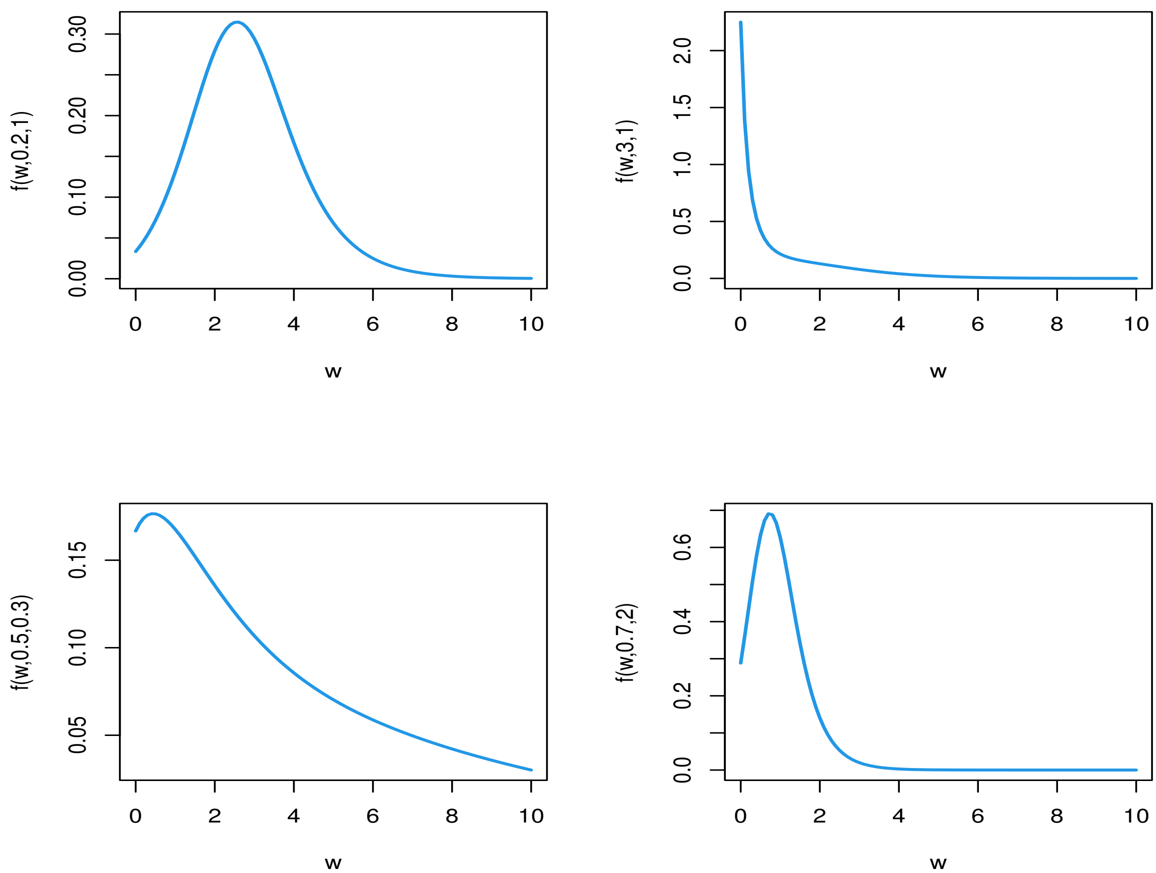


Figure 2. **HRF of NLI**(ϵ, δ)**.**

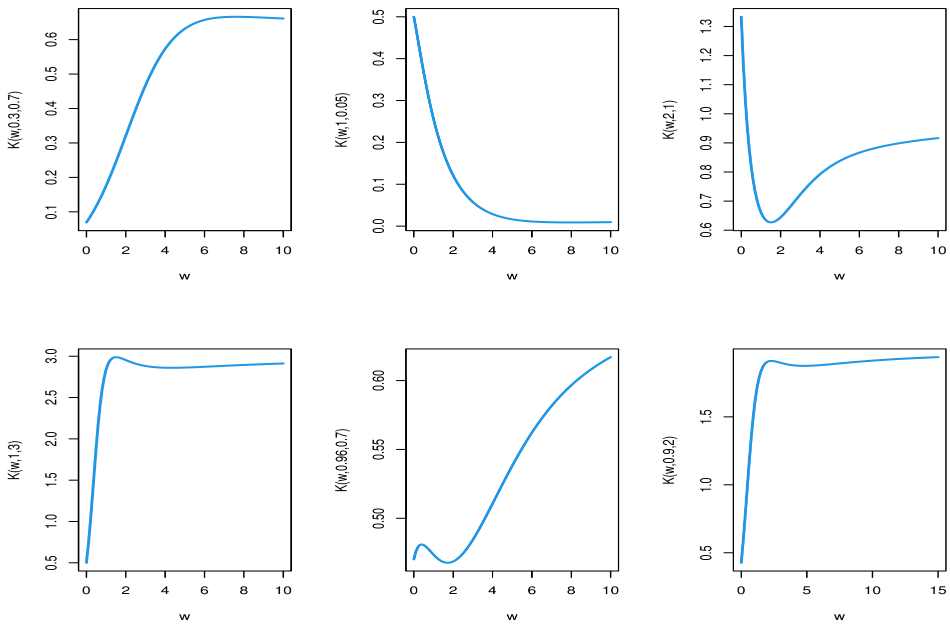
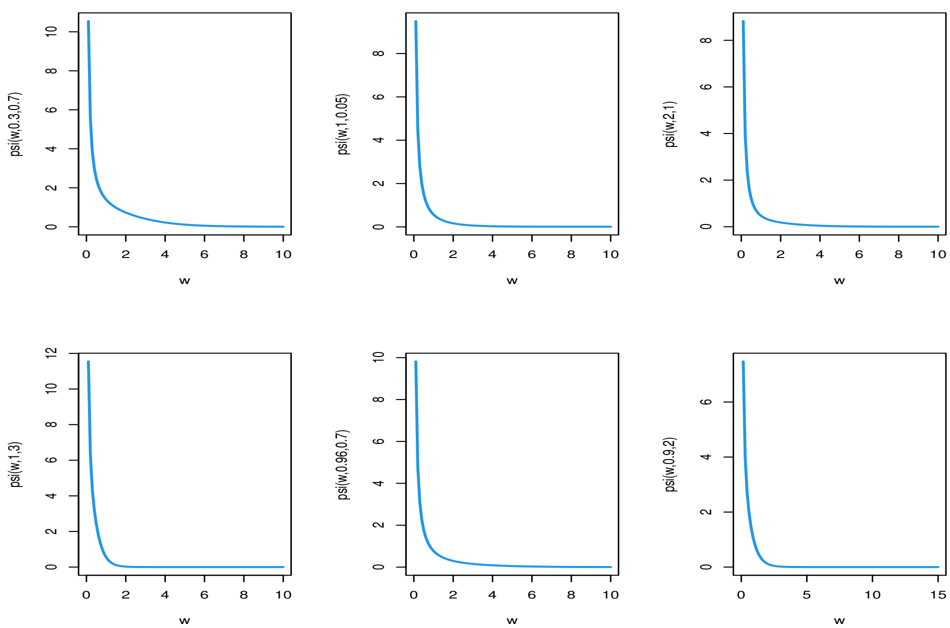


Figure 3. **RHRF of NLI**(ϵ, δ)**.**



Our main goal in the present study is making and study of a new mixture of two Lindley distributions with only two parameters. Figure 1 and fitted densities in the application section show that this model is right-skewed and left-skewed. Figure 2 shows that the hazard rate function of this model can be increasing, decreasing, upside-down, bathtub, and bathtub-upside-down shapes. It would eventuallybe clear that the proposed family of distributions have more flexible hazard rate function. We can use this model in reliability and life-time data modelling.

The remainder of this study is provided as follows: different statistical properties are derived in Section 2. In Section 3, estimation methods are presented and compared with a simulation study. In section 4, some application has been studied. In Section 5, some interesting remarks are discussed.

2. Some important statistical Properties

In this section, we derived some important statistical properties.

2.1. Quantile function

Suppose U uniform random variable on $(0, 1)$ interval, then the root of equation

$$u = \left[1 - H_{\epsilon}(w)\mathrm{e}^{-\epsilon\,\mathrm{w}}\right]\left[1 - H_{\epsilon}(w)\mathrm{e}^{-\epsilon\,\mathrm{w}} + \mathrm{H}_{\delta}(\mathrm{w})\mathrm{e}^{-\delta\,\mathrm{w}}\right]^{-1} \tag{6}$$

has cdf (2). For $\epsilon = \delta$, this equation has a closed form as

$$X_u = \frac{-1}{\epsilon} - 1 - \frac{1}{\epsilon}W((u-1)(1+\epsilon)\mathrm{e}^{-\epsilon-1}; -1). \tag{7}$$

where $W(.; -1)$ corresponds to the negative branch of the Lambert function. For $\delta \neq \epsilon$, we can obtain the quantile function by solving the equation $F_{(\epsilon,\delta)}(w) = u$. We can use uniroot syntax in R software to solve this equation. Here is the R code to solve the equation $F(x) = u$ for the given CDF function:

```
# Define the CDF function F(x)
F <- function(x, epsilon, delta) {
  numerator <- 1 - (1 + epsilon * x / (1 + epsilon)) * exp(-epsilon * x)
  denominator <- 1 - (1 + epsilon * x / (1 + epsilon)) * exp(-epsilon * x) +
  (1 +delta * x / (1 + delta)) * exp(-delta * x)
  return(numerator / denominator)
}

# Function to solve: F(x) - u = 0
solve_equation <- function(u, epsilon, delta, lower = 0.0001, upper = 1000) {
  # Define the function for uniroot, which finds the root of F(x) - u = 0
  root_function <- function(x) F(x, epsilon, delta) - u
  # Use uniroot to find the root in the specified interval [lower, upper]
  result <- uniroot(root_function, c(lower, upper))
  return(result$root)
}

# Example usage:
epsilon <- 0.1  # Set your epsilon value
delta <- 0.2    # Set your delta value
u <- runif(1)   # Generate a random value from uniform distribution U(0,1)
```

```
# Solve the equation F(x) = u
x_solution <- solve_equation(u, epsilon, delta)
print(x_solution)
```

3. Asymptotics

Let $W \sim NLi(\epsilon, \delta)$, when $w \to 0^+$, the limiting properties of the CDF, PDF, and HRF can be written as

$$F_{\epsilon,\delta}(w) \sim \epsilon\, w,$$
$$f_{\epsilon,\delta}(w) \sim \epsilon,$$
$$K_{\epsilon,\delta}(w) \sim \frac{\epsilon}{1-\epsilon\, w}.$$

Let $W \sim NLi(\epsilon, \delta)$, when $w \to \infty$, the limiting properties of the CDF, PDF, and HRF can be written as

$$F_{\epsilon,\delta}(w) \sim 1 - \frac{\delta\, \mathrm{e}^{-\delta\, \mathrm{w}}\, \mathrm{w}}{\delta+1},$$
$$f_{\epsilon,\delta}(w) \sim \frac{\delta^2\, \mathrm{e}^{-\delta\, \mathrm{w}}\, \mathrm{w}}{\delta+1},$$
$$K_{\epsilon,\delta}(w) \sim \delta.$$

3.1. Moments

Let $X \sim NLi(\epsilon, \delta)$,$\mu'_s = E(W^s)(s > 0)$ is obtained as

$$\begin{aligned}
\mathrm{E}(W^s) &= n\int_0^\infty [1-F(z)]z^{s-1}dz = n\int_0^\infty z^{n-1}H(z;\delta)\mathrm{e}^{-\delta\,\mathrm{t}}\left[1-\mathrm{H(z;\epsilon)}\mathrm{e}^{-\epsilon\,\mathrm{z}}+\mathrm{H(z;\delta)}\mathrm{e}^{-\delta\,\mathrm{z}}\right]^{-1}\mathrm{dz} \\
&= \sum_{i_1=0}^{\infty} n\int_0^\infty z^{n-1}H(z;\delta)\mathrm{e}^{-\delta\,\mathrm{z}}\left[\mathrm{H(t;\epsilon)}\mathrm{e}^{-\epsilon\,\mathrm{z}}-\mathrm{H(z;\delta)}\mathrm{e}^{-\delta\,\mathrm{z}}\right]^{\mathrm{i_1}}\mathrm{dz} \\
&= \sum_{i_1=0}^{\infty}\sum_{j_1=0}^{i_1}\sum_{k_1=0}^{j_1+1}\sum_{l_1=0}^{i_1-j_1} \frac{w_{i_1,j_1,k_1,l_1}\,\Gamma(n+k_1+l_1+1)}{(\epsilon(i_1-j_1)+\delta(j_1+1))^{n+k_1+l_1+1}}
\end{aligned}$$

where

$$w_{i_1,j_1,k_1,l_1} = \frac{(-1)^{j_1}\,\epsilon^{l_1}\,\delta^{l_1}\binom{i_1}{j_1}\binom{j_1+1}{k_1}\binom{i_1-j_1}{l_1}}{(1+\delta)^{k_1}\,(1+\epsilon)^{l_1}}$$

and $\Gamma(\epsilon) = \int_0^\infty \mathrm{e}^{-\mathrm{z}}\,\mathrm{z}^{\epsilon-1}$ show the gamma function(for any $\epsilon > 0$).

Let $\mu_W = E(W), \sigma_W^2 = E[(W-\mu_W)^2], skew(X) = E[(\frac{W-\mu_W}{\sigma_W})^3], kurt(W) = [(\frac{W-\mu_W}{\sigma_W})^4] - 3$ denote the mean, variance, skewness and kurtosis of X. Now, we calculate these measures numerically for some value of parameters, and then we can compute kurtosis, skewness, variance and mean. Figure 3 shows the measures.

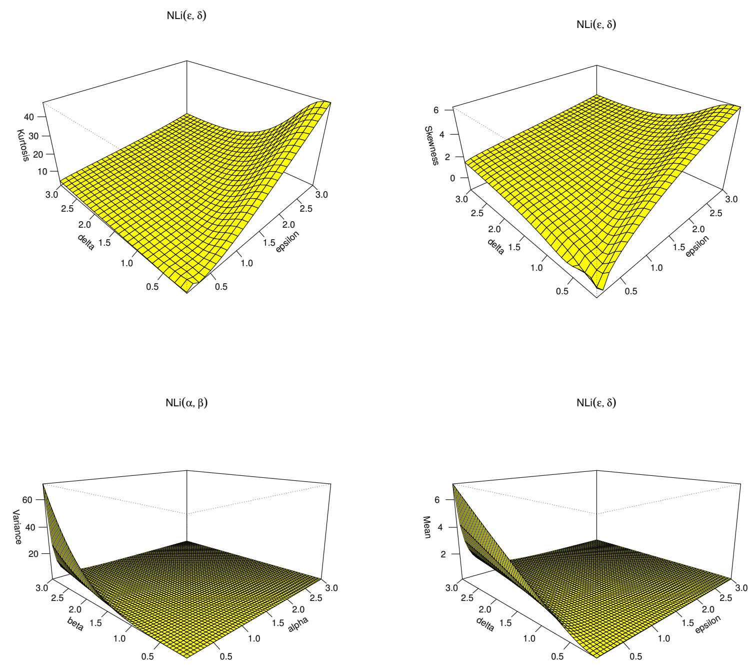


Figure 4. Visualizing Kurtosis, skewness, variance and mean 3D plots of NLi distribution as a function of (ϵ, δ)

3.2. Extreme Value

If $\bar{W} = \frac{W_1 + \ldots + W_n}{n}$ show the sample mean, then using the central limit theorem, $\frac{\sqrt{n}(\bar{W} - E(W))}{\sqrt{\mathrm{Var}(W)}}$ approaches the standard normal distribution as $n \to \infty$. Let $W_{n:n}$ and $W_{1:n}$ show the maximum and minimum of sample $\{W_1, W_2, \cdots W_n\}$. Let $\tau(z) = \frac{1}{\delta}$, we obtain following equations for the (2)

$$\lim_{z \to 0} \frac{F(z\,w)}{F(z)} = w, \tag{8}$$

and

$$\lim_{z \to \infty} \frac{1 - F(z + w\,\tau(z))}{1 - F(z)} = \mathrm{e}^{-w}. \tag{9}$$

Thus, using Theorem 1.6.2 in Leadbetter et al. (2012), there must be norming constants $a_n^{(1)} > 0, \quad a_n^{(2)}, \quad a_n^{(3)} > 0$ and $a_n^{(4)}$ such that

$$Pr\left[a_n^{(1)}(M_{n:n} - a_n^{(2)}) \leq w\right] \to \mathrm{e}^{-\mathrm{e}^{-\mathrm{w}}},$$

and

$$Pr\left[a_n^{(3)}(m_n - a_n^{(4)}) \leq w\right] \to 1 - \mathrm{e}^{-\mathrm{w}},$$

as $n \to \infty$. The form of the norming constants can also be determined.

4. Estimation

4.1. Maximum-likelihood estimation

Now we estimate the parameters using the maximum likelihood estimates (MLEs) for the NLi(ϵ, δ) distribution with complete samples. Suppose $w_1, \ldots, w_m$ be the observed value for a random sample of size m from the NLi(δ, η) distribution. We can write the log-likelihood function of the vector of parameters $\Theta = (\epsilon, \delta)^T$ as

$$\begin{aligned} l(\Theta) &= -\delta \sum_{i_1=1}^{m} w_{i_1} + \sum_{i_1=1}^{m} \log \left\{ h_\epsilon(w_{i_1}) H_\delta(w_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{h}_\delta(\mathrm{w}_{\mathrm{i}_1})(1 - \mathrm{H}_\epsilon(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}}) \right\} \\ &- 2 \sum_{i_1=1}^{n} \log \left[1 - H_\epsilon(w_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{H}_\delta(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\delta\, \mathrm{w}_{\mathrm{i}_1}} \right]. \end{aligned} \tag{10}$$

We can maximize the log-likelihood function r directly by using statistical or mathematical software like R, MATLAB, and MATHEMATICA. The components of the score vector $U(\Theta)$ are obtained by

$$\begin{aligned} U_\epsilon(\Theta) &= \sum_{i_1=1}^{m} \frac{h_\epsilon(w_{i_1})^{(\epsilon)} H_\delta(w_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} - \mathrm{w}_{\mathrm{i}_1} \mathrm{h}_\epsilon(\mathrm{w}_{\mathrm{i}_1}) \mathrm{H}_\delta(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}}}{h_\epsilon(w_{i_1}) H_\delta(w_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{h}_\delta(\mathrm{w}_{\mathrm{i}_1})(1 - \mathrm{H}_\epsilon(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}})} \\ &- \sum_{i_1=1}^{m} \frac{h_\delta(w_{i_1}) H_\epsilon(w_{i_1})^{(\epsilon)} \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{w}_{\mathrm{i}_1} \mathrm{h}_\delta(\mathrm{w}_{\mathrm{i}_1}) \mathrm{H}_\epsilon(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}}}{h_\epsilon(w_{i_1}) H_\delta(w_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{h}_\delta(\mathrm{w}_{\mathrm{i}_1})(1 - \mathrm{H}_\epsilon(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}})} \\ &+ 2 \sum_{i_1=1}^{m} \frac{H_\epsilon(w_{i_1})^{(\epsilon)} \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} - \mathrm{w}_{\mathrm{i}_1} \mathrm{H}_\epsilon(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}}}{1 - H_\epsilon(w_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{H}_\delta(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\delta\, \mathrm{w}_{\mathrm{i}_1}}}. \end{aligned} \tag{11}$$

$$\begin{aligned} U_\delta(\Theta) &= -\sum_{i_1=1}^{m} w_{i_1} + \sum_{i_1=1}^{m} \frac{h_\epsilon(w_{i_1}) H_\delta(w_{i_1})^{(\delta)} \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{h}_\delta(\mathrm{w}_{\mathrm{i}_1})^{(\delta)}(1 - \mathrm{H}_\epsilon(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}})}{h_\epsilon(w_{i_1}) H_\delta(w_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{h}_\delta(\mathrm{w}_{\mathrm{i}_1})(1 - \mathrm{H}_\epsilon(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}})} \\ &- 2 \sum_{i_1=1}^{m} \frac{H_\delta(w_{i_1})^{(\delta)} \mathrm{e}^{-\delta\, \mathrm{w}_{\mathrm{i}_1}} - \mathrm{w}_{\mathrm{i}_1} \mathrm{H}_\delta(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\delta\, \mathrm{w}_{\mathrm{i}_1}}}{1 - H_\epsilon(w_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{w}_{\mathrm{i}_1}} + \mathrm{H}_\delta(\mathrm{w}_{\mathrm{i}_1}) \mathrm{e}^{-\delta\, \mathrm{w}_{\mathrm{i}_1}}} \end{aligned} \tag{12}$$

where

$$\begin{aligned} h_\epsilon(w_{i_1})^{(\epsilon)} &= \frac{\partial h_\epsilon(w_{i_1})}{\partial \epsilon} = \frac{w_{i_1}}{(1+\epsilon)^2}, \\ h_\delta(w_{i_1})^{(\delta)} &= \frac{\partial h_\delta(w_{i_1})}{\partial \delta} = \frac{w_{i_1}}{(1+\delta)^2}, \\ H_\epsilon(w_{i_1})^{(\epsilon)} &= \frac{\partial H_\epsilon(w_{i_1})}{\partial \epsilon} = \frac{\epsilon(\epsilon+2)(1+w_{i_1})}{(1+\epsilon)^2}, \\ H_\delta(w_{i_1})^{(\delta)} &= \frac{\partial H_\delta(w_{i_1})}{\partial \delta} = \frac{\delta(\delta+2)(1+w_{i_1})}{(1+\delta)^2}. \end{aligned}$$

4.2. Weighted Least Squares, Cramér–von Mises and Anderson Darling estimators

Let $v_1, v_2, \cdots v_m$ denote the ordered sample of $w_1, w_2, \cdots w_m$. Swain et al. (1988) introduced the Weighted Least Squares estimators (WLSE). The WLSEs minimized the following functions:

$$\mathrm{WLSE}_{(\delta,\eta)} = \sum_{i_1=1}^{m} \left(\frac{1 - H_\epsilon(v_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{v}_{\mathrm{i}_1}}}{1 - H_\epsilon(v_{i_1}) \mathrm{e}^{-\epsilon\, \mathrm{v}_{\mathrm{i}_1}} + \mathrm{H}_\delta(\mathrm{v}_{\mathrm{i}_1}) \mathrm{e}^{-\delta\, \mathrm{v}_{\mathrm{i}_1}}} - \frac{i_1}{m+1} \right)^2 \times \frac{(m+1)^2(m+2)}{i_1(m-i_1+1)}.$$

Choi and Bulgren (1968) introduced the Cramér–von Mises Estimator (CME) . The CMEs are minimized in the following functions:

$$\mathrm{CME}_{(\delta,\eta)} = \sum_{i_1=1}^{m} \left(\frac{1 - H_\epsilon(v_{i_1})\mathrm{e}^{-\epsilon\,\mathrm{v}_{\mathrm{i}_1}}}{1 - H_\epsilon(v_{i_1})\mathrm{e}^{-\epsilon\,\mathrm{v}_{\mathrm{i}_1}} + \mathrm{H}_\delta(\mathrm{v}_{\mathrm{i}_1})\mathrm{e}^{-\delta\,\mathrm{v}_{\mathrm{i}_1}}} - \frac{2\,i_1 - 1}{2\,m} \right)^2 + \frac{1}{12\,m}.$$

Anderson and Darling (1952) introduced the Anderson-Darling (ADE) estimator. The ADEs are minimized in the following functions:

$$\begin{aligned} \mathrm{ADE}_{(\delta,\eta)} &= -\sum_{i_1=1}^{m} \frac{2i_1 - 1}{m} \log\left(\frac{1 - H_\epsilon(v_{i_1})\mathrm{e}^{-\epsilon\,\mathrm{v}_{\mathrm{i}_1}}}{1 - H_\epsilon(v_{i_1})\mathrm{e}^{-\epsilon\,\mathrm{v}_{\mathrm{i}_1}} + \mathrm{H}_\delta(\mathrm{v}_{\mathrm{i}_1})\mathrm{e}^{-\delta\,\mathrm{v}_{\mathrm{i}_1}}} \right) \\ &- \sum_{i_1=1}^{m} \frac{2i_1 - 1}{m} \log\left(\frac{H_\delta(v_{i_1})\mathrm{e}^{-\delta\,\mathrm{v}_{\mathrm{i}_1}}}{1 - H_\epsilon(v_{i_1})\mathrm{e}^{-\epsilon\,\mathrm{v}_{\mathrm{i}_1}} + \mathrm{H}_\delta(\mathrm{v}_{\mathrm{i}_1})\mathrm{e}^{-\delta\,\mathrm{v}_{\mathrm{i}_1}}} \right) - m. \end{aligned}$$

5. Simulation

In this section, we study and compare the behaviour of different estimators with a simulation study. For three subsets of parameters $\Theta = (\epsilon, \delta) = (0.5, 0.7), (0.5, 1.5), (0.9, 0.4), (1.5, 1), (1.2, 0.8)$, we performed the simulation analysis based on the following steps.

1. Let $u_1, u_2, \cdots, u_m$ be a random sample with size m ($m = 30, 80, \cdots 480$), and then we obtained the sample of NLi by solving the equation $F(w_i) = u_i$. We repeat this step M=2000 times..
2. We computed different estimators for the M=2000 samples, say $\left(\hat{\epsilon}_{l_1}, \hat{\delta}_{l_1}\right)$ for $l_1 = 1, 2, ..., 2000$.
3. Compute the bias and mean squared errors with the following equations.

$$\begin{aligned} Bias(\hat{\epsilon}) &= (1/2000)\sum_{l_1=1}^{2000}(\hat{\epsilon}_{l_1} - \epsilon),\ MSE(\hat{\epsilon}) = (1/2000)\sum_{l_1=1}^{2000}(\hat{\epsilon}_{l_1} - \epsilon)^2, \\ Bias(\hat{\delta}) &= (1/2000)\sum_{l_1=1}^{2000}(\hat{\delta}_{l_1} - \delta),\ MSE(\hat{\delta}) = (1/2000)\sum_{l_1=1}^{2000}(\hat{\delta}_{l_1} - \delta)^2. \end{aligned}$$

The estimated Biases and MSEs are calculated and given in Tables 1-10. The figures 5,6,7,8 and 9 show the estimated Biases and MSEs for a given subset of parameters versus sample size. For estimating parameter ϵ, for all methods of estimation, the biases are almost negative, and the WLSE method performs better. However, for all methods, the difference between Biases are very small, and the absolute value of biases are approch to zero by increasing the sample size. For estimating δ, the Biases are almost negative, and MLE performs better; for all methods, the differences between Biases are very small. The MSEs of both parameters converge to zero as the sample size increases. For a large sample size, for all methods, the differences between the MSEs are very small. All computations in this section and the next section are done with the R software. Table 11 compares the various methods of estimation.

Table 1. Empirical Biases for NLi($\epsilon = 0.5, \delta = 0.7$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	-0.02002	0.00519	-0.00422	-0.01674	0.01797	-0.01614	-0.00516	0.01094
80	-0.02531	-0.01657	-0.01865	-0.02230	-0.01147	-0.02270	-0.02011	-0.01511
130	-0.02190	-0.01685	-0.01738	-0.02014	-0.01381	-0.02046	-0.01970	-0.01620
180	-0.01911	-0.01612	-0.01636	-0.01843	-0.01443	-0.01824	-0.01777	-0.01505
230	-0.01446	-0.01162	-0.01181	-0.01348	-0.01587	-0.01975	-0.01940	-0.01781
280	-0.01554	-0.01296	-0.01289	-0.01336	-0.01427	-0.01788	-0.01783	-0.01733
330	-0.01384	-0.01137	-0.01126	-0.01185	-0.01505	-0.01866	-0.01869	-0.01844
380	-0.01240	-0.01059	-0.01049	-0.01099	-0.01386	-0.01643	-0.01643	-0.01583
430	-0.01371	-0.01204	-0.01201	-0.01253	-0.01110	-0.01346	-0.01342	-0.01286
480	-0.01163	-0.01019	-0.01008	-0.01030	-0.01296	-0.01497	-0.01503	-0.01471

Table 2. Empirical MSEs for NLi($\epsilon = 0.5, \delta = 0.7$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.02385	0.02295	0.02211	0.02253	0.01648	0.01700	0.01556	0.02122
80	0.00725	0.00723	0.00717	0.00781	0.00458	0.00518	0.00496	0.00593
130	0.00438	0.00433	0.00433	0.00475	0.00270	0.00303	0.00296	0.00353
180	0.00310	0.00313	0.00311	0.00344	0.00195	0.00215	0.00212	0.00246
230	0.00239	0.00241	0.00240	0.00260	0.00161	0.00183	0.00180	0.00203
280	0.00192	0.00190	0.00190	0.00206	0.00122	0.00138	0.00138	0.00160
330	0.00169	0.00168	0.00167	0.00180	0.00110	0.00125	0.00124	0.00143
380	0.00145	0.00146	0.00146	0.00159	0.00088	0.00102	0.00102	0.00121
430	0.00130	0.00130	0.00130	0.00141	0.00082	0.00091	0.00091	0.00105
480	0.00117	0.00116	0.00116	0.00124	0.00077	0.00084	0.00084	0.00097

Table 3. Empirical Biases for NLi($\epsilon = 0.5, \delta = 1.5$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.00125	0.03809	0.02467	0.00875	0.09735	0.02013	0.04806	0.08980
80	-0.01402	-0.00308	-0.00458	-0.01068	0.02700	0.00408	0.00783	0.02467
130	-0.01374	-0.00670	-0.00723	0-.01062	0.00475	-0.00973	-0.00845	0.00072
180	-0.01495	-0.01069	-0.01098	-0.01441	0.00168	0-.00719	-0.00646	0.00172
230	-0.01246	-0.00895	-0.00874	-0.01039	-0.00454	-0.01193	-0.01223	-0.00760
280	-0.00913	-0.00644	-0.00632	-0.00780	-0.01158	-0.01709	-0.01728	-0.01305
330	-0.01192	-0.00914	-0.00889	-0.01003	-0.00914	-0.01507	-0.01545	-0.01256
380	-0.01157	-0.00889	-0.00865	-0.00925	-0.01209	-0.01778	-0.01819	-0.01644
430	-0.00913	-0.00753	-0.00737	-0.00831	-0.01332	-0.01660	-0.01681	-0.01427
480	-0.01231	-0.01054	-0.01047	-0.01140	-0.00712	-0.01084	-0.01091	-0.00852

Table 4. Empirical MSEs for NLi($\epsilon = 0.5, \delta = 1.5$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.02945	0.03526	0.03278	0.03333	0.10509	0.09514	0.09552	0.12696
80	0.00808	0.00875	0.00869	0.00991	0.02890	0.02917	0.02916	0.03665
130	0.00505	0.00538	0.00537	0.00607	0.01572	0.01687	0.01663	0.02054
180	0.00353	0.00367	0.00365	0.00413	0.01087	0.01137	0.01129	0.01386
230	0.00303	0.00315	0.00312	0.00353	0.00881	0.00932	0.00921	0.01092
280	0.00232	0.00242	0.00241	0.00264	0.00702	0.00764	0.00757	0.00878
330	0.00201	0.00207	0.00207	0.00236	0.00612	0.00650	0.00652	0.00773
380	0.00177	0.00182	0.00182	0.00204	0.00499	0.00541	0.00542	0.00636
430	0.00158	0.00166	0.00165	0.00185	0.00452	0.00499	0.00496	0.00587
480	0.00141	0.00147	0.00147	0.00170	0.00413	0.00445	0.00445	0.00531

Table 5. Empirical Biases for NLi($\epsilon = 0.9, \delta = 0.4$))

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.00082	-0.00689	-0.02152	-0.04464	-0.01477	-0.03472	-0.02903	-0.02400
80	-0.03170	-0.04031	-0.04342	-0.05043	-0.02079	-0.02980	-0.02842	-0.02802
130	-0.03053	-0.03840	-0.03956	-0.04486	-0.01738	-0.02397	-0.02337	-0.02358
180	-0.02853	-0.03482	-0.03551	-0.03883	-0.01680	-0.02133	-0.02093	-0.02122
230	-0.02009	-0.02612	-0.02643	-0.03029	-0.01584	-0.01992	-0.01974	-0.02015
280	-0.01620	-0.02304	-0.02329	-0.02546	-0.01371	-0.01768	-0.01759	-0.01834
330	-0.02074	-0.02775	-0.02791	-0.02945	-0.01405	-0.01798	-0.01795	-0.01891
380	-0.01722	-0.02207	-0.02219	-0.02408	-0.01304	-0.01596	-0.01589	-0.01640
430	-0.02171	-0.02619	-0.02624	-0.02791	-0.01191	-0.01460	-0.01454	-0.01502
480	-0.01693	-0.02062	-0.02075	-0.02189	-0.01235	-0.01458	-0.01456	-0.01497

Table 6. Empirical MSEs for NLi($\epsilon = 0.9, \delta = 0.4$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.12438	0.09595	0.09197	0 .08546	0.00304	0.00366	0.00328	0.00398
80	0.03093	0.02903	0.02862	0.02969	0.00132	0.00172	0.00164	0.00188
130	0.01729	0.01684	0.01680	0.01712	0.00084	0.00105	0.00103	0.00118
180	0.01218	0.01198	0.01191	0.01233	0.00066	0.00080	0.00078	0.00089
230	0.00959	0.00937	0.00936	0.00950	0.00055	0.00068	0.00068	0.00079
280	0.00744	0.00731	0.00728	0.00744	0.00043	0.00053	.00053	0.00063
330	0.00629	0.00622	0.00620	0.00635	0.00041	0.00052	0.00052	0.00062
380	0.00570	0.00562	0.00560	0.00571	0.00033	0.00041	0.00041	0.00048
430	0.00498	0.00497	0.00496	0.00516	0.00029	0.00036	0.00036	0.00043
480	0.00440	0.00440	0.00440	0.00449	0.00029	0.00035	0.00035	0.00041

Table 7. Empirical Biases for NLi($\epsilon = 1.5, \delta = 1$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.09189	0.12545	0.09764	0.05290	0.07029	0.03212	0.04702	0.07579
80	-0.01141	-0.00242	-0.00787	-0.02059	0.00322	-0.01422	-0.01005	-0.00312
130	-0.03008	0-.02490	-0.02759	-0.03420	-0.01446	-0.02539	-0.02328	-0.02000
180	-0.03978	-0.03669	-0.03784	-0.04498	-0.01711	-0.02521	-0.02425	-0.02164
230	-0.03978	-0.03868	-0.03885	-0.04191	-0.01872	-0.02499	-0.02454	-0.02353
280	-0.02811	-0.02630	-0.02696	-0.03149	-0.02253	-0.02831	-0.02768	-0.02623
330	-0.03281	-0.03279	-0.03272	-0.03400	-0.01957	-0.02543	-0.02538	-0.02558
380	-0.03845	-0.03876	-0.03863	-0.04090	-0.01936	-0.02526	-0.02521	-0.02550
430	-0.03094	-0.03150	-0.03151	-0.03241	-0.02184	-0.02727	-0.02706	-0.02783
480	-0.03772	-0.03760	-0.03766	-0.03960	-0.01814	-0.02307	-0.02287	-0.02301

Table 8. Empirical MSEs for NLi($\epsilon = 1.5, \delta = 1$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.48979	0.38058	0.35815	0.32827	0.04897	0.05030	0.04850	0.06823
80	0.08593	0.08158	0.08055	0.08137	0.01090	0.01109	0.01096	0.01336
130	0.05043	0.04847	0.04827	0.04918	0.00554	0.00599	0.00585	0.00693
180	0.03620	0.03495	0.03478	0.03548	0.00383	0.00411	0.00405	0.00482
230	0.02750	0.02665	0.02646	0.02714	0.00305	0.00334	0.00330	0.00390
280	0.02263	0.02194	0.02194	0.02240	0.00251	0.00283	0.00280	0.00321
330	0.01768	0.01725	0.01716	0.01770	0.00227	0.00250	0.00249	0.00293
380	0.01803	0.01736	0.01731	0.01771	0.00187	0.00211	0.00212	0.00251
430	0.01312	0.01277	0.01274	0.01310	0.00173	0.00200	0.00199	0.00232
480	0.01322	0.01291	0.01288	0.01333	0.00150	0.00172	0.00171	0.00199

Table 9. Empirical Biases for NLi($\epsilon = 1.2, \delta = 0.8$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.05080	0.02649	0.17022	0.04671	-0.00099	0.03348	0.05247	0.04159
80	-0.01509	-0.01991	0.04820	-0.00906	-0.03769	-0.02765	-0.02250	-0.03052
130	-0.02190	-0.02502	0.02761	-0.01848	-0.04721	-0.04080	-0.03921	-0.04286
180	-0.02434	-0.02551	0.02096	-0.01963	-0.03827	-0.03475	-0.03420	-0.03523
230	-0.02623	-0.02704	0.01640	-0.02172	-0.02933	-0.02531	-0.02456	-0.02569
280	-0.02341	-0.02387	0.01294	-0.01923	-0.03294	-0.03128	-0.03111	-0.03132
330	-0.02609	-0.02577	0.01096	-0.02082	-0.03050	-0.02860	-0.02862	-0.02847
380	-0.02502	-0.02446	0.00983	-0.01961	-0.03489	-0.03380	-0.03371	-0.03325
430	-0.02326	-0.02261	0.00837	-0.01846	-0.02944	-0.02881	-0.02885	-0.02826
480	-0.02044	-0.02040	0.00817	-0.01716	-0.03308	-0.03174	-0.03173	-0.03157

Table 10. Empirical MSEs for NLi($\epsilon = 1.2, \delta = 0.8$)

m	$MLE(\hat{\epsilon})$	$WLSE(\hat{\epsilon})$	$ADE(\hat{\epsilon})$	$CME(\hat{\epsilon})$	$MLE(\hat{\delta})$	$WLSE(\hat{\delta})$	$ADE(\hat{\delta})$	$CME(\hat{\delta})$
30	0.35665	0.17022	0.15993	0.15456	0.02964	0.02992	0.02821	0.04022
80	0.05135	0.04820	0.04778	0.04853	0.00535	0.00579	0.00554	0.00670
130	0.02875	0.02761	0.02744	0.02866	0.00336	0.00352	0.00348	0.00399
180	0.02176	0.02096	0.02080	0.02138	0.00245	0.00279	0.00274	0.00319
230	0.01707	0.01640	0.01637	0.01652	0.00193	0.00223	0.00220	0.00253
280	0.01342	0.01294	0.01290	0.01337	0.00152	0.00176	0.00175	0.00209
330	0.01124	0.01096	0.01090	0.01116	0.00147	0.00165	0.00165	0.00190
380	0.01012	0.00983	0.00981	0.01006	0.00125	0.00147	0.00146	0.00170
430	0.00858	0.00837	0.00834	0.00863	0.00107	0.00126	0.00125	0.00149
480	0.00845	0.00817	0.00817	0.00848	0.00098	0.00111	0.00111	0.00131

Table 11. Comparison of Estimation Methods Based on Bias and MSE for NLi Distribution

Parameter Setting	**Best Bias** ($\hat{\epsilon}$)	**Best Bias** ($\hat{\delta}$)	**Best MSE** ($\hat{\epsilon}$)	**Best MSE** ($\hat{\delta}$)	**Overall Best**
$(\epsilon = 0.5,\ \delta = 0.7)$	ADE / CME	ADE	ADE	ADE	**ADE**
$(\epsilon = 0.5,\ \delta = 1.5)$	ADE	ADE	ADE	ADE	**ADE**
$(\epsilon = 0.9,\ \delta = 0.4)$	CME	MLE	CME	MLE	CME ($\hat{\epsilon}$), MLE ($\hat{\delta}$)
$(\epsilon = 1.5,\ \delta = 1)$	CME / ADE	ADE	CME	ADE	**ADE / CME**
$(\epsilon = 1.2,\ \delta = 0.8)$	ADE / CME	ADE	ADE	ADE	**ADE**

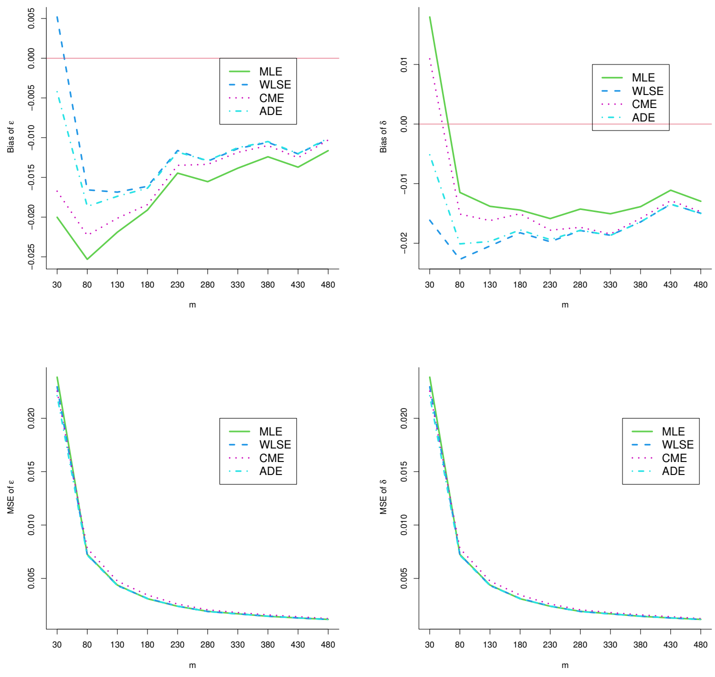


Figure 5. Bias and MSE of estimations for parameter values $(\epsilon, \delta) = (0.5, 0.7)$

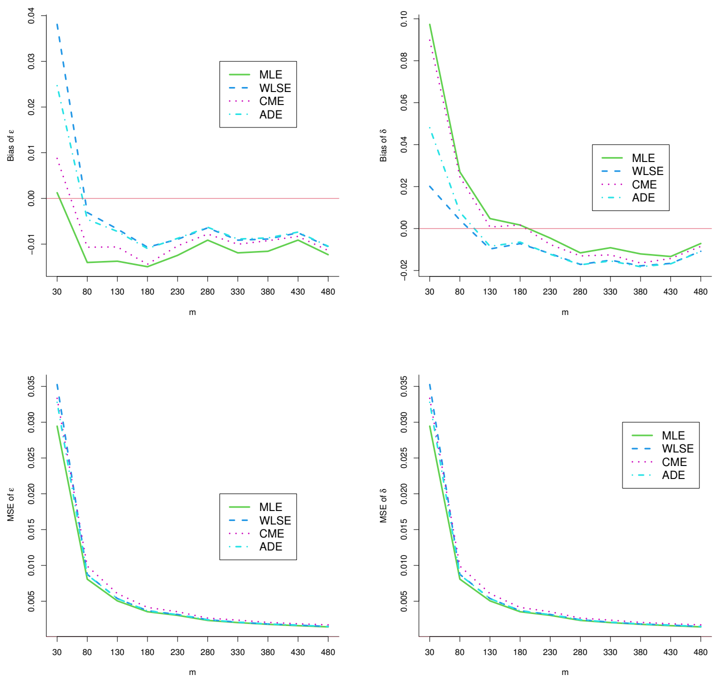


Figure 6. Bias and MSE of estimations for parameter values $(\epsilon, \delta) = (0.5, 1.5)$

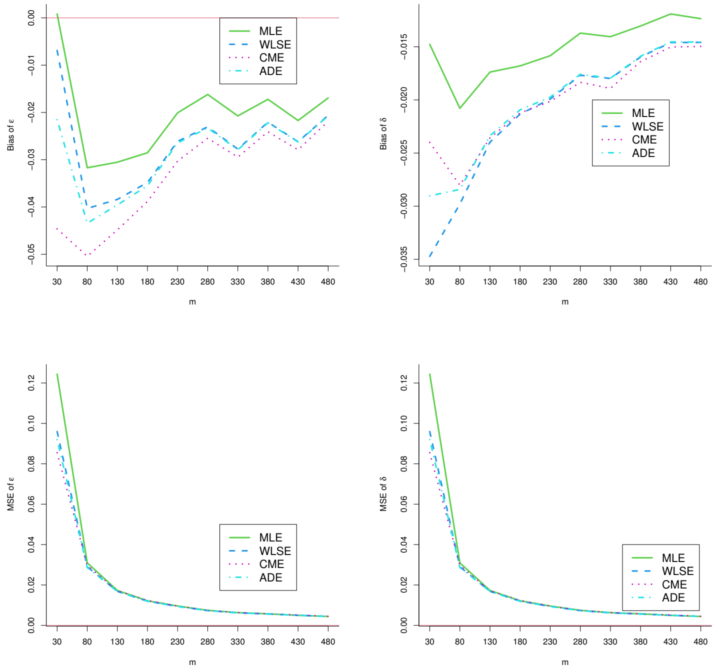


Figure 7. Bias and MSE of estimations for parameter values $(\epsilon, \delta) = (0.9, 0.4)$

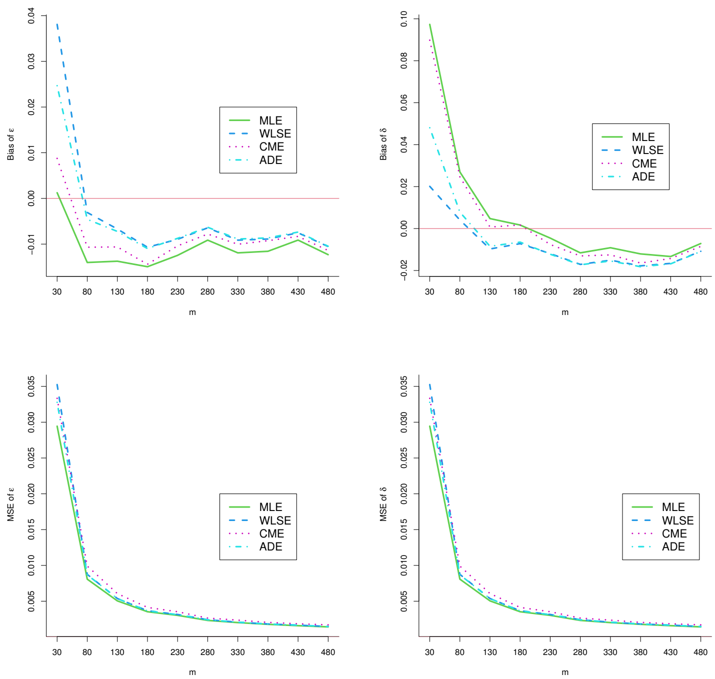


Figure 8. Bias and MSE of estimations for parameter values $(\epsilon, \delta) = (1.5, 1)$

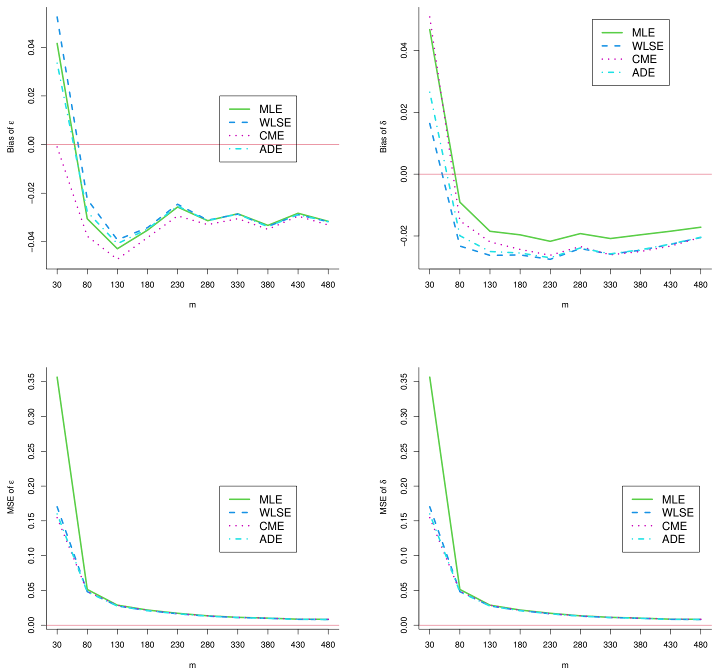


Figure 9. Bias and MSE of estimations for parameter values $(\epsilon, \delta) = (1.2, 0.8)$

6. Applications

In this part, we show the potential of the NLi distribution by fitting it to five real data sets. For each of these real data sets, we employed the Cramér–von Mises (W) and Anderson–Darling (A) statistics, as well as the AIC and BIC, to assess model performance. The parameters were estimated via maximum likelihood estimation.

We used Weibull distribution (W), Li, GL, PL, the Topp-Leone Lindley distribution (TL-Li) (Al-Shomarni et al.,2016), OLL-PL, BLi, KwLi, OLL-Li, Generalized Exponential model (GE) (Gupta and Kundu, 1999), OBu-Li, and Gamma distributions.

The first data are related to the exceedances of flood peaks of the Wheaton River; these data sets were analyzed by Cordeiro et al (2013). The second data are related to total milk production. These data were analysed by Cordeiro et al. (2012) for beta power distribution. The third data sets show the breaking stress of carbon fibers; this data was analyzed by Cordeiro et al. (2011). The fourth data sets are related to the time between the failure of secondary reactor pumps, which was analyzed by Bebington et al. (2007). The fifth data set corresponds to the survival times of patients, which was studied by Gomes et al. (2012).

Table 13 shows the descriptive statistics for five real data sets. The second and third data sets are left-skew and the others are right-skewed. The second data set has negative kurtosis, and the others have positive kurtosis. We obtained the MLE estimators for five real data sets using R statistical software. The estimated parameters (with related standard errors) and other information measures are reported in tables 13, 14, 15, 16, and 17. For each data set, NLi(ϵ, δ) is better than the other 10 competitive models. Figures 10-14 show the estimated HRF, TTT plots and estimated pdf with a histogram. Based on the numerical results presented in Tables I–V, the NLi distribution provides the minimum values for W^*, A^*, and the AIC and BIC information criteria.It is clear that the NLi(ϵ, δ) distribution is a suitable choice for modelling upside-down- bathtub hazard rate and increasing hazard rate function of data sets.

Table 12. Descriptive statistics for data set I, II, and III.

data	kurtosis	skewness	variance	mean
data set I	2.727	1.441	151.221	12.204
data set II	−0.363	−0.330	0.036	0.468
data set III	0.126	−0.128	0.794	2.759
data set IV	0.243	1.276	3.727	1.577
data set V	0.766	1.322	608.719	24.437

Table 13. Information for data set I

model	estimated parameters (se)			W^*	A^*	AIC	BIC
NLi(ϵ, δ)	0.449 (0.103)	0.127 (0.013)		**0.058**	**0.328**	**501.81**	**506.36**
Li(ϵ)	0.153 (0.012)			0.139	0.852	530.42	532.70
PL(ϵ, δ)	0.338 (0.055)	0.699 (0.056)		0.150	0.866	508.44	512.99
OLL-Li(ϵ, δ)	0.183 (0.022)	0.612 (0.066)		0.100	0.620	506.02	510.58
OLL-PL$(\epsilon, \delta, \gamma)$	0.154 (0.091)	1.073 (0.244)	0.558 (0.177)	0.093	0.592	507.93	514.76
OBu-Li$(\epsilon, \delta, \gamma)$	0.200 (0.092)	0.611 (0.067)	0.907 (0.457)	0.098	0.617	507.99	514.82
KwLi$(\epsilon, \delta, \gamma)$	0.130 (0.294)	0.505 (0.087)	0.783 (1.889)	0.131	0.819	511.33	518.16
BLi(ϵ, δ)	0.333 (0.272)	0.555 (0.098)	0.274 (0.240)	0.123	0.766	510.20	517.03
TLLi(ϵ, δ)	0.062 (0.008)	0.474 (0.069)		0.128	0.819	509.73	514.29
Ga(ϵ, δ)	0.838 (0.121)	14.549 (2.812)		0.130	0.751	506.68	511.24
W(ϵ, δ)	0.109 (0.030)	0.901 (0.085)		0.137	0.785	506.99	511.55
GE(ϵ, δ)	0.072 (0.011)	0.828 (0.123)		0.128	0.742	506.58	511.14
GL(ϵ, δ)	0.104 (0.014)	0.508 (0.076)		0.132	0.822	509.34	513.90

Table 14. Information for data set II

model	estimated parameters (se)			W^*	A^*	AIC	BIC
NLi(ϵ,δ)	0.362 (0.082)	7.028 (0.698)		$\mathbf{0.133}$	$\mathbf{0.959}$	$\mathbf{-43.14}$	$\mathbf{-37.97}$
Li(ϵ)	2.708 (0.216)			0.573	3.529	47.40	50.07
PL(ϵ,δ)	6.051 (0.792)	2.571 (0.211)		0.226	1.487	-39.26	-33.92
OLL-Li(ϵ,δ)	2.114 (0.080)	2.432 (0.207)		0.413	2.649	-24.78	-19.44
OLL-PL(ϵ,δ,γ)	18.653 (7.446)	4.206 (0.649)	0.525 (0.105)	0.151	0.927	-44.58	-36.56
OBu-Li(ϵ,δ,γ)	0.733 (0.465)	2.262 (0.173)	36.900 (88.873)	0.168	1.132	-41.39	-33.37
KwLi(ϵ,δ,γ)	0.850 (0.232)	2.524 (0.268)	54.424 (37.966)	0.265	1.732	-35.02	-27.00
BLi(ϵ,δ)	0.281 (0.251)	3.130 (0.431)	89.350 (136.63)	0.474	2.965	-21.37	-13.35
TLLi(ϵ,δ)	2.654 (0.198)	3.500 (0.539)		0.677	4.119	-9.31	-3.96
Ga(ϵ,δ)	3.689 (0.483)	0.127 (0.017)		0.603	3.703	-14.84	-9.503
W(ϵ,δ)	5.381 (0.779)	2.601 (0.209)		0.231	1.523	-38.69	-33.34
GE(ϵ,δ)	4.200 (0.372)	3.713 (0.565)		0.729	4.401	-6.07	-0.73
GL(ϵ,δ)	4.808 (0.382)	3.576 (0.549)		0.697	4.126	-8.06	-2.72

Table 15. Information for data set III

model	estimated parameters (se)			W^*	A^*	AIC	BIC
NLi(ϵ,δ)	0.068 (0.021)	1.685 (0.199)		$\mathbf{0.077}$	$\mathbf{0.415}$	$\mathbf{175.10}$	$\mathbf{179.48}$
Li(ϵ)	0.590 (0.053)			0.213	1.148	246.76	248.95
PL(ϵ,δ)	0.124 (0.031)	2.510 (0.208)		0.091	0.496	175.61	179.99
OLL-Li(ϵ,δ)	0.488 (0.017)	2.964	 0.313	0.16	0.845	179.99	184.37
OLL-PL(ϵ,δ,γ)	0.113 0.080	2.601 0.713	0.950 0.352	0.090	0.494	177.59	184.16
OBu-Li(ϵ,δ,γ)	0.318 0.129	2.453 0.317	5.373 8.358	0.075	0.433	177.15	183.72
KwLi(ϵ,δ,γ)	0.243 0.126	2.905 0.568	63.119 91.84	0.096	0.530	178.15	184.72
BLi(ϵ,δ)	0.016 0.007	3.700 0.623	2150.537 2024.484	0.147	0.786	181.24	187.81
TLLi(ϵ,δ)	0.712 (0.058)	6.338 (1.488)		0.290	1.578	190.09	194.47
Ga(ϵ,δ)	7.478 (1.275)	0.368 (0.064)		0.244	1.321	186.33	190.71
W(ϵ,δ)	0.021 (0.008)	3.441 (0.323)		0.092	0.526	176.13	180.51
GE(ϵ,δ)	1.007 (0.100)	9.199 (2.149)		0.347	1.908	194.74	199.12
GL(ϵ,δ)	1.246 (0.108)	7.041 (1.672)		0.308	1.684	191.59	195.97

Table 16. Information for data set IV

model	estimated parameters (se)			W^*	A^*	AIC	BIC
NLi(ϵ,δ)	1.972 (0.623)	0.706 (0.156)		$\mathbf{0.043}$	$\mathbf{0.292}$	$\mathbf{66.74}$	$\mathbf{69.01}$
Li(ϵ)	0.957 (0.150)			0.102	0.635	72.61	73.74
PL(ϵ,δ)	1.195 (0.212)	0.725 (0.112)		0.073	0.473	69.49	71.76
OLL-Li(ϵ,δ)	1.043 (0.208)	0.709 (0.128)		0.103	0.631	70.34	72.61
OLL-PL(ϵ,δ,γ)	1.209 0.163	0.379 0.838	2.060 4.766	0.043	0.323	71.16	74.57
OBu-Li(ϵ,δ,γ)	0.531 0.489	0.690 0.106	2.176 2.170	0.087	0.543	71.60	75.01
KwLi(ϵ,δ,γ)	0.457 1.409	0.610 0.145	1.659 5.611	0.107	0.661	72.94	76.35
BLi(ϵ,δ)	1.277 1.670	0.638 0.191	0.525 5.611	0.104	0.643	72.79	76.19
TLLi(ϵ,δ)	0.427 (0.098)	0.576 (0.153)		0.118	0.717	71.46	73.73
Ga(ϵ,δ)	0.745 (0.188)	2.115 (0.739)		0.078	0.501	69.51	71.78
W(ϵ,δ)	0.765 (0.0182)	0.807 (0.129)		0.065	0.431	69.02	71.29
GE(ϵ,δ)	0.515 (0.154)	0.739 (0.193)		0.080	0.511	69.56	71.83
GL(ϵ,δ)	0.725 (0.178)	0.612 (0.164)		0.108	0.662	70.97	73.24

Table 17. Information for data set V

model	estimated parameters (se)			W^*	A^*	AIC	BIC
NLi(ϵ,δ)	0.143 (0.028)	0.060 (0.008)		$\mathbf{0.041}$	$\mathbf{0.275}$	$\mathbf{404.52}$	$\mathbf{408.26}$
Li(ϵ)	0.078 (0.008)			0.094	0.566	416.29	418.16
PL(ϵ,δ)	0.184 (0.046)	0.747 (0.070)		0.069	0.431	407.13	410.88
OLL-Li(ϵ,δ)	0.086 (0.011)	0.685 (0.086)		0.093	0.552	407.70	411.44
OLL-PL(ϵ,δ,γ)	0.205 (0.221)	0.709 (0.389)	1.072 (0.766)	0.065	0.415	409.12	414.74
OBu-Li(ϵ,δ,γ)	0.045 (0.031)	0.661 (0.074)	2.096 (1.562)	0.070	0.431	408.61	414.25
KwLi(ϵ,δ,γ)	0.431 (0.002)	0.839 (0.294)	0.110 (0.016)	0.078	0.476	409.47	415.09
BLi(ϵ,δ)	0.431 (0.002)	0.839 (0.253)	0.110 (0.016)	0.081	0.492	409.02	414.64
TLLi(ϵ,δ)	0.033 (0.005)	0.521 (0.096)		0.117	0.692	409.48	413.23
Ga(ϵ,δ)	1.044 (0.108)	23.404 (5.369)		0.074	0.451	406.76	410.51
W(ϵ,δ)	0.039 (0.017)	1.007 (0.111)		0.074	0.452	406.82	410.56
GE(ϵ,δ)	0.042 (0.007)	1.053 (0.202)		0.073	0.447	406.75	410.49
GL(ϵ,δ)	0.057 (0.009)	0.568 (0.108)		0.101	0.606	408.67	412.42

Figure 10. **fitted hrf and pdf for the first data.**

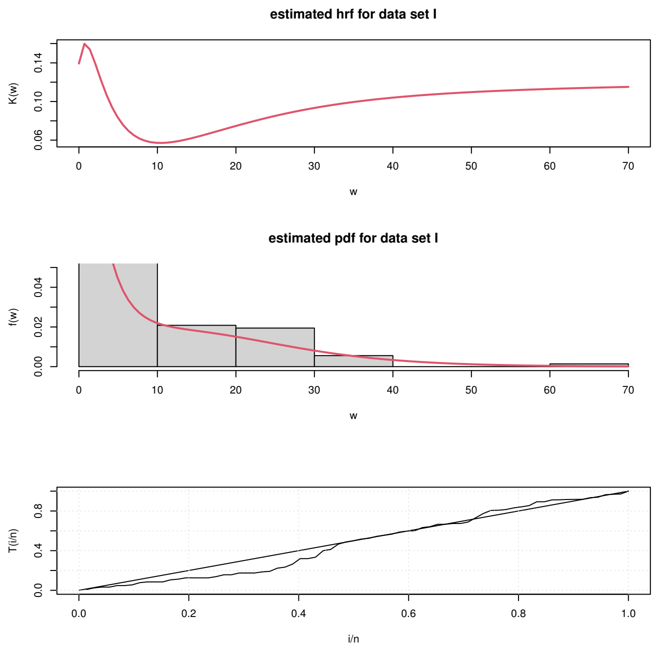


Figure 11. **fitted hrf and pdf for the second data.**

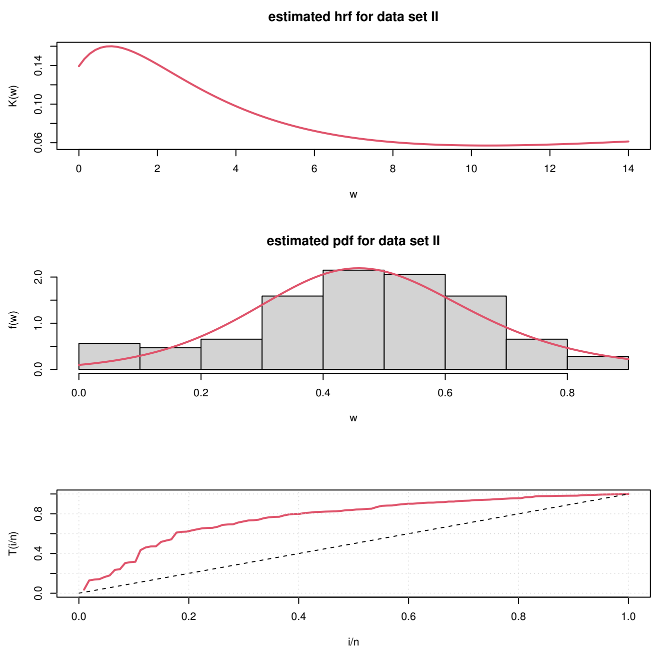


Figure 12. **fitted hrf and pdf for the third data.**

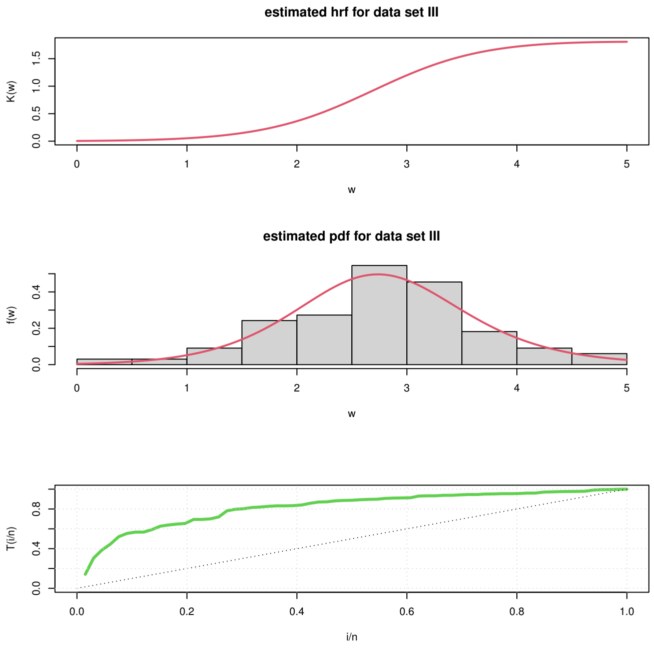


Figure 13. **fitted hrf and pdf for the forth data.**

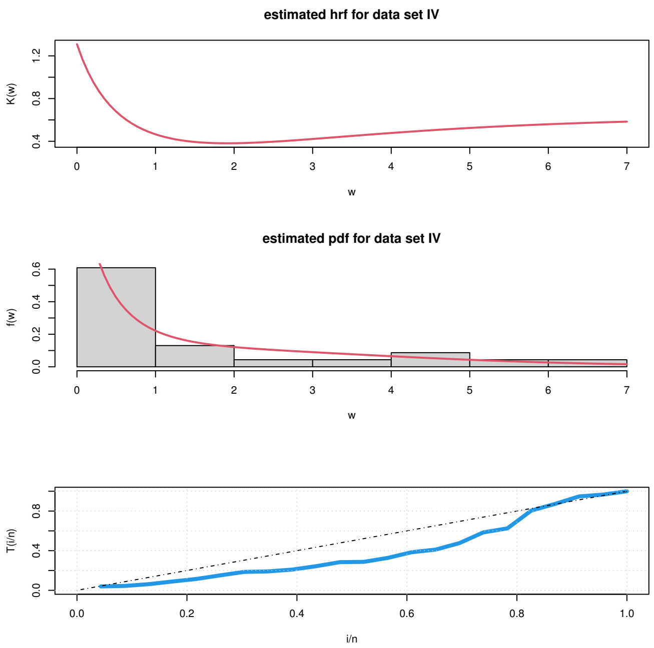
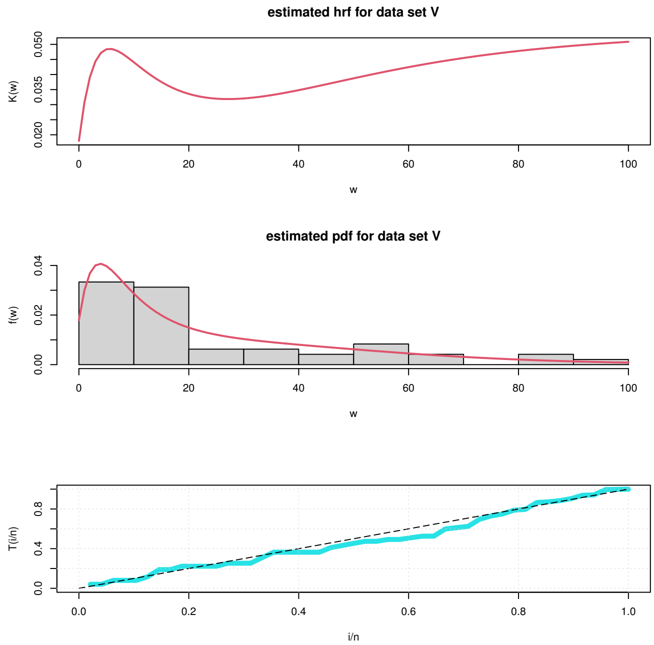


Figure 14. **fitted hrf and pdf for the fifth data.**



7. Conclusions

A new distribution with two parameters by compounding two Lindley distributions is suggested and studied in detail. Statistical properties of the new model are derived. We compared four methods of estimation by simulation study. Finally, we compared the fit of the proposed model with some well known distribution. Numerical results show that the new model performs better than other competitive models. As future work, we can study a bivariate version of this model. We can study the discrete version of this paper. Also, we can study this model with various types of censored data.

REFERENCES

1. M. Alizadeh, S. M. T. K. MirMostafaee and I. Ghosh, *A new extension of power Lindley distribution for analyzing bimodal data*, Chilean Journal of Statistics (ChJS), 8 (1), 2017.
2. A. Al-Shomrani, et al., *Topp–Leone family of distributions: Some properties and application*, Pakistan Journal of Statistics and Operation Research, 443–451, 2016.
3. G. Altun, et al., *Odd Burr Lindley distribution with properties and applications*, Hacettepe Journal of Mathematics and Statistics, 46 (2), 255–276, 2017.
4. T. W. Anderson and D. A. Darling, *Asymptotic theory of certain goodness of fit criteria based on stochastic processes*, The Annals of Mathematical Statistics, 193–212, 1952.
5. N. Balakrishnan, *Order statistics from the half logistic distribution*, Journal of Statistical Computation and Simulation, 20 (4), 287–309, 1985.
6. M. Bebbington, C.-D. Lai and R. Zitikis, *A flexible Weibull extension*, Reliability Engineering & System Safety, 92 (6), 719–726, 2007.
7. R. C. H. Cheng and N. A. K. Amin, *Estimating parameters in continuous univariate distributions with a shifted origin*, Journal of the Royal Statistical Society: Series B, 45 (3), 394–403, 1983.
8. K. Choi and W. G. Bulgren, *An estimation procedure for mixtures of distributions*, Journal of the Royal Statistical Society: Series B, 30 (3), 444–460, 1968.
9. G. M. Cordeiro and M. De Castro, *A new family of generalized distributions*, Journal of Statistical Computation and Simulation, 81 (7), 883–898, 2011.

10. G. M. Cordeiro and R. S. Brito, *The beta power distribution*, 88–112, 2012.
11. G. M. Cordeiro and A. J. Lemonte, *The β-Birnbaum–Saunders distribution: An improved distribution for fatigue life modeling*, Computational Statistics & Data Analysis, 55 (3), 1445–1461, 2011.
12. G. M. Cordeiro, E. M. M. Ortega and D. C. C. da Cunha, *The exponentiated generalized class of distributions*, Journal of Data Science, 11 (1), 1–27, 2013.
13. M. E. Ghitany, B. Atieh and S. Nadarajah, *Lindley distribution and its application*, Mathematics and Computers in Simulation, 78 (4), 493–506, 2008.
14. M. E. Ghitany, et al., *Power Lindley distribution and associated inference*, Computational Statistics & Data Analysis, 64, 20–33, 2013.
15. J. U. Gleaton and J. D. Lynch, *Properties of generalized log-logistic families of lifetime distributions*, Journal of Probability and Statistical Science, 4 (1), 51–64, 2006.
16. M. I. Gomes, M. Ferreira and V. Leiva, *The extreme value Birnbaum–Saunders model, its moments and an application in biometry*, Biometrical Letters, 49 (2), 81–94, 2012.
17. R. D. Gupta and D. Kundu, *Generalized exponential distributions*, Australian & New Zealand Journal of Statistics, 41 (2), 173–188, 1999.
18. S. B. Kang and J. I. Seo, *Estimation in an exponentiated half logistic distribution under progressively Type-2 censoring*, Communications for Statistical Applications and Methods, 18 (5), 657–666, 2011.
19. F. Merovci and V. K. Sharma, *The Beta-Lindley distribution: Properties and applications*, Journal of Applied Mathematics, Article ID 198951, 2014.
20. F. Merovci and V. K. Sharma, *The Kumaraswamy-Lindley distribution: Properties and applications*, 2014.
21. D. N. P. Murthy, M. Xie and R. Jiang, *Weibull models*, John Wiley & Sons, 2004.
22. S. Nadarajah, H. S. Bakouch and R. Tahmasbi, *A generalized Lindley distribution*, Sankhya B, 73 (2), 331–359, 2011.
23. J. J. Swain, S. Venkatraman and J. R. Wilson, *Least-squares estimation of distribution functions in Johnson's translation system*, Journal of Statistical Computation and Simulation, 29 (4), 271–297, 1988.

Appendix: R codes for applications

```
#————————————————————————————————
x <- scan()
1.7
2.2
14.4
1.1
0.4
20.6
5.3
0.7
1.9
13.0
12.0
9.3
1.4
18.7
8.5
25.5
11.6
14.1
22.1
1.1
2.5
14.4
1.7
37.6
0.6
2.2
```

```
39.0
0.3
15.0
11.0
7.3
22.9
1.7
0.1
1.1
0.6
9.0
1.7
7.0
20.1
0.4
2.8
14.1
9.9
10.4
10.7
30.0
3.6
5.6
30.8
13.3
4.2
25.5
3.4
11.9
21.5
27.6
36.4
2.7
64.0
1.5
2.5
27.4
1.0
27.1
20.2
16.8
5.3
9.7
27.5
2.5
27.0

hist(x, prob = TRUE)
#----------------------------------
```

8. Weibull Models: P155

```
pdf_NHL <- function(x, epsilon, delta) {
  A <- 1 - (1 + x * (epsilon / (epsilon + 1))) * exp(-epsilon * x)
  B <- 1 - (1 + x * (delta / (delta + 1))) * exp(-delta * x)
  C <- epsilon^2 * (1 + x) * exp(-epsilon * x) / (1 + epsilon)
  D <- delta^2 * (1 + x) * exp(-delta * x) / (1 + delta)

  return((C * (1 - B) + D * A) / (A + 1 - B)^2)
}
```

9. Maximum Likelihood Estimation

```
library("GenSA")
library("AdequacyModel")

fit.NHL <- function(data, density) {
  minusllike <- function(x)
    -sum(log(density(data, x[1], x[2])))

  lower <- c(0.001, 0.001)
  upper <- c(1000, 1000)

  out <- GenSA(
    lower = lower,
    upper = upper,
    fn = minusllike,
    control = list(verbose = TRUE, max.time = 2)
  )

  return(out[c("value", "par", "counts")])
}

fit.NHL(x, pdf_NHL)
```

10. CDF, PDF, and Goodness-of-Fit

```
cdf_NHL <- function(par, x) {
  epsilon <- par[1]
  delta <- par[2]

  A <- 1 - (1 + x * (epsilon / (epsilon + 1))) * exp(-epsilon * x)
  B <- 1 - (1 + x * (delta / (delta + 1))) * exp(-delta * x)

  return(A / (A + 1 - B))
}
```

```
pdf_NHL <- function(par, x) {
  epsilon <- par[1]
  delta <- par[2]

  A <- 1 - (1 + x * (epsilon / (epsilon + 1))) * exp(-epsilon * x)
  B <- 1 - (1 + x * (delta / (delta + 1))) * exp(-delta * x)
  C <- epsilon^2 * (1 + x) * exp(-epsilon * x) / (1 + epsilon)
  D <- delta^2 * (1 + x) * exp(-delta * x) / (1 + delta)

  return((C * (1 - B) + D * A) / (A + 1 - B)^2)
}

goodness.fit(
  pdf = pdf_NHL,
  cdf = cdf_NHL,
  starts = c(0.4493486, 0.1279458),
  data = x,
  method = "B",
  domain = c(0, 70),
  mle = NULL
)
```