STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, August 2026, pp 1865–1883.
Published online in International Academic Press (www.IAPress.org)

Inference based on Dual Generalized Order Statistics from Unit Teissier Distribution

Zakir Ali [1], Zaki Anwar[2], Nadeem Ahmad [3,*], Iftkhar Khan [3]

[1] *Department of Community Medicine*
MM Medical College and Hospital Kumarhatti-Solan, Himanchal Pradesh (India)
[2] *Department of Statistics and Operations Research, Aligarh Muslim University (Women's College), Aligarh (202002), India*
[3] *Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh (202002), India*

Abstract In this study, we utilize dual generalized order statistics (*dgos*) to explore the moment characteristics of the unit Teissier (UT) distribution. Using this method, we obtain exact and explicit expressions for single and product moments and develop recurrence relations for both single and product moments. Furthermore, we provide a characterization of the UT distribution, along with additional results related to the moments of record values and reversed order statistics (*os*). We estimate the unknown parameter of the UT distribution using the maximum likelihood estimation (MLE) method based on *dgos*. The effectiveness of the derived maximum likelihood estimates (MLEs) is assessed through extensive simulation studies, which emphasize various moments and their corresponding relative mean squared errors (MSEs). This research enhances the understanding of the UT distribution's properties and offers valuable insights into its parameter estimation using *dgos*.

Keywords Dual generalized order statistics, record values, single moments, recurrence relations, characterization, Unit Teissier distribution, maximum likelihood estimation.

AMS 2010 subject classifications 62G30, 62E10, 65C10

DOI: 10.19139/soic-2310-5070-3137

1. Introduction

Ordered random variables (*rvs*) arise in many real-world situations. We also see the application of ordered *rvs* in our daily lives. For example, we may be interested in arranging prices of commodities or arranging data sets in ascending or descending order. Some basic statistical model measures based on the concept of ordered *rvs* are the percentile, the range, the median, etc.Random variables are sequenced according to several models, based on the type of ordering employed. [1] was the first to develop the concept of generalized order statistics (*gos*). Order statistics and record values are extensively utilized in statistical modeling and inference; both frameworks characterize ordered *rvs* organized by magnitude. We present a number of additional models of ordered *rvs*, for example; Pfeifer's record model, sequential order statistics, order statistics with non-integral sample size, k-th record values, and k-records from non-identical distributions. These models have numerous interpretations in reliability theory. Generalized order statistics (*gos*) provide a unified framework that represents multiple models of ordered *rvs*. Through providing the marginal and joint densities. [1] determined several properties of *gos*.

*Correspondence to: Nadeem Ahmad (Email: gi4085@myamu.ac.in). Department of Statistics and Operations Research Aligarh Muslim University, Aligarh (202002), India.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

This framework cannot accommodate random variables with decreasing order. As a result, it is not suitable for analyzing models such as reversed *os* and lower record values. [2] introduced the concept of *dgos*. The *dgos* framework offers a unified approach for studying decreasingly *rvs*, including reversed *os*, lower k-record values, and lower Pfeifer records.: The random variables $W(r,n,\tilde{m},k), r=1,2...n$ are said to be *dgos* from the distribution $F(w)$, and probability distribution function (*pdf*) $f(w)$ if their joint *pdf* is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[F\left(w_i\right)\right]^{m_i} f\left(w_i\right)\right)\left[F\left(w_n\right)\right]^{k-1} f\left(w_n\right). \tag{1}$$

defined on the cone $F^{-1}(1) > w_1 \geq w_2 \geq \ldots \geq w_n > F^{-1}(0)$ of $\Re^n$. where $\gamma_j = k+(n-j)(m+1) > 0$ for all j, $1 \leq j \leq n$ k is a positive integer and $m \geq 1$. If $m=0$ and $k=1$, $W(r,n,\tilde{m},k)$ reduces to reversed *os*. $W(r,n,\tilde{m},k)$ reduces to lower records when $k=1$ and $m \to -1$, of the independent and identically distributed *iid* random variables with *cdf* $F(w)$ and corresponding *pdf f(w)*. Other special cases encompassed within this framework include sequential *os* and progressively Type II censored *os*.

Case I: $m_i = m, i = 1,2...n-1$
The *pdf* of r-th $dgos$ $W(r,n,m,k)$, is provided by

$$f_{W(r,n,m,k)}(w) = \frac{C_{r-1}}{(r-1)!}[F(w)]^{\gamma_r-1} f(w) g_m^{r-1}(F(w)), \tag{2}$$

and the joint *pdf* of r-th $dgos$ $W(r,n,m,k)$ and s-th $dgos$ $W(s,n,m,k), 1 \leq r < s \leq n$, is given by

$$\begin{aligned} f_{W(r,n,m,k),\mathrm{W}(s,n,m,k)}(w,z) =& \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(w)]^m g_m^{r-1}(F(w)) \\ & \times \left[h_m(F(z)) - h_m(F(w))\right]^{s-r-1}\left[\bar{F}(z)\right]^{\gamma_s-1} f(w) f(z), w < z, \end{aligned} \tag{3}$$

where

$$C_{r-1} = \prod_{i=1}^{r}\gamma_i, \quad \gamma_i = k+(n-i)(m+1),$$

$$h_m(w) = \begin{cases} \frac{w^{m+1}}{m+1} & , \quad m \neq -1 \\ \ln w & , \quad m = -1 \end{cases}$$

and

$$\begin{aligned} g_m(w) &= h_m(1) - h_m(w), w \in [0,1) \\ &= \begin{cases} \frac{[1-w^{m+1}]}{m+1} & , \quad m \neq -1 \\ -\ln w & , \quad m = -1 \end{cases} \end{aligned}$$

The recurrence relations the moments utilized to compute higher-order moments are another main topic of this study. The importance of recurrence relations and their usefulness are highlighted in calculating moments of higher orders. They effectively simplify both the time complexity and the processes required to obtain a generalized form for each order of the function under examination. They also add in distribution characterization. Characterization is a crucial tool for identifying sample characteristics and making it easier to identify population distributions. The method showed adaptability and broader statistical modeling applicability. For detailed study see; [3], studied the distributional characteristics of *dgos* and used these to illustrate the uniform distribution, increasing awareness of statistical theory. Exponentiated Weibull distribution to investigate the *dgos* for descendingly ordered *rvs* that provide recurrence relations for their moments, and examining related scenarios presented by [4].

The exact moments for generalized and *dgos* were obtained from a large class of distributions discussed by [5]. [6], constructed characterization relationships, explained implications for *os* and lower records, and used notions of *dgos* to obtain a general continuous distribution. [7], investigated characterizations of the distribution using these relations and hazard functions. It also derived the explicit forms and recurrence relations for moments of *dgos* from the Weibull gamma distribution, including examples for reversed *os* and lower records. [8], explored the recurrence relations and explicit algebraic formulations for moments of *dgos* from J-shaped distributions, generalizing Zghoul's findings. General recurrence relations for single and product moments of *dgos* of different distributions were developed. The unification of models for decreasing ordered *rvs* was addressed by [9]. The exponentiated Ralyeigh distribution's exact formulations and recurrence relations for moments of *dgos* express characterization through reduced moments and emphasize single and product moments, lower record values, and conditional expectations are presented by [10]. [11], explored the recurrence relations among moments of *dgos* from a generalized class of exponentiated distributions. Recursive calculation of single and product moments is made possible by relationships for moments of *dgos* in a transmuted exponential *(TEx)* model, which also include distribution characterizations derived by [12]. [13], introduced structural inference to the Frechet distribution using *dgos*, analyzing statistical measures through Monte Carlo simulations aimed at coverage rates and symmetry of structural intervals. The approach is explained with a numerical example, and structural probability ranges for parameters and the extreme value distribution's dependability are derived. [14] provided a crucial characterization of the exponential distribution, introduced minimal variance linear unbiased estimators for its parameters, and talked about the distributional features of *gos* from a two-parameter exponential distribution. [15], focused on the particular instances for k-th record values, developed a recurrence relation for moments of *gos* including for Pareto and Burr distributions. Unified distributional results and moment expression *gos*, with applications to Pareto, Power, and Weibull distributions, are presented by [16]. Recurrence relation under doubly truncated distribution for *gos* moments. Additionally, it handles *os* moments and records values as specific examples of *gos* discussed by [17].[18], developed the doubly truncated distribution class's recurrence relations for the moment and conditional moment generating function of *gos*, including the Gompertz, Pareto, and Weibull distribution characterizations. Recurrence relations for *os*, k-th records, sequential *os*, and single and product moments of *gos* from a doubly truncated Weibull distribution presented by [19]. Generalized order Statistics as a framework for *os* and record values, enabling product and single moments from different distributions with explicit expressions, and examining associated theorems and specific examples established by [20]. [21], explored ordinary *os* and higher k-records while focusing on the linear exponential distribution, more especially the exponential and Rayleigh distributions, and determining recurrence relations for single and product moments of *gos*. Recurrence relations were derived, and their characterization through conditional moments was discussed. Recurrence relations and explicit formulas for the single and product moments of lower generalized *os* were obtained from a power function distribution using *gos* as a framework for ordered *rvs* by [37]. The ratio and inverse moments of the Burr distribution were obtained using the hyper geometric function, with particular attention to the cases where $m_1 = ... = m_{n-1} = m$ and $\gamma_i \neq \gamma_j$. It also addressed deductions about these cases explored by [22]. The inverse Burr distribution's formulas for moments of lower *gos* also covered the MLE and distinguished methods using simulations provided by [23]. [24], derived single, product, and conditional moments of *gos*, including means, variances, and covariance under progressive Type II censoring, were derived from the exponential distribution. [25], stipulated a clear expression and recurrence relations for the Ailamujia distribution's single and product moments of *gos*. This also generates computational results for *os* and record value. [26], explored the Weibull generalized exponential distribution(*WGED*) using *gos*, derived relationships for its *mgf*, and presented MLE for its parameters. Explaining the structural characteristics of the generalized Pareto distribution, such as the quantile function, moments, and entropy metrics, researched by [27]. [36], investigated recurrence relations for records and *os* specialization and single and product moments of *gos* from the Marshall-Olkin extended family of life distribution. It created an equation and used a recurrence relation for sample sizes to characterize these distributions. Expressions for moments of *os* from the Pareto Weibull distribution that emphasize the importance of *orv's* in domains like sports, stability, and seismology, and allow for the computation of numerous statistical measures were presented by [28].

[29], explored the Benktander Type II distribution's moments, describing them using generalized *os*, and developed specific formulas for single and product moments and recurrence relations. A description of the distribution

MLEs for simulation and parameter estimation is also presented. [30] proposes more properties and simulations which offer a more flexible model for modeling lifetime data. [31] proposes the XLindley distribution (XLD) is a new probability distribution created by mixing exponential and Lindley distributions, with properties including stochastic ordering, quantile functions, and parameter estimation methods. [32] introduced a new distribution named as the Pseudo Lindley Distribution (PsLD) as a generalization of the Lindley distribution (LD). [33] proposed a new Zeghdoudi distribution (ZD) with moment and maximum likelihood estimation methods, demonstrating superior fit to real data compared to other one and two-parameter distributions. [33] presents a novel probability distribution, namely the new XLindley distribution, derived from a unique combination of exponential and gamma distributions through a special mixture formulation. [35] proposes a new family of continuous distributions with one extra shape parameter called the generalized Zeghdoudi distributions (GZD).

French biologist Georges [38] developed the Teissier distribution *(TD)*, a continuous probability distribution, to simulate animal mortality rates from aging. Its *pdf* and *cdf* are defined by

$$f(z;\theta)=\theta\left(e^{\theta z}-1\right)e^{(\theta z-e^{\theta z}+1)},\quad z>0,\quad \theta>0. \tag{4}$$

and

$$F(z;\theta)=1-e^{(\theta z-e^{\theta z}+1)},\quad z>0,\quad \theta>0. \tag{5}$$

Furthermore, it is unique to classical lifetime distributions like gamma and Weibull because it has a heavier tail. Despite its early significance, the *TD* generally declined in the statistical literature until it received renewed attention for theoretical and applied research.

[39] presented the UT distribution, a bounded probability distribution that is particularly optimized for modeling data on the unit interval $(0,1)$. It is derived from the *TD*. Entropy measures, quantile functions, and moments of the UT distribution are among the statistical properties thoroughly studied in this work. The mode is estimated and validated using various estimation techniques, including Bayesian and maximum likelihood approaches. Real-world data applications and simulation research demonstrate that the UT distribution is a good candidate for modeling properties and comparable bounded data.

Mathematically, the UT distribution is obtained by applying the definition of the *pdf* provided by (4) to the transformation $W=e^{-Z}$. The following formalized its definition.

The UT distribution's *pdf* is given by

$$f(w)=\theta\left(w^{-\theta}-1\right)w^{-(\theta+1)}e^{-w^{-\theta}+1},\qquad 0<w<1,\quad \theta>0. \tag{6}$$

The associated *cdf* is expressed by

$$F(w)=w^{-\theta}e^{-w^{-\theta}+1},\qquad 0<w<1,\quad \theta>0. \tag{7}$$

In view of (6) and (7), we get

$$f(w)=\frac{\theta\left(w^{-\theta}-1\right)}{w}F(w). \tag{8}$$

The reliability function of the UT distribution is

$$Q(w)=1-F(w)=1-w^{-\theta}e^{-w^{-\theta}+1}.$$

The hazard rate function of the UT distribution is

$$H(w)=\frac{f(w)}{Q(w)}.$$

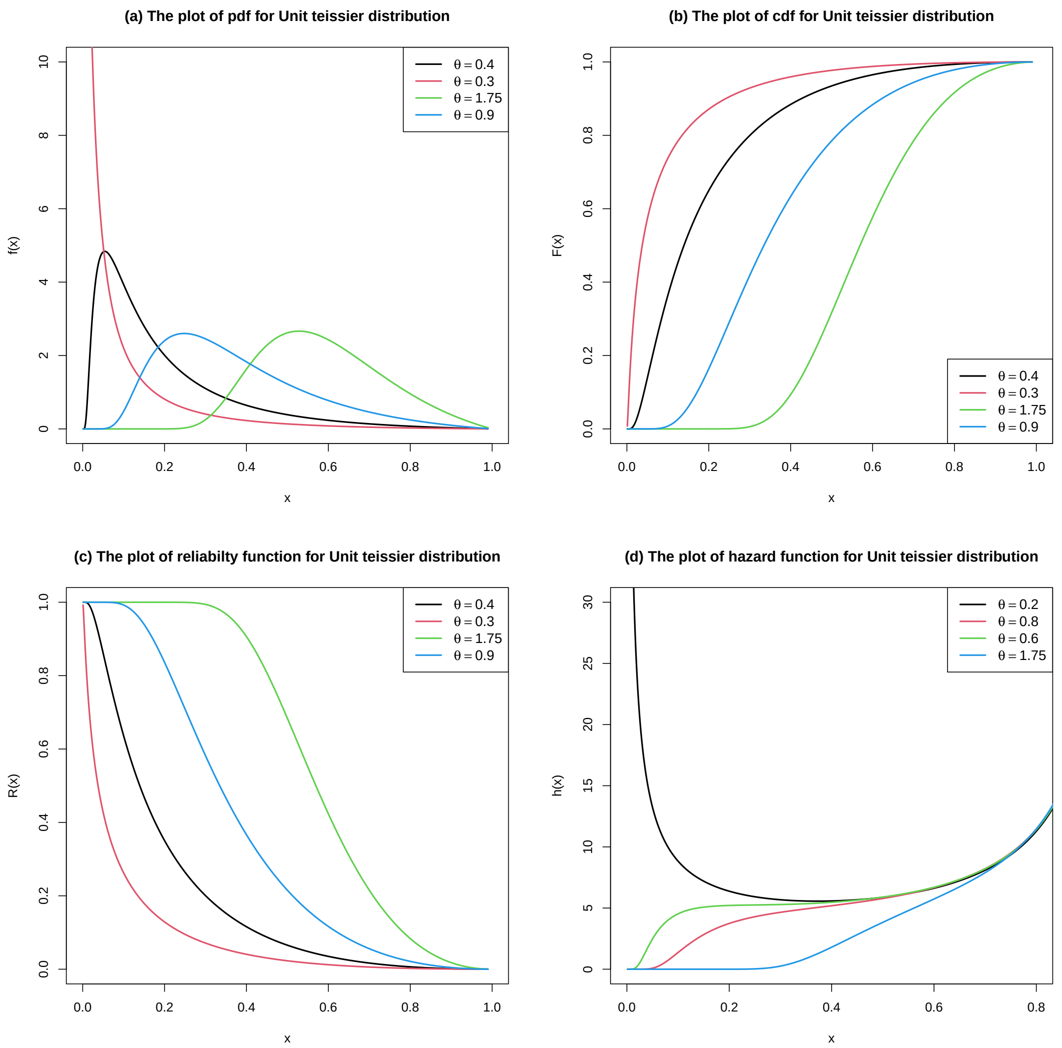

Figure 1. Plots of the p.d.f., c.d.f., reliability function, and hazard rate function at different parameter values.

Figures (a) and (b) show the density and distribution functions of the UT distribution in various possible shapes. The plots demonstrate the flexibility of the UT distribution, showing its ability to model data with various shapes (unimodal, decreasing) and hazard rates (increasing, bathtub-shaped). The behavior of the hazard rate functions and the reliability of the UT distribution are displayed in Figures (c) and (d) above.

The main contributions of this study significantly expand understanding of moment properties and estimation approaches for the UT distribution across *dgos*, reversed order statistics, and k-th lower record values. By establishing a rigorous foundation, these results enable more effective statistical analysis and modeling of distributions with similar properties, with broad implications for both theoretical advancement and practical application.

The structure of the paper is organized as follows. Section 2 presents the derivation of exact and explicit expressions for single moments and the recurrence relations of *dgos* from the UT distribution. Section 3 provides the derivation of precise and explicit recurrence relations for product moments. In Section 4, we demonstrate the characterization of the UT distribution. subsection 5.1 derived the MLE of UT distribution baased on *dgos*. In

Section 6, we have discussed a simulation study for os, and record values. Ultimately, a conclusion is provided within Section 7.

2. Single moments and relations

Theorem 2.1 From UT distribution as given in (6) for $1 \leq r \leq n,\ k \geq 1$ and $\delta = 1, 2, \ldots$

$$E\left[W^{\delta}(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{\nu=0}^{r-1} (-1)^{\nu} \binom{r-1}{\nu} e^{\gamma_{r-\nu}} \times \left[(\gamma_{r-\nu})^{\left(\frac{\delta-\theta(\gamma_{r-\nu}+1)}{\theta}\right)} \Gamma\left(\left(\frac{-\delta+\theta(\gamma_{r-\nu}+1)}{\theta}\right), \gamma_{r-\nu}\right) - (\gamma_{r-\nu})^{\left(\frac{\delta-\theta\gamma_{r-\nu}}{\theta}\right)} \Gamma\left(\left(\frac{-\delta+\theta\gamma_{r-\nu}}{\theta}\right), \gamma_{r-\nu}\right)\right], \quad m \neq -1 \tag{9}$$

$$E\left[W^{\delta}(r,n,m,k)\right] = \frac{\theta k^{r}}{(r-1)!} \sum_{\nu=0}^{\infty} \sum_{\rho=0}^{\nu+r-1} b_a\,(r-1)\,(-1)^{\rho} \binom{\nu+r-1}{\rho} e^{\rho+k} \left[(\rho+k)^{\frac{\delta-\theta(\delta+k+1)}{\theta}} \Gamma - \left(\frac{\delta-\theta\,(\rho+k+1)}{\theta}\right), (\rho+k) - (\rho+k)^{\frac{\delta-\theta(\rho+k)}{\theta}} \Gamma - \left(\frac{\delta-\theta\,(\rho+k)}{\theta}\right)\right], \quad m \to -1 \tag{10}$$

Proof: For **m** $\neq$ **-1**
From (2), we have

$$E\left[W^{\delta}(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!} \int_0^1 w^{\delta} [F(w)]^{\gamma_r - 1} f(w) g_m^{r-1}(F(w)) dw. \tag{11}$$

On expanding $g_m^{r-1}(F(w)) = \left[\frac{1}{m+1}(1-[F(w)]^{m+1})\right]^{r-1}$ using binomial expansion in (11) and using (8), we get

$$E\left[W^{\delta}(r,n,m,k)\right] = \frac{\theta C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{\nu=0}^{r-1} (-1)^{\nu} \binom{r-1}{\nu} e^{\gamma_{r-\nu}} \left[I_1 - I_2\right] \tag{12}$$

where

$$I_1 = \int_0^1 w^{\delta-\theta(\gamma_{r-\nu}+1)-1} e^{-w^{-\theta}\gamma_{r-\nu}} dw$$

$$I_2 = \int_0^1 w^{\delta-\theta(\gamma_{r-\nu}-1)} e^{-w^{-\theta}\gamma_{r-\nu}} dw$$

Setting $u = w^{-\theta}$ in I_1, I_2 and simplifying using generalized exponential function,

$$E_n(u) = \int_1^{\infty} \frac{e^{-ut}}{t^n} dt = u^{n-1}\Gamma(1-n,\ u).$$

we get,

$$I_1 = \frac{1}{\theta}\left\{ (\gamma_{r-\nu})^{\left(\frac{\delta-\theta(\gamma_{r-\nu}+1)}{\theta}\right)} \Gamma\left(\left(\frac{-\delta+\theta(\gamma_{r-\nu}+1)}{\theta}\right), \gamma_{r-\nu}\right)\right\}$$

$$I_2 = \frac{1}{\theta}\left\{(\gamma_{r-\nu})^{\left(\frac{\delta-\theta\gamma_{r-\nu}}{\theta}\right)}\Gamma\left(\left(\frac{-\delta+\theta\gamma_{r-\nu}}{\theta}\right), \gamma_{r-\nu}\right)\right\}$$

Substituting I_1 and I_2 in (12), we get (9).
For m$\rightarrow -1$, we have

$$E\left[W^{\delta}(r,n,-1,k)\right] = E\left[(W^{(k)}_{W'_{(r)}})^{\delta}\right] = \frac{k^r}{(r-1)!}\int_0^1 w^{\delta}[F(w)]^{k-1}f(w)\left[-log(F(w)\right]^{r-1}dw. \tag{13}$$

where $E\left[(W^{(k)}_{W'_{(r)}})^{\delta}\right]$ represents the $\delta-th$ moments of the $k-th$ lower record values
Applying logarithmic expansion and (8) in (13)

$$[-ln(1-y)]^h = \left(\sum_{x=1}^{\infty}\frac{y^x}{x}\right)^h = \sum_{x=0}^{\infty}c_x(h)y^{x+h}$$

where $c_x(h)$ denotes the coefficient of y^{c+h} in the expansion of $\left(\sum_{x=1}^{\infty}\frac{y^x}{x}\right)^h$, we get

$$E\left[(W^{(k)}_{W'_{(r)}})^{\delta}\right] = \frac{\theta k^r}{(r-1)!}\sum_{\nu=0}^{\infty}\sum_{\rho=0}^{\nu+r-1}b_{\nu}\,(r-1)\,(-1)^{\rho}\binom{\nu+r-1}{\rho}e^{\gamma_r+b}\left[I_1-I_2\right] \tag{14}$$

where

$$I_1 = \int_0^1 w^{\delta-\theta(\rho+k+1)-1}e^{-w^{-\theta}(\rho+k)}dw$$

$$I_2 = \int_0^1 w^{\delta-\theta(\rho+k)-1}e^{-w^{-\theta}(\rho+k)}dw$$

Setting $u = w^{-\theta}$ in I_1, I_2 After applying the generalized exponential function for simplification, we obtain

$$I_1 = \frac{1}{\theta}\left\{(\rho+k)^{\left(\frac{\delta-\theta(\rho+k+1)}{\theta}\right)}\Gamma\left(-\left(\frac{\delta-\theta(\rho+k+1)}{\theta}\right), \rho+k\right)\right\}$$

$$I_2 = \frac{1}{\theta}\left\{(\rho+k)^{\left(\frac{\delta-\theta(\rho+k)}{\theta}\right)}\Gamma\left(-\left(\frac{\delta-\theta(\rho+k)}{\theta}\right), \rho+k\right)\right\}$$

Substituting I_1 and I_2 in (14), we get (10).

Special Cases:
a. If m = 0 and k = 1 in equation (9), the single moments of the reversed *os* for the UT distribution are derived as follows.

$$\begin{aligned}E\left[W^{\delta}_{r:n}\right] =& C_{r:n}\sum_{\nu=0}^{r-1}(-1)^{\nu}\binom{r-1}{\nu}e^{n-r+\nu+1}\\ &\times\left[(n-r+\nu+1)^{\left(\frac{\delta-\theta(n-r+\nu+2)}{\theta}\right)}\Gamma\left(\left(\frac{-\delta+\theta(n-r+\nu+2)}{\theta}\right), n-r+\nu+1\right)\right.\\ &\left.-(n-r+\nu+1)^{\left(\frac{\delta-\theta(n-r+\nu+1)}{\theta}\right)}\Gamma\left(\left(\frac{-\delta+\theta(n-r+\nu+1)}{\theta}\right), n-r+\nu+1\right)\right].\end{aligned}$$

b. Substituting $k=1$ into Equation (10) yields the moments of the lower record values for the UT distribution as follows.

$$E\left[W^{\delta}(r,n,-1,1)\right]=\frac{\theta}{(r-1)!}\sum_{\nu=0}^{\infty}\sum_{\rho=0}^{\nu+r-1}b_a\left(r-1\right)(-1)^{\rho}\binom{\nu+r-1}{\rho}e^{\rho+1}$$
$$\left[(\rho+1)^{\frac{\delta-\theta(\rho+2)}{\theta}}\Gamma-\left(\frac{\delta-\theta\left(\rho+2\right)}{\theta}\right),(\rho+1)\right.$$
$$\left.-(\rho+1)^{\frac{\delta-\theta(\rho+1)}{\theta}}\Gamma-\left(\frac{\delta-\theta\left(\delta+1\right)}{\theta}\right)\right].$$

Table 1. Means and variances of *dgos* obtained from the UT distribution

$n,\ r=1,2,3,4,5,6,7,8$ and $(m=1,\ \theta=4.5,\ k=2)$.

n	r	1	2	3	4	5	6	7	8
1	E(W)	0.84894							
	V(W)	0.00412							
2	E(W)	0.88553	0.81235						
	V(W)	0.00263	0.00293						
3	E(W)	0.90344	0.84972	0.79366					
	V(W)	0.00197	0.00202	0.16377					
4	E(W)	0.91468	0.86972	0.82972	0.78164				
	V(W)	0.00159	0.0016	0.00165	0.00199				
5	E(W)	0.92261	0.88295	0.84987	0.81629	0.77297			
	V(W)	0.00133	0.00133	0.00133	0.00141	0.00176			
6	E(W)	0.92861	0.89262	0.86361	0.83613	0.80637	0.76629		
	V(W)	0.00116	0.00115	0.00114	0.00115	0.00125	0.0016		
7	E(W)	0.93336	0.90012	0.87388	0.84992	0.82578	0.79861	0.7609	
	V(W)	0.00102	0.00102	0.001	0.00099	0.00103	0.00113	0.00147	
8	E(W)	0.93725	0.90617	0.88198	0.86039	0.83946	0.81757	0.79229	0.75642
	V(W)	0.00091	0.00091	0.00089	0.00088	0.00089	0.00093	0.00103	0.00137

Table 2. Mean and variance of the reversed *os* for the UT distribution for

$n,\ r=1,2,3,4,5,6,7,8$ and $\theta=4.5$.

n	r	1	2	3	4	5	6	7	8
1	E(W)	0.80453							
	V(W)	0.00605							
2	E(W)	0.84894	0.76013						
	V(W)	0.00412	0.00404						
3	E(W)	0.87129	0.80425	0.73807					
	V(W)	0.00319	0.00298	0.00311					
4	E(W)	0.88553	0.82854	0.77996	0.72410				
	V(W)	0.00263	0.00244	0.00235	0.00259				
5	E(W)	0.89570	0.84489	0.80402	0.76392	0.71415			
	V(W)	0.00225	0.00209	0.00197	0.00196	0.00225			
6	E(W)	0.90344	0.85697	0.82071	0.78732	0.75221	0.70654		
	V(W)	0.00197	0.00184	0.00172	0.00166	0.00169	0.00201		
7	E(W)	0.90961	0.86643	0.83334	0.80388	0.77491	0.74313	0.70044	
	V(W)	0.00175	0.00165	0.00154	0.00146	0.00144	0.00151	0.00183	
8	E(W)	0.91468	0.87411	0.84338	0.81659	0.79117	0.76515	0.73579	0.69539
	V(W)	0.00159	0.00150	0.00139	0.00132	0.00128	0.00129	0.00136	0.00170

Table 3. The skewness and kurtosis measures of reversed *os* for the UT distribution

$n,\ r = 1, 2, 3, 4, 5, 6, 7, 8$ and $\theta = 4.5$.

n	r	1	2	3	4	5	6	7	8
1	Skewness	0.00006							
	Kurtosis	2.49325							
2	Skewness	0.01614	0.01412						
	Kurtosis	2.38906	2.89989						
3	Skewness	0.04063	0.00355	0.01244					
	Kurtosis	2.67642	2.62416	2.92883					
4	Skewness	0.05828	0.0003	0.00771	0.00533				
	Kurtosis	2.36472	2.58944	3.05826	3.1749				
5	Skewness	0.07945	0.0003	0.00358	0.00789	0.00287			
	Kurtosis	2.76079	2.55606	3.02922	3.07264	3.04544			
6	Skewness	0.10421	0.00458	0.00125	0.00538	0.00977	0.00088		
	Kurtosis	3.74236	3.69836	3.13044	3.11336	2.17262	3.07723		
7	Skewness	0.09981	0.00965	0.00015	0.00125	0.00887	0.00323	0.0002	
	Kurtosis	2.33268	3.46139	3.35108	4.23997	3.02506	2.95034	2.76995	
8	Skewness	0.11611	0.01375	0.00007	0.00054	0.00089	0.00611	0.00493	0.00034
	Kurtosis	2.67516	3.905	2.88457	4.13512	5.09662	3.3342	2.81295	3.15067

Table 4. The skewness and kurtosis of *dgos* obtained from the UT distribution

$n,\ r = 1, 2, 3, 4, 5, 6, 7, 8$ and $(m = 1,\ \theta = 4.5,\ k = 2)$.

n	r	1	2	3	4	5	6	7	8
1	Skewness	0.01614							
	Kurtosis	2.38906							
2	Skewness	0.05828	0.00008						
	Kurtosis	2.36472	2.78352						
3	Skewness	0.10421	0.0038	33.63539					
	Kurtosis	3.74236	3.17979	22.74857					
4	Skewness	0.11611	0.00819	0.00009	0.00095				
	Kurtosis	2.67516	2.31916	2.24455	2.77649				
5	Skewness	0.12304	0.01198	0.00012	0.00003	0.00312			
	Kurtosis	1.76675	2.10992	1.66703	1.83918	3.27209			
6	Skewness	0.16379	0.01577	0.00357	0.00066	0.00108	0.0039		
	Kurtosis	3.28116	2.06971	3.5592	1.71367	4.03727	2.37983		
7	Skewness	0.14128	0.02896	0.00833	0.00116	0.00218	0.00005	0.00697	
	Kurtosis	0.96792	2.68603	4.03855	2.37054	5.25331	1.65493	2.82742	
8	Skewness	0.17893	0.04032	0.01565	0.00001	0.00069	0.00004	0.00072	0.01091
	Kurtosis	3.72925	5.47616	5.58501	-0.38046	0.56887	1.16019	3.15796	3.32923

Table 5. Moments and properties of the UT distribution obtained from record data.

r	$E(W)$	$E(W^2)$	$E(W^3)$	$E(W^4)$	$Variance$
			$\theta = 2$		
1	0.62106	0.40365	0.27362	0.19269	0.01794
2	0.50561	0.26597	0.14549	0.08268	0.01033
3	0.44127	0.20139	0.0951	0.04646	0.00667
4	0.39808	0.16317	0.0689	0.02998	0.00470
5	0.36626	0.13765	0.05311	0.02105	0.00350
6	0.34145	0.11931	0.04268	0.01564	0.00272
7	0.32133	0.10543	0.03534	0.01211	0.00218
8	0.30456	0.09455	0.02993	0.00966	0.00179
9	0.29029	0.08576	0.0258	0.0079	0.00149
10	0.27793	0.07851	0.02255	0.00659	0.00126
			$\theta = 2.5$		
1	0.68059	0.47707	0.34384	0.25434	0.01387
2	0.57766	0.34227	0.20797	0.12955	0.00858
3	0.51832	0.2745	0.14857	0.08218	0.00584
4	0.4775	0.23229	0.11516	0.0582	0.00428
5	0.44684	0.20297	0.09375	0.04404	0.00330
6	0.42254	0.18118	0.07886	0.03485	0.00264
7	0.40257	0.16423	0.06791	0.02847	0.00217
8	0.38573	0.15061	0.05954	0.02383	0.00182
9	0.37124	0.13937	0.05292	0.02033	0.00155
10	0.35858	0.12992	0.04757	0.01761	0.00134
			$\theta = 3$		
1	0.75736	0.58241	0.45451	0.35972	0.00881
2	0.67397	0.46016	0.31827	0.22297	0.00592
3	0.62401	0.39367	0.25111	0.16197	0.00428
4	0.58867	0.34983	0.20989	0.12716	0.00330
5	0.56154	0.31797	0.18158	0.10458	0.00264
6	0.53965	0.2934	0.16073	0.08873	0.00218
7	0.52139	0.27368	0.14465	0.07698	0.00183
8	0.50577	0.25739	0.13181	0.06793	0.00159
9	0.49217	0.24362	0.12128	0.06073	0.00139
10	0.48016	0.23178	0.11248	0.05488	0.00123

We calculate and display the moments, variances, skewness, and kurtosis of *dgos*, reverse order statistics, and lower record values in Table 1, 2, 3, 4 and 5 respectively. From Table 1 and 2 it can be seen clearly that for large sample size the variance of distribution is decreasing. Again, it can be seen from Table 5, variance for large sample size, is decreasing in case of lower record data.

Theorem 2.2 Let W be a random variable following the UT distribution as specified in (6). For $1 \leq r < n$ and $\delta = 1, 2, 3, \ldots$, the following recurrence relation is met.

$$E\left[W^{\delta}(r,n,m,k)\right] = \gamma_r\left\{\frac{\theta}{\delta-\theta}E\left[w^{\delta-\theta}(r,n,m,k)\right] + \frac{\theta}{\delta}E\left[W^{\delta}(r,n,m,k)\right]\right\} - \frac{(\gamma_r+1)C_{r-1}}{C_{r-1}^{(k+1,m)}} \left\{\frac{\theta}{\delta-\theta}E\left[W^{p-\theta}(r-1,n,m,k+1)\right] + \frac{\theta}{\delta}E\left[W^{\delta}(r-1,n,m,k+1)\right]\right\}. \tag{15}$$

Proof: We have, from (11)

$$E\left[W^{\delta}(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!}\int_0^1 w^{\delta}[F(w)]^{\gamma_r-1}f(w)g_m^{r-1}(F(w))dw.$$

Using (8), we get

$$E\left[W^{\delta}(r,n,m,k)\right] = \frac{\theta C_{r-1}}{(r-1)!}\left[I_1 - I_2\right] \tag{16}$$

where

$$I_1 = \int_0^1 w^{\delta-\theta-1}[F(w)]^{\gamma_r}g_m^{r-1}(F(w))dw$$

$$I_2 = \int_0^1 w^{\delta-1}[F(w)]^{\gamma_r}g_m^{r-1}(F(w))dw)$$

Applying integration by parts with $w^{\delta-\theta-1}$ as the Integration and the remaining term as the differentiable component, we obtain

$$\begin{aligned} I_1 = &\frac{r-1}{\delta-\theta}\int_0^1 w^{\delta-\theta}[F(w)]^{\gamma_r+m}f(w)g_m^{r-2}(F(w))dw \\ &-\frac{\gamma_r}{p-\theta}\int_0^1 w^{\delta-\theta}[F(w)]^{\gamma_r-1}f(w)g_m^{r-1}(F(w))dw \end{aligned}$$

Similarly I_2

$$\begin{aligned} I_2 = &\frac{r-1}{\delta}\int_0^1 w^{\delta}[F(w)]^{\gamma_r+m}f(w)g_m^{r-2}(F(w))dw \\ &-\frac{\gamma_r}{\delta}\int_0^1 w^{\delta}[F(w)]^{\gamma_r-1}f(w)g_m^{r-1}(F(w))dw \end{aligned}$$

Substituting I_1, I_2 in (16), we get (15).

Special Cases

a. In place of $m = 0$ and $k = 1$ into Equation (15) simplifies the recurrence relation to that of single moments of reversed *os*.

$$\begin{aligned} E\left[W^{\delta}_{r:n}\right] =&\theta(n-r+1)\left\{\frac{1}{\delta-\theta}E\left[w^{\delta-\theta}_{r:n}\right] + \frac{1}{\delta}E\left[W^{\delta}_{r:n}\right]\right\} - \frac{(n-r+1)(n-r+2)}{n+1} \\ &\left\{\frac{1}{\delta-\theta}E\left[W^{\delta-\theta}_{r-1:n+1}\right] + \frac{1}{\delta}E\left[W^{\delta}_{r-1:k+1}\right]\right\}. \end{aligned}$$

b. The recurrence relation in Equation (15) transforms into the recurrence relation for single moments of the k-th record values when $m = -1$.

$$\begin{aligned} E\left[(W^{(k)}_{W_{(r)}})^{\delta}\right] =&\theta k\left\{\frac{1}{\delta-\theta}E\left[(W^{(k)}_{W_{(r)}})^{\delta-\theta}\right] + \frac{1}{\delta}E\left[(W^{(k)}_{w_{(r)}})^{\delta}\right]\right\} - \frac{k^r}{(k+1)^{r-1}} \\ &\left\{\frac{1}{\delta-\theta}E\left[(W^{(k+1)}_{W_{(r)}})^{\delta-\theta}\right] + \frac{1}{\delta}E\left[(W^{(k+1)}_{W_{(r)}})^{\delta}\right]\right\}. \end{aligned}$$

3. Relations for product moments

Theorem 3.1 Let W be a random variable (*rv*) following the UT distribution as defined in (6). For $1 \leq r < s \leq n-1$ and $\delta, j = 1, 2, 3, \ldots,$ the following recurrence relation holds

$$\begin{aligned}
&E\left[W^{\delta}(r,n,m,k)Z^{j}(r,n,m,k)\right]\\
&= \theta\gamma_s\left\{\frac{1}{j-\theta}E\left[W^{\delta}(r,n,m,k)Z^{j-\theta}(r,n,m,k)\right] - \frac{1}{j}\ E\left[w^{\delta}(r,n,m,k)Z^{j}(r,n,m,k)\right]\right\}\\
&-\frac{(\gamma_s+1)C_{s-1}}{C_{s-1}^{(k+1,m)}}\left\{\frac{1}{j-\theta}E\left[W^{\delta}(r-1,n,m,k+1)Z^{j-\theta}(r-1,n,m,k+1)\right]\right.\\
&\left.-\frac{1}{j}E\left[W^{\delta}(r-1,n,m,k+1)Z^{j}(r-1,n,m,k+1)\right]\right\}.
\end{aligned} \tag{17}$$

Proof: Using (3), we get

$$\begin{aligned}
E\left[W^{\delta}(r,n,m,k)Z^{j}(r,n,m,k)\right] =&\frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^1\int_0^z w^{\delta}z^{j}[F(w)]^m f(w)g_m^{r-1}(F(w))\\
&\times\left[h_m(F(z))-h_m(F(w))\right]^{s-r-1}[F(z)]^{\gamma_s-1}f(z)dwdz.
\end{aligned} \tag{18}$$

Using (8) in (18), we get

$$\begin{aligned}
&E\left[W^{\delta}(r,n,m,k)Z^{j}(r,n,m,k)\right]\\
&=\frac{\theta C_{s-1}}{(r-1)!(s-r-1)!}\int_0^1 w^{\delta}[F(w)]^m g_m^{r-1}(F(w))f(w)I_1(z)dw - \frac{\theta C_{s-1}}{(r-1)!(s-r-1)!}\\
&\times\int_0^1\int_0^z w^{\delta}z^{j-1}[F(w)]^m f g_m^{r-1}(F(w))\left[h_m(F(z))-h_m(F(w))\right]^{s-r-1}f(w)[F(z)]^{\gamma_s}dwdz.
\end{aligned} \tag{19}$$

$$\begin{aligned}
&E\left[W^{\delta}(r,n,m,k)Z^{j}(r,n,m,k)\right]\\
&=\frac{\theta C_{s-1}}{(r-1)!(s-r-1)!}\int_0^1 w^{\delta}[F(\delta)]^m g_m^{r-1}(F(w))\left[h_m(F(z))-h_m(F(w))\right]^{s-r-1}I_1(z)dw\\
&-\frac{\theta C_{s-1}}{(r-1)!(s-r-1)!}\int_0^1 w^{\delta}[F(w)]^m g_m^{r-1}(F(w))\left[h_m(F(z))-h_m(F(w))\right]^{s-r-1}I_2(z)dw.
\end{aligned} \tag{20}$$

where

$$I_1(z)=\int_0^z z^{j-\theta-1}\left[h_m(F(z))-h_m(F(w))\right]^{s-r-1}[F(z)]^{\gamma_s}dz$$

and

$$I_2(z)=\int_0^z z^{j-1}\left[h_m(F(z))-h_m(F(w))\right]^{s-r-1}[F(z)]^{\gamma_s}dz.$$

Integrating by parts treating $z^{j-\theta-1}$ for integration and rest for differentiation,we get

$$\begin{aligned}
I_1(z)=&\frac{\gamma_s}{j-\theta}\int_0^z z^{j-\theta}\left[h_m(F(z))-h_m(F(w))\right]^{s-r-1}[F(z)]^{\gamma_s-1}f(z)dz - \frac{(s-r-1)}{j-\theta}\\
&\times\int_0^z z^{j-\theta}\left[h_m(F(z))-h_m(F(w))\right]^{s-r-2}[F(z)]^{\gamma_s}f(z)dz.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2(z)=&\frac{\gamma_s}{j}\int_0^z z^{j}\left[h_m(F(z))-h_m(F(w))\right]^{s-r-1}[F(z)]^{\gamma_s-1}f(z)dz - \frac{(s-r-1)}{j}\\
&\times\int_0^z z^{j}\left[h_m(F(z))-h_m(F(w))\right]^{s-r-2}[F(z)]^{\gamma_s}f(z)dz.
\end{aligned}$$

Substituting $I_1(z)$ and $I_2(z)$ in (20) and after simplification, we get (17).

Special cases:
a. The recurrence relation for the product moments of reversed os is derived by substituting $m = 0$ and $k = 1$ into Equation (17).

$$E\left[W_{r:n}^{\delta} Z_{s:n}^{j}\right]$$
$$= (n-s+1)\left\{\frac{\theta}{j-\theta}E\left[W_{r:n}^{\delta} Z_{s:n}^{j-\theta}\right] + \frac{\theta}{j}E\left[W_{r:n}^{\delta} Z_{s:n}^{j}\right]\right\} - \frac{(n-s+1)(n-s+2)}{n+1}$$
$$\times \left\{\frac{\theta}{j-\theta}E\left[W_{r:n+1}^{\delta} X_{s-1:n+1}^{j-\theta}\right] + \frac{\theta}{j}E\left[W_{r:n+1}^{\delta} Z_{s-1:n+1}^{j}\right]\right\}.$$

b.The recurrence relation given in (17) simplifies to the recurrence relation for the product moments of the $(k)-th$ record values when $m = -1$ is applied.

$$E\left[(W_{W'_{(r)}}^{(k)})^{\delta}(Z_{Z'_{(s)}}^{(k)})^{j}\right]$$
$$= k\left\{\frac{\theta}{j-\theta}E\left[(W_{W'_{(r)}}^{(k)})^{\delta}(Z_{Z'_{(s)}}^{(k)})^{j-\theta}\right] + \frac{\theta}{j}E\left[(W_{W'_{(r)}}^{(k)})^{\delta}(Z_{Z'_{(s)}}^{(k)})^{j}\right]\right\} - \frac{k^{s}}{(k+1)^{s-1}}$$
$$\times \left\{\frac{\theta}{j-\theta}E\left[(W_{W'_{(r)}}^{(k+1)})^{\delta}(Z_{Z'_{(s-1)}}^{(k+1)})^{j-\theta}\right] + \frac{\theta}{j}E\left[(W_{W'_{(r)}}^{(k+1)})^{\delta}(Z_{Z'_{(s-1)}}^{(k+1)})^{j}\right]\right\}.$$

4. Characterization

Theorem 4.1 Let W be a continuous random variable as defined in (6) and (7); then the following relation holds

$$E\left[W^{\delta}(r,n,m,k)\right] = \gamma_r\left\{\frac{\theta}{\delta-\theta}E\left[W^{\delta-\theta}(r,n,m,k)\right] + \frac{\theta}{\delta}E\left[W^{\delta}(r,n,m,k)\right]\right\} - \frac{(\gamma_r+1)C_{r-1}}{C_{r-1}^{(k+1,m)}}$$
$$\left\{\frac{\theta}{\delta-\theta}E\left[W^{\delta-\theta}(r-1,n,m,k+1)\right] + \frac{\theta}{\delta}E\left[W^{\delta}(r-1,n,m,k+1)\right]\right\}. \tag{21}$$

If and only if

$$F(w) = w^{-\theta}e^{-w^{-\theta}+1}, \quad 0 < w < 1, \quad \theta > 0.$$

Proof: The necessary condition easily follows from Theorem ??. If the relation (21) is satisfied, substituting (2) into (21) we get

$$\frac{C_{r-1}}{(r-1)!}\int_0^1 w^{\delta}[F(w)]^{\gamma_r-1}f(w)g_m^{r-1}(F(w))dw$$
$$= \frac{\theta C_{r-1}(r-1)}{(\delta-\theta)(r-1)!}\int_0^1 w^{\delta-\theta}[F(w)]^{\gamma_r+m}f(w)g_m^{r-2}(F(w))dw$$
$$- \frac{\theta C_{r-1}\gamma_r}{(\delta-\theta)(r-1)!}\int_0^1 w^{\delta-\theta}[F(w)]^{\gamma_r-1}f(w)g_m^{r-1}(F(w))dw \tag{22}$$
$$- \frac{\theta C_{r-1}(r-1)}{\delta(r-1)!}\int_0^1 w^{\delta}[F(w)]^{\gamma_r+m}f(w)g_m^{r-2}(F(w))dw$$
$$+ \frac{\theta C_{r-1}\gamma_r}{\delta(r-1)!}\int_0^1 w^{\delta}[F(w)]^{\gamma_r-1}f(w)g_m^{r-1}(F(w))dw.$$

Equation (22) can be written as

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_0^1 w^{\delta}[F(w)]^{\gamma_r-1}f(w)g_m^{r-1}(F(w))dw\\
&=\frac{\theta C_{r-1}(r-1)}{(\delta-\theta)(r-1)!}\int_0^1 w^{\delta-\theta}[F(w)]^{\gamma_r+m}f(w)g_m^{r-2}(F(w))dw\\
&-\frac{\theta C_{r-1}}{(\delta-\theta)(r-1)!}\int_0^1 w^{\delta-\theta}g_m^{r-1}(F(w))\left(\frac{d}{dw}[F(w)]^{\gamma_r}\right)dw\\
&-\frac{\theta C_{r-1}(r-1)}{\delta(r-1)!}\int_0^1 w^{\delta}[F(w)]^{\gamma_r+m}f(w)g_m^{r-2}(F(w))dw\\
&+\frac{\theta C_{r-1}}{\delta(r-1)!}\int_0^1 w^{\delta}g_m^{r-1}(F(w))\left(\frac{d}{dw}[F(w)]^{\gamma_r}\right)dw.
\end{aligned}\tag{23}$$

Subsequently, by integrating the second and fourth terms on the right-hand side of (23) and simplifying, we obtain

$$\frac{C_{r-1}}{(r-1)!}\int_0^1 w^{\delta}[F(w)]^{\gamma_r-1}g_m^{r-1}(F(w))\left(f(w)-\frac{\theta\left(w^{-\theta}-1\right)}{w}F(w)\right)dw=0.\tag{24}$$

Thereafter, utilising a generalized version of the Muntz-Szasz theorem [40] in (24), we get

$$f(w)-\frac{\theta\left(w^{-\theta}-1\right)}{w}F(w)=0$$

$$f(w)=\frac{\theta\left(w^{-\theta}-1\right)}{w}F(w)$$

$$\frac{f(w)}{F(w)}=\frac{\theta\left(w^{-\theta}-1\right)}{w}$$

which prove that,

$$F(w)=w^{-\theta}e^{-w^{-\theta}+1},\quad 0<w<1,\quad \theta>0.$$

5. Parameter Estimation

5.1. Maximum likelihood Estimation

In this section, we derive MLEs for the shape parameter θ of the UT distribution treating the *dgos* parameters m and k as known constants based on the chosen sampling scheme ($m=0, k=1$for reversed order statistics) ($m=-1, k=1$for k-th lower record values). Let $W(1,n,m,k), W(2,n,m,k), \ldots, W(n,n,m,k)$ be the m-*dgos*. Considering (1), the likelihood function can be defined as

$$\begin{aligned}
L(\theta|w)=&k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left(w_i^{-\theta}e^{-w_i^{-\theta}+1}\right)^m\theta\left(w_i^{-\theta}-1\right)w_i^{-(\theta+1)}e^{-w_i^{-\theta}+1}\right)\\
&\times\left(\left(w_n^{-\theta}e^{-w_n^{-\theta}+1}\right)^{k-1}\theta\left(w_n^{-\theta}-1\right)w_n^{-(\theta+1)}e^{-w_n^{-\theta}+1}\right)
\end{aligned}$$

After simplification and taking the log on both sides, we get

$$\begin{aligned}
\ln L(\theta|w)=&\ln k+n\ln\theta+\sum_{j=1}^{n-1}\ln\gamma_j+(-m\theta-(\theta+1))\sum_{i=1}^{n-1}\ln w_i+(m+1)\sum_{i=1}^{n-1}\left(1-w_i^{-\theta}\right)\\
&+\sum_{i=1}^{n}\ln(w_i^{-\theta}-1)+(-\theta(k-1)-(\theta+1))\ln w_n+k\left(1-w_n^{-\theta}\right).
\end{aligned}\tag{25}$$

Special Cases
a. If $m = 0$ and $k = 1$ the *dgos* which reduces to the case of reversed *os*. Therefore, Equations (25) can be used to calculate the MLEs of the parameters within the reversed *os* framework, given by

$$\begin{aligned}\ln L(\theta|w) =& n\ln\theta + \sum_{j=1}^{n-1}\ln(n-j+1) - (\theta+1)\sum_{i=1}^{n}\ln w_i \\ &+ \sum_{i=1}^{n}\ln(w_i^{-\theta}-1) + \left(1-w_n^{-\theta}\right).\end{aligned}$$

b. For $m = -1$ and $k = 1$ the *dgos* which reduces to the $k-th$ lower record values. Therefore, Equations (25) can be applied to obtain the MLEs of the parameters using lower record values.

$$\ln L(\theta|w) = n\ln\theta - \sum_{i=1}^{n}\ln w_i + \sum_{i=1}^{n}\ln(w_i^{-\theta}-1) - \theta\ln w_n + \left(1-w_n^{-\theta}\right).$$

6. Simulation Study

This section examines the simulation results related to the mathematical findings presented in the previous sections. Numerical computations are conducted for general order statistics, as well as specific cases such as reversed *os* and lower record values.
The contents of Table 6 and 7 pertain to the maximum likelihood (M.L.) estimates of unknown parameters in the context of the Benktander Type II distribution. Specifically, these tables provide information on the estimators employed for the unknown parameters are evaluated in terms of their average estimate (A.E.), average bias (A.B.) and mean squared error (M.S.E.). The algorithm utilized to generate the data in Table 6 and 7 are described as follows.
1) We produce a random sample of size ”n” from the Benktander Type II distribution, using a predefined value of an unknown parameter (referred to as ”a”).
2) The maximum likelihood estimate, denoted as $\hat{a}$, for the unknown parameter a is computed using the random sample generated in step (1) and employing the method described in subsection 5.1.
3) The aforementioned process is iterated 1000 times, resulting in 1000 values of $\hat{a}$, denoted as $\hat{a_1}$, $\hat{a_2}$, $\hat{a_3}$,..., $a_{\hat{1000}}$.
4) The obtained values of $\hat{a_i}$, where i takes values from 1 to 1000, are utilized to calculate the average estimate (A.E.), average bias (A.B.), and mean squared error (M.S.E.) using the following equations.

$$A.E. = \frac{1}{1000}\sum_{i=1}^{1000}\hat{a}_i, \quad A.B. = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{a}_i - a), \quad M.S.E. = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{a}_i - a)^2.$$

where $\hat{a}_i$ is the MLE of a at the i-th repetition.

Table 6 represents the A.E., A.B., and M.S.E. of the UT based on reversed *os* at different parameters with different sample sizes for $\theta \in \{(1.0,\ 1.50,\ 2.0)\}$ and n $\in \{8, 16, 24, 32, 40, 48, 55, 64, 72, 80\}$. Table 7 represent the A.E., A.B., and M.S.E. of the UT based on lower record values *os* at different parameters with different sample sizes for $\theta \in \{(1.0,\ 1.50,\ 2.0)\}$ and n $\in \{2, 3, 4, 5\}$. [41] derived MLE of inverted topp-leone distribution based on record values as sample size increases the MSEs decreases.

Table 6. Parameter estimation of the UT distribution using reversed *os*.

n	A.E.	A.B.	M.S.E.
	$\theta = 1.0$		
8	1.04114	0.04114	0.02712
16	1.02023	0.02023	0.01136
24	1.01417	0.01417	0.00744
32	1.00924	0.00924	0.00523
40	1.00781	0.00781	0.00421
48	1.00622	0.00622	0.00351
55	1.00610	0.00610	0.00302
64	1.00514	0.00514	0.00242
72	1.00431	0.00431	0.00208
80	1.00367	0.00367	0.00182
		$\theta = 1.5$	
8	1.56171	0.06171	0.06103
16	1.53034	0.03034	0.02557
24	1.52126	0.02125	0.01674
32	1.51386	0.01386	0.01176
40	1.51172	0.01172	0.00947
48	1.50933	0.00933	0.00790
55	1.50916	0.00915	0.00680
64	1.50772	0.00772	0.00545
72	1.50648	0.00648	0.00469
80	1.50550	0.00550	0.00409
		$\theta = 2.0$	
8	2.08228	0.08228	0.10850
16	2.04045	0.04045	0.04545
24	2.02834	0.02834	0.02977
32	2.01848	0.01848	0.02091
40	2.01562	0.01562	0.01683
48	2.01244	0.01244	0.01405
55	2.01221	0.01221	0.01208
64	2.01029	0.01029	0.00968
72	2.00863	0.00863	0.00834
80	2.00734	0.00734	0.00726

Table 7. Parameter estimation of the UT distribution using lower record values.

n	A.E.	A.B.	M.S.E.
		$\theta = 1.0$	
2	1.2344	0.2344	0.2779
3	1.1225	0.1225	0.0992
4	1.0955	0.0955	0.0605
5	1.0697	0.0697	0.0404
		$\theta = 1.5$	
2	1.8053	0.3053	0.5592
3	1.6820	0.1820	0.2344
4	1.6422	0.1422	0.1483
5	1.6060	0.1060	0.0886
		$\theta = 2.0$	
2	2.4427	0.4427	1.1457
3	2.2384	0.2384	0.4294
4	2.1718	0.1718	0.2374
5	2.1229	0.1229	0.1399

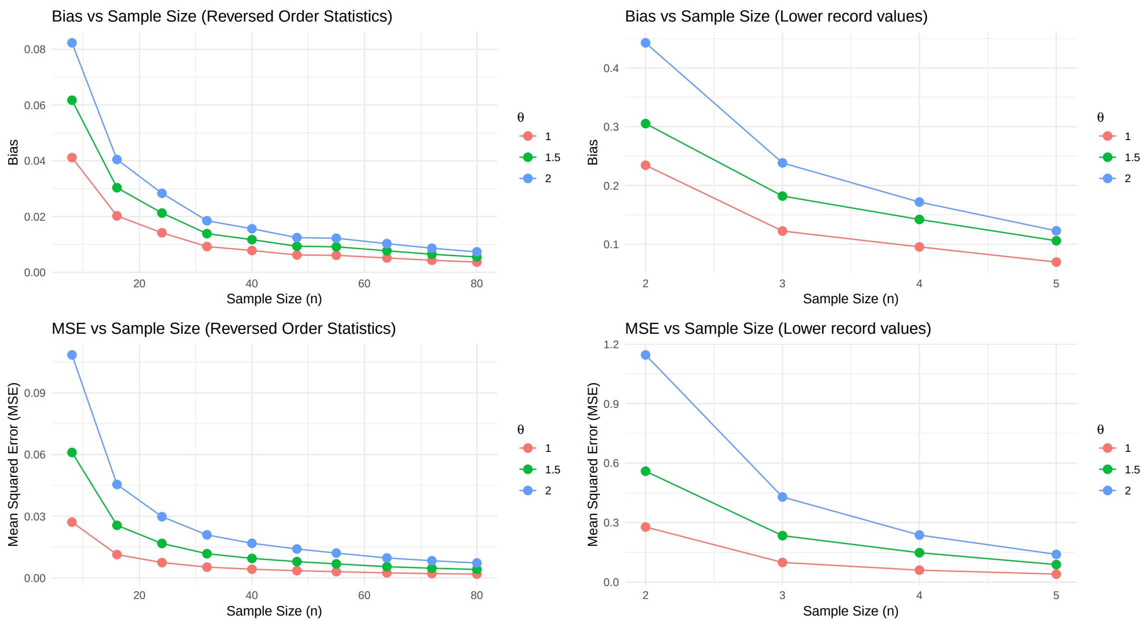

Figure 2. Plots of bias vs sample size and MSE vs sample size for reversed order statistics and record values at different parameter values.

7. Conclusions

In this study, explicit expressions for the single moments of *dgos* have been derived. These expressions are formulated for the general case and for specific cases such as reversed *os* and lower record values. Furthermore, recurrence relations for evaluating higher-order moments, including both single and product moments, have been established and analyzed within the frameworks of reversed *os* and lower record values. We have also formulated the mathematical expressions necessary to obtain the MLEs of the unknown parameter θ for both reversed *os*s and lower record values. For moment calculations, `MATLAB` was utilized, while the `nleqslv` in **R** was employed for MLE. The application of these computational methods improved the accuracy and reliability of our parameter estimates. Simplified expressions for specific special cases were also provided. Additionally, the paper presents several results concerning the characterization of the distribution. To complement the theoretical developments, we carried out a comprehensive simulation study to evaluate the performance of the proposed methods.

Specifically, we investigated MLEs, biases, MSEs, and moments of the distribution across different sample sizes and parameter configurations. The results indicate that, as the sample size increases, both the biases and MSEs of the parameter estimates decrease, thereby confirming the consistency and efficiency of the estimators.

Acknowledgment

The authors are thankful to the referred reviewers and Editor for their comments and suggestions to improve the paper.

REFERENCES

1. Kamps, Udo, *A concept of generalized order statistics*, Journal of Statistical Planning and Inference, vol. 48, no. 1, pp. 1–23, 1995.

2. Kamps, Udo, *Dual generalized order statistics*, Metron, vol. 61, no. 1, pp. 13–26, 2003.
3. Ahsanullah, M, *A characterization of the uniform distribution by dual generalized order statistics*, Communications in Statistics - Theory and Methods, vol. 33, no. 12, pp. 2921–2928, 2004.
4. Khan, RU and Anwar, Zaki and Athar, Haseeb, *Recurrence relations for single and product moments of dual generalized order statistics from exponentiated Weibull distribution*, Aligarh J. Statist, vol. 28, pp. 37–45, 2008.
5. Khan, AH and Anwar, Z and Athar, H, *Exact moments of generalized and dual generalized order statistics from a general form of distributions*, J. Statist. Science, vol. 1, no. 1, pp. 27–44, 2009.
6. Khan, AH and Anwar, Zaki and Chishti, Safia, *Characterization of continuous distributions through conditional expectation of functions of dual generalized order statistics*, Pakistan Journal of Statistics, vol. 26, no. 4, 2010.
7. Mahmoud, MAW and Abdel-Aty, Y and Mohamed, NM and Hamedani, Gholamhossein *Recurrence relations for moments of dual generalized order statistics from Weibull gamma distribution and Its characterizations*, Journal of Statistics Application & Probability, vol. 3, no. 2, pp. 189–199, 2014.
8. Khan, RU and Khan, MA, *PraDual generalized order statistics from family of J-shaped distribution and its characterization*, Journal of King Saud University-Science vol. 27, no. 4, pp. 285–291, 2015.
9. Saran, Jagdish and Pushkarna, Narinder and Tiwari, Rashmi, *Recurrence relations for single and product moments of dual generalized order statistics from a general class of distributions*, Journal of Statistical Theory and Applications, vol. 14, no. 2, pp. 123–130, 2015.
10. Khan, MAR and Khan, RU and Singh, B, *Relations for moments of dual generalized order statistics from exponentiated Rayleigh distribution and associated inference*, Journal of Statistical Theory and Applications, vol. 18, no. 4, pp. 402–415, 2019.
11. Athar, Haseeb and Alharbi, Yousef F and Fawzy, Mohamad A, *A study on moments of dual generalized order statistics from exponentiated generalized class of distributions*, Pakistan Journal of Statistics and Operation Research, vol. 17, no. 3, pp. 531–544, 2021.
12. Sobhi, AL and Mashail, M, *Moments of dual generalized order statistics and characterization for transmuted exponential model*, Computational Journal of Mathematical and Statistical Sciences, vol. 1, no. 1, pp. 42–50, 2022.
13. Maswadah, M, *Structural inference on the Fré chet distribution based on the dual generalized order statistics*, Research Square, 2023.
14. Ahsanullah, M, *,Generalized order statistics from exponential distribution*, Journal of Statistical planning and Inference, vol. 85, no. 1-2, pp. 85–91, 2000.
15. Pawlas, Piotr and Szynal, Dominik, *Recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto, and Burr distributions*, Communications in Statistics-theory and Methods, vol. 30, no. 4, pp. 739–746, 2001.
16. Udo Kamps and Erhard Cramer, *On distributions Of generalized order statistics*, Statistics vol. 35, no. 3, pp. 269–280, 2001.
17. Ahmad, Abd El-Baset A and Fawzy, Mohammad A, *Recurrence relations for single moments of generalized order statistics from doubly truncated distributions*, Journal of statistical planning and inference, vol. 117, no. 2, pp. 241–249, 2003.
18. Al-Hussaini, Essam K and Ahmad, Abd EL-Baset A and Al-Kashif, MA, *Recurrence relations for moment and conditional moment generating functions of generalized order statistics*, Metrika, vol. 61, no. 2, pp. 199–220, 2005.
19. Khan, RU and Anwar, Zaki and Athar, Haseeb, *Recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution*, Aligarh J. Statist, vol. 27, pp. 69–79, 2007.
20. Faizan, Muhammad and Athar, Haseeb, *Moments of generalized order statistics from a general class of distributions*, Journal of Statistics, vol. 15, pp. 36–43, 2008.
21. Ahmad, ABD EL-Baset A, *Single and product moments of generalized order statistics from linear exponential distribution*, Communications in Statistics—Theory and Methods, vol. 37, no. 8, pp. 1162–1172, 2008.
22. Khan, AH and Khan, MJS, *On Ratio and Inverse Moment of Generalized Order Statistics From Burr Distribution.*, Pakistan Journal of Statistics, vol. 28,no. 1, 2012.
23. Kumar, Devendra, *Lower generalized order statistics based on inverse Burr distribution*, American Journal of Mathematical and Management Sciences, vol. 35, no. 1, pp. 15–35, 2016.
24. Kumar, Devendra and Dey, Sanku, *Relations for moments of generalized order statistics from extended exponential distribution*, American Journal of Mathematical and Management Sciences, vol. 36, no. 4, pp. 378–400, 2017.
25. Gupta, Neetu and Anwar, Zaki and Dar, Aijaz Ahmad, *Moment properties of generalized order statistics from Ailamujia distribution*, International Journal of Computational and Theoretical Statistics, vol. 5, no. 02, pp. 115–122, 2018.
26. Gupta, Neetu and Jamal, Qazi Azhad, *Inference for Weibull generalized exponential distribution based on generalized order statistics*, Journal of Applied Mathematics and Computing, vol. 61, no. 1, pp. 573–592, 2019.
27. Malik, Mansoor Rashid and Kumar, Devendra, *Generalized Pareto distribution based on generalized order statistics and associated inference*, Statistics in Transition New Series, vol. 20, no. 3, 2019.
28. Anwar, Z and Ali, Z and Faizan, M and Khan, MA, *A note on moments of order statistics from the Pareto–Weibull distribution*, Annals of Mathematical Sciences and Applications, vol. 8,no. 1, pp. 175–198, 2023.
29. Anwar, Zaki and Ali, Zakir and Faizan, Mohd and Khan, Iftkhar *Moments and Inferences based on Generalized Order Statistics from Benktander Type II Distribution*, Statistics, Optimization & Information Computing, vol. 13, no. 3, pp. 1299–1319, 2025.
30. Nedjar, Sihem and Zeghdoudi, Halim, *On gamma Lindley distribution: Properties and simulations*, Journal of Computational and Applied Mathematics, vol. 298, pp. 167–174, 2016.
31. Chouia, Sarra and Zeghdoudi, Halim, *The XLindley distribution: properties and application*, Journal of Statistical Theory and Applications, vol. 20, no. 2, pp. 318–327, 2021.
32. Zeghdoudi, Halim and Nedjar, Sihem, *A pseudo Lindley distribution and its application*, Afrika Statistika, vol. 11, no. 1, pp. 923–932, 2016.
33. Messaadia, Hamouda and Zeghdoudi, Halim, *Zeghdoudi distribution and its applications*, International Journal of Computing Science and Mathematics, vol. 9, no. 1, pp. 58–65, 2018.
34. Bouchahed, Lahsen and Zeghdoudi, Halim, *A new and unified approach in generalizing the Lindley's distribution with applications*, Statistics, vol. 61, 2018

35. Khodja, Nawel and Gemeay, Ahmed M and Zeghdoudi, Halim and Karakaya, Kadir and Alshangiti, Arwa M and Bakr, Mahmoud E and Balogun, Oluwafemi Samson and Muse, Abdisalam Hassan and Hussam, Eslam, *Modeling voltage real data set by a new version of Lindley distribution*, IEEE Access, vol. 11, pp. 67220–67229, 2023
36. Abdul-Moniem, IB, *Recurrence Relations for Moments of Generalized Order Statistics from Marshall–Olkin Extended Family of Life Distributions and its Characterization*, Sohag J. Math, vol. 6, no. 3, pp. 45–50, 2019.
37. Athar, Haseeb and Faizan, Mohd, *Moments of lower generalized order statistics from power function distribution and its characterization*, International Journal of Statistical Sciences, vol. 11, pp. 125–134, 2011.
38. Georges Teissier, *Recherches sur le vieillissement et sur les lois de la mortalité*, Annales de physiologie et de physicochimie biologique, vol. 10, no. 1, pp. 237–284, 1934.
39. Krishna, Anuresha and Maya, Radhakumari and Chesneau, Christophe and Irshad, Muhammed Rasheed, *The unit Teissier distribution and its applications*, Mathematical and Computational Applications, vol. 27, no. 1, pp. 12, 2022.
40. Hwang, JS and Lin, GD, *On a generalized moment problem. II*, Proceedings of the American Mathematical Society, vol. 91, no. 4, pp. 577–580, 1984.
41. Khan, MJS and Ansari, Farhan and Azhad, Qazi J and Kabdwal, Naresh Chandra, *Moments and inferences of inverted topp-leone distribution based on record values*, International Journal of System Assurance Engineering and Management, vol. 15, no. 6, pp. 2623–2633, 2024.