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A New Modified Lifetime Distribution via a series-parallel system Approach: Mathematical Formulation, Estimation with Simulation Study under Progressive Type II Censoring Scheme and Real Data Validation

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Abstract Motivated by the series-parallel arrangement of a system, a new modified lifetime distribution is introduced by compounding the Alpha Power Modified Weibull Distribution with the Geometric Distribution twice, named Alpha Power Modified Weibull-2 Geometric Distribution. The new family can allow various hazard rate curves that compete well with other alternatives in fitting real data. We derive formal expressions for its moments, generating function, mean residual lifetime, and other reliability functions. Keeping the perspective of the importance of censoring in reliability and lifetime studies, the maximum likelihood estimation method under progressive type II censoring is used to estimate the model parameters. To investigate the performance of these methods, a simulation study is administered. Finally, application of the distribution to 2 real data sets is presented to illustrate the flexibility of the proposed distribution for both the uncensored and censored samples, and also for a comparative study with other competing distributions.

Keywords Compounding, Failure Times, Maximum Likelihood Estimation, Lifetime Distribution, System, Series-Parallel Structure

AMS 2010 subject classifications 60-XX, 60E05

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1. Introduction

At times, the existing classical distribution fail to provide a good fit to real data. To circumvent this constraint, research efforts have led to the development of methods for generating new distributions or extending the existing distribution by augmenting the current well-known distribution with additional parameter(s). Among these methods, compounding of distribution is a notable approach.This method leads to the generation of new distributions characterized by enhanced flexibility and broader properties, making them well-suited for modeling a wide range of complex phenomena. It also results in generalized forms of established distribution families.

In the context of engineering, reliability, biological, demographic and physical studies, failures in system components can occur due to exposure to several risk factors. In statistical literature, a system that fails when the minimum (Maximum) lifetime among all risks is realized is known as a series (parallel) system.

In series system, the system will fail at the failure of the first unit, due to it structural dependency or at a risk with minimum lifetime when looking at the risk factors exposed to the system. Let ”n” be the number of units/risks to a system, with X_i (i=1,...,n) as the corresponding lifetime, then the series system will fail at min$(X_1, .., X_n)$.

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Now, if n is a discrete random variable independent and identically distributed (i.i.d.) with probability mass function (pmf) P(N=n), X_i following any distribution with cumulative distribution (cdf) F(x_i) , then a new lifetime distribution can arise by compounding P(N=n) and F(x_i). [2] was the first to proposed a way to add a parameter to the lifetime distribution by Adopting this approach. Among such compound distributions introduced recently is the exponential-geometric distribution with decreasing hazard rate developed by compounding geometric and exponential distributions (EG) by [3]. The exponential-Poisson (EP) and exponential logarithmic (EL) distributions introduced by [6] and [7], respectively having a decreasing hazard rates. Similarly, we have the Weibull-geometric [9], Weibull-Poisson [10], Exponential-Weibull [11], Modified Weibull geometric [12], Additive Weibull-geometric [14], Exponentiated Transmuted Weibull geometric [17],Exponentiated Inverse Weibull-geometric [20], Weilbull lindley [21] and Inverse Weibull-geometric [22], Ishita Power series [24], Inverse Lindley Power Series [27], Pareto poisson [30] , unit Gombertz Power Series[32], Zhang-Power Series [33], Alpha Power Modified Weibull-geometric Distribution [35] and many more.

In parallel system, the system will fail at the failure of the last functioning unit, due to it structural dependency or at a risk with maximum lifetime when looking at the risk factors exposed to the system. Let "n" be the number of units / risks to a system, with X_i (i=1,...,n) as the corresponding lifetime, then the parallel system will fail at max($X_1, .., X_n$). Now, if n is a discrete random variable independent and identically distributed (i.i.d.) with pmf P(N=n), X_i following any distribution with cdf F(x_i) , then a new lifetime distribution can arise by compounding P(N=n) and F(x_i). Some of the distributions proposed and studied using this approach of compounding are the generalised exponential geometric [8], Poisson half-logistic[13], complementary exponentiated burr XII Poisson [19], generalised Lindley-Weibull [25], geometric-mixture exponential[31], compound truncated Poisson log-normal distribution[34].

A new system can always be produced by combining the model of series and parallel system. one such system is the Series-parallel, which is essentially a combination of multiple parallel systems arranged in the structure of a series system . Some of the recent introduced distributions following this approach are the unified lifetime distribution [15], geometric Power Lindley Poisson [16], Doubly Poisson-exponential [18], exponential geometric power series [23], power series exponential power series[28]and Poisson-geometric Lomax [29].

In this paper, we used such a concept to generate a new distribution in which the number of units considered in the system is random. We call the new distribution the Alpha Power Modified Weibull- 2 Geometric Distribution (APMW-2G). The Distribution shows multiple hazard rate shapes, which is one of the motivations of the APMW2G Distribution. Another motivation of the APMW-2G Distribution is an approach based on the failure of the system itself.

The rest of the paper is organized as follows: In Section 2, we introduce the new Alpha Power Modified Weibull-2 Geometric (APMW-2G) Distribution by compounding the Alpha Power Modified Weibull distribution and the geometric distribution. In Section 3, we discuss some of its statistical properties. In Section 4, the estimation of the parameters is performed by the method of maximum likelihood estimation. In Section 5, a simulation study is performed to illustrate the behavior of the MLEs in terms of different sample sizes n and various sets of parameter values. Three real data sets are studied to demonstrate the importance of the distribution in section 6. Finally, we conclude the paper in section 7.

2. The Alpha Power Modified Weibull-2 Geometric Distribution (APMW-2GD)

In this section, we introduce the Alpha Power Modified Weibull-2 Geometric Distribution by considering the Alpha Power Modified Weibull and the Geometric Distribution for compounding based on series-parallel system. Here we suppose that a system is made of k series subsystem functioning independently at a given time and that the jth series sub-system. j= 1,...,k consist of l_j parallel components with failure times, say X_{1j},...,$X_{l_j j}$, the the whole system will

fail if $min_{1\leq j\leq k}max(X_{1j}, X_{2j}, ...X_{l_j j})$, i.e $min_{1\leq j\leq k}max_{1\leq i\leq l_j}(X_{ij})$ occurs. If k and l_j are the realization of RV K subjecting to a discrete distribution with pmf P(K = k) and L_j subjecting to a discrete distribution with pmf P(L_j = l_j) respectively, and (X_{ij} are independent and identically distributed (iid) RVs subjecting to a continuous lifetime distribution with pdf f(x) and cdf F(x), then a new lifetime distribution can arise by compounding P(K = k), P(L_j = l_j) and F(x). As the system considered is series-parallel system, let us consider T= $min_{1\leq j\leq k}max_{1\leq i\leq l_j}(X_{ij})$, where X_{ij} are iid Random Variables following Alpha Power Modified Weibull Distribution (APMW), L_j and K be two Random Variables following Geometric Distribution.

The Cumulative distribution function (cdf) of a random variable (say N) following the Geometric Distribution and the probability distribution function (pdf) of the APMW Distribution, respectively, given by:

$$P(N = n) = (1 - p)p^{n-1},\ n = 1, 2, 3, ..., \text{and}\ \ 0 < p < 1 \tag{1}$$

and

$$f_{APMW}(x, \theta) = f_X(x) = \frac{log\alpha}{\alpha - 1}\gamma\lambda^\gamma e^{\gamma-1}\alpha^{1-e^{-(\lambda x)^\gamma}}e^{-(\lambda x)^\gamma}; \text{if}\ \alpha > 0, \alpha \neq 0. \tag{2}$$

Now, for all j, j=1,2,3,...k, let Z_j = $max_{1\leq i\leq l_j}(X_{ij})$. Then the conditional cdf of Z_j given L_j = l_j is given as

$$G(z_j \mid l_j; \tau) = F_X(z_j; \tau)^{l_j} \tag{3}$$

and the marginal cdf of Z_j is given by

$$F_{Z_j}(z_j; \tau, p) = \sum_{l_j=1}^{\infty} G(z_j \mid l_j; \tau)P(L_j = l_j) = \frac{(1 - p)(\alpha^{1-e^{-(\lambda z_j)^\gamma}} - 1)}{(\alpha - 1 - p(\alpha^{1-e^{-(\lambda z_j)^\gamma}} - 1))}, \tag{4}$$

Again, Since T= $min_{1\leq j\leq k}max_{1\leq i\leq l_j}(X_{ij})$= $min_{1\leq j\leq k}Z_j$, the conditional cdf of T given K=k, is given by

$$G(t \mid k; \tau, p) = 1 - [1 - F_Z(t; \tau)]^k = 1 - \left[\frac{\alpha - \alpha^{1-e^{-(\lambda t)^\gamma}}}{\alpha - 1 - p(\alpha^{1-e^{-(\lambda t)^\gamma}} - 1)}\right]^k \tag{5}$$

Finally, we have the cdf of T, which is the cdf of the desired APMW-2G distribution obtained as

$$F_{APMW2G}(t; \tau, p, q) = \frac{(1 - p)\left(1 - \alpha^{1-e^{-(\lambda x)^\gamma}}\right)}{\left(1 - \alpha - p + \alpha q + (p - q)\alpha^{1-e^{-(\lambda x)^\gamma}}\right)}, \tag{6}$$

and the corresponding pdf of the APMW-2G distribution as

$$f_{APMW2GD}(t; \tau, p, q) = \frac{(1 - p)(\alpha + q - q\alpha - 1)log\alpha\gamma\lambda^\gamma t^{\gamma-1}\alpha^{1-e^{-(\lambda t)^\gamma}}e^{-(\lambda t)^\gamma}}{\left[1 - \alpha - p + \alpha q + (p - q)\alpha^{1-e^{-(\lambda t)^\gamma}}\right]^2}, \tag{7}$$

Where, t > 0, $\tau = (\alpha, \gamma, \lambda) > 0, 0 \leq p \leq 1$ and $0 \leq q \leq 1$.

The survival function and the hazard rate function (hrf) of the APMW-2G distribution are given, respectively by

$$S_{APMW2GD}(t; \tau, p, q) = \frac{(q - 1)\left(\alpha - \alpha^{1-e^{-(\lambda t)^\gamma}}\right)}{1 - \alpha - p + \alpha q + (p - q)\alpha^{1-e^{-(\lambda x)^\gamma}}} \tag{8}$$

and

$$h_{APMW2GD}(t;\tau,p,q) = \frac{(1-p)(\alpha+q-q\alpha-1)log\alpha\gamma\lambda^{\gamma}e^{\gamma-1}\alpha^{1-e^{-(\lambda t)^{\gamma}}}e^{-(\lambda t)^{\gamma}}}{(q-1)\left(\alpha-\alpha^{1-e^{-(\lambda t)^{\gamma}}}\right)\left(1-\alpha-p+\alpha q+(p-q)\alpha^{1-e^{-(\lambda t)^{\gamma}}}\right)}, \tag{9}$$

Where, t > 0, $\tau = (\alpha, \gamma, \lambda) > 0, 0 \leq p \leq 1$ and $0 \leq q \leq 1$.

In addition to the above mathematical derivation, Figure 1 and 2 represent the graphical representation of the pdf and the hrf, showing the possible shapes for some selected values of the parameters, respectively. Fig. 1 and 2 show that the APMW-2G Distribution has a very supple and agile trait, with the PDF having diverse unimodal shapes and the HRF with increasing, decreasing, bathtub and upside-down bathtub shapes.

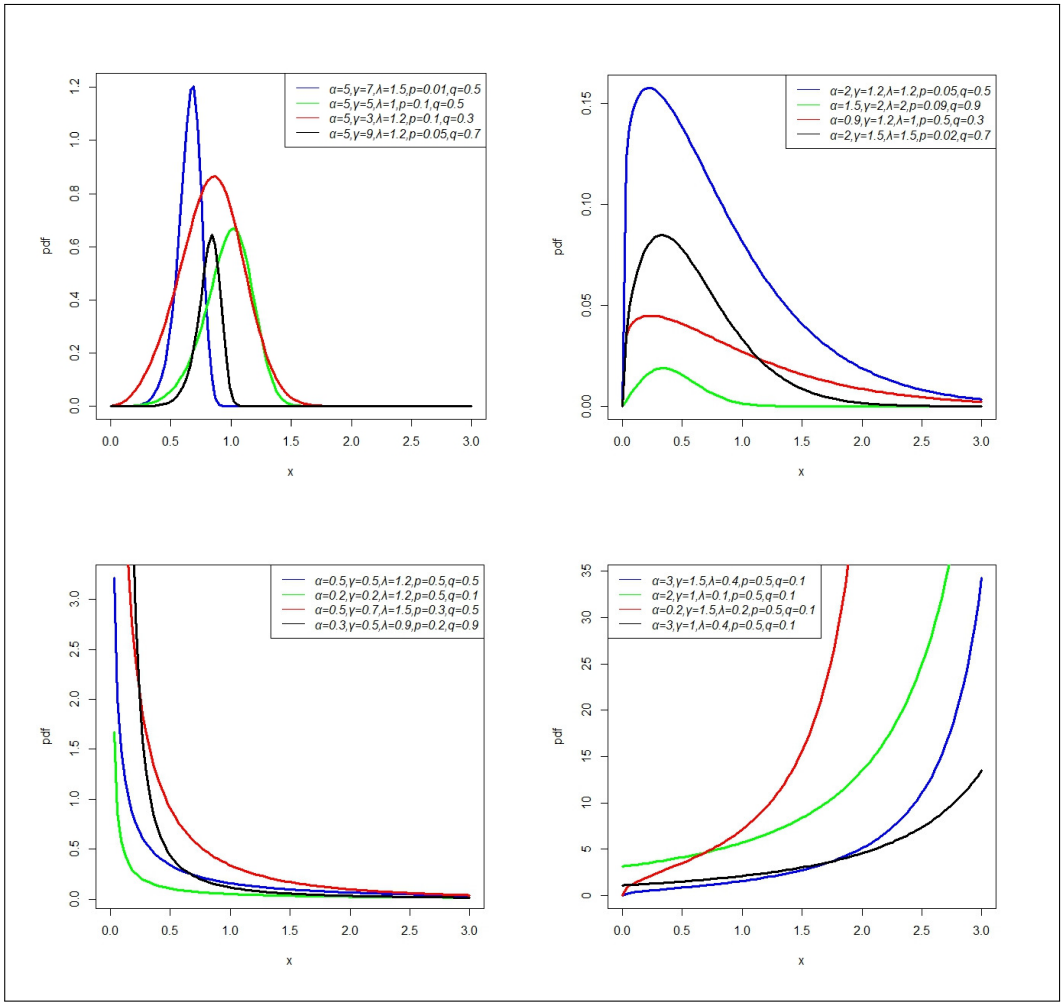


Figure 1. Plots of the pdfs of the APMW-2G distribution for three different set of parameters values

3. Characteristic Properties

In this section, various statistical properties of the APMW-2G distribution have been studied

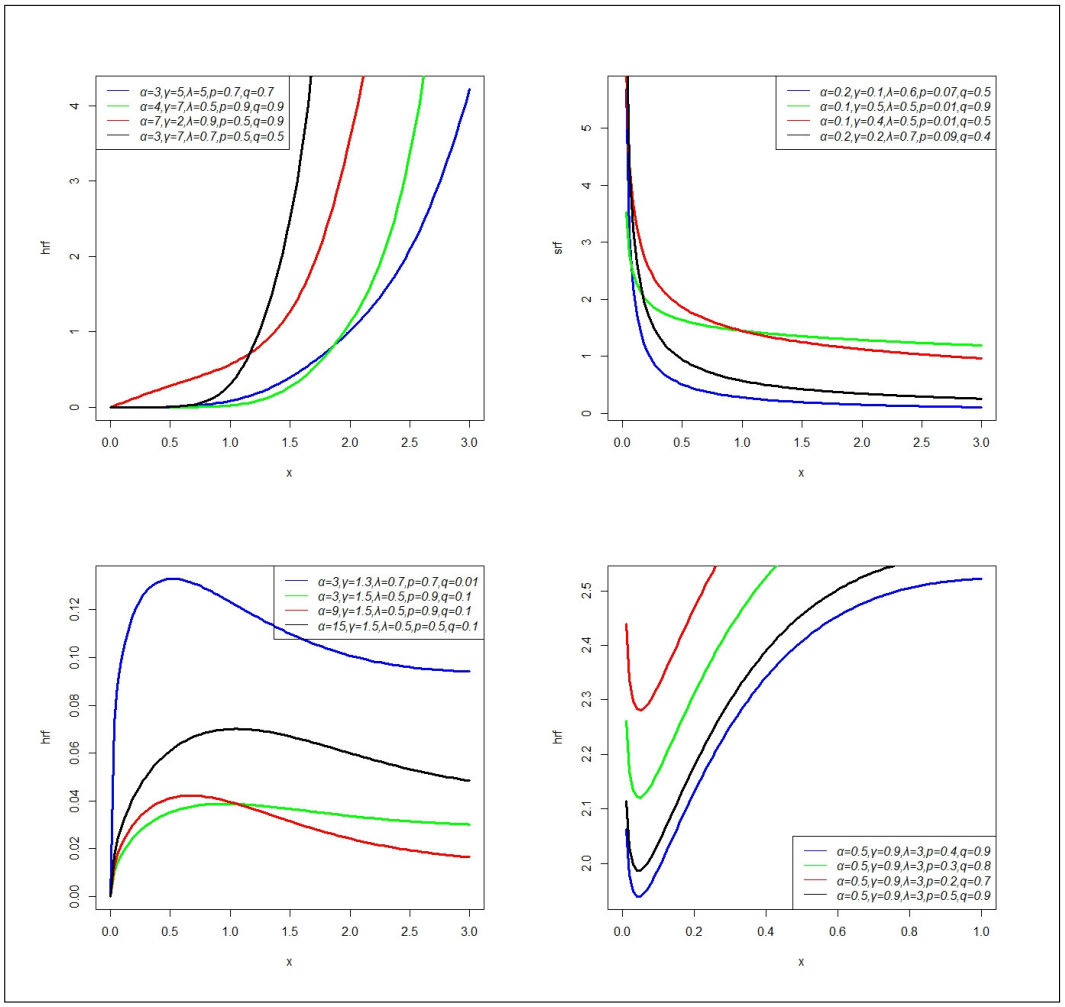


Figure 2. Plots of the hrfs of APMW-2G distribution for three different set of parameters values

3.1. Quantiles

The quantile function of a random variable T is the inverse of its distribution function. Thus, the quantile function denoted by t_u, for $0 < u < 1$, can be easily obtained from Eq. 6 and is obtained as

$$T_u = \left[\frac{-1}{\lambda^\gamma} log \left(1 - \frac{log\left(\frac{u-u\alpha-p\alpha+\alpha qu+p-1}{p-1-up+uq} \right)}{log\alpha} \right) \right]^{\frac{1}{\gamma}} \tag{10}$$

Hence, the first (25 percentile), second (median or 50 percentile) and the third (75 percentile) quantile of the distribution can be obtained by simply putting u= 0.25, 0.5 and 0.75, respectively, in the above equation.

Table 1 presents numerical calculations for the first, second and third quartiles of the APMW-2G Distribution for different parameter values.

3.2. Moments

Let the RV t have APMW-2GD with the PDF as given in Eq. 7. Then, for r=1,2,...., the r^{th} moment of T is given by

Table 1. The first, second and third quartiles of the APMW-2GD for different parameter value

parmeters					$x_{0.25}$	$x_{0.50}$	$x_{0.75}$
2.5	2	1.5	0.01	0.1	0.4191988	0.6279803	0.8570249
3					0.4357251	0.6460256	0.8738867
4.5					0.4733167	0.6850399	0.9093174
2.5	3	1.5	0.01	0.1	0.4893087	0.6406197	0.7881917
	6				0.5711443	0.6535134	0.7248870
	9				0.6013585	0.6578687	0.7049362
2.5	2	3	0.01	0.1	0.2095994	0.3139902	0.4285125
		4.5			0.1397329	0.2093268	0.2856750
		6			0.1047997	0.1569951	0.2142562
2.5	2	1.5	0.1	0.1	0.4356429	0.6475626	0.8767463
			0.3		0.4807627	0.6997454	0.9282367
			0.7		0.6475626	0.8767463	1.0944732
2.5	2	1.5	0.01	0.01	0.4356429	0.6475626	0.8767463
				0.3	0.3777345	0.5771402	0.8046484
				0.7	0.2596167	0.4191988	0.6279803

$$\mu'_r = \left(\int_0^\infty t^r f(t)\, dt \right) \tag{11}$$

Substituting Eq. 7 in the above equation, we get

$$\mu'_r = \left[\int_0^\infty t^r \frac{(1-p)(\alpha+q-q\alpha-1) log\alpha\gamma\lambda^\gamma t^{\gamma-1}\alpha^{1-e^{-(\lambda t)^\gamma}} e^{-(\lambda t)^\gamma}}{\left[1-\alpha-p+\alpha q+(p-q)p\alpha^{1-e^{-(\lambda t)^\gamma}}\right]^2}\, dt \right], \tag{12}$$

$$\mu'_r = \frac{(1-p)(\alpha+q-q\alpha-1)\Gamma(\frac{r}{\gamma}+1)}{\lambda^{(\gamma)^{\frac{r}{\gamma}}}} \sum_{i=0}^{\infty}(i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha+p-q\alpha)^m (q-p)^{(i-m)} \alpha^{(i-m+1)} \sum_{v=0}^{\infty} \frac{(-log)^{(v+1)}(i-m+1)^v}{v!(v+1)^{\frac{r}{\gamma}+1}}. \tag{13}$$

(The complete and thorough derivation is provided in the appendix.)

In addition, using the first four cumulants denoted by C_r, the coefficient of skewness and kurtosis can also be calculated from the ordinary moments of T, where

$$C_r = \mu'_r - \sum_{i=0}^{r-1} \binom{r-1}{i-1} C_i \mu'_{r-i} \tag{14}$$

Therefore we have,
$C_1 = \mu'_1$ (mean)
$C_2 = \mu'_2 - (\mu'_1)^2$ (variance)
$C_3 = \mu'_3 - 3\mu'_2\mu'_1 + (\mu'_1)^3$

Table 2. The second,third, fourth and fifth moment of the APMW-2GD for different parameter value

parmeters					μ_2'	μ_3'	μ_4'	μ_5'
2.5	2	1.5	0.01	0.1	0.3942291	0.3669536	0.3808985	0.4322525
3					0.4212548	0.396287	0.4144089	0.472755
4.5					0.4738958	0.4558191	0.4841497	0.5584644
2.5	3	1.5	0.01	0.1	0.340469	0.2628194	0.2159713	0.1869689
	6				0.3181267	0.2169605	0.1513196	0.1076177
	9				0.3178663	0.2120845	0.1431674	0.09766107
2.5	2	3	0.01	0.1	0.09855728	0.0458692	0.02380616	0.01350789
		4.5			0.04380324	0.01359087	0.00470245	0.001778817
		6			0.02463932	0.00573365	0.001487885	0.000422122
2.5	2	1.5	0.05	0.1	0.4524634	0.4231262	0.4407169	0.5014042
			0.09		0.5278195	0.4963266	0.5190727	0.5923343
			0.1		0.5501804	0.5181513	0.5425168	0.6196119
2.5	2	1.5	0.01	0.01	0.5501804	0.5181513	0.5425168	0.6196119
				0.3	0.1973406	0.1801057	0.1842773	0.2069315
				0.7	0.04376065	0.03896267	0.03916034	0.04341341

$$C_4 = \mu_4' - 4\mu_3'\mu_1' - 3(\mu_2')^2 + 12\mu_2'(\mu_1')^2 - 6(\mu_1')^4$$

Now, the coefficients of skewness (denoted by CS) and kurtosis (denoted by CK) is respectively obtained, as
i. CS=$\frac{C_3}{C_2^{3/2}}$ and
ii. CK=$\frac{C_4}{C_2^2}$

The first four moments of the APMW-2G distribution, as well as the mean, variance skewness and kurtosis for different parameter values are presented in Table 2 and 3. From the table we observe that
(i) The First four moments increase by increasing α and p and decrease with increasing λ, γ and q.
(ii) The mean increase by increasing α and p and decrease with increasing γ and q.
(iii) The variance increase by increasing of only α and decrease with increasing λ, γ, p and q.
(iv) The proposed APMW-2G distribution is suitable for modeling both positively and negatively skewed data sets.
(v) The proposed APMW-2G distribution is suitable for modeling both platykurtic and leptokurtic data sets.

3.3. Sensitivity Analysis of Parameters

Apart from the numerical computational analysis to describe the characteristics of the distribution, A sensitivity analysis using the Latin Hypercube Sampling (LHS) is employed, in which, through systematic sampling, simulation, and 3D-plots, we visualize parameter interaction and give us the key insight into how these parameters influence the shape and reliability properties of the distribution.

From Fig. 3, we see that the hazard rate remains generally low across all parameter spaces, with most values below 0.02. Parameters α, λ and γ show minimal influence, producing only slight fluctuations. The $\alpha - \gamma$ interaction has a moderate effect, occasionally raising hazard rates toward the upper bound. The strongest impact arises from the p–q space, where hazard rates are more scattered and high values occur more frequently. In summary, p and q are the primary drivers of hazard variability, α and γ contribute moderately, and λ plays a limited role. Risk modeling should therefore prioritize accurate calibration of p and q to capture hazard dynamics better.

Coming to the Skewness, as from Fig. 4 we see, that λ has minimal influence, with values staying low except for rare outliers. αcontributes moderately, especially when combined with γ, where higher values increase asymmetry.

Table 3. The mean, variance, skewness and kurtosis value of the APMW-2GD for different parameter value

parmeters					mean	variance	skewness	kurtosis
2.5	2	1.5	0.01	0.1	0.4881611	0.1559278	0.3616958	2.372258
3					0.5136386	0.1574302	0.2912037	2.349467
4.5					0.559413	0.1609529	0.1647346	2.323779
2.5	3	1.5	0.01	0.1	0.4771901	0.1127586	-0.1918346	1.88178
	6				0.4789176	0.08876467	-0.7720605	1.98893
	9				0.4827436	0.08482493	-0.9415425	2.108169
2.5	2	3	0.01	0.1	0.2440806	0.03898196	0.3616958	2.372258
		4.5			0.1627204	0.01732532	0.3616958	2.372258
		6			0.1220403	0.00974549	0.3616958	2.372258
2.5	2	1.5	0.05	0.1	0.5567182	0.1425282	0.2329651	2.544759
			0.09		0.6445791	0.1123373	0.2997071	2.95627
			0.1		0.6704805	0.1006363	0.4484463	3.020234
2.5	2	1.5	0.01	0.01	0.6704805	0.1006363	0.4484463	3.020234
				0.3	0.2510578	0.1343106	1.282383	3.665404
				0.7	0.05761883	0.04044072	3.907855	18.96664

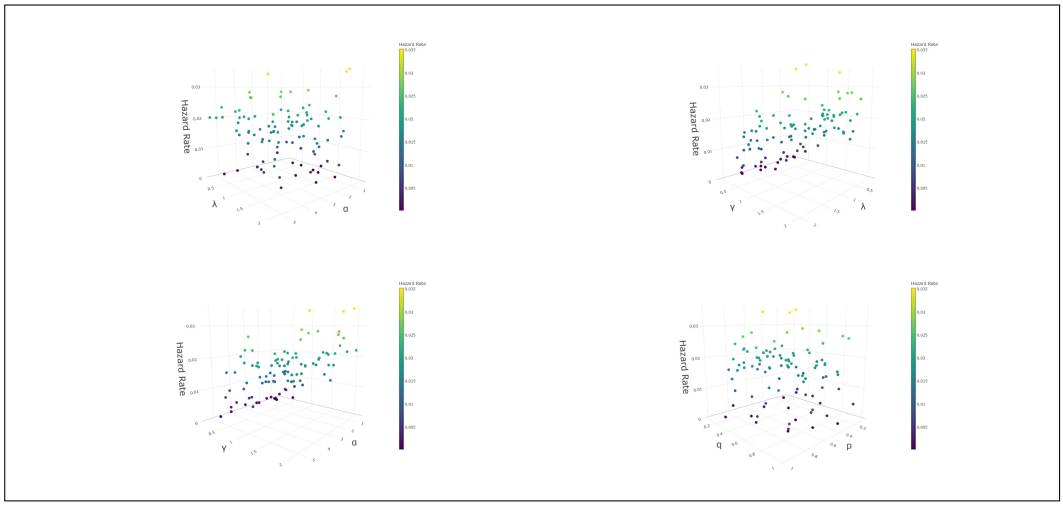


Figure 3. 3D-Plot of the Hazard Rate as a function of the parameters

γ plays a stronger role, often driving sharp spikes in skewness and creating long-tailed behavior. The most significant effects come from p and q, which produce the widest variation and extreme skewness values above 20–25. Overall, skewness is weakly influenced by λ, moderately shaped by α and γ, and dominated by p and q, making them the primary drivers of asymmetry and tail risk.

The kurtosis analysis, Fig. 5 shows that λ has minimal impact, with only rare outliers. α and γ have a moderate influence, with higher values sometimes driving large increases in tail heaviness. The strongest effects come from p and q, which produce the widest variation and extreme peaks exceeding, indicating a high potential for heavy-tailed behavior. Overall, kurtosis is weakly affected by λ, moderately by α and γ, and dominated by p and q.

The practical interpretations of the distribution's parameters can be summarized as follows:

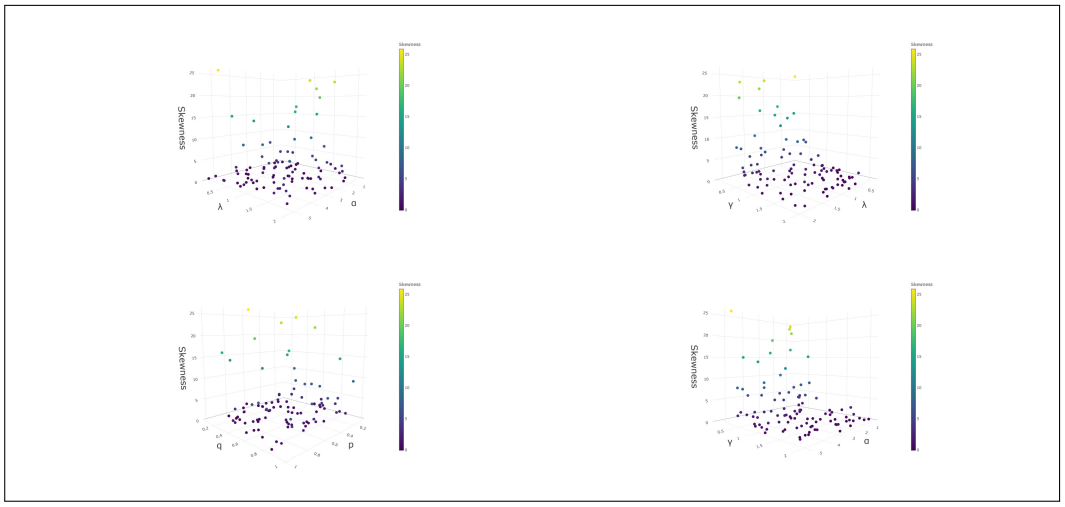


Figure 4. 3D-Plot of the Skewness as a function of the parameters

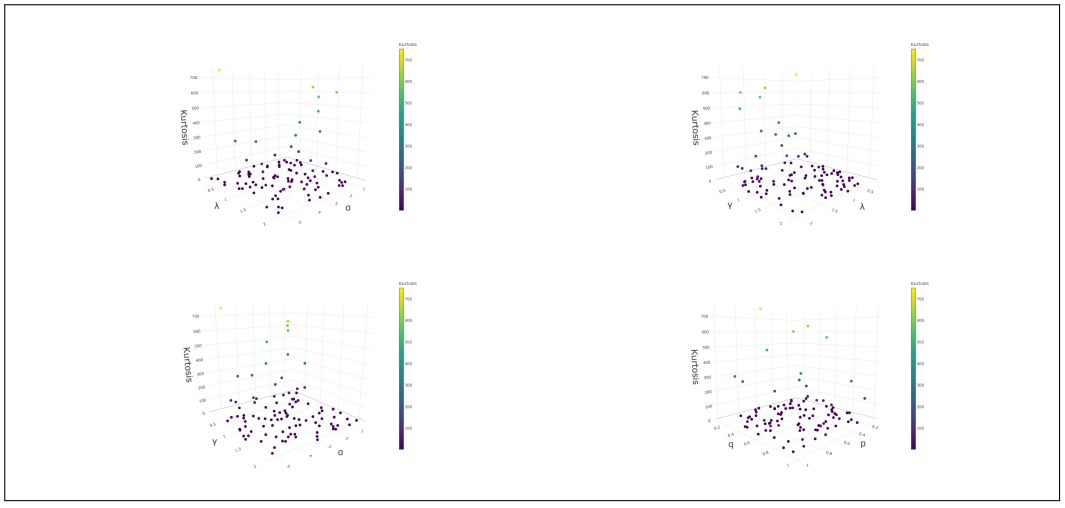


Figure 5. 3D-Plot of the Kurtosis as a function of the parameters

p: A dominant shape parameter that strongly controls hazard variability, skewness, kurtosis, and tail behaviour, governing the extremal structure and overall asymmetry of the distribution.

q: Similar to p, it is a dominant shape parameter with major influence on tail risk, asymmetry, and peakedness, making it crucial in shaping the distribution's extremal and asymmetric behaviour.

α: A moderate shape-adjustment parameter that affects hazard rates and increases skewness and kurtosis, especially through interaction, regulating how quickly risk escalates and how the tails evolve.

γ: Another secondary shape modifier that moderately influences hazard rate and tail behavior, contributing to changes in skewness and kurtosis and shaping the distribution's asymmetry in combination with α .

λ: A scale-regulation or damping parameter that stabilizes the distribution by reducing the likelihood of extreme values, exerting minimal influence on skewness, kurtosis, or overall shape.

3.4. Moment Generating Function

The Moment Generating Function of a random variable T plays a very crucial role in Probability theory and statistics, providing the basis for an alternative route of analysis, enabling us to derive moments and probability distribution with ease, as the name suggests. The MGF of T is defined as the expected value of e^{tz}, given by

$$M_T(z) = \left(\int_0^\infty e^{tz} f(t)\, dt \right) \tag{15}$$

Using the Maclaurin series expansion $e^{tx} = \sum_{l=0}^{\infty} \frac{(tx)^l}{l!}$, we have

$$M_T(z) = \left(\int_0^\infty \sum_{l=0}^{\infty} \frac{(tz)^l}{l!} f(t)\, d \right).$$

In a way similar to the moments, using the series representation in Eq. 46, the binomial expansion, and the series representation in Eq. 12, we have

$$\begin{aligned} M_T(z) = (1-p)(\alpha + q - q\alpha - 1) \sum_{i=0}^{\infty} (i+1) \sum_{m=0}^{i} \binom{i}{m} (\alpha + p - q\alpha)^m (q-p)^{(i-m)} \alpha^{(i-m+1)} \\ \sum_{v=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Gamma(\frac{l}{\gamma} + 1)(-log)^{(v+1)} (i-m+1)^v t^l}{\lambda^{(\gamma)^{\frac{l}{\gamma}}} v! l! (v+1)^{\frac{l}{\gamma}+1}}. \end{aligned} \tag{16}$$

3.5. Mean Deviation

The Mean Deviation about the mean and the median gives us insight into the variability of a population. Now, let us denote the mean and median of the APMW-2G distribution by μ and M, respectively. The Mean deviation about the mean and the mean deviation about the median, denoted by $D_\mu(T)$ and $D_M(T)$ respectively, can be calculated as,

$$\begin{aligned} D_\mu(T) = E(|T-\mu|) &= \int_0^\infty |T-\mu| f(t) dt \\ &= 2 \int_0^\mu (\mu - t) f(t) dt \\ &= 2\mu F(\mu) - 2 \int_0^\mu t f(t) dt \end{aligned} \tag{17}$$

$$\begin{aligned} D_M(T) = E(|T-M|) &= \int_0^\infty |T-M| f(t) dt \\ &= \mu - 2 \int_0^M t f(t) dt \end{aligned} \tag{18}$$

For short, let us set ϕ=(μ or M) and $\psi(\phi)$=$\int_0^\phi t f(t) dt$. Therefore, using the series representation given in Eq. 46, the binomial expansion and the series representation given in Eq. 12, we have

$$\psi(\phi) = \frac{(1-p)(\alpha+q-q\alpha-1)}{\lambda^{(\gamma)^{\frac{1}{\gamma}}}} \sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha+p-q\alpha)^m(q-p)^{(i-m)}$$
$$\alpha^{(i-m+1)}\sum_{v=0}^{\infty}\frac{(-log)^{(v+1)}(i-m+1)^v\gamma\left((\frac{1}{\gamma}+1),(v+1)\lambda^\gamma\phi^\gamma\right)}{v!(v+1)^{\frac{1}{\gamma}+1}} \tag{19}$$

By substituting the above in Eq. 17 and Eq. 18, the mean deviation about the mean and about the median of APMW-2G distribution is respectively obtained as

$$D_\mu(T) = 2\mu\frac{(1-p)\left(1-\alpha^{1-e^{-(\lambda t)^\gamma}}\right)}{\left(1-\alpha-p+\alpha q+(p-q)\alpha^{1-e^{-(\lambda t)^\gamma}}\right)} - 2\frac{(1-p)(\alpha+q-q\alpha-1)}{\lambda^{(\gamma)^{\frac{1}{\gamma}}}}$$
$$\sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha+p-q\alpha)^m(q-p)^{(i-m)}\alpha^{(i-m+1)}$$
$$\sum_{v=0}^{\infty}\frac{(-log)^{(v+1)}(i-m+1)^v\gamma\left((\frac{1}{\gamma}+1),(v+1)\lambda^\gamma\mu^\gamma\right)}{v!(v+1)^{\frac{1}{\gamma}+1}} \tag{20}$$

$$D_M(T) = \mu - 2\frac{(1-p)(\alpha+q-q\alpha-1)}{\lambda^{(\gamma)^{\frac{1}{\gamma}}}}\sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha+p-q\alpha)^m(q-p)^{(i-m)}$$
$$\alpha^{(i-m+1)}\sum_{v=0}^{\infty}\frac{(-log)^{(v+1)}(i-m+1)^v\gamma\left((\frac{1}{\gamma}+1),(v+1)\lambda^\gamma M^\gamma\right)}{v!(v+1)^{\frac{1}{\gamma}+1}} \tag{21}$$

3.6. Incomplete Moments

Incomplete moments are easily interpreted and form natural building blocks from which measure of inequality may be constructed.

The h^{th} incomplete moment of the APMW-2G distribution is define by

$$I(x;h) = \int_0^x t^h f(t)\,dt \tag{22}$$

Similarly using the series representation as given in Eq. 46 and Eq. 49 and the binomial expansion, the h^{th} incomplete momemt of T following APMW-2G distribution is obtained as

$$I(x;h) = \frac{(1-p)(\alpha+q-q\alpha-1)}{\lambda^{(\gamma)^{\frac{h}{\gamma}}}}\sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha+p-q\alpha)^m(q-p)^{(i-m)}$$
$$\alpha^{(i-m+1)}\sum_{v=0}^{\infty}\frac{(-log)^{(v+1)}(i-m+1)^v\gamma\left((\frac{h}{\gamma}+1),(v+1)(\lambda x)^\gamma\right)}{v!(v+1)^{\frac{h}{\gamma}+1}}, \tag{23}$$

where, γ(a,t) is the lower incomplete gamma function.

The first two incomplete moments, obtained by setting h=0 and h=1, respectively, offer particularly valuable insights into the shape of a distribution in various applications. The first incomplete moment is also closely associated with the Bonferroni and Lorenz curves, as well as with the concepts of mean residual life and mean waiting time.

3.7. Conditional Moments

The conditional moment of the random variable T is defined by

$$E\left[T^n/T>n\right]=\frac{1}{S(t)}\int_t^{\infty}t^n f(t)\,dt \tag{24}$$

Now substituting from Eq. 7 and Eq. 8, the conditional moment of the APMW-2G distribution is obtained as

$$E\left[T^n/T>n\right]=\frac{(1-p)(\alpha+q-q\alpha-1)}{\lambda^{(\gamma)^{\frac{n}{\gamma}}}}\sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha+p-q\alpha)^m(q-p)^{(i-m)}$$
$$\alpha^{(i-m+1)}\sum_{v=0}^{\infty}\frac{(-log)^{(v+1)}(i-m+1)^v\gamma\left((\frac{n}{\gamma}+1),(v+1)(\lambda t)^{\gamma}\right)}{v!(v+1)^{\frac{n}{\gamma}+1}}, \tag{25}$$

where, Γ(a,x) is the upper incomplete gamma function.

3.8. Mean Residual Life

In the context of life testing situation or reliability studies, the expected additional lifetime that a component has survived until time "z" is a function z, called the mean residual life $\mu_R(z)$ and is defined as

$$\mu_R(z)=\frac{1}{S(z)}\left(\int_z^{\infty}tf(t)dt-z\right) \tag{26}$$

$$\mu_R(t)=\frac{1}{S(z)}\left(\int_0^{\infty}tf(t)dt-\int_0^{z}tf(t)dt-z\right) \tag{27}$$
$$=\frac{1}{S(z)}\left(\mu_1'-I(z,1)-t\right), \tag{28}$$

where, S(z) is the survival function defined in Eq. 8, μ_1' is the mean whixh can be ontained by setting r=1 in Eq. 13 and I(z,1) is the First Incomplete Moments obtained by setting h=1 in Eq. 23.

3.9. Measures of Inequality and uncertainty

In this subsection, we derive the Lorenz and Bonferroni curves as measures of inequality, and also compute the Rényi entropy as an important measure of uncertainty.

3.9.1. Lorenz and Bonferroni curve The Lorenz and Bonferroni curve for a random variable T is obtained as,

$$L(t)=\frac{I(x;1)}{E(t)}\quad\text{and}\quad B(t)=\frac{L(t)}{E(t)}, \tag{29}$$

Where, I(t;1) is the first incomplete moment, E(t) is the mean, which is also the first moment.

3.9.2. Renyi Entropy The entropy of a random variable T with density function f(t) quantifies the degree of uncertainty or randomness in a system. One commonly used measure of entropy is Rényi entropy, defined as follows

$$I_R(\beta)=\frac{1}{1-\beta}log\left(\int_0^{\infty}(f(t))^{\beta}dt\right) \tag{30}$$

Substituting Eq. 7 in the above equation, we have

$$I_R(\beta) = \frac{1}{1-\beta} log\left((1-p)(\alpha+q-q\alpha-1)^{\beta}(log\alpha\gamma\lambda^{\gamma})^{\beta}\right.$$
$$\left.\int_0^{\infty}\left(\frac{x^{\gamma-1}\alpha^{1-e^{-(\lambda x)^{\gamma}}}e^{-(\lambda x)^{\gamma}}}{\left[1-\alpha+-p\alpha q+(p-q)\alpha^{1-e^{-(\lambda x)^{\gamma}}}\right]^2}\right)^{\beta} dx \tag{31}$$

Using the binomial expansion and the series expansion as given in Eq. 12 we have,

$$I_R(\beta) = \frac{\beta}{1-\beta}\left(log(1-p)+log(\alpha+q-q\alpha-1)+log(log\alpha\gamma\lambda^{\gamma})+log\alpha\right)$$
$$+\frac{1}{1-\beta}log\left[\sum_{m=0}^{2\beta}\sum_{k=0}^{\infty}\frac{-(-log\alpha)^k\Gamma\left(\frac{\beta(\gamma-1)}{\gamma}+1\right)}{\binom{m}{2\beta}(1-\alpha-p+q\alpha)^m(p-q)^m\gamma\left(\lambda^{\gamma}(k(\beta-m)-\beta)^{\frac{\beta(\gamma-1)}{\gamma}+1}\right.}\right]. \tag{32}$$

3.10. Order Statistics

Let $T_1, T_2, ..., T_r$ be a random sample of size n following APMW-2G distribution and let $T_{1:r}, T_{2:r}, ..., T_{r:r}$ be their corresponding order statistics. The pdf of the i^{th} order statistic, $T_{i:r}$ denoted by $f_{i:r}(t)$ is given by

$$f_{i:r}(t) = \frac{1}{B(i, r-i+1)} f(t)F(t)^{i-1}(1-F(t))^{r-i}$$

By using the binomial expansion in the above equation we have

$$f_{i:r}(t) = \frac{1}{B(i, r-i+1)}\sum_{l=0}^{k-i} C_l^{k-i}(-1)^l f(t)F(t)^{l+i-1} \tag{33}$$

Substituting the value from Eq. 6 and Eq. 7 in the above equation and using the binomial expansion, the pdf of the i^{th} order statistics is obtained as

$$f_{i:r}(t) = \frac{1}{B(i, r-i+1)}\sum_{l=0}^{k-i}\sum_{m=0}^{l+i-1} C_l^{k-i}C_m^{l+i-1}(-1)^{l+m}(1-p)(\alpha+q-q\alpha-1)$$
$$\frac{\alpha^{m(1-e^{-(\lambda t)^{\gamma}})}+(1-e^{-(\lambda t)^{\gamma}})}{(1-\alpha-p+\alpha q+(p-q)\alpha^{1-e^{-(\lambda t)^{\gamma}}})^{l+i-1}} log\alpha\gamma\lambda^{\gamma}t^{\gamma-1}e^{-(\lambda t)^{\gamma}}. \tag{34}$$

The associate cdf of the i^{th} order statistic, $T_{i:r}$ denoted by $F_{i:r}(t)$ is given by

$$F_{i:r}(t) = \sum_{i=0}^{k} C_l^k F(t)^i(1-F(t))^{k-i} = \sum_{i=0}^{k}\sum_{l=0}^{k-i}(-1)^l C_l^k C_l^{k-i}F(t)^{(i+l)}$$
$$= \sum_{i=0}^{k}\sum_{l=0}^{k-i}(-1)^l C_l^k C_l^{k-i}\left[\frac{(1-p)\alpha-\alpha^{1-e^{-(\lambda t)^{\gamma}}}}{1-\alpha-p+\alpha q+(p-q)\alpha^{1-e^{-(\lambda t)^{\gamma}}}}\right]^{(i+l)} \tag{35}$$

The pdf and cdf of the smallest $T_{1:r}$ and largest $T_{r:r}$ order statistics can be obtained by simply setting i=1 and i=r, respectively, in Eq. 34 and Eq. 35.

4. Parameter Estimation under Progressive type-II censoring

In reliability and lifetime studies, it is often impractical—or even impossible—to observe the complete lifetimes of all experimental units under investigation. To address this limitation, censoring techniques play a critical role in experimental design and analysis. Censoring refers to the deliberate termination or withdrawal of certain test units from observation, either at predetermined inspection times or at random points during the study, before the occurrence of their actual failure.

The adoption of censoring strategies offers several key advantages. First, it significantly reduces the duration of the experiment, allowing researchers to obtain meaningful results without waiting for all units to fail. Second, it contributes to cost efficiency, since maintaining long-duration experiments often involves considerable resources, labour, and operational expenses. Furthermore, censoring enables investigators to capture valuable statistical information about survival behaviour and failure patterns while maintaining methodological rigour, even when complete lifetime data are unavailable. By incorporating censoring into experimental frameworks, researchers can achieve a balance between practical feasibility and statistical validity, thereby enhancing both the efficiency and applicability of reliability studies in engineering, biomedical research, and other fields where lifetime analysis is essential.

Progressive Type II censoring generalises the classical Type II scheme by allowing predetermined numbers of surviving units to be withdrawn immediately after each observed failure at prefixed inspection times.As discuss by [4], in this design, the number of failures is fixed in advance, while the experiment duration remains random, depending on when the final failure occurs. This approach balances analytical convenience with practical flexibility, making it especially useful in reliability and lifetime studies where observing all unit failures is impractical. Under this scheme, from a t,otal of n units placed simultaneously under life test, only m($<$ n)are completely observed until failure. Then given a censoring plan R=$(R_1, R_2, ..., R_m)$, at the time $t_{1:m:n}$ of the first failure, R_1 units are randomly censored from the (n-1) surviving units.At the time $t_{2:m:n}$ of the second failure, R_2 units are randomly censored from the (n-2-R_1) surviving units and its continue until at the time $t_{m:m:n}$ of the m^{th} failure, all the remaining $R_m = n - m - R_1 - ... - R_{m-1}$ surviving units are censored. The ordered failure times $T_{1:m:n;R_1} \leq T_{2:m:n;R_2} \leq ... \leq T_{m:m:n;R_m}$ are the progressive type II censored samples.

4.1. Maximum likelihood estimation

Let t = $(t_{1:m:n}, t_{2:m:n;}...t_{m:m:n;})$ with $t_{1:m:n} \leq t_{2:m:n} \leq ... \leq t_{m:m:n}$ be the m observations under progressive type ii censoring from a sample of size n drawn from the APMW-2G distribution with cdf and pdf given in Eq. 6 and Eq. 7 respectively. The likelihood function based on the progressive type ii censored sample x is given by

$$L_{P-II}(\theta, p, q; t_{i:m:n}) = A \prod_{i=1}^{m} f(t_{i:m:n})[1 - F(t_{i:m:n})]^{R_i} \tag{36}$$

Where $A = n(n-1-R_1)(n-2-R_1-R_2)...(n-m+1-R_1-...R_{m-1})$
Using Eq. 5 and Eq.6 and taking log on both side we obtained the log likelihood function as

$$\begin{aligned} l_{P-II}(\theta, p; x_{i:m:n}) = logA + mlog(\alpha + q - q\alpha - 1) + mloglog\alpha + mlog(1-p) \\ + mlog\left(\lambda^{\gamma}\gamma\right) + (n-m)log\alpha - log\alpha \sum_{i=1}^{m} e^{-(\lambda t_{i:m:n})^{\gamma}} - \lambda^{\gamma} \sum_{i=1}^{m} t_{i:m:n}^{\gamma} \\ + (\gamma - 1) \sum_{i=1}^{m} logt_{i:m:n} + \sum_{i=1}^{m} R_i log(q-1) + \sum_{i=1}^{m} R_i log\left(1 - \alpha^{-e^{-(\lambda t_{i:m:n})^{\gamma}}}\right) \\ - \sum_{i=1}^{m} (2 + R_i) log\left(1 - \alpha - p + q\alpha + (p-q)\alpha^{1-e^{-(\lambda t_{i:m:n})^{\gamma}}}\right). \end{aligned} \tag{37}$$

Now differentiating Eq. w.r.t α,λ, γ and p respectively and equating them to zero, we obtain the MLEs of α, λ, γ ,p and q.

$$\begin{aligned}\frac{\delta l_{PC}(\theta,p,q)}{\delta\alpha} &= \frac{m(1-p)}{(\alpha+q-q\alpha-1)}+\frac{m}{\alpha log\alpha}-\frac{1}{\alpha}\sum_{i=1}^{m}e^{-(\lambda t_{1:m:n})^{\gamma}}\\ &+\frac{(n-m)}{\alpha}+\sum_{i=1}^{m}\frac{R_i\left(\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)(e^{-(\lambda t_{1:m:n})^{\gamma}})}{\left(1-\alpha^{-e^{-(\lambda t_{i:m:n})^{\gamma}}}\right)}\\ &-\sum_{i=1}^{m}(2+R_i)\frac{p-1+\left(p\alpha^{-e^{-(\lambda t_{1:m:n})^{\gamma}}}(e^{-(\lambda t_{1:m:n})^{\gamma}})\right)}{\left(1-\alpha-p-q\alpha+(p-q)\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)}=0\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{\delta l_{P-II}(\theta,p,q)}{\delta\lambda} &= \frac{m\gamma}{\lambda}+log\alpha\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{m}e^{-(\lambda t_{1:m:n})^{\gamma}}t_{1:m:n}^{\gamma}-\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{m}t_{1:m:n}^{\gamma}\\ &+log\alpha\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{m}\frac{R_i t_{1:m:n}^{\gamma}\left(\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)(e^{-(\lambda t_{1:m:n})^{\gamma}})}{\left(1-\alpha^{-e^{-(\lambda t_{i:m:n})^{\gamma}}}\right)}\\ &+(p-q)log\alpha\gamma\lambda^{(\gamma-1)}\sum_{i=1}^{m}(2+R_i)\frac{\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}t_{1:m:n}^{\gamma}}{\left(1-\alpha-p+q\alpha+(p-q)\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)}=0\end{aligned} \tag{39}$$

$$\begin{aligned}\frac{\delta l_{P-II}(\theta,p)}{\delta\gamma} &= \frac{m}{\gamma}+mlog\lambda+log\alpha\lambda^{\gamma}\sum_{i=1}^{m}e^{-(\lambda t_{1:m:n})^{\gamma}}t_{1:m:n}^{\gamma}log(\lambda t_{1:m:n})- \qquad (40)\\ &\sum_{i=1}^{m}(\lambda t_{1:m:n})^{\gamma}log(\lambda t_{1:m:n})+\sum_{i=1}^{n}logt_{1:m:n}\\ &+p\sum_{i=1}^{m}\frac{(2+R_i)log\alpha\alpha^{-e^{-(\lambda t_{1:m:n})^{\gamma}}}e^{-(\lambda t_{1:m:n})^{\gamma}}(\lambda t_{1:m:n})^{\gamma}log(\lambda t_{1:m:n})}{\left(1-\alpha-p+q\alpha+(p-q)\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)}\\ &+log\alpha\lambda^{\gamma}\sum_{i=1}^{m}\frac{R_i t_{1:m:n}^{\gamma}log(\lambda t_{1:m:n})\left(\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)(e^{-(\lambda t_{1:m:n})^{\gamma}})}{\left(1-\alpha^{-e^{-(\lambda t_{i:m:n})^{\gamma}}}\right)}=0 \qquad (41)\end{aligned}$$

$$\frac{\delta l_{P-II}(\theta,p,q)}{\delta p}=\frac{m}{(p-1)}-\sum_{i=1}^{m}\frac{(2+R_i)\left(\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}-1\right)}{\left(1-\alpha-p+q\alpha+(p-q)\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)}=0 \tag{42}$$

$$\begin{aligned}\frac{\delta l_{P-II}(\theta,p,q)}{\delta p} &= \frac{m\alpha}{(1-\alpha-q+q\alpha)}+\sum_{i=1}^{m}\frac{R_i}{(q-1)}\\ &-\sum_{i=1}^{m}\frac{(2+R_i)\left(\alpha-\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)}{\left(1-\alpha-p+q\alpha+(p-q)\alpha^{1-e^{-(\lambda t_{1:m:n})^{\gamma}}}\right)}=0\end{aligned} \tag{43}$$

The maximum likelihood estimates (MLEs) of (α, λ, γ,p) for the progressive type ii censored samples are the simultaneous solutions of the Eq. 38, Eq. 39, Eq. 41 , Eq. 42 and Eq. 43. The above mentioned equations are not in

closed form, thus numerical technique like the Newton-Raphson method can be used to obtained the MLEs. Also, there are many well-established packages in R language that can be used to obtained the maximization of Eq. 37.

The Standard Error and the ACIs can be obtained in the similar way as the uncencored samples. Hence, for any arbitrary $0 < \tau < 1$, the $100(1-\tau)\%$ ACIs of the unknown parameters can be determined as follows:

$$(\hat{\alpha}) \pm z_{\frac{\tau}{2}} \sqrt{var(\hat{\alpha})}, (\hat{\lambda}) \pm z_{\frac{\tau}{2}} \sqrt{var(\hat{\lambda})}, (\hat{\gamma}) \pm z_{\frac{\tau}{2}} \sqrt{var(\hat{\gamma})}, (\hat{p}) \pm z_{\frac{\tau}{2}} \sqrt{var(\hat{p})}, \tag{44}$$

Where, $z_{\frac{\tau}{2}}$ is the upper $(\frac{\tau}{2})^{th}$ percentile point of the standard normal distribution.

5. Simulation Analysis

In this section, we will use the algorithm presented in [4] to simulate progressive type II censored samples from the APMW-2G distribution. Finally, a simulation study will be conducted to evaluate the behaviour of the MLEs.

5.1. Generation method

A progressive type II censored sample from the APMW-2G distribution can be generated using the following simple algorithm, as given in [4]
(i) Generate m independent Uniform (0,1) observations $W_1, W_2, ..., W_m$.
(ii) Set $V_i = W_i^{1/(\sum_{j=m-i+1}^{m} R_j)}$ for i=1,2,...,m, R_j is the censoring plan
(iii) Set $U_{i:m:n} = 1 - V_m V_{m-1} ... V_{m-i+1}$ for i=1,2,...,3. Then $U_{1:m:n}, U_{2:m:n}, ..., U_{m:m:n}$ is the required progressive type II censored sample from the Uniform (0,1) distribution.
(iv) Finally, we set $T_{i:m:n} = F^{-1}(U_{i:m:n})$, for i = 1,2,..,m, where $F^{-1}(.)$ is the inverse cumulative distribution function of the lifetime Distribution under consideration. Then $T_{1:m:n}, T_{1:m:n}, ..., T_{1:m:n}$ is the required progressive type II censored sample from the Uniform (0,1) distribution.

Thus,for APMW-2G distribution we have,

$$F^{-1}(U) = \left[\frac{-1}{\lambda^{\gamma}} log \left(1 - \frac{log\left(\frac{u - u\alpha - p\alpha + \alpha q u + p - 1}{p - 1 - up + uq} \right)}{log\alpha} \right) \right]^{\frac{1}{\gamma}} \tag{45}$$

5.2. Simulation study

In this subsection, we conduct a simulation study to assess the behavior of the MLEs for progressive type II censored data in terms of different censoring schemes, sample sizes, and censored values.

The following are the censoring schemes (CS) applied in the generation of the the samples.
(i) CS1: R_i =1, i=1,2,...,n-m,
R_i =o otherwise
(ii) CS2:R_i =n-m, i=1,
R_i =o otherwise
(ii) CS3:R_i =n-m, i=m,
R_i =o otherwise

The value of m have been taken to represent 50 percent, 75 percent and 100 percent of the sample size. We have considered 2 different parameter sets of value as follows:

Set 1 : $\alpha = 1.5$, $\lambda = 2$, $\gamma = 0.9$, p = 0.5 , q = 0.3
Set 2 : $\alpha = 0.9$, $\lambda = 1.5$, $\gamma = 2.5$, p = 0.2, q = 0.7
Set 3: $\alpha = 2$, $\lambda = 7$, $\gamma = 1.5$, p = 0.3, q = 0.5

Table 4. MLE, Bias and MSE for Set 1 parameter value

			MLE					Bias					MSE				
n	m	cs	α	λ	γ	p	q	α	λ	γ	p	q	α	λ	γ	p	q
	50		1.1421	2.0604	0.9860	0.5836	0.3990	-0.3579	0.0604	0.0860	0.0836	0.0990	0.6020	0.2804	0.1703	0.0821	0.0789
100	75	CS-1	1.1806	2.0808	0.9675	0.5666	0.3782	-0.3194	0.0808	0.0675	0.0666	0.0782	0.5431	0.2497	0.1421	0.0799	0.0627
	100		1.1618	2.0737	0.9533	0.5602	0.3652	-0.3382	0.0737	0.0533	0.0602	0.0652	0.3132	0.2148	0.1139	0.0753	0.0576
	150		1.2935	1.9863	0.9581	0.5731	0.3585	-0.2065	-0.0137	0.0581	0.0731	0.0585	0.7891	0.1136	0.0889	0.0723	0.0480
300	225	CS-1	1.2884	2.0109	0.9466	0.5805	0.3492	-0.2116	0.0109	0.0466	0.0805	0.0492	0.6621	0.1132	0.0703	0.0664	0.0455
	300		1.3030	2.0086	0.9306	0.5638	0.3365	-0.1970	0.0086	0.0306	0.0638	0.0365	0.4165	0.0894	0.0481	0.0596	0.0448
	50		1.3152	2.0790	1.0079	0.557	0.3562	-0.1847	0.0790	0.1079	0.05762	0.0562	0.7876	0.3611	0.1804	0.0823	0.0705
100	75	CS-2	1.1230	2.0526	0.9743	0.6035	0.3966	-0.3769	0.0526	0.0743	0.1035	0.0966	0.7519	0.2183	0.1441	0.0725	0.0628
	100		1.1736	2.0896	0.9497	0.5352	0.3538	-0.3263	0.0896	0.0497	0.0352	0.0538	0.7296	0.1266	0.1251	0.0642	0.0539
	150		1.3767	1.9521	0.9943	0.6029	0.3424	-0.1232	-0.0478	0.0943	0.1029	0.0424	0.9185	0.1186	0.1005	0.0640	0.0381
300	225	CS-2	1.3133	2.0468	0.9166	0.5442	0.3309	-0.1866	0.0468	0.0166	0.0442	0.0309	0.7840	0.1141	0.0441	0.0607	0.0397
	300		1.3964	1.989	0.9280	0.5683	0.3484	-0.1035	-0.0103	0.0280	0.0683	0.0484	0.5440	0.0779	0.0410	0.0659	0.0465
	50		1.2690	2.0155	0.9617	0.5835	0.4280	-0.2309	0.0155	0.0617	0.0835	0.1280	0.8267	0.2846	0.2143	0.09023	0.1152
100	75	CS-3	1.1760	2.0119	0.9769	0.5653	0.3758	-0.3239	0.0119	0.0769	0.0653	0.07586	0.7192	0.2287	0.1547	0.0822	0.0656
	100		1.1736	2.0096	0.9497	0.5352	0.3538	-0.3263	0.0096	0.0497	0.0352	0.05380	0.5296	0.22660	0.12515	0.07421	0.0539
	150		1.2904	1.9953	0.9252	0.5723	0.4205	-0.2096	-0.0047	0.0252	0.0723	0.1205	0.9839	0.1325	0.1398	0.0779	0.0892
300	225	CS-3	1.3569	2.0065	0.9399	0.5586	0.3550	-0.1431	0.0065	0.0399	0.0586	0.0550	0.9123	0.1227	0.0815	0.0687	0.0492
	300		1.3030	2.0086	0.9306	0.5638	0.3365	-0.1970	0.0086	0.0306	0.0638	0.0365	0.8165	0.0894	0.0481	0.0596	0.0448

Table 5. MLE, Bias and MSE for Set 2 parameter value

			MLE					Bias					MSE				
n	m	cs	α	λ	γ	p	q	α	λ	γ	p	q	α	λ	γ	p	q
	50		0.5106	1.5380	1.6619	0.4063	0.6561	-0.3894	0.0380	-0.8381	0.2063	-0.0439	0.5637	0.0597	0.9542	0.0997	0.0892
100	75	CS-1	0.4880	1.5510	1.9001	0.3764	0.6485	-0.4120	0.0510	-0.5999	0.1764	-0.0515	0.4646	0.0401	0.4018	0.0818	0.0735
	100		0.5017	1.5535	1.8908	0.6638	0.3862	-0.3983	0.0535	-0.6092	0.1862	-0.0362	0.2911	0.0365	0.4154	0.0834	0.0675
	150		0.5210	1.4926	1.7815	0.3956	0.6295	-0.3790	-0.0074	-0.7185	0.1956	-0.0705	0.6158	0.0235	0.6211	0.0853	0.0891
300	225	CS-1	0.4803	1.5235	1.9537	0.3827	0.6375	-0.4197	0.0235	-0.5463	0.1827	-0.0625	0.5588	0.0180	0.3114	0.0814	0.0644
	300		0.5171	1.5397	1.9559	0.3587	0.6684	-0.3829	0.0397	-0.5441	0.1587	-0.0316	0.3924	0.0154	0.3094	0.0740	0.0495
	50		0.5451	1.5849	1.8702	0.3658	0.6635	-0.3548	0.0849	-0.6297	0.1658	-0.0364	0.5679	0.0775	0.45348	0.0991	0.0758
100	75	CS-2	0.4634	1.530	1.9181	0.3983	0.6515	-0.4365	0.03022	-0.5818	0.1983	-0.0484	0.4640	0.04408	0.3695	0.0826	0.0645
	100		0.4484	1.5709	1.8596	0.3816	0.6467	-0.4515	0.0709	0.39928	0.1816	-0.0532	0.4447	0.0455	0.28201	0.07809	.05002
	150		0.5245	1.5144	1.9372	0.3668	0.6066	-0.3754	0.0144	-0.5627	0.1668	-0.0933	0.6464	0.0207	0.3368	0.0719	0.0825
300	225	CS-2	0.5254	1.5376	1.9523	0.3667	0.6357	-0.3745	0.0376	-0.5476	0.1667	-0.06421	0.5944	0.02048	0.3100	0.0768	0.0714
	300		0.4785	1.5356	1.9520	0.3556	0.6656	-0.4214	0.0356	-0.5479	0.1556	-0.0343	0.5441	0.0124	0.3136	0.0693	0.0470
	50		0.4841	1.5378	1.3815	0.4106	0.6865	-0.4159	0.0378	-1.1185	0.2106	-0.0135	0.6312	0.0580	1.7615	0.0995	0.0944
100	75	CS-3	0.4419	1.5010	1.5642	0.4269	0.6515	-0.4581	0.0010	-0.9358	0.2269	-0.0485	0.6680	0.0443	1.2339	0.1052	0.0832
	100		0.5017	1.5535	1.8908	0.3862	0.6638	-0.3983	0.0535	-0.6092	0.1862	-0.0362	0.4911	0.0365	0.4154	0.0834	0.0675
	150		0.4599	1.5041	1.4136	0.4122	0.6902	-0.4401	0.0041	-1.0864	0.2122	-0.0098	0.6324	0.0237	1.6099	0.1001	0.0850
300	225	CS-3	0.4359	1.4939	1.6449	0.3913	0.6458	-0.4641	-0.0061	-0.8551	0.1913	-0.0542	0.6345	0.0257	0.9342	0.0886	0.0809
	300		0.5171	1.5397	1.9559	0.3587	0.6684	-0.3829	0.0397	-0.5441	0.1587	-0.0316	0.5924	0.0154	0.3094	0.0740	0.0495

The average maximum likelihood estimates (MLEs), biases, and mean squared errors (MSEs) corresponding to the parameter values in Set 1 and Set 2 are presented in Table 4 and 6, respectively. The findings based on these results can be summarised as follows:

i) For both sets, the average MLEs are highly consistent with the true parameter values considered in the study.

ii) The average bias approaches zero as the sample size n increases.

iii) The MSEs exhibit a decreasing trend with an increase in the sample size n.

Overall, these results demonstrate the unbiasedness and consistency of the proposed MLEs.

Table 6. MLE, Bias and MSE for Set 3 parameter value

			MLE					Bias					MSE				
n	m	cs	α	λ	γ	p	q	α	λ	γ	p	q	α	λ	γ	p	q
	50		1.0942	6.9619	1.5333	0.5134	0.4263	-0.9057	-0.03803	0.0333	0.2134	-0.0736	1.3758	1.1579	0.0380	0.1216	0.0806
100	75	CS-1	1.0398	6.9266	1.5286	0.5174	0.4114	-0.9601	-0.0733	0.0286	0.2174	-0.0885	0.5324	0.8167	0.0223	0.1177	0.0664
	100		1.0460	6.9556	1.5300	0.4936	0.4092	-0.9539	-0.04430	0.0300	0.1936	-0.0907	0.3210	0.5914	0.0136	0.1122	0.0628
	150		2.1453	6.7001	1.5278	0.3174	0.5642	-0.8546	-0.2998	0.0278	0.2174	-0.1357	1.5922	1.0210	0.0166	0.1186	0.0741
300	225	CS-1	2.1540	6.8575	1.5181	0.3077	0.4558	-0.8459	-0.1424	0.0181	0.1477	0.1441	1.2680	0.9537	0.0091	0.0948	0.0583
	300		1.9988	6.9456	1.5018	0.3569	0.5154	-0.8011	-0.0543	0.0018	0.1569	-0.0845	0.5524	0.6684	0.0052	0.0811	0.0532
	50		1.8829	7.0319	1.5262	0.3897	0.5234	0.2386	0.0437	-0.1843	0.1378	-0.2091	0.7329	0.8537	0.7325	0.1938	0.0431
100	75	CS-2	2.0523	7.7071	1.5167	0.4166	0.5351	-0.4120	0.0510	-0.5999	0.1764	-0.0515	0.4646	0.0401	0.4018	0.0818	0.0735
	100		2.0016	7.0031	1.5153	0.4072	0.5090	-0.3983	0.0535	-0.6092	0.1862	-0.0362	0.2911	0.0365	0.3154	0.0434	0.0675
	150		2.1356	6.7522	1.5310	0.3644	0.4789	0.1898	-0.2477	0.0310	-0.2644	-0.02107	0.7525	1.1321	0.0167	0.1252	0.0636
300	225	CS-2	2.3615	6.9062	1.5178	0.3404	0.4926	0.1637	-0.0642	0.0178	0.2404	0.0301	0.4569	1.0814	0.0123	0.1087	0.0582
	300		2.0083	6.9069	1.5014	0.4463	0.5749	0.0636	-0.0930	0.0014	0.2400	0.0749	0.3030	0.4728	0.0041	0.1008	0.0055
	50		1.6829	7.0042	1.5262	0.4166	0.5234	0.1829	0.0042	0.0262	0.3166	0.0234	2.6348	2.0033	0.0261	0.1542	0.0787
100	75	CS-3	1.2836	7.0319	1.5158	0.3897	0.5090	-0.2163	0.0319	0.0158	0.2897	0.0090	1.5629	1.5378	0.0197	0.1455	0.0661
	100		1.3517	7.0307	1.5167	0.4072	0.5351	-0.1482	0.0307	0.0167	0.3072	0.0351	0.9787	0.8961	0.0050	0.1130	0.0656
	150		1.8085	6.7887	1.4737	0.4148	0.5929	0.30858	-0.2112	-0.0262	0.3148	0.0929	2.3062	0.9185	0.06190	0.1565	0.0966
300	225	CS-3	1.5393	6.9576	1.4913	0.3435	0.5194	0.0393	-0.0423	-0.0086	0.2435	0.0194	2.0682	0.8272	0.0210	0.1115	0.0676
	300		2.0603	6.9069	1.5014	0.3463	0.5749	0.060362	-0.09306	0.00149	0.246312636	0.07497	0.4569	0.47289	0.00418	0.10087	0.0557

Table 7. Data set 1

0.04, 1.866,2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82, 3.00, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663

6. Applications to real data set

We assess the efficiency of the proposed APMW-2G distribution using two datasets, each examined in both uncensored and censored cases. For comparison, we also consider the Alpha Power Modified Weibull–Geometric (APMWG), Alpha Power Modified Weibull (APMW), Doubly Poisson Exponential (DPE), Exponential Geometric Poisson (EGP), Exponential Geometric–Geometric (EGG), Poisson Geometric Lomax (PGL), and the Weibull distribution.

We compute the negative log-likelihood (-l), Akaike Information Criterion (AIC), small-sample corrected AIC (AICc), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criterion (HQIC), and the Kolmogorov–Smirnov (KS) test statistic with its p-value for the comparative study. The preferred model is the one that minimizes -l, AIC, AICc, BIC, HQIC, and the KS statistic (with a correspondingly larger p-value).

6.1. Application I

For the first application, we have considered a real-life data set from [5] which represents the failure time of a particular windshield model. The data was also considered by [18]. The data set 1 is provided in Table 7.

6.1.1. Uncensored data Following [1], the Total Time on Test (TTT) plot for the uncensored data, presented in Fig. 6, indicates an inverted bathtub–shaped hazard rate. This pattern suggests that the APMW-2G distribution provides an appropriate fit for these data. The maximum likelihood estimates (MLEs) along with their corresponding standard errors (SEs), as well as the model selection criteria—the negative log-likelihood (-l), Akaike Information

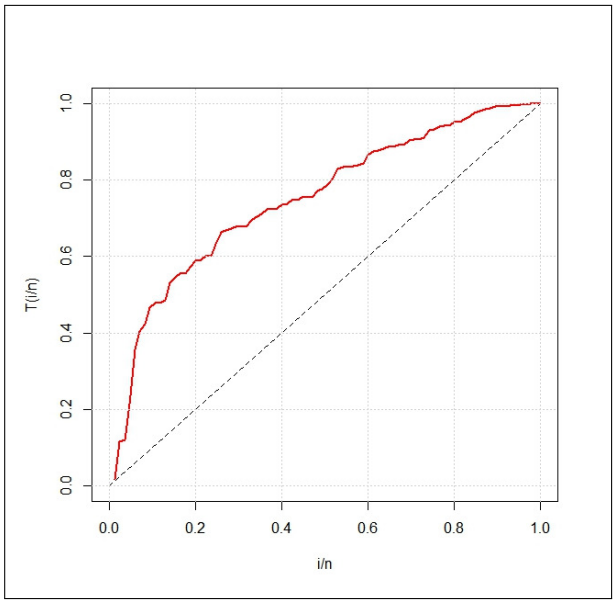


Figure 6. TTT plot for data set 1

Table 8. MLEs (SE) for all the distributions for data set 1

Distribution	α	γ	λ	p	q
APMW-2G	1.3230 (0.4936)	1.0905 (0.9742)	1.1775 (0.879)	0.9743 (0.0862)	0.4279 (0.3401)
APMWG	1.5670 (1.2064)	1.9657 (0.9723)	0.3800 (0.1149)	0.0100 (0.8745)	-
APMW	0.9860 (0.6927)	1.9650 (3.479)	0.3585 (0.7861)	-	-
DPE	0.0100 (0.2348)	2.9670 (0.8148)	0.7048(7.0116)	-	-
EGP	5.0000 (0.4897)	0.4560(0.0108)	-	0.9153 (0.9928)	-
EGG	2.1249 (0.0045)	0.5777 (0.8791)	-	-	-
PGL	1.6210 (0.7843)	0.0950 (0.7421)	3.0000 (4.8720)	0.0168 (0.9234)	-
W	1.94699 (0.2276)	0.432 (0.7054)	-	-	-

Table 9. Analytical measures for data set 1 for all the competing distributions

Distribution	-l	AIC	AICC	BIC	HQIC	K-S	p-value
APMW-2G	129.3769	268.7538	269.5133	280.9671	273.6664	0.0775	0.6866
APMWG	132.2576	272.5152	273.0152	282.2858	276.4452	0.1012	0.3493
APMW	133.6379	273.2757	273.572	280.6037	276.2232	0.1194	0.1771
DPE	150.7381	307.4761	307.7724	314.8041	310.4236	0.9588	< 0.05
EGP	145.5194	297.0388	297.3351	304.3667	299.9863	0.1822	0.0071
EGG	153.62	311.2401	311.3864	316.1254	313.2051	0.5910	< 0.05
PGL	187.5147	383.0294	383.5294	392.8001	386.9595	0.1248	<0.05
W	139.9211	283.8422	283.9885	288.7275	285.8072	0.79364	< 0.05

Criterion (AIC), corrected AIC (AICc), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criterion (HQIC), and the Kolmogorov–Smirnov (KS) statistic with its associated p-value—are reported in Tables 8 and 9, respectively for the APMW-2G and all competing distributions under the uncensored case.

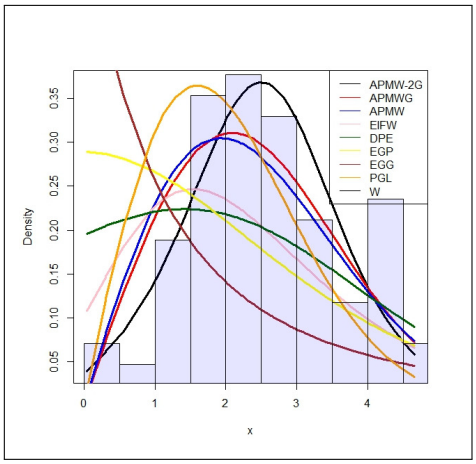


Figure 7. estimated densities with histogram for data set 1.

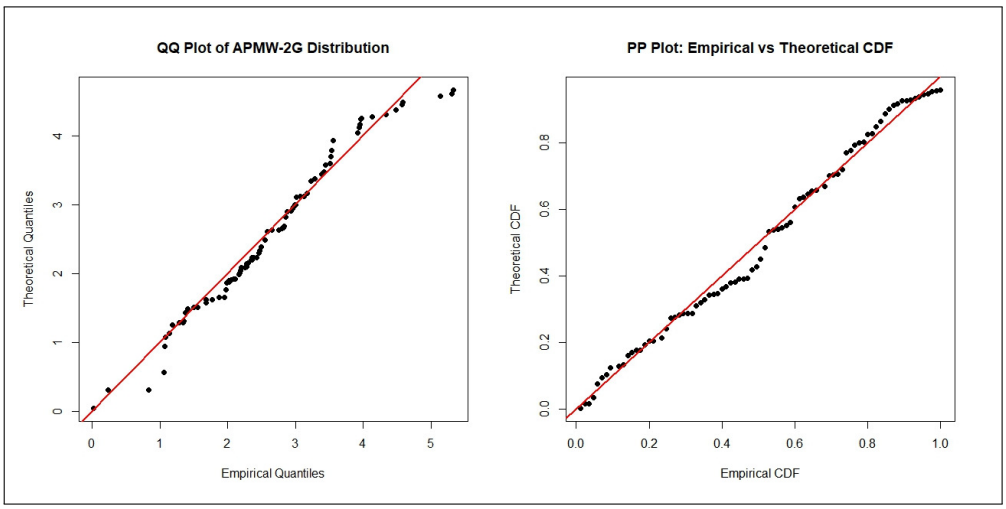


Figure 8. (a)QQ-plot; (b) PP-plot of APMW-2G distribution for data set 1.

From Table 9, it is observed that the APMW-2G distribution yields the lowest values of the negative log-likelihood (-l), AIC, AICc, BIC, HQIC, and KS statistic, along with the highest corresponding p-value, among all the competing distributions considered. These analytical criteria collectively indicate that the APMW-2G distribution provides the best fit for Dataset 1. Fig. 7 present the estimated probability density functions (PDFs) fitted to Dataset 1 for all the competing models. Furthermore, Fig. 8 displays the Q–Q and P–P plots of the proposed distribution for the same dataset, confirming its suitability and adequacy in modeling the data.

6.1.2. censored data Applying Progressive type II censoring with CS1 to the 85 observations given in table 6 we obtain the progressive type II censored data of sample size m= 50 in Table 10 below.

Based on the progressive Type-II censored sample, a comprehensive comparative analysis was conducted. The TTT plot in Fig. 9 indicates an inverted bathtub–shaped hazard rate, suggesting that the APMW-2G distribution provides an appropriate fit for these data. The maximum likelihood estimates (MLEs) and their corresponding standard errors (SEs), along with model selection criteria—including the negative log-likelihood (-l), Akaike Information Criterion (AIC), corrected AIC (AICc), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criterion (HQIC), and the Kolmogorov–Smirnov (KS) statistic with its associated p-value—are

Table 10. Progressive type II censored sample to Data set 1

0.040, 0.301, 0.309 ,0.557, 0.943 ,1.070, 1.124, 1.248, 1.281, 1.281, 1.303, 1.432, 1.480, 1.505, 1.506 ,1.568, 1.615 ,1.619 ,1.652, 1.652, 1.757, 1.866 ,1.876, 1.899, 1.911 ,1.912 ,1.914 ,1.981 ,2.010 ,2.038, 2.085 ,2.089 ,2.097 ,2.135 ,2.154, 2.190, 2.194, 2.223 ,2.224 ,2.229, 2.300, 2.324, 2.385 ,2.481, 2.610, 2.625 ,2.632, 2.646, 2.661, 2.688

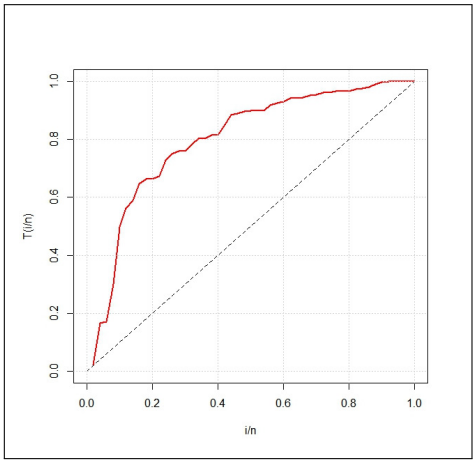


Figure 9. TTT plot for the progressive type II censored sample to data set 1

Table 11. MLEs (SE) for all the distributions for the progressive type II censored sample to data set 1

Distribution	α	γ	λ	p	q
APMW-2G	1.0003 (0.1457)	1.5460 (0.3467)	1.1945 (0.4519)	0.9768 (0.7124)	0.2957 (1.0071)
APMWG	3.000 (7.2349)	2.7588 (0.9418)	0.5603 (0.7460)	0.0107 (0.00148)	-
APMW	2.0614 (0.4218)	2.7539 (0.4100)	0.5615 (0.1749)	-	-
DPE	0.0103 (0.123)	3.0214 (0.7891)	1.0049 (0.4613)	-	-
EGP	1.5680 (0.4582)	2.000 (1.2481)	-	0.9875 (0.0745)	-
EGG	2.0000 (0.9147)	0.8323 (0.9919)	-	-	-
PGL	2.1705 (1.2272)	0.0116 (0.2495)	3.0000 (0.1380)	0.0092 (0.4281)	-
W	3.0513 (0.1190)	0.5065 (1.0140)	-	-	-

reported in Tables 11 and 12, respectively, for the APMW-2G and all competing distributions under the progressive Type-II censored case.

From Table 12, it is evident that the APMW-2G distribution attains the lowest values of the negative log-likelihood (-l), AIC, AICc, BIC, HQIC, and KS statistic, along with the highest corresponding p-value, among all competing distributions considered. These analytical criteria collectively indicate that the APMW-2G distribution provides the best fit for Dataset 1. Fig. 10 illustrates the estimated probability density functions (PDFs) fitted to the progressive Type-II censored sample of Dataset 1 for all competing models. Furthermore, Fig. 11 presents the Q–Q and P–P plots of the proposed distribution for the same dataset, confirming its suitability and adequacy in modelling the data.

Table 12. Analytical measures for the progressive type II censored sample to data set 1 for all the competing distributions

Distribution	-l	AIC	AICC	BIC	HQIC	K-S	p-value
APMW-2G	45.8381	101.6763	103.0399	111.2364	105.3168	0.0814	0.8943
APMWG	50.13864	108.2773	109.1662	115.9254	111.1897	0.13479	0.3237
APMW	50.10761	106.2152	[106.737	111.9513	108.3996	0.13456,	0.3257
DPE	68.80142	143.6028	144.1246	149.3389	145.7872	0.976,	< 0.05
EGP	49.17436	104.3487	104.8705	110.0848	106.533	0.13958,	0.2842
EGG	71.6336	147.2672	147.5225	151.0912	148.7234	0.60902,	<0.05
PGL	89.45674	186.9135	187.8024	194.5616	189.8259	0.958,	< 0.05
W	52.0303	108.0606	108.3159	111.8846	109.5168	0.91872,	< 0.05

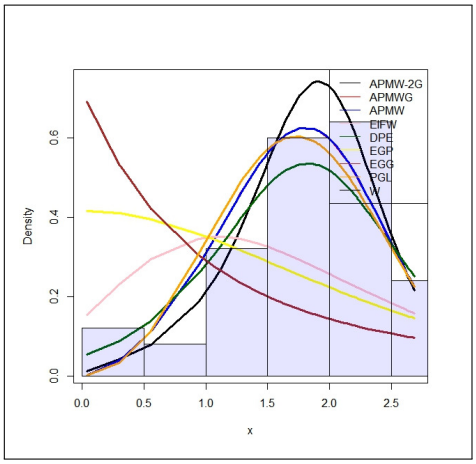


Figure 10. estimated densities with histogram for the progressive type II censored sample to data set 1.

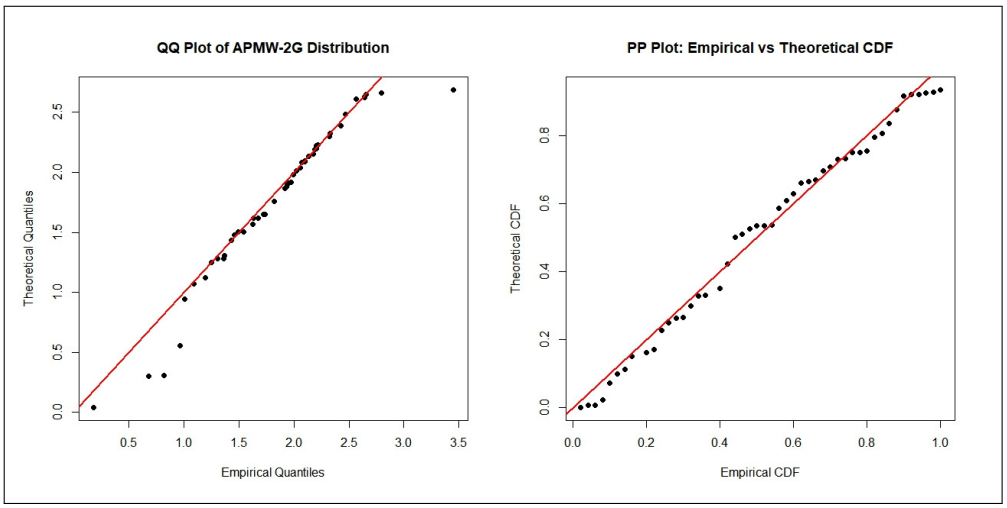


Figure 11. (a)QQ-plot; (b) PP-plot of APMW-2G distribution for the progressive type II censored sample to data set 1.

Table 13. Data set 2

275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106 ,300 ,300 ,212 ,300, 300, 300, 2 ,261, 293, 88, 247 ,28 ,143, 300, 23, 300 ,80, 245, 266

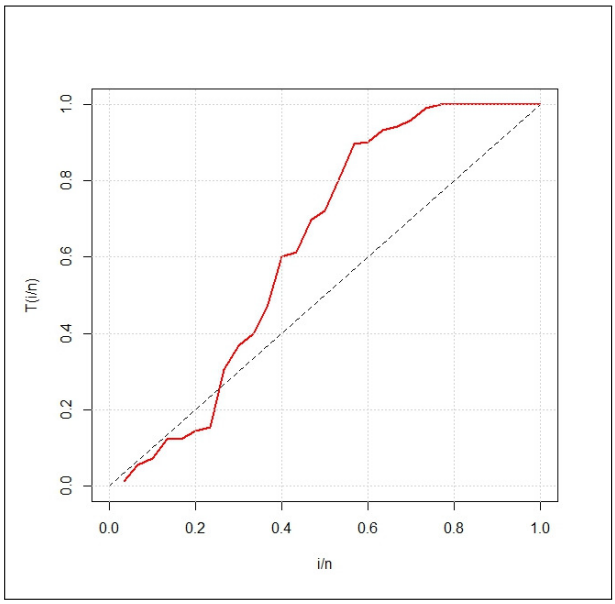


Figure 12. TTT plot for data set 2

6.2. Application 2

For the second application, we have considered the behavior of 30 devices inserted into the primary electrical system after a series of determined conditions and time had elapsed. These data were presented by [26]. The result of the failure cycles are presented in Table 13 below:

6.2.1. uncensored data AS shown in Fig. 12 the Total Time on Test (TTT) plot indicates that the data have a behavior similar to that of a bathtub curve. Therefore, it is submitted to the analysis with the distributions established. The maximum likelihood estimates (MLEs) with their standard errors (SEs), together with the model comparison metrics—the negative log-likelihood (-l), AIC, AICc, BIC, HQIC, and the Kolmogorov–Smirnov (KS) statistic with its p-value—for the APMWG and all competing distributions for the uncensored case are reported in Tables 13and 14.

From Table 15, it is evident that the APMW-2G distribution achieves the lowest values of the negative log-likelihood (-l), AIC, AICc, BIC, HQIC, and KS statistic, along with the highest corresponding p-value, among all competing distributions considered. These analytical measures collectively confirm that the APMW-2G distribution provides the best fit for Dataset 2. Fig. 13 illustrates the estimated probability density functions (PDFs) fitted to Dataset 2 for all competing models. Furthermore, Fig. 14 presents the Q–Q and P–P plots of the proposed distribution for the same dataset, further reaffirming its suitability and adequacy in modeling the data.

6.2.2. censored data Applying Progressive type II censoring with CS1 to the 30 observations given in Table 13 we obtain the progressive type II censored data of sample size m= 20 in Table 16 below.

Based on the progressive Type-II censored sample of the dataset, a comprehensive comparative analysis has been conducted. The TTT plot in Fig. 15 indicates a bathtub-shaped hazard rate, suggesting that the APMW-2G distribution provides an appropriate fit for these data. The maximum likelihood estimates (MLEs) and their

Table 14. MLEs (SE) for all the distributions for data set 2

Distribution	α	γ	λ	p	q
APMW-2G	0.9334(0.2416)	0.9552 (0.1702)	0.0116 (0.8457)	0.8530 (1.8731)	0.1527 (0.1694)
APMWG	3.000 (0.36190)	1.2748 (0.1376)	0.0048 (0.5957)	0.5240 (0.6025)	-
APMW	0.0965 (1.1278)	1.3292 (0.6487)	0.00291 (0.2621)	-	-
DPE	0.1601(0.1379)	3.000 (0.1201)	0.0091 (0.3858)	-	-
EGP	0.9780 (0.2409)	0.0196 (0.2340)	-	0.9833 (0.0779)	-
EGG	2.4920 (1.5192)	0.0126 (0.0167)	-	-	-
PGL	0.2719 (0.0164)	0.0015 (0.1071)	3.103 (5.9970)	0.0176 (0.4790)	-
W	0.9195 (0.3371)	0.00913 (0.1110)	-	-	-

Table 15. Analytical measures for data set 2 for all the competing distributions

Distribution	-l	AIC	AICC	BIC	HQIC	K-S	p-value
APMW-2G	182.7517	375.5035	378.0035	382.5094	377.7447	0.18698	0.245
APMWG	184.7883	377.5767	379.1767	383.1815	379.3697	0.21773	0.1163
APMW	184.8256	375.6511	376.5742	379.8547	376.9959	0.20575	0.1577
DPE	187.7230	381.4459	382.3690	385.6495	382.7907	5.9251	<0.05
EGP	188.6536	383.3073	384.2304	387.5109	384.6520	0.23947	0.06408
EGG	187.9082	379.8163	380.2608	382.6187	380.7129	0.60279	<0.05
PGL	212.8012	433.6024	435.2024	439.2071	435.3954	0.91906	<0.05
W	189.1447	382.2895	382.7339	385.0919	383.1860	1.0120	<0.05

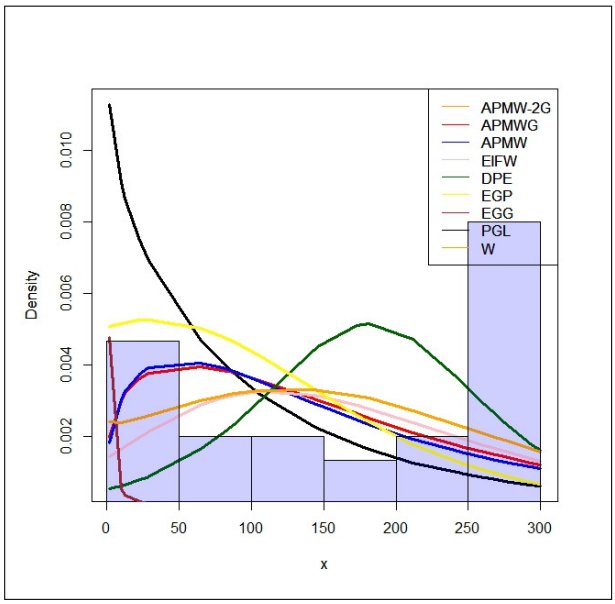


Figure 13. estimated densities with histogram for data set 2.

Table 16. Progressive type II censored sample to Data set 2

2 ,10 , 13 , 23 , 23 , 28 ,30 ,65 , 80 , 88, 106, 143, 147, 173 ,181 ,212 ,245 ,247 ,261,266

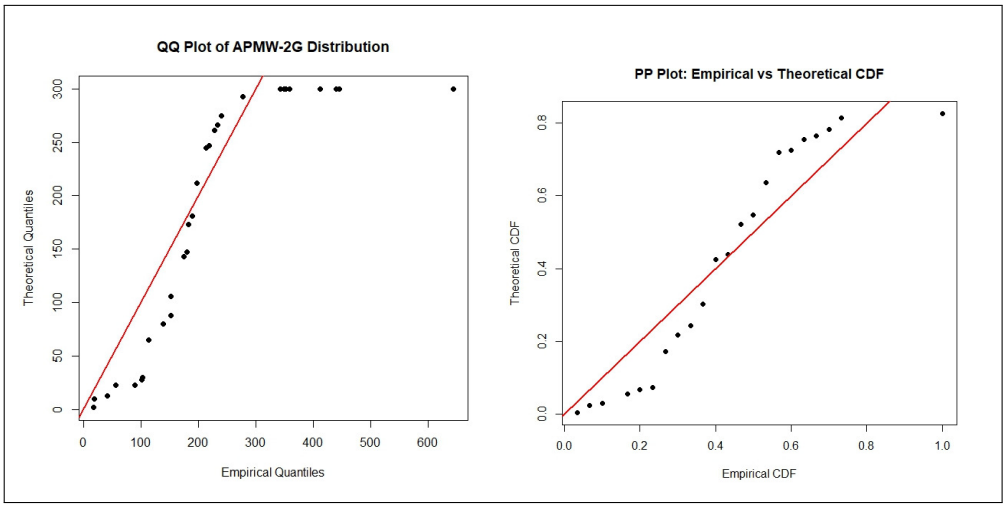


Figure 14. (a)QQ-plot; (b) PP-plot of APMW-2G ditribution for data set 2.

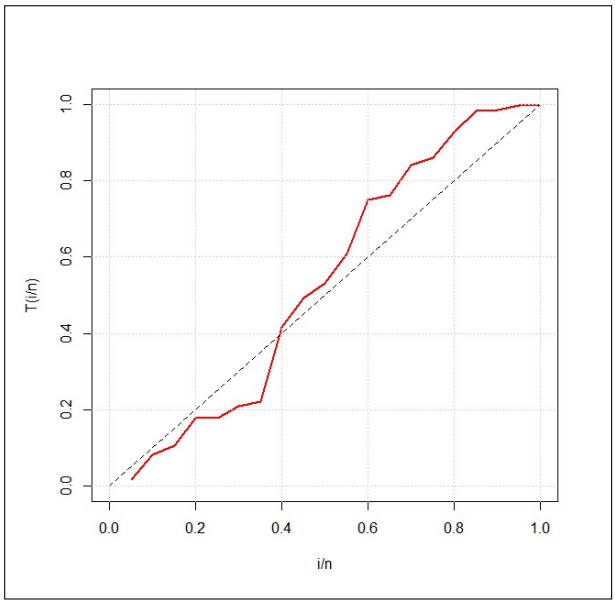


Figure 15. TTT plot for the progressive type II censored sample to data set 2

corresponding standard errors (SEs), along with model selection criteria—including the negative log-likelihood (-l), Akaike Information Criterion (AIC), corrected AIC (AICc), Bayesian Information Criterion (BIC), Hannan–Quinn Information Criterion (HQIC), and the Kolmogorov–Smirnov (KS) statistic with its associated p-value—are reported in Tables 17 and 18, respectively, for the APMW-2G and all competing distributions under the censored case.

From Table 18, it is evident that the APMW-2G distribution attains the lowest values of the negative log-likelihood (-l), AIC, AICc, BIC, HQIC, and KS statistic, along with the highest corresponding p-value, among all competing distributions considered. These analytical measures collectively indicate that the APMW-2G distribution provides the best fit for the progressive Type-II censored sample of Dataset 2. Fig. **??** presents the estimated probability density functions (PDFs) fitted to the progressive Type-II censored sample of Dataset 2 for all

Table 17. MLEs (SE) for all the distributions for the progressive type 2 censored sample to data set 2

Distribution	α	γ	λ	p	q
APMW-G	0.8701 (0.1184)	0.9315 (0.5134)	0.0125(1.5147)	0.7771 (0.4518)	0.4877 (0.2325)
APMWG	3.0401 (0.1312)	0.6461 (0.0688)	0.0231 (0.1076)	0.0101 (0.0296)	-
APMW	3.5791 (0.0347)	0.5861 (0.2482)	0.0291 (0.1468)	-	-
DPE	3.1001 (0.1255)	3.0301 (0.1191)	0.00751 (0.5631)	-	-
EGP	0.6481 (5.9024)	0.0241 (0.1387)	-	0.9361 ((0.4460)	-
EGG	0.9436 (0.2763)	0.0158 (9.1040)	-	-	-
PG	L0.3101 (1.0342)	0.0111 (0.0257)	3.001 (0.1379)	0.0092 (0.1569)	-
W	0.47181 (1.1714)	0.04321 (0.7875)	-	-	-

Table 18. Analytical measures for all the distributions for the progressive type 2 censored sample to data set 2

Distribution	-l	AIC	AICC	BIC	HQIC	K-S	p-value
APMW-2G	115.0121	240.0241	244.3099	245.0028	240.996	0.16252	0.6663
APMWG	118.2621	244.5243	247.1909	248.5072	245.3018	0.27883,	0.08922
APMW	119.0907	244.1814	245.6814	247.1686	244.7645	0.29729,	0.0583
DPE	118.2642	242.5285	244.0285	245.5157	243.1116	0.26484	0.1209
EGP	120.2608	246.5216	248.0216	249.5088	247.1047	0.26345,	0.1245
EGG	120.3536	244.7073	245.4131	246.6987	245.096	0.34497	0.01713
PGL	129.7307	[1] 267.4613	270.128	271.4442	268.2388	0.956	< 0.05
W	127.6195	259.2391	259.9449	261.2305	259.6278	0.99777	< 0.05

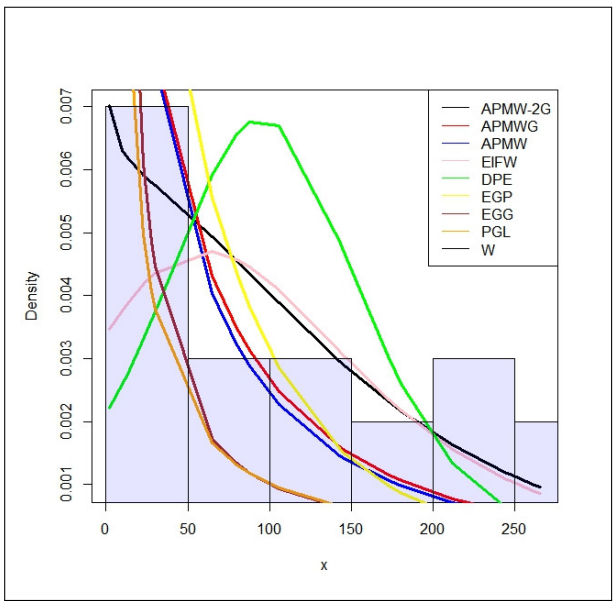


Figure 16. estimated densities with histogram for the progressive type II censored sample to data set 1.

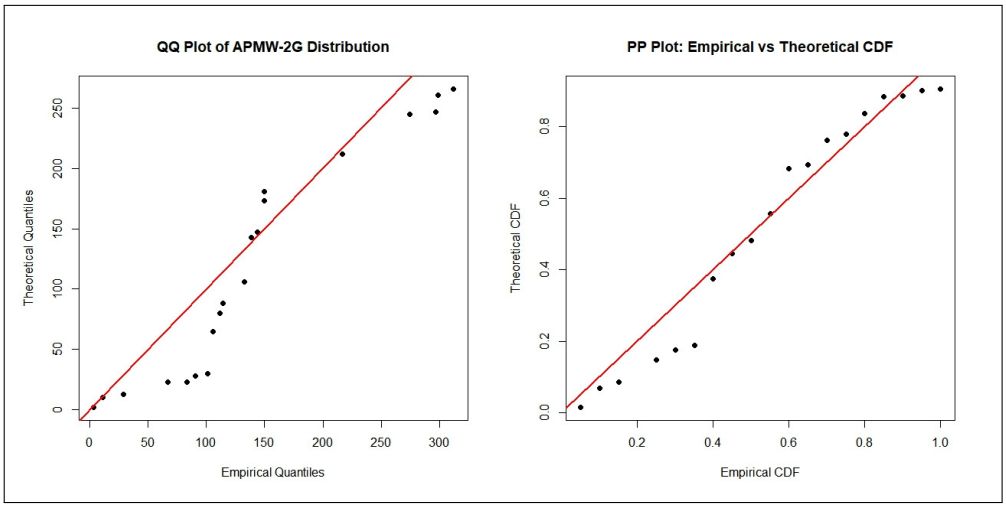


Figure 17. (a)QQ-plot; (b) PP-plot of APMW-2G distribution for the progressive type II censored sample to data set 2.

competing models. Furthermore, Fig. 17 displays the Q–Q and P–P plots of the proposed distribution for the same dataset, confirming its suitability and adequacy in modeling the censored data.

7. Conclusion

In this study, we introduce a new five-parameter lifetime model, the Alpha Power Modified Weibull–2G (APMW-2G) distribution. Key statistical properties—including moments, the moment-generating function, mean residual life, and measures of inequality and uncertainty—are derived. Parameters are estimated under progressive Type-II censoring using the maximum likelihood method, and their performance is evaluated through Monte Carlo simulations. Analyses of two real datasets confirm the adequacy of the proposed model for both censored and uncensored cases. The TTT plots further indicate that APMW-2G provides an appropriate fit. Overall, APMW-2G outperforms the base Alpha Power Modified Weibull (APMW) and several competing distributions, including Alpha Power Modified Weibull–Geometric (APMWG), Doubly Poisson Exponential (DPE), Exponential Geometric Poisson (EGP), Exponential Geometric–Geometric (EGG), Poisson Geometric Lomax (PGL), and the Weibull distributions.

Future research may explore bivariate and multivariate extensions, alternative estimation techniques, and compounding schemes based on different system failure structures to develop broader classes of lifetime distributions.

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Appendix

If $|z| < 1$ and v > 0, we have the series representation

$$(1-z)^{-v} = \sum_{i=0}^{\infty} \frac{\Gamma(v+i) z^i}{\Gamma(v) i!} \tag{46}$$

Using the above equation in Eq. 12 we have,

$$\mu'_r = \left[\int_0^\infty t^r(1-p)(\alpha+q-q\alpha-1)log\alpha\gamma\lambda^\gamma t^{\gamma-1}\alpha^{1-e^{-(\lambda t)^\gamma}}e^{-(\lambda t)^\gamma} \sum_{i=0}^{\infty}\frac{(2+i)}{\Gamma 2i!}\left[1-\alpha-p+\alpha q+(p-q)p\alpha^{1-e^{-(\lambda t)^\gamma}}\right]^i dt \right. \tag{47}$$

Again, using the binomial expansion

$$(a+b)^n = \sum_{m=0}^{n}\binom{i}{m}a^m b^{n-m}$$

we have

$$\mu'_r = (1-p)(\alpha+q-q\alpha-1)\sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha+q-q\alpha)^m(q-p)^{i-m} \int_0^\infty t^r log\alpha\gamma\lambda^\gamma t^{\gamma-1}\left[\alpha^{1-e^{-(\lambda t)^\gamma}}\right]^{i-m+1}e^{-(\lambda t)^\gamma}dt \tag{48}$$

We also have the series representation

$$\alpha^{-w} = \sum_{v=0}^{\infty}\frac{-(-log\alpha)^v w^v}{v!} \tag{49}$$

Incorporating the above given equation to the term $\left[\alpha^{1-e^{-(\lambda t)^\gamma}}\right]^{i-m+1}$, we have

$$\mu'_r = (1-p)(\alpha+q-q\alpha-1)\sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha+q-q\alpha)^m(q-p)^{i-m}\alpha^{i-m+1} \sum_{k=0}^{\infty}\frac{(-log\alpha)^{k+1}(i-m+1)^k}{k!}\int_0^\infty t^{r+\gamma-1}\gamma\lambda^\gamma e^{-(\lambda t)^\gamma}dt \tag{50}$$

Finally, integrating the integral part of the above equation we have,

$$\mu'_r = \frac{(1-p)(\alpha+q-q\alpha-1)\Gamma(\frac{r}{\gamma}+1)}{\lambda^{(\gamma)\frac{r}{\gamma}}}\sum_{i=0}^{\infty}(i+1)\sum_{m=0}^{i}\binom{i}{m}(\alpha+p-q\alpha)^m(q-p)^{(i-m)}\alpha^{(i-m+1)} \sum_{v=0}^{\infty}\frac{(-log)^{(v+1)}(i-m+1)^v}{v!(v+1)^{\frac{r}{\gamma}+1}}. \tag{51}$$

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