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Alpha Power Transformed Extended Inverse Lindley Distribution: Properties and its Applications

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Abstract Variable transformation is a fundamental concept in probability and distribution theory and has been widely employed to enhance the flexibility of statistical models. The alpha power transformation (APT) is a useful technique for extending existing distributions to provide better fits to real-life data. In this article, we introduce a new four-parameter distribution, termed the alpha power transformed extended inverse Lindley distribution (APTEILD). Several distributional and statistical properties of the proposed model are derived, including the probability density function, cumulative distribution function, hazard rate function, moment generating function, quantile function, median, mode, order statistics, and entropy measures. Parameter estimation is carried out using five estimation methods: maximum likelihood estimation, maximum product spacing, ordinary least squares, weighted least squares, and the Cramér–von Mises approach. A Monte Carlo simulation study is conducted to generate random samples from the proposed distribution and to examine the finite-sample performance of the estimators. The practical applicability of the APTEILD model is illustrated using three real-life datasets. In addition, likelihood ratio tests are performed to assess the adequacy of the proposed model in comparison with its nested counterparts.

Keywords Alpha power transformation, Extended inverse Lindley Distribution, hazard function, survival function, Cramér–von Mises

AMS 2010 subject classifications 62E15, 62F35

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1. Introduction

In recent years, considerable attention has been devoted to the development of novel continuous probability distributions for lifetime data analysis, with the primary objective of improving model flexibility in capturing varying degrees of skewness and kurtosis. Lindley [1] introduced a probability distribution in the context of Bayesian statistics by constructing a mixture of the exponential distribution with parameter θ and the gamma distribution with parameters $(2, \theta)$. The resulting model, obtained by mixing these two distributions in specific proportions, is referred to as the Lindley distribution. The cumulative distribution function (cdf) and probability density function (pdf) of the LD are given by:

$$F(y;\theta) = 1 - \left(1 + \frac{\theta(1+y)}{(1+\theta)}\right) e^{-\theta y} \quad \text{and}$$

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$$f(y;\theta) = \frac{(\theta^2(1+y)e^{-\theta y})}{\theta+1}$$

A well-known limitation of the Lindley distribution in modeling lifetime data is its strictly increasing failure rate function, which restricts its applicability in situations where more flexible hazard shapes are required. Consequently, enhancing the adaptability of the model through suitable generalizations has become an important area of research. [2] investigated several statistical properties of the Lindley distribution and showed that, under certain conditions, it can provide a better fit than the exponential distribution. Owing to its analytical tractability and practical relevance, the Lindley distribution has attracted increasing attention in applied sciences in recent years. In this context, [3] introduced the generalized Lindley distribution (GLD). Subsequently, [4] proposed the extended Lindley distribution, while Shanker et al. [5] developed the two-parameter Lindley distribution. Furthermore, Ghitany et al. [6] introduced the power Lindley distribution, further expanding the flexibility of the Lindley family. In the literature, many researchers have developed the inverse version of some of the existing distributions to show non-monotone shapes of the hazard function. [7] introduced the inverse Weibull distribution in the context of survival data analysis. Subsequently, [8] proposed the one-parameter inverse Lindley distribution for modelling stress–strength reliability. This model was later extended by [9] to a two-parameter form, known as the generalized inverse Lindley distribution. These generalizations have demonstrated greater flexibility than several competing models and offer enhanced adaptability for modeling real-world reliability and lifetime data.

Mahdavi and Kundu [10] developed a new family of distributions known as Alpha Power Transformation (APT) technique, adding a new parameter α to the base distribution to increase the flexibility of the model. Given the base cdf $G(x)$ and pdf $g(x)$ of a baseline model with random variable X, the new cdf $F_{\text{APT}}(x)$ is expressed as follows:

$$F_{\text{APT}}(x) = \frac{\alpha^{G(x)} - 1}{\alpha - 1}, \quad x > 0, \alpha > 0, \alpha \neq 1 \tag{1}$$

and

$$F_{\text{APT}}(x) = G(x), \text{ for } \alpha = 1.$$

The corresponding probability density function $f_{\text{APT}}(x)$ of the transformed random variable X is given by:

$$f_{\text{APT}}(x) = \frac{\log \alpha}{\alpha - 1}\alpha^{G(x)}g(x), \quad x > 0, \alpha > 0, \alpha \neq 1 \tag{2}$$

and

$$f_{\text{APT}}(x) = g(x), \text{ for } \alpha = 1$$

Mahdavi and Kundu [10] first applied the alpha power transformation to the one-parameter exponential distribution and proposed the two-parameter alpha power exponential (APE) distribution, along with a comprehensive investigation of its distributional properties. Since then, this transformation technique has been widely employed to construct more flexible extensions of classical distributions. Notable examples include the alpha power-transformed Weibull [11], alpha power-transformed generalized exponential [12], alpha power-transformed inverse Lindley [13], alpha power-transformed inverse Lomax [14], alpha power-transformed Weibull–Fréchet [15], alpha power-transformed extended power Lindley [16], alpha power exponentiated inverse Rayleigh [17], and the alpha power transformation of the Lindley distribution [18]. Further developments include the alpha power-transformed beta distribution [19] and the alpha power-transformed generalized Pareto distribution [20]. In addition, several recent studies have proposed new distributions based on similar alpha power transformation frameworks, such as the alpha power Pareto distribution [21], a generalized alpha power family with applications to cancer data [22], the alpha power-transformed extended exponential distribution [23], a new extended alpha power-transformed family [24], the alpha power-transformed log-logistic distribution with applications to breaking stress data [25], the two-parameter alpha power-transformed Aradhana distribution [26], the alpha power Weibull transformation obtained by combining the APT and Weibull models [27], the modified alpha power-transformed Weibull distribution with engineering applications [28], the alpha power-transformed exponentiated Perks distribution [29], the alpha power-transformed XLindley distribution [30], the

alpha power-transformed generalized Weibull distribution [31], the three-parameter alpha power-transformed logistic distribution [32], and the alpha power modified Weibull–geometric distribution [33].

In this paper, we introduce the alpha power transformed extended inverse Lindley distribution (APTEILD) by applying the alpha power transformation to a three-parameter inverse Lindley model. Existing inverse Lindley–type distributions often suffer from limited flexibility in capturing complex hazard rate behaviors and extreme skewness, particularly when modeling data exhibiting nonmonotonic or upside-down bathtub hazard shapes. The inclusion of an additional shape parameter, α, in the proposed four-parameter APTEILD effectively overcomes these limitations by providing greater control over tail behaviour and hazard rate dynamics. Consequently, the APTEILD accommodates upside-down bathtub hazard functions while offering an improved fit for highly skewed lifetime data. Motivated by recent developments in alpha power–based models, this study aims to develop a more flexible and versatile distribution capable of outperforming existing competing models. The enhanced flexibility of the probability density, cumulative distribution, survival, and hazard rate functions makes the APTEILD a competitive alternative to classical lifetime distributions. For instance, lifetime data arising in biomedical studies often exhibit strong right-skewness and an upside-down bathtub hazard rate, reflecting an initial increase in risk followed by a gradual decline. In such situations, the extended inverse Lindley distribution and several existing alpha power–transformed models may fail to simultaneously capture the peak hazard and tail behaviour, resulting in poor overall fit. By incorporating the additional shape parameter α, the proposed APTEILD provides greater adaptability in modeling both the hazard rate and tail behaviour, thereby offering a more accurate and flexible framework for such data. Its practical usefulness is demonstrated through applications to real data sets, where it consistently provides superior fit in reliability and survival analysis. Table 1 summarizes practical guidelines for selecting lifetime models based on data characteristics. While simpler models are adequate for mildly skewed data with monotonic hazard rates, the proposed APTEILD is preferable for strongly skewed data and nonmonotonic (upside-down bathtub) hazard behavior due to its enhanced flexibility and tail control.

Table 1. Selecting appropriate Lindley-type lifetime models based on data characteristics

Data Characteristics	**Model**	**Justification**
Mild skewness; monotonic hazard	Power Lindley / Extended Inverse Lindley	Simple structure is sufficient to capture basic hazard patterns
Moderate skewness; limited tail effects	Alpha power–transformed Lindley-type models	Additional shape flexibility improves goodness-of-fit
Strong right-skewness; heavy-tailed data	APTEILD	Extra shape parameter provides better control over tail behavior
Upside-down bathtub hazard	APTEILD	Effectively captures nonmonotonic hazard rate behavior

The three-parameter extended inverse Lindley distribution [34] for a random variable X is defined by the following cdf and pdf

$$G(x;\theta,\beta,\delta) = \left[1 + \frac{\theta\beta}{\theta+\beta}\frac{1}{x^{\delta}}\right] e^{\frac{-\theta}{x^{\delta}}} \tag{3}$$

$$g(x;\theta,\beta,\delta) = \frac{\delta\theta^2}{\theta+\beta}\left[\frac{\beta+x^{\delta}}{x^{2\delta+1}}\right] e^{\frac{-\theta}{x^{\delta}}} \tag{4}$$

where $x>0, \theta>0, \beta>0, \delta>0$.
The corresponding survival and hazard functions are, respectively, given by

$$S(x;\beta,\theta,\delta) = 1 - \left[1 + \frac{\theta\beta}{\theta+\beta}\frac{1}{x^{\delta}}\right] e^{\frac{-\theta}{x^{\delta}}} \tag{5}$$

$$h(x;\beta,\theta,\delta) = \frac{\delta\theta^2(\beta+x^{\delta})}{x^{\delta+1}\left[(\theta+\beta)x^{\alpha}(e^{\frac{\theta}{x^{\delta}}}-1)-\beta\theta\right]} \tag{6}$$

where $(\theta, \beta, \delta) > 0$. Here, we use this distribution as a base distribution to develop the new alpha power transformed extended inverse Lindley distribution by adding one more parameter.
The following sections of the paper are organized as follows: Sections 2 and 3 introduce the proposed APTEILD distribution and present its main statistical properties. Section 4 discusses several parameter estimation methods for the APTEILD model. A Monte Carlo simulation study is reported in Section 5 to examine the performance of the estimators. In Section 6, three real-life datasets are analyzed to illustrate the applicability of the proposed distribution, and the results of the associated hypothesis tests are also presented. Finally, concluding remarks are provided in Section 7.

2. Alpha Power Transformed Extended Inverse Lindley Distribution

Using equations (1) and (2), we derived a new distribution from the baseline distribution of the random variable X whose pdf g(x) and cdf G(x) are given in (3) and (4).
Definition : Let X be a random variable having APTEILD. Then its cdf and pdf, by adding an extra parameter $\alpha(\alpha > 0, \alpha \neq 1)$ are given by

$$F(x;\alpha,\theta,\beta,\delta) = \begin{cases} \frac{\alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta}{x^\delta}}}-1}{\alpha-1} & \text{if } \alpha > 0, \alpha \neq 1 \\ \left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta}{x^\delta}} & \text{if } \alpha = 1 \end{cases} \tag{7}$$

$$f(x;\alpha,\theta,\beta,\delta) = \begin{cases} \frac{log(\alpha)}{\alpha-1}\frac{\delta\theta^2}{\theta+\beta}\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}}\alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta}{x^\delta}}}, & \text{if } \alpha > 0, \alpha \neq 1 \\ \frac{\delta\theta^2}{\theta+\beta}\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}}, & \text{if } \alpha = 1 \end{cases} \tag{8}$$

The parameter α gives the overall shape and tail behaviour of the distribution. The parameter θ controls the scale and rate of decay of the density. The parameter β functions as a shape parameter that influences the spread and tail thickness of the distribution, while the parameter δ regulates the curvature of the density, affecting skewness and the sharpness of the mode. Hence, each parameter is associated with a specific distributional feature, facilitating the interpretation and practical application of the model.

Particular Cases: Some of the novel distributions are special cases of APTEILD distribution, i.e
(a)If $\alpha = 1$, then the APTEILD reduces to the three-parameter extended inverse Lindley distribution(EILD) proposed by [34].

$$f(x;\theta,\beta,\delta) = \frac{\delta\theta^2}{\theta+\beta}\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}}. \tag{9}$$

(b) If $\alpha = 1, \beta = 1$,then APTEILD dsitribution becomes two parameter Generalized inverse Lindley distribution(GILD) proposed by [9].

$$f(x;\theta,\delta) = \frac{\delta\theta^2}{\theta+1}\left[\frac{1+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}}. \tag{10}$$

(c) If $\alpha = 1, \beta = 1, \delta = 1$ then APTEILD assumes the one parameter inverse Lindley distribution(ILD) introduced by [8].

$$f(x;\theta) = \frac{\theta^2}{\theta+1}\left[\frac{1+x}{x^3}\right]e^{\frac{-\theta}{x}}. \tag{11}$$

(d)If $\beta = 1, \delta = 1$ then APTEILD distribution reduces to alpha power transformed inverse Lindley distribution (APTILD) proposed by [13].

$$f(x;\alpha,\theta) = \frac{log(\alpha)}{\alpha-1}\frac{\theta^2}{\theta+1}\left[\frac{1+x}{x^3}\right]e^{\frac{-\theta}{x}}\alpha^{\left[1+\frac{\theta}{\theta+1}\frac{1}{x}\right]e^{\frac{-\theta}{x}}}. \tag{12}$$

(e)If $\beta = 1$ then APTEILD becomes alpha power transformed generalized inverse Lindley distribution (APTGILD) of the form

$$f(x;\alpha,\theta,\delta) = \frac{log(\alpha)}{\alpha-1}\frac{\delta\theta^2}{\theta+1}\left[\frac{1+x^{\delta}}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^{\delta}}}\alpha^{\left[1+\frac{\theta}{\theta+1}\frac{1}{x^{\delta}}\right]e^{\frac{-\theta}{x^{\delta}}}}. \tag{13}$$

(f) If $\alpha = e, \delta = 1, \beta = 1$ then APTEILD reduces to the inverse poisson-Lindley distribution(IP-L) proposed by [13].

$$f(x;\theta) = \frac{1}{e-1}\left(\frac{\theta^2}{\theta+1}\right)\left(\frac{1+x}{x^3}\right)e^{\frac{-\theta}{x}}e^{\left(1+\frac{\theta}{(\theta+1)x}\right)e^{\frac{-\theta}{x}}}. \tag{14}$$

(g) If $\alpha = 1, \beta = 0$, then APTEILD becomes the inverse weibull distribution (IW) discussed by [34].

$$f(x;\theta,\delta) = \frac{\delta\theta}{x^{\delta+1}}e^{\frac{-\theta}{x^{\delta}}}. \tag{15}$$

Figures 1 and 2 illustrate the shapes of the pdf of the APTEILD model for different values of the shape parameter α, while keeping θ=1.5 and δ=1 fixed, and considering two different values of the scale parameter β. For $\beta = 1.5$ (Figure 1), the pdf exhibits a unimodal and right-skewed behaviour for all considered values of $\alpha \leq 1$ and $\alpha \geq 1$. As α increases, the peak of the distribution decreases and shifts slightly to the right, indicating a reduction in concentration around the mode and a heavier right tail. In contrast, smaller values of α produce sharper peaks, reflecting higher probability mass near the origin. The case α=1 serves as a baseline model and displays comparatively heavier tails than those corresponding to $\alpha < 1$.

For $\beta = 5$ (Figure 2), similar qualitative features are observed; however, increasing β leads to a flattening of the pdf and a slower decay in the tail, demonstrating the strong influence of β on the dispersion of the distribution. The combined effects of α and β provide a wide range of density shapes, including highly peaked, moderately skewed, and heavy-tailed forms. These plots confirm that the APTEILD model is highly flexible, capable of modelling diverse data patterns commonly encountered in lifetime and reliability studies. The sensitivity of the pdf to changes in α and β highlights the suitability of the proposed model for fitting data with varying degrees of skewness and tail behaviour.

3. Statistical Properties

In this section, we discuss some theoretical properties of the newly proposed APTEIL distribution.

3.1. Survival and Hazard Functions

Proposition 1. The survival and hazard functions of the new APTEILD with parameter $\alpha > 0, \alpha \neq 1$ are expressed as

$$S_{APTEILD}(x;\alpha,\theta,\beta,\delta) = \frac{\alpha\left(1-\alpha^{-\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^{\delta}}\right)e^{\frac{-\theta}{x^{\delta}}}}\right)}{\alpha-1} \tag{16}$$

$$h_{APTEILD}(x;\alpha,\theta,\beta,\delta) = \frac{log(\alpha)(\delta\theta^2)(\beta+x^{\delta})\alpha^{2A(x)-1}}{(\theta+\beta)x^{(2\delta+1)}\alpha^{A(x)}-1}e^{\frac{-\theta}{x^{\delta}}} \tag{17}$$

where $A(x) = 1 + (\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^{\delta}})e^{\frac{-\theta}{x^{\delta}}}$

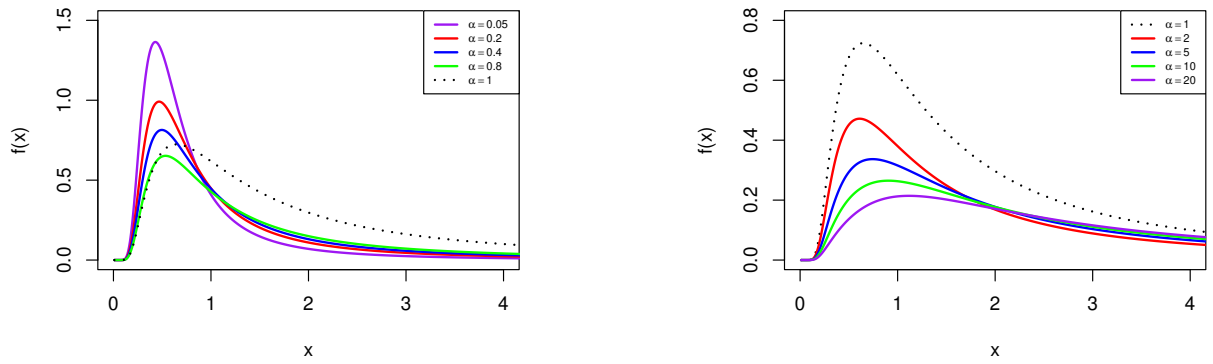


Figure 1. Shapes of the pdf of the APTEILD model under different values of α and fixed values of $\beta = 1.5$, $\theta = 1.5$, and $\delta = 1$

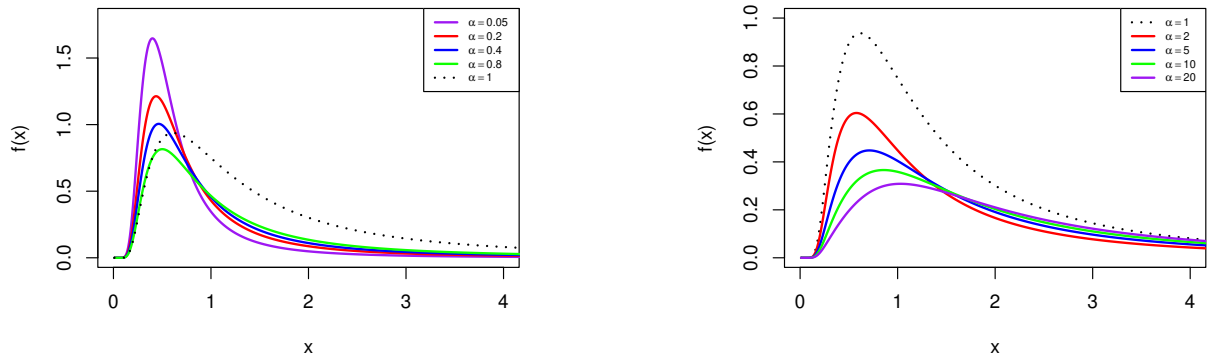


Figure 2. Shapes of the pdf of the APTEILD model under different values of α and fixed values of $\beta = 5$, $\theta = 1.5$, and $\delta = 1$.

Proof. The survival function of the new APTEILD is as follows

$$\begin{aligned} S_{APTEILD}(x;\alpha,\theta,\beta,\delta) &= 1 - F_{APTEILD}(x;\alpha,\beta,\delta,\theta) \\ &= 1 - \frac{\alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^{\delta}}\right]e^{\frac{-\theta}{x^{\delta}}}} - 1}{\alpha - 1} \\ &= \frac{\alpha\left(1 - \alpha^{-\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^{\delta}}\right)e^{\frac{-\theta}{x^{\delta}}}}\right)}{\alpha - 1} \end{aligned}$$

The hazard function is expressed as

$$\begin{aligned} h_{APTEILD}(x;\alpha,\theta,\beta,\delta) &= \frac{f_{APTEILD}(x;\beta,\alpha,\theta,\delta)}{S_{APTEILD}(x;\beta,\theta,\delta,\alpha)} \\ &= \frac{\frac{log(\alpha)}{\alpha-1}\frac{\delta\theta^2}{\theta+\beta}\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}}\alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta}{x^\delta}}}}{\alpha\left(1-\alpha^{-\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^\delta}\right)e^{\frac{-\theta}{x^\delta}}}\right)/(\alpha-1)} \\ &= \frac{log(\alpha)\delta\theta^2(\beta+x^\delta)e^{\frac{-\theta}{x^\delta}}\alpha^{\left(1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right)e^{\frac{-\theta}{x^\delta}}}}{\alpha\left(1-\alpha^{-\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^\delta}\right)e^{\frac{-\theta}{x^\delta}}}\right)(\theta+\beta)(x^{2\delta+1})} \\ &= \frac{log(\alpha)\delta\theta^2(\beta+x^\delta)e^{\frac{-\theta}{x^\delta}}\alpha^{A(x)}}{\alpha\left(1-\alpha^{-A(x)}\right)(\theta+\beta)(x^{2\delta+1})} \\ &= \frac{log(\alpha)(\delta\theta^2)(\beta+x^\delta)\alpha^{2A(x)-1}}{(\theta+\beta)x^{(2\delta+1)}\alpha^{A(x)}-1}e^{\frac{-\theta}{x^\delta}} \end{aligned}$$

where $A(x) = 1 + (\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^\delta})e^{\frac{-\theta}{x^\delta}}$

If we put $\alpha = 1$, then it reduces to the hazard function of the extended Lindley distribution. Figures 3 and 4 illustrate the hazard rate function of the APTEILD model for different values of α under fixed parameter settings. The hazard function exhibits an upside-down bathtub shape, indicating an initial increase in risk followed by a gradual decline. Changes in α mainly affect the height and location of the hazard peak, capturing heterogeneity in risk or treatment effects in biomedical studies. The parameters $\theta$and β control the baseline risk and persistence of the hazard, respectively, while δ regulates the rate at which the hazard decreases over time, allowing the model to represent diverse aging and survival patterns.

3.2. Quantile Function and Median

Using the quantile transformation of like in [35], we get the inverse of the cdf F^{-1} to a r.v which is uniformly distributed on (0,1). The quantile function of the new APTEIL distribution of a random variable X is $Q(u) = F^{-1}(x)$, where

$$u = \frac{\alpha^{\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^\delta}e^{\frac{-\theta}{x^\delta}}\right)}-1}{\alpha-1}$$

Multiplying both sides by $\alpha - 1$ and adding 1, we get

$$\alpha^{\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^\delta}e^{\frac{-\theta}{x^\delta}}\right)} = 1 + u(\alpha-1)$$

$$\implies 1 + \frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^\delta}e^{\frac{-\theta}{x^\delta}} = log_\alpha(1+u(\alpha-1))$$

$$\implies \frac{\theta\beta}{\theta+\beta}ze^{-\theta z} = A \text{ , where } A = log_\alpha(1+u(\alpha-1)) - 1 \text{ and } z = \frac{1}{x^\delta}$$

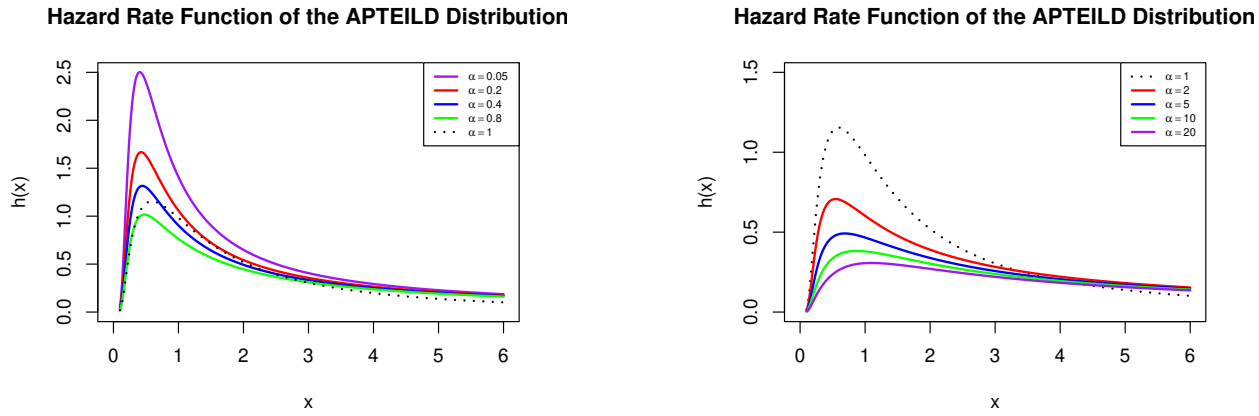


Figure 3. Shapes of the hazard function of the APTEILD model under different values of α and fixed values of $\beta = 0.5$, $\theta = 1.0$, and $\delta = 1$

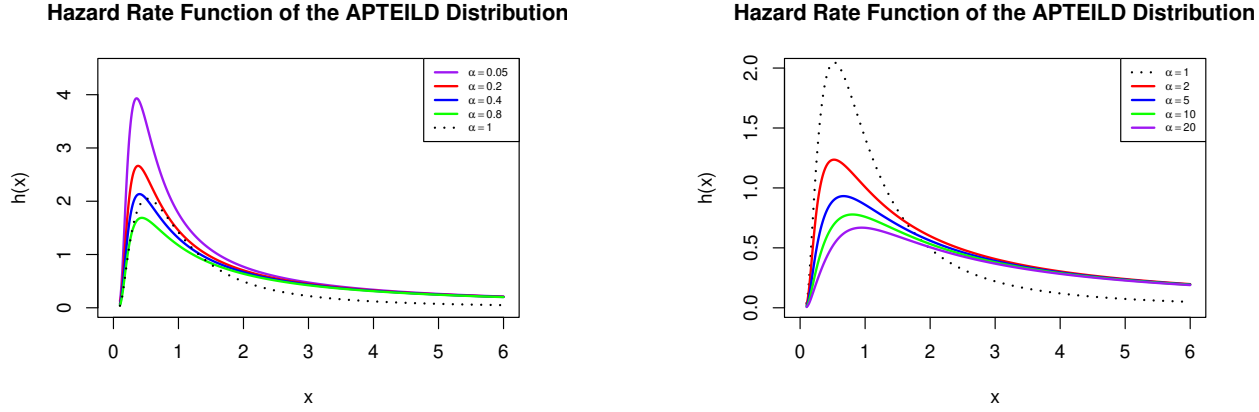


Figure 4. Shapes of the hazard function of the APTEILD model under different values of α and fixed values of $\beta = 5$, $\theta = 1.0$, and $\delta = 1$.

$$\begin{aligned}
&\implies ze^{-\theta z} = \frac{A(\theta+\beta)}{\theta\beta} \\
&\implies -\theta z e^{-\theta z} = \frac{-A(\theta+\beta)}{\beta} \text{ (Multiplying both sides by}(-\theta)) \\
&\implies z = -\frac{1}{\theta} W_{-1}\left(-\frac{A(\theta+\beta)}{\beta}\right) \text{ (using Lambert } W_{-1} \text{ function)} \\
&\implies x = \left(-\frac{1}{\theta} W_{-1}\left(-\frac{A(\theta+\beta)}{\beta}\right)\right)^{-\frac{1}{\delta}}
\end{aligned}$$

Therefore,the quantile function is

$$Q_x(u) = \left(-\frac{1}{\theta} W_{-1}\left(-\frac{(log_\alpha(1+u(\alpha-1))-1)(\theta+\beta)}{\beta}\right)\right)^{-\frac{1}{\delta}} \tag{18}$$

If we put u= $1/2$ in equation (18), we get the median function for the new APTEILD :

$$Median = \left(-\frac{1}{\theta} W_{-1}\left(-\frac{\left(\log_{\alpha} \frac{\alpha+1}{2} - 1\right)(\theta+\beta)}{\beta} \right) \right)^{-\frac{1}{\delta}} \tag{19}$$

3.3. Mode

The mode of APTEILD is expressed as

$$\begin{aligned} Mode &= \frac{d}{dx} f(x;\alpha,\beta,\delta,\theta) \\ &= \frac{d}{dx} \frac{log(\alpha)}{\alpha-1} \frac{\delta\theta^2}{\theta+\beta} \left[\frac{\beta+x^{\delta}}{x^{2\delta+1}} \right] e^{\frac{-\theta}{x^{\delta}}} \alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^{\delta}}\right]e^{\frac{-\theta}{x^{\delta}}}} \\ &= \frac{log(\alpha)}{\alpha-1} \frac{\delta\theta^2}{\theta+\beta} \frac{d}{dx} \left[\left[\frac{\beta+x^{\delta}}{x^{2\delta+1}} \right] e^{\frac{-\theta}{x^{\delta}}} \alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^{\delta}}\right]e^{\frac{-\theta}{x^{\delta}}}} \right] \end{aligned}$$

$$Mode = \frac{log(\alpha)}{\alpha-1} \frac{\delta\theta^2}{\theta+\beta} \frac{e^{-2x^{-\delta}\theta} x^{-2-4\delta} \alpha^{\frac{e^{-x^{-\delta}\theta} x^{-\delta}(\beta\theta+x^{\delta}(\beta+\theta))}{\beta+\theta}} \left(-e^{-x^{-\delta}} x^{\delta}(\beta+\theta)(x^{2\delta}(1+\delta)-\beta\delta\theta+x^{\delta}(\beta+2\beta\delta-\delta\theta))(x^{\delta}+\beta)^2\delta\theta^2 log(\alpha) \right)}{\beta+\theta}$$

Since the pdf of the APTEILD model does not admit a closed-form expression for the mode, it is obtained numerically. The mode is computed by maximizing the log-pdf over the positive real line using one-dimensional numerical optimization. Specifically, a bounded optimization algorithm such as the Brent method is employed with an initial search interval chosen to cover the region where the density is non-negligible. Convergence is declared when the relative change in the objective function is less than 10^{-6}. The resulting numerical values of the mode for selected parameter combinations are reported in Table 2.

Table 2. Mode of APTEILD for different values of $\alpha, \beta, \theta, \delta$.

α	β	θ	δ	Mode
1.5	0.5	0.5	1.5	0.3544
2.5	2.5	1.5	2	0.8088
3.5	1.2	2.5	1	0.9802
0.5	2.5	2.5	2	1.0693

3.4. Moments and Moment-Generating Functions

The r^{th} raw moments of the proposed distribution (8) can be derived as

$$\begin{aligned} \mu_r^{'} = E(X^r) &= \int_0^{\infty} X^r f(x) dx \\ &= \int_0^{\infty} X^r \frac{log(\alpha)}{\alpha-1} \frac{\delta\theta^2}{\theta+\beta} \left[\frac{\beta+x^{\delta}}{x^{2\delta+1}} \right] e^{\frac{-\theta}{x^{\delta}}} \alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^{\delta}}\right]e^{\frac{-\theta}{x^{\delta}}}} dx \\ &= \frac{log(\alpha)}{\alpha-1} \frac{\delta\theta^2}{\theta+\beta} \int_0^{\infty} X^r \left[\frac{\beta+x^{\delta}}{x^{2\delta+1}} \right] e^{\frac{-\theta}{x^{\delta}}} \alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^{\delta}}\right]e^{\frac{-\theta}{x^{\delta}}}} dx \end{aligned}$$

By using power series rule, we expand this term

$$\alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^{\delta}}\right]e^{\frac{-\theta}{x^{\delta}}}} = \sum_{k=0}^{\infty} \frac{(log\alpha)^k \left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^{\delta}}\right)^k e^{\frac{-\theta k}{x^{\delta}}}}{k!}$$

Now ,

$$\mu_r^{'} = \frac{log(\alpha)}{\alpha-1}\frac{\delta\theta^2}{\theta+\beta}\int_0^\infty X^r\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}}\sum_{k=0}^{\infty}\frac{(log\alpha)^k\left(1+\frac{\theta\beta}{\theta+\beta}.\frac{1}{x^\delta}\right)^k e^{\frac{-\theta k}{x^\delta}}}{k!}dx$$

$$\mu_r^{'} = \sum_{k-0}^{\infty}\frac{(log\alpha)^{k+1}}{k!}\frac{1}{\alpha-1}\frac{\delta\theta^2}{\theta+\beta}\int_0^\infty X^r\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}(1+k)}\left(1+\frac{\theta\beta}{\theta+\beta}.\frac{1}{x^\delta}\right)^k dx$$

After using binomial expansion, we get

$$\left(1+\frac{\theta\beta}{\theta+\beta}.\frac{1}{x^\delta}\right)^k = \sum_{i=0}^{\infty}\binom{k}{i}\left(\frac{\theta\beta}{(\theta+\beta)x^\delta}\right)^i$$

Next,

$$\begin{aligned}\mu_r^{'} &= \sum_{k-0}^{\infty}\frac{(log\alpha)^{k+1}}{k!}\frac{1}{\alpha-1}\frac{\delta\theta^2}{\theta+\beta}\int_0^\infty X^r\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}(1+k)}\sum_{i=0}^{\infty}\binom{k}{i}\left(\frac{\theta\beta}{(\theta+\beta)x^\delta}\right)^i dx\\ &= \sum_{k-0}^{\infty}\sum_{i=0}^{\infty}\binom{k}{i}\frac{(log\alpha)^{k+1}}{k!}\frac{1}{\alpha-1}\frac{\delta\theta^2}{\theta+\beta}\int_0^\infty X^r\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]e^{\frac{-\theta}{x^\delta}(1+k)}\left(\frac{\theta\beta}{(\theta+\beta)x^\delta}\right)^i dx\end{aligned}$$

After solving the integral part, we get the r^{th}moments i.e

$$\mu_r^{'} = \frac{1}{\alpha-1}\theta^{\frac{r}{\theta}}\sum_{k-0}^{\infty}\sum_{i=0}^{\infty}\binom{k}{i}\frac{(log\alpha)^{k+1}}{k!}\frac{\beta^i}{(\theta+\beta)^{i+1}}(1+k)^{\frac{r}{\delta}-2-i}\times\left\{\beta\Gamma\left(i+2-\frac{r}{\delta}\right)+\theta(1+k)\Gamma\left(i+1-\frac{r}{\delta}\right)\right\} \quad (20)$$

The mean of new APTEIL distribution is

$$E(X)=\mu_1^{'} = \frac{\theta^{\frac{1}{\theta}}}{\alpha-1}\sum_{k-0}^{\infty}\sum_{i=0}^{\infty}\binom{k}{i}\frac{(log\alpha)^{k+1}}{k!}\frac{\beta^i(1+k)^{\frac{1}{\delta}-2-i}}{(\theta+\beta)^{i+1}}\times\left\{\beta\Gamma\left(i+2-\frac{1}{\delta}\right)+\theta(1+k)\Gamma\left(i+1-\frac{1}{\delta}\right)\right\} \quad (21)$$

The moment generating function(m.g.f) of the APTEILD distribution is given by

$$M_x(s)=E(e^{sx})=\int_0^\infty e^{sx}f(x)dx$$

using the power series expansion of $e^{sx}=\sum_{k=0}^{\infty}\frac{(sx)^r}{r!}=\sum_{r=0}^{\infty}\frac{s^rx^r}{r!}$, we get

$$M_x(s) = \int_0^\infty\sum_{r=0}^{\infty}\frac{s^rx^r}{r!}f(x)dx \quad (22)$$

$$= \sum_{r=0}^{\infty}\frac{s^r}{r!}\int_0^\infty x^rf(x)dx \quad (23)$$

$$= \sum_{r=0}^{\infty}\frac{s^r}{r!}\frac{\theta^{\frac{r}{\theta}}}{\alpha-1}\sum_{k-0}^{\infty}\sum_{i=0}^{\infty}\binom{k}{i}\frac{(log\alpha)^{k+1}}{k!}\frac{\beta^i(1+k)^{\frac{r}{\delta}-2-i}}{(\theta+\beta)^{i+1}}\times\left[\beta\Gamma\left(i+2-\frac{r}{\delta}\right)+\theta(1+k)\Gamma\left(i+1-\frac{r}{\delta}\right)\right] \quad (24)$$

3.5. Stochastic Ordering

Comparing two random variables (or their distributions) to determine which one tends to yield larger values is known as stochastic ordering. We compare the complete distributions rather than individual realizations.Let X and Y are independent random variables with cdf F_X and F_Y respectively, then X is said to be smaller than Y in the context of

- **stochastic order**: $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.

- **hazard rate order**: $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.
- **mean residual life order**: $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ for all x.
- **likelihood ratio order**: $(X \leq_{lr} Y)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

We are aware of the following consequences by Shaked and Shanthikumar [41]:

$$X \leq_{lr} Y \;\Rightarrow\; X \leq_{hr} Y \;\Rightarrow\; X \leq_{mrl} Y$$

$$\Downarrow$$

$$X \leq_{st} Y$$

Proposition 2: Let $X \sim APTEILD(\alpha_1, \theta_1, \beta, \delta)$ and $Y \sim APTEILD(\alpha_2, \theta_2, \beta, \delta)$.If $\alpha_1 = \alpha_2 = \alpha$ and $\theta_1 \geq \theta_2$ then $X \leq_{lr} Y, X \leq_{hr} Y, X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof: Using the pdf in equation (8), the likelihood ratio is

$$\frac{f_x(x)}{f_y(x)} = \frac{\frac{log(\alpha_1)}{\alpha_1 - 1}\frac{\delta\theta_1^2}{\theta_1+\beta}\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right] e^{\frac{-\theta_1}{x^\delta}} \alpha_1^{\left[1+\frac{\theta_1\beta}{\theta_1+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta_1}{x^\delta}}}}{\frac{log(\alpha_2)}{\alpha_2 - 1}\frac{\delta\theta_2^2}{\theta_2+\beta}\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right] e^{\frac{-\theta_2}{x^\delta}} \alpha_2^{\left[1+\frac{\theta_2\beta}{\theta_2+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta_2}{x^\delta}}}}$$

$$= \frac{log(\alpha_1)(\alpha_2-1)(\theta_2+\beta)\alpha_1^{\left[1+\frac{\theta_1\beta}{\theta_1+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta_1}{x^\delta}}} e^{\frac{\theta_2-\theta_1}{x^\delta}}}{log(\alpha_2)(\alpha_1-1)(\theta_1+\beta)\alpha_2^{\left[1+\frac{\theta_2\beta}{\theta_2+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta_2}{x^\delta}}}}$$

If we put $\alpha_1 = \alpha_2 = \alpha$, we get

$$\frac{f_x(x)}{f_y(x)} = \frac{\theta_1^2(\theta_2+\beta)}{\theta_2^2(\theta_1+\beta)} e^{\frac{\theta_2-\theta_1}{x^\delta}} \alpha^{\left[\left(1+\frac{\theta_1\beta_1}{\theta_1+\beta_1}\frac{1}{x_1^\delta}\right)e^{\frac{-\theta_1}{x^{\delta_1}}} - \left(1+\frac{\theta_2\beta_2}{\theta_2+\beta_2}\frac{1}{x_2^\delta}\right)e^{\frac{-\theta_2}{x^{\delta_2}}}\right]}$$

Thus,

$$\frac{d}{dx}\log\left(\frac{f_Y(x)}{f_X(x)}\right) = \frac{d}{dx}\left[\log\left(\frac{\theta_1^2(\theta_2+\beta)}{\theta_2^2(\theta_1+\beta)}\right) + \frac{\theta_2-\theta_1}{x^\delta} + \log(\alpha)\cdot\Delta(x)\right],$$

where

$$\Delta(x) = \left(1+\frac{\theta_1\beta}{\theta_1+\beta}\cdot\frac{1}{x^\delta}\right)\exp\left(-\frac{\theta_1}{x^\delta}\right) - \left(1+\frac{\theta_2\beta}{\theta_2+\beta}\cdot\frac{1}{x^\delta}\right)\exp\left(-\frac{\theta_2}{x^\delta}\right).$$

Since $\theta_1 \geq \theta_2$, we have $\frac{\theta_2-\theta_1}{x^\delta} \leq 0$. Also, $\Delta(x) \leq 0$, because the exponential term with larger θ decays faster and makes the expression non-positive. Thus, when $\alpha_1 = \alpha_2 = \alpha$ and $\theta_1 \geq \theta_2$, it follows that $\frac{d}{dx}log\frac{f_x(x)}{f_y(x)} \leq 0$, which implies that $X \leq_{lr} Y, X \leq_{hr} Y, X \leq_{mrl} Y$ and $X \leq_{st} Y$.

3.5.1. Order statistics Let $X_{(1,n)}, X_{(2,n)}\cdots X_{(n,n)}$ denote the order statistics of the random variable $X_1, X_2 \cdots X_n$. Using the c.d.f and p.d.f given in (7) and (8) we compute the r^{th} order statistics are given by

$$f_y(y) = \frac{n!}{(r-1)!(n-r)!}F^r(y)(1-F^r(y))^{n-r}f(y)$$

$$= \frac{n!}{(r-1)!(n-r)!}\sum_{u=0}^{n-r}(-1)^u\binom{n-r}{u}F^{r-1+u}(y)f(y)$$

and

$$F_y(Y) = \sum_{k=r}^{n} \binom{n}{k} F^k(y)[1-F(y)]^{n-k}$$
$$= \sum_{k=r}^{n}\sum_{u=0}^{n-k} \binom{n}{k}\binom{n-k}{u}(-1)^u F^{k+u}(y)$$

Now,

$$f_y(y) = \frac{\frac{log\alpha}{\alpha-1}(\frac{\delta\theta^2}{\theta+\beta})n!}{(r-1)!(n-r)!}\sum_{u=0}^{n-r}(-1)^u\binom{n-r}{u}\left(\frac{\beta+y^\delta}{y^{2\delta+1}}\right)e^{\frac{-\theta}{y^\delta}}\alpha^{\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{y^\delta}\right)e^{\frac{-\theta}{y^\delta}}}[C(y)]^{r-1+u}$$

and

$$F_y(y) = \sum_{k=r}^{n}\sum_{u=0}^{n-k}\binom{n}{k}\binom{n-k}{u}(-1)^u[C(y)]^{j+u}$$

where $C(y) = \frac{\alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{y^\delta}\right]e^{\frac{-\theta}{y^\delta}}}-1}{\alpha-1}$.

3.6. Entropy Metric

3.6.1. Rényi Entropy The variation of the unpredictability in the distribution of any random variable is measured by Rényi entropy [36]. It offers crucial tools for analyzing changes in evolution over time and indicating variability in distributions at certain points in time. Rényi introduced an expression for the entropy function. so it is called Rényi entropy. The Rényi entropy function for our new APTEILD distribution is defined as

$$Re(\tau) = \frac{1}{1-\tau}log\left[\int_0^\infty f^\tau(x)dx\right]; \tau > 0, \tau \neq 0$$

$$= \frac{1}{1-\tau}log\left[\int_0^\infty \left(\frac{log(\alpha)}{\alpha-1}\right)^\tau\left(\frac{\delta\theta^2}{\theta+\beta}\right)^\tau\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]^\tau e^{\frac{-\theta\tau}{x^\delta}}\alpha^{\tau\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta}{x^\delta}}}dx\right]$$
$$= \frac{1}{1-\tau}log\left[\left(\frac{log(\alpha)}{\alpha-1}\right)^\tau\left(\frac{\delta\theta^2}{\theta+\beta}\right)^\tau\int_0^\infty\left[\frac{\beta+x^\delta}{x^{2\delta+1}}\right]^\tau e^{\frac{-\theta\tau}{x^\delta}}\alpha^{\tau\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta}{x^\delta}}}dx\right]$$

The integral is highly non-trivial and likely does not admit a simple closed-form solution for parameters. Numerical methods might be necessary for specific regimes.

3.6.2. Shannon Entropy It gives the uncertainty or average information content in a random variable (or probability distribution). Shannon [37] introduced this entropy function, so it is called Shannon entropy. The Shannon entropy function for the APTEILD is defined as

$$H(f) = -E[lnf(x)] = \int_0^\infty f(x)lnf(x)dx$$

Now, $lnf(x) = ln\left(\frac{log\alpha}{\alpha-1}\right) + ln(\frac{\delta\theta^2}{\theta+\beta}) + ln\left(\frac{\beta+x^\delta}{x^{2\delta+1}}\right) - \frac{\theta}{x^\delta} + ln\alpha\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^\delta}\right)e^{-\frac{\theta}{x^\delta}}$

Then,

$$H(f) = -\left[ln\left(\frac{log\alpha}{\alpha-1}\right) + ln(\frac{\delta\theta^2}{\theta+\beta}) + E\left(ln\left(\frac{\beta+x^\delta}{x^{2\delta+1}}\right)\right) - \theta E\left(\frac{1}{x^\delta}\right) + ln\alpha E\left[\left(1+\frac{\theta\beta}{\theta+\beta}\cdot\frac{1}{x^\delta}\right)e^{-\frac{\theta}{x^\delta}}\right]\right]$$

Closed-form expressions for the Shannon and Rényi entropies of the APTEILD distribution are not available due to the complexity of the density function. Therefore, entropy measures are evaluated numerically. The integrals are approximated using numerical quadrature techniques based on Gauss–Kronrod integration over a suitably truncated domain to ensure convergence. Table 3 presents the computed Shannon and Rényi entropy values for various parameter settings. All numerical computations are performed with a relative error tolerance less than 10^{-6}. Also, we present the plot of Rényi and Shannon

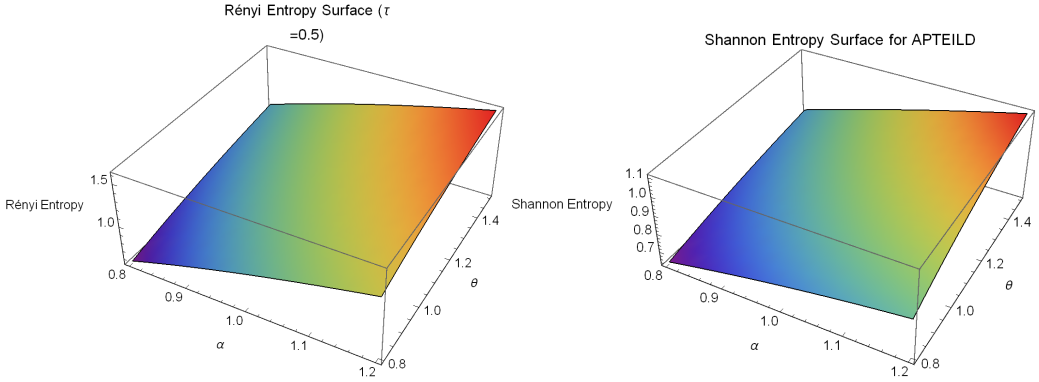


Figure 5. Plots of Rényi and Shannon entropy for fixed values $\beta = 0.5$ and $\delta = 2$, with varying α and θ.

entropy for fixed values $\beta = 0.5$ and $\delta = 2$, with varying α and θ in Figure 5.

For large values of x, the pdf of the APTEILD model exhibits exponential decay due to the term $e^{\frac{-\theta}{x^\delta}}$, ensuring rapid convergence of the entropy integrals. This asymptotic behaviour justifies the domain truncation used in numerical integration and significantly reduces computational burden. Although closed-form asymptotic approximations are not derived explicitly, the tail behaviour confirms the numerical stability and efficiency of the adopted computational procedures.

Table 3. Shannon and Rényi entropy values for different parameters

α	θ	β	δ	Shannon	Rényi $\tau = 0.5$	Rényi $\tau = 2$
0.8	0.8	0.5	2	0.7447	1.8093	0.3248
1.0	0.8	0.5	2	0.8053	1.8682	0.3857
1.2	0.8	0.5	2	0.8539	1.9150	0.4350
0.8	1.5	0.5	2	1.1616	2.2250	0.7409
1.0	1.5	0.5	2	1.2226	2.2841	0.8025

4. Parameter Estimation

4.1. Maximum Likelihood Estimation

Let $X_1, X_2 \cdots X_n$ be independent and identically distributed r.v with p.d.f $f_{APTEILD}(x; \alpha, \beta, \theta, \delta)$.The likelihood function $L(\alpha, \beta, \delta, \theta; x)$ can be maximized using MLE over the parameter domain. The likelihood function is derived as follows

$$L(x; \alpha, \beta, \delta, \theta) = \prod_{i=1}^{n} \frac{log(\alpha)}{\alpha - 1} \frac{\delta\theta^2}{\theta + \beta} \left[\frac{\beta + x^\delta}{x^{2\delta+1}}\right] e^{\frac{-\theta}{x^\delta}} \alpha^{G(x)}$$

where $G(x) = \left[1 + \frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right] e^{-\frac{\theta}{x^\delta}}$ which is the base distribution cumulative distribution function.
The log-likelihood function is expressed by

$$\begin{aligned} lnL(\alpha, \beta, \delta, \theta) =& nlog(log\alpha) - nlog(\alpha - 1) + nlog\delta + 2nlog\theta - nlog(\theta + \beta) + \\ & \sum_{i=1}^{n}[log(\beta + x_i^\delta) - (2\delta + 1)logx_i - \theta x_i^{-\delta}] + (log\alpha)\sum_{i=1}^{n}\left[\left(1 + \frac{\theta\beta}{\theta + \beta}\frac{1}{x^\delta}\right) e^{-\frac{\theta}{x^\delta}}\right] \end{aligned}$$

For compactness, we define for each i,

$$t_i = x_i^{-\delta}, e_i = e^{-\theta t_i}, A(\theta, \beta) = \frac{\theta\beta}{\theta + \beta}, G_i = G(x_i) = (1 + At_i)e_i$$

Then, by partial derivatives of the equation with respect to $\alpha, \beta, \delta, \theta$ and equating to 0, we get the following equations:

$$\frac{\partial logL(\alpha,\beta,\delta,\theta)}{\partial \alpha} = \frac{n}{\alpha log\alpha} - \frac{n}{\alpha - 1} + \frac{1}{\alpha}\sum_{i=1}^{n} G_i = 0 \tag{25}$$

Now, $\frac{\partial A}{\partial \beta} = \frac{\theta^2}{(\theta^2+\beta^2)}$ and $\frac{\partial G_i}{\partial \beta} = \frac{\theta^2}{(\theta+\beta)^2} t_i e_i$

$$\frac{\partial logL(\alpha,\beta,\delta,\theta)}{\partial \beta} = -\frac{n}{\theta+\beta} + \sum_{i=1}^{n}\frac{1}{\beta + x_i^{\delta}} + (log\alpha)\sum_{i=1}^{n}\frac{\theta^2}{(\theta+\beta)^2} t_i e_i = 0 \tag{26}$$

we use $\frac{\partial A}{\partial \theta} = \frac{\beta^2}{(\theta+\beta)^2}$ and $\frac{\partial G_i}{\partial \theta} = e_i\left[t_i \frac{\beta^2}{(\theta+\beta)^2} - t_i(1+At_i)\right]$ Then,

$$\frac{\partial logL(\alpha,\beta,\delta,\theta)}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta+\beta} - \sum_{i=1}^{n} t_i + (log\alpha)\sum_{i=1}^{n} e_i\left[t_i\frac{\beta^2}{(\theta+\beta)^2} - t_i(1+At_i)\right] = 0 \tag{27}$$

Next,

$$\frac{\partial}{\partial \delta} log(\beta + x_i^{\delta}) = \frac{x_i^{\delta} log x_i}{\beta + x_i^{\delta}}, \frac{\partial t_i}{\partial \delta} = -t_i log x_i, \frac{\partial e_i}{\partial \delta} = \theta t_i log x_i e_i$$

Also,

$$\frac{\partial G_i}{\partial \delta} = e_i t_i log x_i[-A + \theta(1+At_i)]$$

Therefore,

$$\frac{\partial logL(\alpha,\beta,\delta,\theta)}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^{n}\left[\frac{x_i^{\delta} log x_i}{\beta + x_i^{\delta}} - 2log x_i + \theta t_i log x_i\right] + (log\alpha)\sum_{i=1}^{n} e_i t_i log x_i[-A + \theta(1+At_i)] = 0 \tag{28}$$

Solving all the equations from (25-28). simultaneously, we obtain the estimates of α, β, θ, and δ. Since an analytical solution is not feasible, the Newton–Raphson model is utilized to obtain the solutions for the aforementioned equations.

4.2. Method of Maximum Product Spacing Estimators

[42] introduced the maximum product spacing (MPS) method as an alternative to MLE for the estimation of parameters of continuous univariate distributions. [43] independently developed the same method as an approximation to the Kullback-Leibler measure of information.For $x > 0$, the cdf of the APTEILD distribution with parameter vector $\boldsymbol{\psi} = (\alpha, \theta, \beta, \delta)$ is given by

$$F(x;\alpha,\theta,\beta,\delta) = \frac{\alpha^{A(x)} - 1}{\alpha - 1}, \qquad \alpha > 0,\ \alpha \neq 1, \tag{29}$$

where

$$A(x) = \left[1 + \frac{\theta\beta}{\theta+\beta}\, x^{-\delta}\right] \exp\left(-\frac{\theta}{x^{\delta}}\right). \tag{30}$$

For notational convenience, we define

$$B(x) = \exp\left(-\frac{\theta}{x^{\delta}}\right), \qquad C(x) = 1 + \frac{\theta\beta}{\theta+\beta}\, x^{-\delta}, \tag{31}$$

so that $A(x) = C(x)B(x)$.

Let $x_{1:n} < .... < x_{n:n}$ be the order statistics of a random sample of size n from APTEIL distribution. Let us also define the uniform spacing of a random sample from the APTEIL distribution as:

$$D_j(\psi) = F(x_{j:n}|\psi) - F(x_{j-1:n}|\psi)$$

, $j = 1, \ldots, n+1$ where

$$F(x_{0:n}|\psi) = 0 \quad \text{and} \quad F(x_{n+1:n}|\psi) = 1$$

clearly

$$\sum_{j=1}^{n+1} D_j(\psi) = 1$$

The maximum product spacing estimators (MPSE) $\hat{\alpha}_{MPSE}$, $\hat{\theta}_{MPSE}$,$\hat{\beta}_{MPSE}$ and $\hat{\delta}_{MPSE}$ of the unknown parameter α , θ, β and $\delta$are obtained by maximising the function with respect to α,θ,β and δ.

$$G(\psi) = \left[\prod_{j=1}^{n+1} D_j(\psi)\right]^{\frac{1}{n+1}}$$

or equivalently, by maximizing the function

$$H(\psi) = \frac{1}{n+1}\sum_{j=1}^{n+1} \log D_j(\psi)$$

The estimators $\hat{\alpha}_{MPSE}$,$\hat{\theta}_{MPSE}$,$\hat{\beta}_{MPSE}$ and $\hat{\delta}_{MPSE}$ of the parameters α,θ,β and δ can also be obtained by solving the non linear equations

$$\frac{\partial}{\partial\alpha}H(\psi) = \frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{D_j(\psi)}[\Delta_1(x_{j:n}|\psi) - \Delta_1(x_{j-1:n})|\psi] = 0$$

$$\frac{\partial}{\partial\theta}H(\psi) = \frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{D_j(\psi)}[\Delta_2(x_{j:n}|\psi) - \Delta_2(x_{j-1:n})|\psi] = 0$$

$$\frac{\partial}{\partial\beta}H(\psi) = \frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{D_j(\psi)}[\Delta_3(x_{j:n}|\psi) - \Delta_3(x_{j-1:n})|\psi] = 0$$

$$\frac{\partial}{\partial\delta}H(\psi) = \frac{1}{n+1}\sum_{j=1}^{n+1}\frac{1}{D_j(\psi)}[\Delta_4(x_{j:n}|\psi) - \Delta_4(x_{j-1:n})|\psi] = 0$$

then we can write

$$\Delta_1(x_{j:n}|\psi) = \frac{\alpha^{A(x)}(\alpha-1)A(x)\log(\alpha) - \left(\alpha^{A(x)}-1\right)}{(\alpha-1)^2}. \tag{32}$$

$$\Delta_2(x_{j:n}|\psi) = \frac{\alpha^{A(x)}\log(\alpha)}{\alpha-1}\,B(x)\left[\frac{\beta^2}{(\theta+\beta)^2}x^{-\delta} - \frac{1}{x^\delta}\left(1+\frac{\theta\beta}{\theta+\beta}x^{-\delta}\right)\right]. \tag{33}$$

$$\Delta_3(x_{j:n}|\psi) = \frac{\alpha^{A(x)}\log(\alpha)}{\alpha-1}\,B(x)\frac{\theta^2}{(\theta+\beta)^2}x^{-\delta}. \tag{34}$$

$$\Delta_4(x_{j:n}|\psi) = \frac{\alpha^{A(x)}\log(\alpha)}{\alpha-1}\,x^{-\delta}\log(x)\,B(x)\left[\theta\left(1+\frac{\theta\beta}{\theta+\beta}x^{-\delta}\right) - \frac{\theta\beta}{\theta+\beta}\right]. \tag{35}$$

[42] showed that maximizing H as a method of parameter estimation is as efficient as MLE estimation and the MPS estimators are consistent under more general conditions than the MLE estimators.

4.3. Methods of Ordinary and Weighted Least-Squares Estimators

The least square estimators and weighted least square estimators were proposed by [44] to estimate the parameters of Beta distributions. The ordinary least-squares estimators $\hat{\alpha}_{OLSE}$,$\hat{\theta}_{OLSE}$,$\hat{\beta}_{OLSE}$ and $\hat{\delta}_{OLSE}$ of the parameters α,θ,β and δ are obtained by minimizing the function:

$$S(\alpha,\theta,\beta,\delta) = \sum_{j=1}^{n}\left[F(x_j:n|\alpha,\theta,\beta,\delta) - \frac{j}{n+1}\right]^2 \tag{36}$$

These estimators can also be obtained by solving the non-linear equations:

$$\sum_{j=1}^{n}\left[F(x_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]\Delta_1(x_j:n|\alpha,\theta,\beta,\delta)=0,$$

$$\sum_{j=1}^{n}\left[F(x_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]\Delta_2(x_j:n|\alpha,\theta,\beta,\delta)=0$$

$$\sum_{j=1}^{n}\left[F(x_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]\Delta_3(x_j:n|\alpha,\theta,\beta,\delta)=0$$

$$\sum_{j=1}^{n}\left[F(x_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]\Delta_4(x_j:n|\alpha,\theta,\beta,\delta)=0$$

where $\Delta_1(x_j:n|\alpha,\theta,\beta,\delta)$,$\Delta_2(x_j:n|\alpha,\theta,\beta,\delta)$,$\Delta_3(x_j:n|\alpha,\theta,\beta,\delta)$ and $\Delta_4(x_j:n|\alpha,\theta,\beta,\delta)$ are given by Eq. (32)–(35) respectively.

The weighted least-squares estimators $\hat{\alpha}_{WLSE}$,$\hat{\theta}_{WLSE}$, $\hat{\beta}_{WLSE}$ and $\hat{\delta}_{WLSE}$ of the parameters α, θ, β and δ are obtained by minimizing the function:

$$W(\alpha,\theta,\beta,\delta)=\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left[F(t_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]^2 \tag{37}$$

These estimators can also be obtained by solving the non-linear equations:

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left[F(t_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]\Delta_1(t_{j:n}|\alpha,\theta,\beta,\delta)=0, \tag{38}$$

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left[F(t_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]\Delta_2(t_{j:n}|\alpha,\theta,\beta,\delta)=0 \tag{39}$$

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left[F(t_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]\Delta_3(t_{j:n}|\alpha,\theta,\beta,\delta)=0 \tag{40}$$

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left[F(t_{j:n}|\alpha,\theta,\beta,\delta)-\frac{j}{n+1}\right]\Delta_4(t_{j:n}|\alpha,\theta,\beta,\delta)=0 \tag{41}$$

where $\Delta_1(.|\alpha,\theta,\beta,\delta)$, $\Delta_2(.|\alpha,\theta,\beta,\delta)$,$\Delta_3(.|\alpha,\theta,\beta,\delta)$ and $\Delta_3(.|\alpha,\theta,\beta,\delta)$ are given by Eq. (32) to Eq. (35) respectively.

4.4. Method of Cramér–von Mises

The Cramér–von Mises (CvM) estimator is a member of the minimum-distance class of goodness-of-fit–based estimation methods. Unlike likelihood-based approaches, this estimator relies solely on the comparison between the theoretical cumulative distribution function (CDF) of the assumed model and the empirical distribution function (EDF) obtained from the data, without imposing additional parametric assumptions. Parameter estimation is achieved by minimizing the overall discrepancy between the theoretical and empirical distributions, measured through a squared distance criterion. Several empirical investigations have demonstrated that the CvM estimator can provide reduced bias and improved performance in finite samples when compared with other distance-based estimators. These advantages make the CvM approach an appealing and robust alternative to maximum likelihood estimation, especially in settings where the likelihood function is analytically intractable, numerically unstable, or highly sensitive to model misspecification. Accordingly, the CvM method has gained considerable attention in the estimation of parameters for complex and flexible distributional models. The corresponding CvM objective function is given by:

$$\begin{aligned}W_n^2(\alpha,\theta,\beta,\delta)&=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(X_{(i)};\alpha,\theta,\beta,\delta)-\frac{2i-1}{2n}\right]^2\\&=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{\alpha^{\left[1+\frac{\theta\beta}{\theta+\beta}\frac{1}{x^\delta}\right]e^{\frac{-\theta}{x^\delta}}}-1}{\alpha-1}-\frac{2i-1}{2n}\right]^2.\end{aligned} \tag{42}$$

The Cramér–von Mises (CvM) estimators of the parameters $\alpha, \theta, \beta, \delta$ are obtained by minimizing the objective function $W_n^2(\alpha, \theta, \beta, \delta)$. Since $W_n^2(\alpha, \theta, \beta, \delta)$ is a highly nonlinear function of the model parameters, analytical solutions are not tractable. Therefore, the CvM estimates are obtained using numerical optimization techniques that minimize the squared distance between the theoretical cumulative distribution function and the empirical distribution function. All computations are performed using numerical optimization routines in the R statistical software.

5. Simulation Study

In this section, a comprehensive Monte Carlo simulation study is conducted to evaluate and compare the finite-sample performance of five estimation techniques for the parameters of the proposed four-parameter APTEIL distribution. Specifically, the Maximum Likelihood (ML), Maximum Product Spacing (MPS), Ordinary Least Squares (OLS), Weighted Least Squares (WLS), and Cramér–von Mises (CvME) estimators are examined. The primary objective of this study is to assess the efficiency, accuracy, and consistency of these estimators under different sampling conditions.

To investigate the effect of sample size on estimator performance, five sample sizes are considered, namely $n = 50$, 100, 200, 300, and 500, which represent scenarios ranging from small to moderately large samples. In addition, three distinct parameter configurations are selected to evaluate the robustness of the estimation procedures across different regions of the parameter space. Random samples are generated using the inverse cumulative distribution function method, which is widely recognized for its numerical stability and effectiveness in simulation-based studies.

For each combination of sample size and parameter configuration, $M = 1000$ Monte Carlo replications are performed. The performance of each estimator is assessed using the bias and the mean squared error (MSE), which are defined as

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta,$$
$$\text{MSE}(\hat{\theta}) = \text{Var}(\hat{\theta}) + \left[\text{Bias}(\hat{\theta})\right]^2.$$

The simulation results corresponding to the three parameter configurations are summarized in Tables 4–6. Each table reports the bias and the corresponding MSE (in parentheses) for all estimation methods across the considered sample sizes. The results indicate that the bias and MSE of all estimators decrease as the sample size increases, demonstrating the consistency of the ML, MPS, OLS, WLS, and CvME estimators for the APTEIL distribution. It is observed that the parameter β exhibits relatively larger bias and MSE across all estimation methods in Table 6, indicating that it is more difficult to estimate accurately, particularly for small to moderate sample sizes. Similar behaviour has been reported in other inverse-type lifetime distributions.

To generate the random sample from the APTEIL distribution, we use the following algorithm:

Step 1: Set the values of $n, \alpha, \beta, \theta, \delta$ and the initial value y_0.

Step 2: Generate U, where $U \sim U(0, 1)$.

Step 3: Use Newton's method

$y^* = y_0 - \frac{F(y_o,\alpha,\beta,\delta,\theta)-U}{f(y_0,\alpha,\beta,\delta,\theta)}$, where $f(y_o, \alpha, \beta, \delta, \theta)$ and $F(y_o, \alpha, \beta, \delta, \theta)$ are cdf and pdf of the APTEILD.

Step 4: If $|y_0 - y*| \leq \epsilon$, then $y = y*$ as a sample from APTEILD. Otherwise, set $y_0 = y*$ and repeat to step 3.

Step 5: Repeat steps 2 to 4 to get the random sample $x_1, x_2 \cdots x_n$.

6. Real-Life Applications

To access the performance, importance and flexibility of the newly developed APTEILD distribution in real life applications, we considered three data sets and compared the fit with some well known existing distributions, i.e., three-parameter extended inverse Lindley distribution(EILD), inverse Lindley distribution(ILD), inverse Poisson-Lindley distribution(IP-L), one-parameter Lindley distribution(L), Power Lindley distribution(PL), two parameter Lindley distribution(TPLD), Weibull distribution(W), Gamma distribution and Exponetiated Lindley distribution. Here, our base model is extended inverse Lindley distribution with three parameter from which we developed the new distribution. The performance of the competing models is assessed using information-based criteria, namely the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC), together with goodness-of-fit measures such as the Kolmogorov–Smirnov (KS) statistic and its associated p-value. Furthermore, parameter estimation for all datasets is carried out using the maximum product spacing (MPS), ordinary least squares (OLS), weighted least squares (WLS), and Cramér–von Mises (CvM) methods, and the

Table 4. Bias and the corresponding MSEs (in parenthesis) for the APTEIL distribution

n	Parameters	ML	MPS	OLS	WLS	CVME
50	$\alpha = 0.1$	0.91065 (2.45296)	1.04134 (3.27106)	1.14705 (3.49958)	1.10879 (3.34751)	1.04490 (3.13909)
	$\theta = 0.5$	0.60578 (2.26217)	0.54147(1.99625)	0.81919 (3.14879)	0.66578 (2.24220)	0.75452 (2.85725)
	$\beta = 1.2$	0.00768 (2.16310)	0.16048(2.59938)	0.02661 (2.24011)	0.19094(2.63135)	0.20470 (2.70857)
	$\delta = 1.0$	0.35531 (2.62931)	0.21228(2.09300)	0.31997 (2.60733)	0.43994 (2.91459)	0.29358 (2.51096)
100	$\alpha = 0.1$	0.88628 (1.98452)	1.03133(3.26870)	1.06752 (3.02563)	0.93122 (2.43431)	0.88750 (2.29333)
	$\theta = 0.5$	0.64279 (2.25600)	0.54519(1.78813)	0.47576 (1.72039)	0.48136 (1.65715)	0.51332 (1.77105)
	$\beta = 1.2$	-0.26803 (1.40270)	-0.16148(1.66696)	-0.12903 (1.64234)	-0.16835 (1.62012)	-0.07241 (1.89823)
	$\delta = 1.0$	0.11941(1.80689)	0.08484(1.77438)	-0.05632 (1.27630)	0.17195 (1.91236)	0.160032 (1.96114)
200	$\alpha = 0.1$	0.84541 (1.70962)	0.92131(2.32745)	0.95527 (2.30306)	0.91788 (2.23338)	0.85604 (2.10322)
	$\theta = 0.5$	0.42676 (1.29161)	0.92131(2.32748)	0.49978 (1.52799)	0.62225 (2.03378)	0.40390 (1.36255)
	$\beta = 1.2$	-0.08961 (1.63161)	-0.19040(1.37382)	-0.28470 (1.26613)	-0.29914 (1.09380)	-0.23911 (1.30475)
	$\delta = 1.0$	-0.06086 (1.21313)	-0.14280(0.98062)	-0.09702 (0.98188)	-0.12702 (0.92531)	0.04208 (1.53771)
300	$\alpha = 0.1$	0.81131 (1.60967)	0.91782(2.05019)	0.83693 (1.81027)	0.89933(2.12855)	0.91676 (2.15230)
	$\theta = 0.5$	0.47184(1.34874)	0.44748(1.34798)	0.56837 (1.70061)	0.38129 (1.13462)	0.46827 (1.37620)
	$\beta = 1.2$	-0.21106 (1.18420)	-0.38953(0.86020)	-0.28072 (1.16054)	-0.25081 (1.17547)	-0.28461 (1.22799)
	$\delta = 1.0$	0.03101 (1.22347)	0.02260(1.41290)	0.07061 (0.99659)	-0.12467 (1.00889)	-0.01067 (1.06973)
500	$\alpha = 0.1$	0.62274 (1.03032)	0.82213(1.58631)	0.81351 (1.64246)	0.76047 (1.31523)	0.78367 (1.54329)
	$\theta = 0.5$	0.33452 (0.83767)	0.43467(1.26910)	0.42914 (1.07452)	0.35696 (0.91839)	0.36358 (0.91897)
	$\beta = 1.2$	-0.03246 (0.91396)	-0.37685(0.80353)	-0.34666 (0.9000)	-0.35127 (0.82138)	-0.44282 (0.75592)
	$\delta = 1.0$	-0.23003 (0.64365)	-0.21881(0.71934)	-0.11499 (0.91045)	-0.06745 (0.92637)	-0.12519 (0.83554)

Table 5. Bias and the corresponding MSEs (in parenthesis) for the APTEIL distribution

n	Parameters	ML	MPS	OLS	WLS	CVME
50	$\alpha = 1.2$	0.19816 (2.29070)	0.13645 (2.27688)	0.05745 (2.06130)	0.02534 (2.10481)	-0.10351 (1.48373)
	$\theta = 1.0$	0.30135 (2.34271)	0.255128(2.20695)	0.35740 (2.36374)	0.39714 (2.62635)	0.288641 (2.25422)
	$\beta = 1.2$	0.19368 (2.44895)	0.239866(2.6000)	0.11256 (2.21485)	0.24692(2.77892)	0.160835 (2.41566)
	$\delta = 0.5$	0.77643 (2.67115)	0.75248(2.63294)	0.94756 (3.56031)	0.79741 (2.79015)	0.81361 (2.99726)
100	$\alpha = 1.2$	0.13415(2.24575)	-0.04607(1.74282)	-0.05791 (1.64754)	-0.05725 (1.71233)	0.04207 (1.94794)
	$\theta = 1.0$	0.10603 (1.43223)	0.15398(1.72372)	0.25199 (2.07491)	0.32363 (2.33159)	0.28163 (2.09341)
	$\beta = 1.2$	0.05062 (1.97581)	0.05192(2.03530)	-0.01726 (1.84400)	-0.08271 (1.61460)	-0.07128 (1.53032)
	$\delta = 0.5$	0.71834(2.47082)	0.76177(2.58643)	0.76944 (2.71004)	0.67112 (2.24691)	0.75595 (2.56732)
200	$\alpha = 1.2$	-0.10345 (1.54169)	-0.02986(1.83922)	-0.08486 (1.58068)	0.03522 (2.01266)	-0.10735 (1.49898)
	$\theta = 1.0$	0.21214 (2.01846)	0.19270(1.91208)	0.13099 (1.70863)	0.18847 (1.95453)	0.16604 (1.84531)
	$\beta = 1.2$	-0.01470 (1.87516)	0.00351(1.88925)	-0.01145 (1.83789)	0.03580 (2.08996)	0.08613 (2.26771)
	$\delta = 0.5$	0.72284 (2.53395)	0.64761(2.13095)	0.63085 (2.02247)	0.65148 (2.14571)	0.81137 (2.86166)
300	$\alpha = 1.2$	0.02162 (1.41009)	0.01811(1.13054)	-0.10769 (1.51292)	-0.07276(1.67621)	-0.06466 (1.72680)
	$\theta = 1.0$	0.13625(1.72377)	0.15742(1.78003)	0.12445 (1.65336)	0.17306 (1.84846)	0.15690 (1.80033)
	$\beta = 1.2$	-0.02398 (1.76389)	-0.00205(1.86454)	-0.09528 (1.56221)	-0.04770 (1.74957)	0.04034 (2.10057)
	$\delta = 0.5$	0.67014 (2.32405)	0.57801(1.77473)	0.67019 (2.23623)	0.58306 (1.79637)	0.67613 (2.28715)
500	$\alpha = 1.2$	-0.01280 (1.32185)	-0.09105(1.64697)	-0.097062 (1.50364)	-0.06326 (1.66040)	-0.02133 (1.89054)
	$\theta = 1.0$	0.15952 (1.73036)	0.15323(1.78334)	0.06427 (1.39650)	0.08523 (1.43587)	0.15431 (1.81882)
	$\beta = 1.2$	-0.11502 (1.49468)	-0.05029(1.81022)	-0.10125 (1.55857)	-0.06294 (1.70508)	-0.05249 (1.80336)
	$\delta = 0.5$	0.66886 (2.25152)	0.50375(1.47504)	0.54048 (1.56001)	0.57433 (1.70217)	0.62112 (2.02773)

resulting fits are evaluated through appropriate goodness-of-fit criteria. Summary descriptive statistics of the datasets are reported in Table 7.

6.1. Dataset I: Annual Maximum Flood Discharges

The first dataset comprises the maximum annual flood discharges of the North Saskatchewan River at Edmonton, measured in thousands of cubic feet per second, and spans a continuous period of 47 years. The data were obtained from [45] and exhibit a pronounced right-skewed distribution. Figure 6 presents several non-parametric plots that illustrate the main characteristics of Dataset I. The parameter estimates of the fitted models, together with the corresponding goodness-of-fit statistics, are

Table 6. Bias and the corresponding MSEs (in parenthesis) for the APTEIL distribution

n	Parameters	ML	MPS	OLS	WLS	CVME
50	$\alpha = 3.0$	-0.10990 (2.41622)	-0.06720 (2.40767)	-0.13576 (2.37357)	-0.08114 (2.38124)	-0.16850 (2.21937)
	$\theta = 1.5$	0.32840 (2.28494)	0.33326(2.19484)	0.343341 (2.33812)	0.45605 (2.82883)	0.43463 (2.71811)
	$\beta = 0.5$	1.47072 (4.81920)	1.31317(3.81634)	1.46461 (4.49264)	1.39044(4.77892)	1.46083 (4.72566)
	$\delta = 1.5$	0.42211 (2.56115)	0.34620(2.31477)	0.50696 (2.79031)	0.33099 (2.33956)	0.42449 (2.56663)
100	$\alpha = 3.0$	-0.11326(2.19876)	-0.16543(2.05214)	-0.19390 (2.09572)	-0.17651 (2.11233)	-0.24818 (1.95748)
	$\theta = 1.5$	0.27568 (2.00712)	0.34862(2.10769)	0.40928 (2.37614)	0.40437 (2.52159)	0.27351 (1.95496)
	$\beta = 0.5$	1.41007 (4.21424)	1.32335(2.88009)	1.42515 (4.34826)	1.42764 (4.34435)	1.42657 (4.17427)
	$\delta = 1.5$	0.35009(2.25682)	0.47187(2.57063)	0.44027 (2.57348)	0.44387 (2.41554)	0.34501 (2.19656)
200	$\alpha = 3.0$	-0.18598 (1.79732)	-0.19475(2.03012)	-0.15558 (1.97569)	-0.15103 (2.04997)	-0.18088 (1.91367)
	$\theta = 1.5$	0.26077 (1.85482)	0.45090(2.58640)	0.43738 (2.35063)	0.35550 (2.27437)	0.32661 (1.99216)
	$\beta = 0.5$	1.37751 (4.08223)	1.24006(3.35319)	1.38152 (3.83789)	1.38229 (4.02363)	1.20918 (3.22007)
	$\delta = 1.5$	0.33492 (2.11663)	0.35823(2.21733)	0.48459 (2.51321)	0.33596 (2.08772)	0.32841 (2.21401)
300	$\alpha = 3.0$	-0.30603 (1.72433)	-0.23696(1.13054)	-0.20360 (1.75232)	-0.12518(2.03443)	-0.25306 (1.85174)
	$\theta = 1.5$	0.19031(1.56817)	0.40602(2.25209)	0.20260 (1.63285)	0.28616 (1.96404)	0.32511 (1.97640)
	$\beta = 0.5$	1.21264 (3.04813)	1.31288 (3.66627)	1.22482(3.14085)	1.35771 (3.93812)	1.29960 (3.66446)
	$\delta = 1.5$	0.27364 (2.10487)	0.25436(1.79963)	0.24446 (1.81004)	0.28893 (1.84717)	0.18223 (1.68893)
500	$\alpha = 3.0$	-0.38807 (1.51600)	-0.29026(1.59661)	-0.27628 (1.77602)	-0.25293 (1.81645)	-0.27875 (1.62995)
	$\theta = 1.5$	0.10392 (1.56133)	0.23877 (1.74390)	0.19955(1.53317)	0.32781 (1.87879)	0.20582 (1.55054)
	$\beta = 0.5$	1.12123 (2.62064)	1.25146(3.18290)	1.21030 (2.97422)	1.22157 (2.90271)	1.20778 (3.10200)
	$\delta = 1.5$	0.15330 (1.48460)	0.24519(1.72566)	0.26630 (1.86155)	0.12514 (1.36417)	0.20821 (1.58962)

Table 7. Descriptive statistics of the datasets

Dataset	**Mean**	**Median**	**Standard deviation**	**Kurtosis**	**Skewness**
Dataset I	51.4951	40.4000	32.3768	2.0685	7.9506
Dataset II	3.2774	2.7635	1.9431	2.7882	16.7036
Dataset III	1.6156	1.5260	0.4818	2.9396	14.4942

reported in Table 8. As observed from this table, the proposed APTEILD distribution outperforms the competing models by attaining the lowest AIC, BIC, and Kolmogorov–Smirnov statistics, along with the highest associated p-value.

The fitted probability density functions and selected competing distributions are displayed in Figure 7. Moreover, Figure 8 illustrates several fitted functions of the APTEILD distribution, including the probability density function, cumulative distribution function, survival function, and P–P plot. The Q–Q plot and the estimated hazard function for Dataset I are shown in Figure 9. The Q–Q plot indicates a satisfactory agreement between the empirical and theoretical quantiles, while the estimated hazard function exhibits a unimodal pattern, highlighting the flexibility of the APTEILD model in capturing non-monotonic risk behavior in flood discharge data. Furthermore, Table 13 reports the K–S and CvM statistics and associated p-values for the APTEILD model across various estimation techniques.

Furthermore, Figure 7 depicts the fitted density curves, while the empirical and fitted survival functions are illustrated in Figure 8.

6.2. Dataset II: New Vehicle Prices Data

The second dataset consists of prices ($\times 10^4$ dollars) of 428 new vehicles for the year 2004, reported in Kiplinger's Personal Finance (December 2003). This dataset exhibits an extreme right-skewed behavior and has been previously analyzed by [46]. Figure 10 presents several non-parametric plots illustrating the characteristics of Dataset II. The parameter estimates of the fitted distributions, along with the corresponding goodness-of-fit statistics, are reported in Table 10. As shown in this table, the proposed APTEILD model outperforms the competing distributions by attaining the lowest AIC, BIC, and K–S statistics, while also yielding the highest associated p-value.

The fitted probability density functions of the APTEILD distribution and selected competing models are displayed in Figure 11. In addition, Figure 12 illustrates several fitted functions of the APTEILD model, including the pdf, cdf, survival function, and P–P plot. The Q–Q plot and the estimated hazard function for Dataset II are presented in Figure 13. The estimated hazard function exhibits an initially increasing pattern followed by a gradual decline, highlighting the flexibility of the APTEILD distribution in capturing non-monotonic hazard rate behavior commonly observed in real-life economic data. Furthermore, Table 9 indicates that all estimation methods provide an adequate fit to the data.

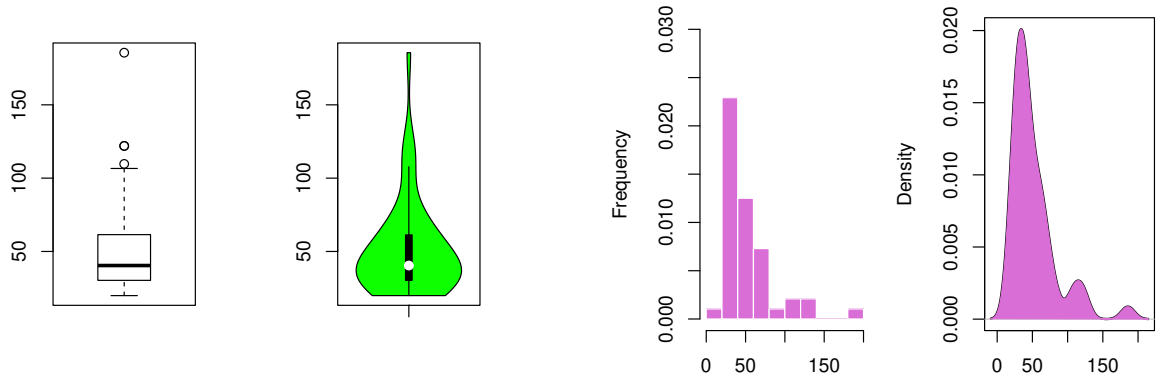


Figure 6. Non-parametric plots for flood discharge Data

Table 8. MLEs(SEs in parentheses) and Model selection criteria for flood discharge data

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\delta}$	AIC	BIC	K-S	P value
APTEILD	0.0100 (0.0157)	148.4130 (24.0478)	0.0100 (24.5060)	1.1609 (0.0810)	441.330	448.815	0.0629	0.9913
EILD	-	148.4131 (73.1641)	0.0100 (18.2117)	1.4073 (0.1608)	455.302	460.916	0.1684	0.1314
ILD	-	40.7703 (5.7502)	-	-	473.587	475.458	0.2862	0.0008
IP-L	-	28.2004 (4.4918)	-	-	481.054	482.926	0.2872	0.0007
L	-	0.0381 (0.0038)	-	-	454.280	456.151	0.1893	0.0642
PL	1.2874 (0.1230)	0.0117 (0.0061)	-	-	450.230	453.973	0.1350	0.3455
TPLD	20.0000 (110.8085)	0.0388 (0.0039)	-	-	455.097	458.839	0.1817	0.0840
W	-	58.3857 (5.0597)	-	1.7724 (0.1777)	455.413	459.155	0.1403	0.3009
Gamma	3.6537 (0.7143)	14.0939 (2.9540)	-	-	447.031	450.773	0.1356	0.3406
ExpLindley	2.5725 (0.7158)	0.0586 (0.0079)	-	-	444.853	448.956	0.1283	0.4081

Table 9. Different estimation methodologies for for the flood discharge data

	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\delta}$	**KS**	**KS-p**	**CvM**	**CvM-p**
MPS	82.3899	142.1349	0.0010	2.2400	0.0593	0.9958	0.0316	0.9717
OLS	89.9999	122.9191	0.0306	2.2395	0.0597	0.9955	0.0315	0.9718
WLS	100.000	142.0821	0.0520	2.2342	0.0640	0.9892	0.0326	0.9681
CvM	100.000	133.0871	0.0001	2.2390	0.0601	0.9950	0.0315	0.9719

6.3. Dataset III: Simulated Glass Fiber Strengths

The third data consists of 63 observations, which are generated to simulated strengths of glass fibers obtained from [47]. The dataset is right skewed. The data characteristics are illustrated through several non-parametric diagnostic plots presented in Figure 14. Parameter estimates of the fitted models, together with the associated goodness-of-fit statistics, are summarized in Table 12. As indicated in this table, the proposed APTEILD model provides the best overall fit among the competing distributions, as evidenced by the smallest values of the AIC, BIC, and Kolmogorov–Smirnov statistics, along with comparatively larger p-values.

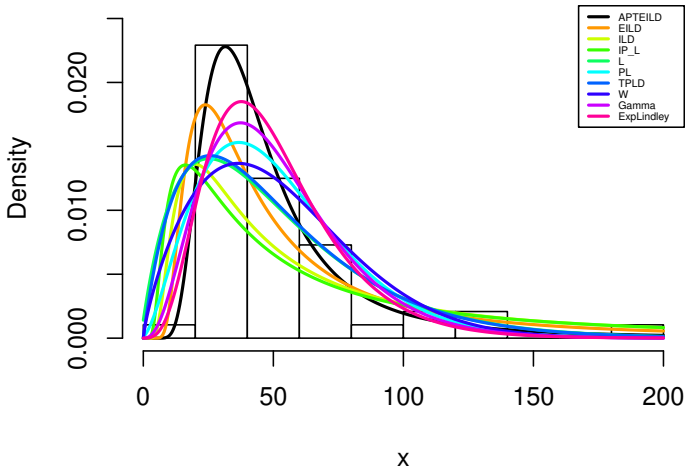


Figure 7. Plots for the fitted APTEILD pdf and other models to the flood discharge data

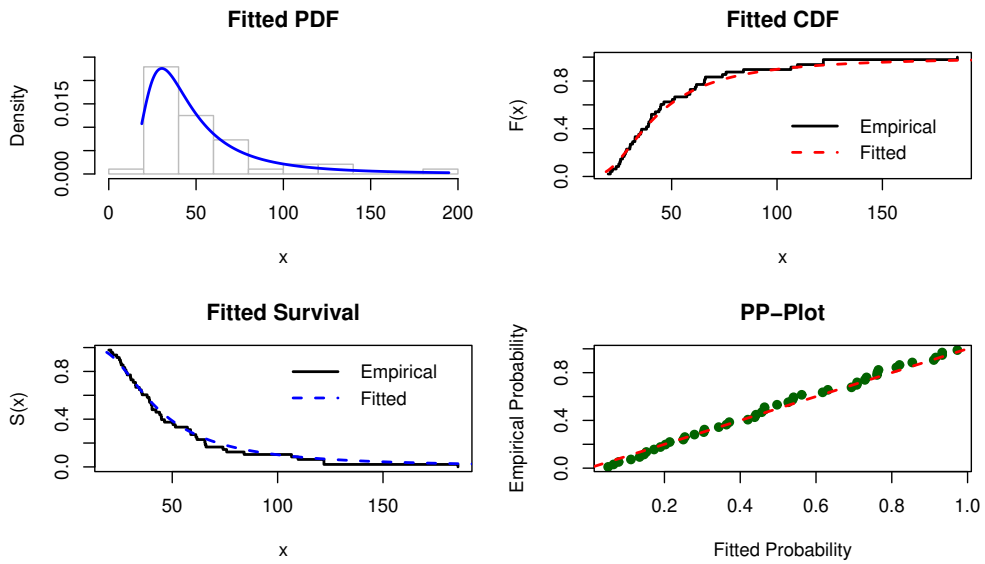


Figure 8. Fitted functions of APTEILD for the flood discharge data

The fitted density functions of the APTEILD model and selected competing distributions are displayed in Figure 15. Moreover, Figure 16 provides graphical representations of the fitted pdf, cdf, survival function, and the corresponding P–P plot of the APTEILD model. The Q–Q plot and the estimated hazard function for Dataset III are shown in Figure 17. The unimodal hazard function highlights its ability to model non-monotonic failure rates in glass fiber strength data. Furthermore, Table 13 summarizes the K–S and CvM goodness-of-fit statistics and their p-values for the APTEILD model under different estimation methods.

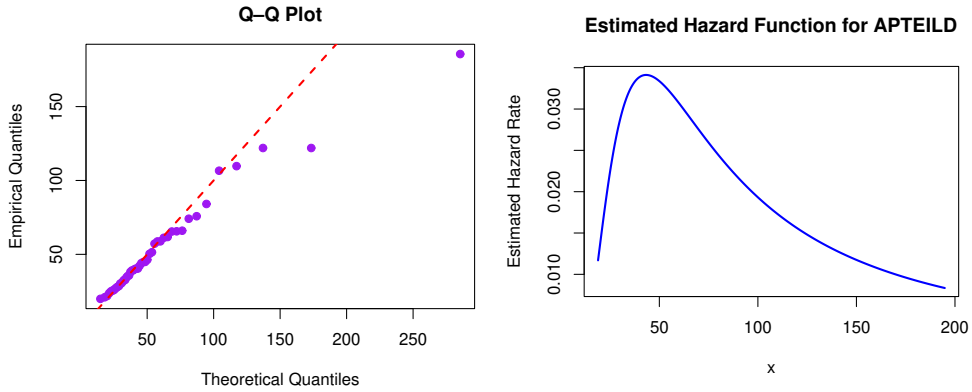


Figure 9. Q-Q plot and estimated hazard plots of APTEILD for the flood discharge data

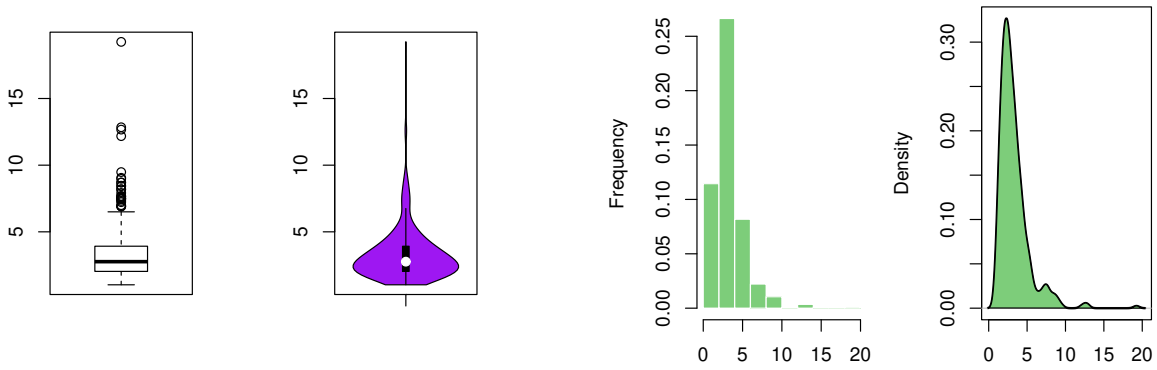


Figure 10. Non-parametric plots for Prices of 2004 New Cars and Trucks Data

6.4. Hypothesis Testing

This section performs the likelihood ratio test (LRT) to discriminate APTEILD$(\alpha,\beta,\theta,\delta)$ with its nested model EILD(θ,β,δ) and ILD(θ) . The statistic considered for this test is

(i) To discriminate ILD(θ) from APTEILD$(\alpha,\beta,\theta,\delta)$ model, the null hypothesis $H_0:\alpha=\beta=\delta=1$ is considered against $H_0:\alpha\neq\beta\neq\delta\neq 1$. Then the test statistic is

$$-2\log(LRT)=-2[\log L(\hat{\theta}_1,\beta=1,\delta=1,\alpha=1|x)-\log L(\hat{\theta}_2,\hat{\beta}_2,\hat{\delta}_2,\hat{\alpha}_2)]\sim\chi^2_3$$

where $\log L(\hat{\theta}_1)$ and $\log L(\hat{\alpha}_2,\hat{\beta}_2,\hat{\theta}_2,\hat{\delta}_2)$ are the MLEs of ILD(θ) and APTEILD$(\alpha,\beta,\theta,\delta)$ distributions, respectively.

(ii) To discriminate EILD(θ,β,δ) from APTEILD$(\alpha,\beta,\theta,\delta)$ model, the null hypothesis $H_0:\alpha=1$ is considered against $H_1:\alpha\neq 1$. Then the test statistic is

$$-2\log(LRT)=-2[\log L(\hat{\theta}_1,\hat{\beta}_1,\hat{\delta}_1,\alpha=1|x)-\log L(\hat{\theta}_2,\hat{\beta}_2,\hat{\delta}_2,\hat{\alpha}_2)]\sim\chi^2_1$$

where $\log L(\hat{\theta}_1,\hat{\beta}_1,\hat{\delta}_1)$ and $\log L(\hat{\alpha}_2,\hat{\beta}_2,\hat{\theta}_2,\hat{\delta}_2)$ are the MLEs of EILD(θ,β,δ) and APTEILD$(\alpha,\beta,\theta,\delta)$ distributions,respectively.

The likelihood ratio test (LRT) results for all three datasets are reported in Table 14. It is evident that, for both null hypothesis, the calculated LRT statistics exceed the corresponding chi-square critical values at the 5% significance level. Consequently, the null hypotheses are rejected, indicating that the data are better described by the proposed novel family of distributions than by the simpler competing models.

Table 10. MLEs(SEs in parentheses) and Model selection criteria for Prices of 2004 New Cars and Trucks Data

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\delta}$	AIC	BIC	K-S	P value
APTEILD	5.0000 (2.2378)	5.4496 (1.2693)	0.0100 (1.0925)	2.6525 (0.1126)	1502.076	1518.312	0.0362	0.6300
EILD	-	6.7834 (0.9136)	0.0100 (0.7807)	2.3166 (0.0850)	1512.520	1524.697	0.0528	0.1841
ILD	-	3.1990 (0.1297)	-	-	1839.346	1843.405	0.2812	0.0000
IP-L	-	2.3408 (0.1032)	-	-	1898.609	1902.668	0.2835	0.0000
L	-	0.5075 (0.0178)	-	-	1758.291	1762.351	0.2226	0.0000
PL	1.4803 (0.0431)	0.2775 (0.0187)	-	-	1628.039	1636.157	0.0911	0.0016
TPLD	20.0000 (9.9423)	0.6013 (0.0209)	-	-	1664.069	1672.187	0.1627	0.0000
W	-	3.7119 (0.1036)	-	1.8389 (0.0596)	1642.420	1650.538	0.0989	0.0005
Gamma	4.0702 (0.2676)	0.8052 (0.0563)	-	-	1559.439	1567.557	0.0688	0.0348
ExpLindley	4.4666 (0.4444)	0.9553 (0.0398)	-	-	1539.599	1547.717	0.0618	0.0762

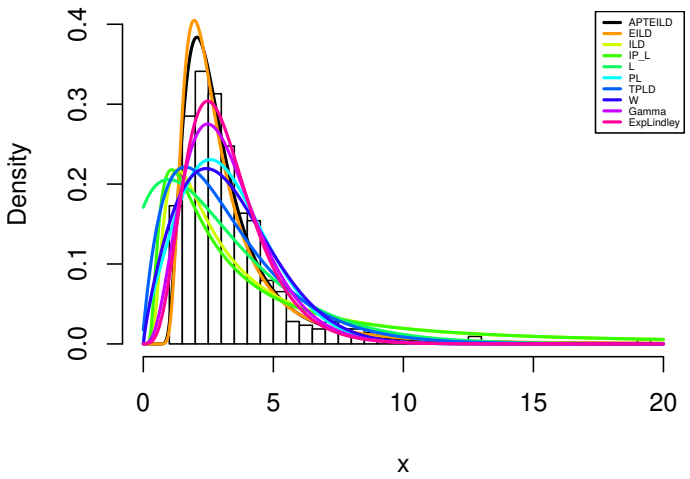


Figure 11. Plots for the fitted APTEILD pdf and other models to the Prices of 2004 New Cars and Trucks Data

Table 11. Different estimation methodologies for Prices of 2004 New Cars and Trucks Data

	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\delta}$	**KS**	**KS-p**	**CvM**	**CvM-p**
MPS	14.4479	4.0689	0.0001	2.6916	0.0275	0.9019	0.0683	0.7620
OLS	21.6095	3.7427	0.0001	2.7075	0.0282	0.8830	0.0541	0.8511
WLS	23.3456	3.6553	0.0001	2.7226	0.0265	0.9231	0.0604	0.8113
CvM	21.3786	3.7927	0.0001	2.7175	0.0277	0.8957	0.0539	0.8521

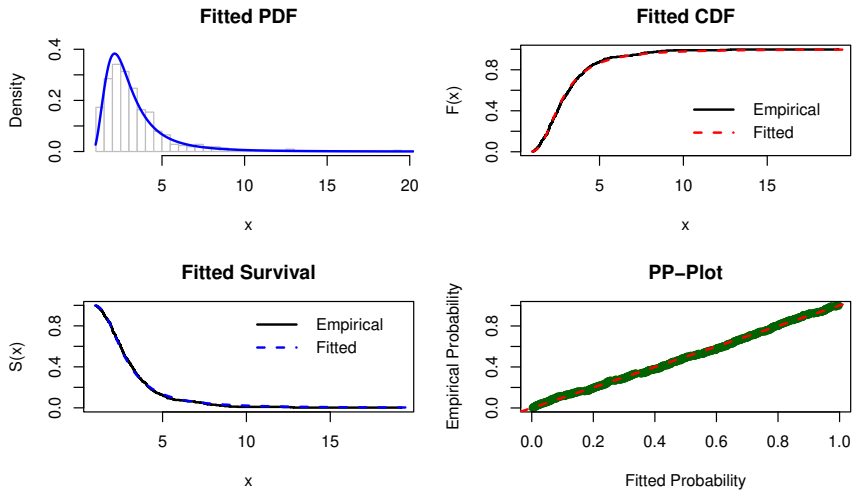


Figure 12. Fitted functions of APTEILD for Prices of 2004 New Cars and Trucks Data

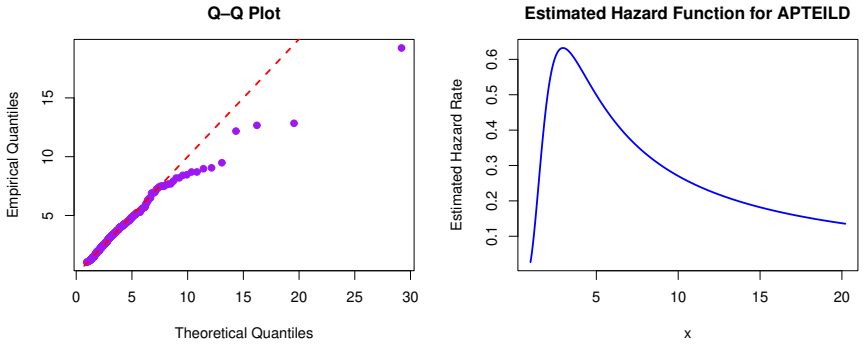


Figure 13. Q-Q plot and estimated hazard plots of APTEILD for the Prices of 2004 New Cars and Trucks Data

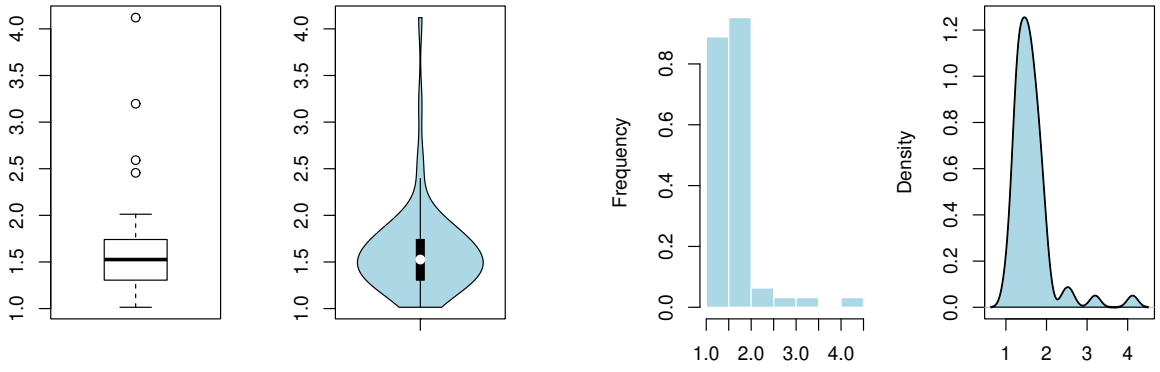


Figure 14. Non-parametric plots for simulated strengths of glass fibers data

Table 12. MLEs(SEs in parentheses) and Model selection criteria for Simulated strengths of glass fibers data

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\delta}$	AIC	BIC	K-S	P value
APTEILD	0.0118 (0.0276)	8.7943 (1.3776)	10.0000 (35.4415)	2.7794 (0.8055)	45.869	54.441	0.0721	0.8756
EILD	-	8.2605 (1.0320)	10.0450 (60.0031)	4.7176 (2.9561)	46.087	54.516	0.0806	0.7778
ILD	-	2.0297 (0.2053)	-	-	180.669	182.812	0.4504	0.0000
IP-L	-	1.4705 (0.1600)	-	-	190.832	192.975	0.4324	0.0000
L	-	0.9382 (0.0889)	-	-	172.952	175.095	0.4347	0.0000
PL	2.5760 (0.1851)	0.4122 (0.0605)	-	-	94.252	98.538	0.1908	0.0175
TPLD	20.0000 (17.2098)	1.2027 (0.1107)	-	-	149.188	153.475	0.3887	0.0000
W	-	1.7875 (0.0784)	-	3.0620 (0.2403)	96.734	101.020	0.2051	0.0084
Gamma	10.0000 (2.2037)	0.1615 (0.0361)	-	-	72.339	76.625	0.1705	0.0453
ExpLindley	20.0000 (8.5736)	2.6334 (0.2946)	-	-	64.985	69.271	0.1713	0.0438

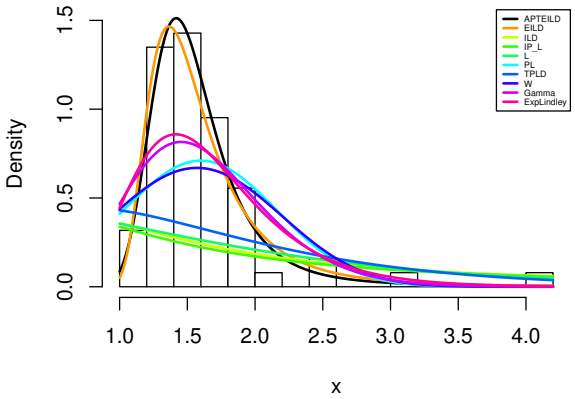


Figure 15. Plots for the fitted APTEILD pdf and other models to the Simulated strengths of glass fibers data

Table 13. Different estimation methodologies for Simulated strengths of glass fibers data

	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\delta}$	**KS**	**KS-p**	**CvM**	**CvM-p**
MPS	0.0109	8.1028	6.3571	2.6815	0.0694	0.9212	0.0538	0.8544
OLS	16.2246	3.9284	3.3561	6.5228	0.0692	0.9231	0.0451	0.9066
WLS	45.7069	2.9809	6.0201	6.5741	0.0728	0.8914	0.0449	0.9075
CvM	14.3238	4.3467	1.9340	6.6741	0.0747	0.8725	0.04415	0.9122

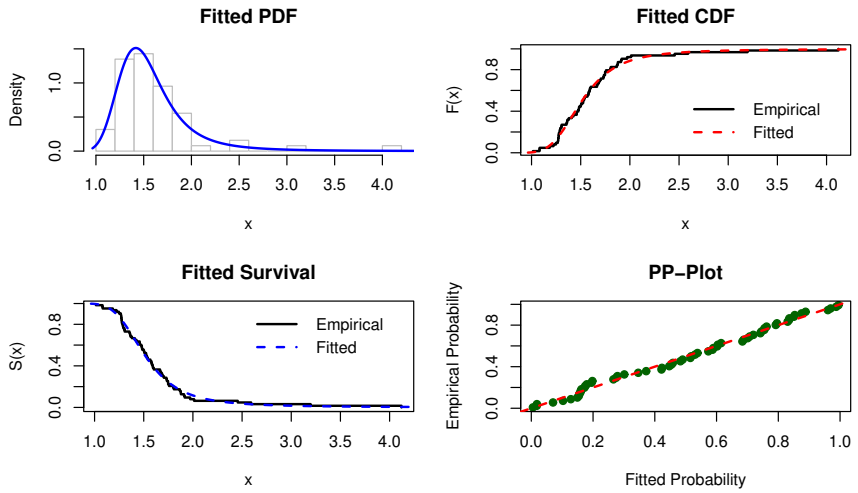


Figure 16. Fitted functions of APTEILD for Simulated strengths of glass fibers data

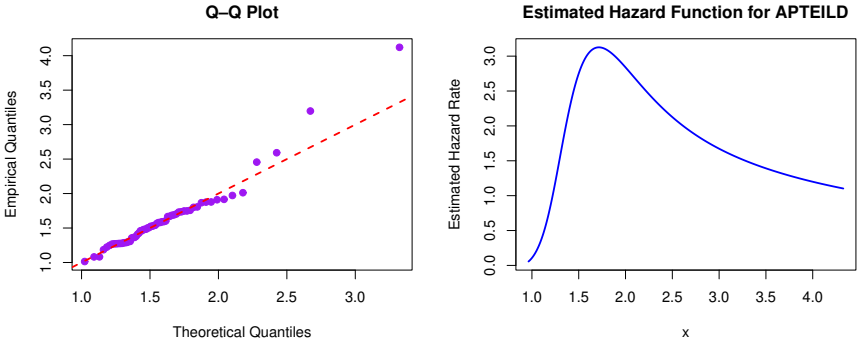


Figure 17. Q-Q plot and estimated hazard plots of APTEILD for simulated strengths of glass fibers data

Table 14. LR test results for the APTEILD model across three datasets

Hypothesis	LRT statistic			d.f.	Critical value at 5%
	Dataset I	Dataset II	Dataset III		
$H_0: \alpha = \beta = \delta = 1$ vs $H_1: H_0$ is False	40.334	345.373	140.620	3	7.814
$H_0: \alpha = 1$ vs $H_1: \alpha \neq 1$	10.542	14.546	6.248	1	3.841

7. Conclusions

In this paper, a new four-parameter inverse lifetime distribution, termed the alpha power transformed extended inverse Lindley distribution (APTEILD), has been introduced. The proposed model extends the three-parameter extended inverse Lindley distribution and offers enhanced flexibility for modeling right-skewed data. Several fundamental distributional properties of the APTEILD model have been derived, including the probability density function, cumulative distribution function, survival function, and hazard rate function. Additional characteristics such as the quantile function, median, mode, moments, entropy measures, and order statistics have also been investigated. The hazard rate function of the APTEILD model exhibits considerable flexibility, enabling it to accommodate a variety of shapes commonly encountered in applications from medical, engineering, and social science studies. Parameter estimation was performed using four estimation techniques: maximum likelihood estimation (MLE), Maximum Product Spacing (MPS), ordinary least squares (OLS), weighted least squares (WLS), and the Cramér–von Mises (CvM) method. An extensive Monte Carlo simulation study was conducted to evaluate and compare the finite-sample performance of these estimators under different sample sizes and parameter settings. The practical relevance of the proposed distribution was further demonstrated through applications to three real-world right-skewed datasets. Model comparison results indicate that the APTEILD distribution consistently outperforms several competing models, as evidenced by smaller values of the AIC, BIC, and Kolmogorov–Smirnov statistics, along with larger associated p-values. In addition, likelihood ratio tests were employed to assess the superiority of the proposed model over its nested counterparts, further confirming its improved goodness of fit. The APTEIL model represents a valuable addition to the family of inverse lifetime distributions and provides a flexible tool for modeling skewed data in diverse practical settings. Future research may focus on extending this work by developing Bayesian estimation procedures and exploring further applications of the model in reliability and survival analysis.

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Appendix

R codes for PDF, CDF, survival function, hazard function, quantile, moments, skewness, kurtosis, moment generating function, simulation and ML estimation are available at https://github.com/sajadulhussain/APTEILD.git.

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