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New Insights into Fixed Points Results in Neutrosophic Metric Like-Spaces and Their Applications

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Abstract This paper introduces and develops the concept of neutrosophic metric-like spaces, extending both intuitionistic fuzzy metric spaces and traditional metric-like spaces. These spaces offer a more comprehensive framework for exploring mathematical structures and their properties. Key topics such as sequence convergence, G-Cauchy sequences, contractive mappings, and fixed-point theorems are examined within this context. To demonstrate the practical significance and utility of the proposed framework, illustrative examples and a discussion of potential applications are provided.

Keywords G-Cauchy sequence, Contractive mappings, Neutrosophic metric-like space, Fixed point, Integral equation.

AMS 2010 subject classifications 47H10; 54H25.

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1. Introduction

Fuzzy set theory, introduced by Zadeh [19] in 1965, has emerged as a powerful mathematical tool for modeling uncertainty and vagueness in various real-world phenomena, moving beyond the limitations of binary logic. Building upon this, Kramosil and Michalek [9] developed fuzzy and statistical metric spaces, which laid a strong foundation for further explorations into generalized metric structures, including intuitionistic and neutrosophic frameworks. George and Veeramani [2] contributed significantly to the field through fixed-point theorems in fuzzy metric spaces, while Grabiec [3] introduced the notion of G-Cauchy sequences, highlighting certain structural limitations, such as the absence of G-completeness and the non-implication of compactness [14, 18, 16].

In response to such limitations, Harandi [5] generalized partial metric spaces by introducing metric-like structures and initiated fixed-point studies therein. Later, Shukla et al. [12] advanced this direction by proposing fuzzy metric-like spaces and deriving fixed-point results under fuzzy contractive conditions. Meanwhile, Atanassov [1] introduced intuitionistic fuzzy sets, which were extended to intuitionistic fuzzy metric spaces by Park [11], establishing a richer framework to handle hesitation degrees in uncertainty modeling. Gregori et al. [4] provided a structural equivalence between intuitionistic and fuzzy metric spaces, which was further extended to intuitionistic fuzzy metric-like spaces by Onbaşıoğlu et al. [10].

To model indeterminacy more comprehensively, Smarandache [13] introduced neutrosophic sets, characterized by independent degrees of truth, indeterminacy, and falsity. Kirisci and Simsek [8] built on this by formulating

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neutrosophic metric spaces, and Sowndrarajan et al. [7] explored their potential via fixed-point results for generalized contraction mappings.

Nevertheless, neutrosophic metric spaces preserve the classical axiom $\mathcal{A}(\zeta,\zeta,\theta)=1$ for the truth-membership function, which may be restrictive in certain contexts. For instance, in evaluating paradoxical evidence in logic systems or measuring the cost of self-comparison in network models, such a condition may not adequately capture the underlying structure. To overcome this limitation, we introduce neutrosophic metric-like spaces, where the axiom is generalized to $\mathcal{A}(\zeta,\zeta,\theta)\leq 1$. This relaxation provides greater flexibility in modeling self-relations and indeterminacy, thereby extending the applicability of neutrosophic analysis.

Motivated by these developments and gaps in the existing literature, this paper focuses on the structural and analytical aspects of intuitionistic and neutrosophic metric-like spaces. It aims to bridge the theoretical gap between existing fixed-point results in classical fuzzy settings and their neutrosophic counterparts. Specifically, we establish new fixed-point theorems under neutrosophic contractive conditions involving classical Cauchy sequences. The novelty of our approach lies in the use of neutrosophic metric-like structures, which accommodate a broader spectrum of uncertainty than traditional fuzzy or intuitionistic frameworks.

In addition, illustrative examples are provided to demonstrate the validity and scope of the theoretical results. Finally, we apply our findings to solve a class of integral equations, thereby emphasizing the practical significance of neutrosophic metric-like spaces in mathematical modeling and analysis.

2. Preliminaries

In this section, we recall some basic notions and definitions that will serve as the foundation for our study.

Definition 2.1. ([10]). Let $\Xi\neq\emptyset$. A function $\rho:\Xi\times\Xi\to\mathbb{R}$ is termed metric-like on Ξ provided that it fulfills the following conditons:
(ML1) $\rho(\zeta,\varphi)=0\Rightarrow\zeta=\varphi$;
(ML2) $\rho(\zeta,\varphi)=\rho(\varphi,\zeta)$;
(ML3) $\rho(\zeta,z)\leq\rho(\zeta,\varphi)+\rho(\varphi,z)$.
The pair (Ξ,ρ) is known a s a metric-like space (MLS) on Ξ.
It is evident that , $\rho(\zeta,\varphi)=\max\{\zeta,\varphi\}$ is regarded as metric-like on $\Xi=[0,\infty)$.

Definition 2.2. ([10]) The 5-tuple $(\Xi,\mathcal{A},\mathcal{B},*,\diamond)$ is said to be an Intuitionistic Fuzzy Metric-Like Space (IFMLS), where Ξ is a non-empty set, * denotes a continuous t-norm, $\diamond$ signifies a continuous t-conorm, and $\mathcal{A}$, $\mathcal{B}$ are fuzzy sets defined on $\Xi\times\Xi\times(0,\infty)$. These elements satisfy the following conditions for all $\zeta,\varphi,\mathfrak{k}\in\Xi$ and $\theta,\varpi>0$

(i) $\mathcal{A}(\zeta,\varphi,\theta)+\mathcal{B}(\zeta,\varphi,\theta)\leq 1$;

(ii) $\mathcal{A}(\zeta,\varphi,\theta)>0$;

(iii) $\mathcal{A}(\zeta,\varphi,\theta)=1\implies\zeta=\varphi$;

(iv) $\mathcal{A}(\zeta,\varphi,\theta)=\mathcal{A}(\varphi,\zeta,\theta)$;

(v) $\mathcal{A}(\zeta,\varphi,\theta)*\mathcal{A}(\varphi,\mathfrak{k},\varpi)\leq\mathcal{A}(\zeta,\mathfrak{k},\theta+\varpi)$;

(vi) $\mathcal{A}(\zeta,\varphi,\cdot)$ is a continuous mapping from $(0,\infty)\to[0,1]$

(vii) $\mathcal{B}(\zeta,\varphi,\theta)<1$;

(viii) $\mathcal{B}(\zeta,\varphi,\theta)=1\implies\zeta=\varphi$;

(ix) $\mathcal{B}(\zeta,\varphi,\theta)=\mathcal{B}(\varphi,\zeta,\theta)$;

(x) $\mathcal{B}(\zeta,\varphi,\theta)\diamond\mathcal{B}(\varphi,\mathfrak{k},\varpi)\geq\mathcal{B}(\zeta,\mathfrak{k},\theta+\varpi)$;

(xi) $\mathcal{B}(\zeta,\varphi,\cdot):(0,\infty)\to[0,1]$ is a continuous mapping.

Definition 2.3. ([16]). The sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ in $(\Xi, \mathcal{A}, \mathcal{B}, *, \diamond)$ termed Cauchy if given any $\epsilon \in (0,1)$ and each $\theta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{A}(\zeta_n, \zeta_m, \theta) > 1-\epsilon$, and $\mathcal{B}(\zeta_n, \zeta_m, \theta) < \epsilon$, for all $n, m \geq n_0$. Alternatively this condition met if $\lim_{m,n\to\infty} \mathcal{A}(\zeta_n, \zeta_m, \theta) = 1$ and $\lim_{m,n\to\infty} \mathcal{B}(\zeta_n, \zeta_m, \theta) = 0$ for all $\theta > 0$. The IFMLS $(\Xi, \mathcal{A}, \mathcal{B}, *, \diamond)$ named to be complete If every Cauchy sequence is converges in $(\Xi, \mathcal{A}, \mathcal{B}, *, \diamond)$.

Definition 2.4. ([16]). A sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ in an IFMS $(\Xi, \mathcal{A}, \mathcal{B}, *, \diamond)$ is referred as G-Cauchy if $\lim_{n\to\infty} \mathcal{A}(\zeta_n, \zeta_{n+p}, t) = 1$ and $\lim_{n\to\infty} \mathcal{B}(\zeta_n, \zeta_{n+p}, t) = 0$ for each $t > 0$ and each $p \in \mathbb{N}$. The space $(\Xi, \mathcal{A}, \mathcal{B}, *, \diamond)$ is described as complete if every G- Cauchy sequence is in Ξ converges.

Definition 2.5. A non-empty set Ξ is given, with $*$ as a continuous t-norm, $\diamond$ as a t-conorm, and $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ as neutrosophic sets on $\Xi \times \Xi \times (0,\infty)$. The following conditions are satisfied for all $\zeta, \varphi, \mathfrak{k} \in \Xi$ and $\theta, \varpi > 0$.
(NML1) $0 \leq \mathcal{A}(\zeta,\varphi,\theta) \leq 1; 0 \leq \mathcal{B}(\zeta,\varphi,\theta) \leq 1; 0 \leq \mathcal{C}(\zeta,\varphi,\theta) \leq 1;$
(NML2) $\mathcal{A}(\zeta,\varphi,\theta) + \mathcal{B}(\zeta,\varphi,\theta) + \mathcal{C}(\zeta,\varphi,\theta) \leq 3;$
(NML3) $\mathcal{A}(\zeta,\varphi,\theta) > 0;$
(NML4) $\mathcal{A}(\zeta,\varphi,\theta) = 1 \Rightarrow \zeta = \varphi;$
(NML5) $\mathcal{A}(\zeta,\varphi,\theta) = \mathcal{A}(\varphi,\zeta,\theta);$
(NML6) $\mathcal{A}(\zeta,\varphi,\theta) * \mathcal{A}(\varphi,\mathfrak{k},\varpi) \leq \mathcal{A}(\zeta,\mathfrak{k},\theta+\varpi);$
(NML7) $\mathcal{A}(\zeta,\varphi,.)$ is continuous a mapping from $(0,\infty) \to (0,1];$
(NML8) $\mathcal{B}(\zeta,\varphi,\theta) < 1;$
(NML9) $\mathcal{B}(\zeta,\varphi,\theta) = 0 \Rightarrow \zeta = \varphi;$
(NML10) $\mathcal{B}(\zeta,\varphi,\theta) = \mathcal{B}(\varphi,\zeta,\theta),$
(NML11) $\mathcal{B}(\zeta,\varphi,\theta) \diamond \mathcal{B}(\varphi,\mathfrak{k},\varpi) \geq \mathcal{B}(\zeta,\mathfrak{k},\theta+\varpi);$
(NML12) $\mathcal{B}(\zeta,\varphi,.) : (0,\infty) \to (0,1]$ is continuous.
(NML13) $\mathcal{C}(\zeta,\varphi,\theta) < 1;$
(NML14) $\mathcal{C}(\zeta,\varphi,\theta) = 0 \Rightarrow \zeta = \varphi;$
(NML15) $\mathcal{C}(\zeta,\varphi,\theta) = \mathcal{C}(\varphi,\zeta,\theta);$
(NML16) $\mathcal{C}(\zeta,\varphi,\theta) \diamond \mathcal{C}(\varphi,\mathfrak{k},\varpi) \geq \mathcal{C}(\zeta,\mathfrak{k},\theta+\varpi);$
(NML17) $\mathcal{C}(\zeta,\varphi,.) : (0,\infty) \to (0,1]$ is continuous. Then $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \diamond)$ is said to be a Neutrosophic Metric-Like Space (NMLS).

Example 2.6. Let $\Xi = \mathbb{R}^+$. Define $r * s = \min\{r, s\}$ and $r \diamond s = \max\{r, s\}$ for all $r, s \in [0,1]$. Consider the neutrosophic sets $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ in $\Xi \times \Xi \times (0,\infty)$ given by

$$\mathcal{A}(\zeta,\varphi,\theta) = \frac{\theta}{\theta + \max\{\zeta,\varphi\}}, \quad \mathcal{B}(\zeta,\varphi,\theta) = \frac{\max\{\zeta,\varphi\}}{\theta + \max\{\zeta,\varphi\}}, \quad \mathcal{C}(\zeta,\varphi,\theta) = \frac{\max\{\zeta,\varphi\}}{\theta},$$

for all $\zeta, \varphi \in \Xi$ and $\theta > 0$.

We check only triangle inequalities (NML6), (NML11) and (NML16), because verification of the other conditions is standard. Let $\zeta, \varphi \in \Xi$ and $\theta, \mathfrak{k}, \varpi$
$\max\{\zeta,\mathfrak{k}\} \leq \max\{\zeta,\varphi\} + \max\{\varphi,\mathfrak{k}\}$
$\Rightarrow \frac{\theta+\varpi}{\theta+\varpi+\max\{\zeta,\mathfrak{k}\}} \geq \min\{\frac{\theta}{\theta+\max\{\zeta,\mathfrak{k}\}}, \frac{\varpi}{\varpi+\max\{\zeta,\mathfrak{k}\}}\}$
$\Rightarrow \mathcal{A}(\zeta,\mathfrak{k},\theta+\varpi) \geq \mathcal{A}(\zeta,\varphi,\theta) * \mathcal{A}(\varphi,\mathfrak{k},\varpi).$
$\frac{\max\{\zeta,\mathfrak{k}\}}{\theta+\varpi+\max\{\zeta,\mathfrak{k}\}} \leq \max\{\frac{\max\{\zeta,\varphi\}}{\theta+\max\{\zeta,\varphi\}}, \frac{\max\{\varphi,\mathfrak{k}\}}{\varpi+\max\{\varphi,\mathfrak{k}\}}\}$
$\Rightarrow \mathcal{B}(\zeta,\varphi,\theta) \diamond \mathcal{B}(\varphi,\mathfrak{k},\varpi) \geq \mathcal{B}(\zeta,\mathfrak{k},\theta+\varpi$
Similarly, $\mathcal{C}(\zeta,\varphi,\theta) \diamond \mathcal{C}(\varphi,\mathfrak{k},\varpi) \geq \mathcal{C}(\zeta,\mathfrak{k},\theta+\varpi)$
Hence, $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \diamond)$ is a NMLS.
Observe that

$$\mathcal{A}(\zeta,\zeta,\theta) = \frac{\theta}{\theta+\zeta} < 1 \quad \text{for } \zeta > 0,$$

which indicates that even the "self-distance" of a non-zero element is associated with a certain degree of uncertainty or cost. This highlights the neutrosophic nature of the model, distinguishing it from the classical metric-type framework.

Definition 2.7. Let $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \diamond)$ be a NMLS and $\{\zeta_n\}$ be sequence in Ξ
(a) $\{\zeta_n\}$ is considered convergent to $\zeta \in \Xi$ if $\lim\limits_{n\to\infty} \mathcal{A}(\zeta_n, \zeta, \theta) = 1$, $\lim\limits_{n\to\infty} \mathcal{B}(\zeta_n, \zeta, \theta) = 0$ and $\lim\limits_{n\to\infty} \mathcal{C}(\zeta_n, \zeta, \theta) = 0$ for all $\theta > 0$.
(b)$\{\zeta_n\}$ in Ξ is referred to as a Cauchy if given any $\epsilon \in (0,1)$ and each $\theta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathcal{A}(\zeta_n, \zeta_m, \theta) > 1 - \epsilon$, $\mathcal{B}(\zeta_n, \zeta_m, \theta) < \epsilon$ and $\mathcal{C}(\zeta_n, \zeta_m, \theta) < \epsilon \; \forall \; m, n \geq n_0$ or equivalently, if $\lim\limits_{m,n\to\infty} \mathcal{A}(\zeta_n, \zeta_m, \theta) = 1$, $\lim\limits_{m,n\to\infty} \mathcal{B}(\zeta_n, \zeta_m, \theta) = 0$ and $\lim\limits_{m,n\to\infty} \mathcal{C}(\zeta_n, \zeta_m, \theta) = 0$ for all $\theta > 0$.
(c) The structure $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \diamond)$ is referred to be complete if every Cauchy sequence ζ_n in Ξ converges to some $\zeta \in \Xi$.

3. Main Results

The core results of this paper are established in this section, highlighting the role of neutrosophic metric-like structures in generalizing and strengthening fixed-point theory.

Definition 3.1. Let $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \diamond)$ be a NMLS. A function $\mathfrak{h} : \Xi \to \Xi$ is termed a neutrosophic contractive if there exists a constant $\mathfrak{d} \in (0,1)$, referred to as the neutrosophic constant of $\mathfrak{h}$ such that the following conditions hold for all $\zeta, \varphi \in \Xi$ and $\theta > 0$.

$$\frac{1}{\mathcal{A}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta)} - 1 \leq \mathfrak{d}\left[\frac{1}{\mathcal{A}(\zeta, \varphi, \theta)} - 1\right] \tag{3.1.1}$$

$$\mathcal{B}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta) \leq \mathfrak{d}\mathcal{B}(\zeta, \varphi, \theta) \text{ and} \tag{3.1.2}$$

$$\mathcal{C}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta) \leq \mathfrak{d}\mathcal{C}(\zeta, \varphi, \theta) \tag{3.1.3}$$

Theorem 3.2. Let $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \diamond)$ be a neutrosophic metric-like space and $\mathfrak{h} : \Xi \to \Xi$ a neutrosophic contractive mapping with neutrosophic contractive constant $\mathfrak{d}$. Let

$$\lim_{\theta\to 0^+} \mathcal{A}(\zeta_n, \zeta_{n+1}, \theta) > 0, \tag{3.2.1}$$

$$\lim_{\theta\to 0^+} \mathcal{B}(\zeta_n, \zeta_{n+1}, \theta) < 1 \text{ and} \tag{3.2.2}$$

$$\lim_{\theta\to 0^+} \mathcal{C}(\zeta_n, \zeta_{n+1}, \theta) < 1, n \in \mathbb{N}. \tag{3.2.3}$$

Then, $\mathfrak{h}$ has a unique fixed point $\zeta \in \Xi$ and $\mathcal{A}(\zeta, \zeta, \theta) = 1, \mathcal{B}(\zeta, \zeta, \theta) = 0$ and $\mathcal{C}(\zeta, \zeta, \theta) = 0$ for all $\theta > 0$.

Proof
Let $\zeta_0 \in \Xi$ be an arbitrary initial element, and define a sequence $\zeta_n \subset \Xi$ recursively by $\zeta_n = \mathfrak{h}(\zeta_{n-1}) \; \forall \; n \in \mathbb{N}$. If there exists an $n \in \mathbb{N}$ such that $\zeta_n = \zeta_{n-1}$, then ζ_n is an invariant point of the map $\mathfrak{h}$. On the other hand, if $\zeta_n \neq \zeta_{n-1} \; \forall \; n \in \mathbb{N}$, we use the contractive condition (3.1.1) for $\theta > 0$ and any $n \in \mathbb{N}$. This gives the following inequality: $\frac{1}{\mathcal{A}(\zeta_n, \zeta_{n+1}, \theta)} - 1 = \frac{1}{\mathcal{A}(\mathfrak{h}(\zeta_{n-1}), \mathfrak{h}(\zeta_n), \theta)} - 1 \leq \mathfrak{d}^n \left[\frac{1}{\mathcal{A}(\zeta_0, \zeta_1, \theta)} - 1\right]$.
Taking the limit as n approaches infinity, we arrive

$$\lim_{n\to\infty} \mathcal{A}(\zeta_n, \zeta_{n+1}, \theta) = 1, \theta > 0. \tag{3.2.4}$$

To establish the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, let us proceed by contradiction. Suppose the opposite, that there exist $\epsilon \in (0, \mathfrak{L})$, with $\mathfrak{L} \leq 1$, $\theta_0 > 0$, and sequences n_l and m_l such that $m_l > n_l > l$ for all $l \in \mathbb{N}$ and

$$\mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \leq \mathfrak{L} - \epsilon, l \in \mathbb{N}, \tag{3.2.5}$$

and

$$\mathcal{A}(\zeta_{m_l - 1}, \zeta_{n_l}, \theta_0) > \mathfrak{L} - \epsilon, l \in \mathbb{N}. \tag{3.2.6}$$

Clearly, from (3.2.5),

$$\lim_{l\to\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \le \mathfrak{L} - \epsilon. \tag{3.2.7}$$

By applying (NML6), for any $l \in \mathbb{N}$ and $p \in (0, \theta_0)$, we have

$$\mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \ge \mathcal{A}(\zeta_{m_l}, \zeta_{m_l-1}, p) * \mathcal{A}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0 - p). \tag{3.2.8}$$

As $p \to 0^+$ in (3.2.8), leveraging the continuity of $\mathfrak{h}$ and (3.2.1), we obtain

$$\begin{aligned}
\lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) &= \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0)\right)\\
&\ge \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{m_l-1}, p) * \mathcal{A}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0 - p)\right)\\
&= \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{m_l-1}, p)\right) * \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0 - p)\right)\\
&= 1 * \lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0)\\
&= \lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0) \ge \mathfrak{L} - \epsilon,
\end{aligned}$$

and, together with (3.2.7), we have $\lim_{l\to\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) = \mathfrak{L} - \epsilon$.

To prove that $\lim_{l\to\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) = \mathfrak{L} - \epsilon$, we proceed using conditions (3.2.1) and (3.2.6) in the following manner:

$$\begin{aligned}
\lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) &= \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0)\right)\\
&\ge \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0 - p) * \mathcal{A}(\zeta_{n_l}, \zeta_{n_l+1}, p)\right)\\
&\ge \lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) * 1\\
&= \lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) = \mathfrak{L} - \epsilon.
\end{aligned}$$

Moreover, By utilizing (3.2.1) and (3.2.6), we get:

$$\begin{aligned}
\mathfrak{L} - \epsilon &= \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0)\right)\\
&\ge \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0 - p) * \mathcal{A}(\zeta_{n_l+1}, \zeta_{n_l}, p)\right)\\
&\ge \lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) * 1\\
&= \lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0).
\end{aligned}$$

Therefore, we have: $\lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) = \mathfrak{L} - \epsilon$.

Now,

$$\begin{aligned}
\lim_{l\to+\infty} \mathcal{A}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) &= \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0)\right)\\
&\ge \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l+1}, \zeta_{m_l}, p) * \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0 - p)\right)\\
&= \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l+1}, \zeta_{m_l}, p)\right) * \lim_{l\to+\infty}\left(\lim_{p\to0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0 - p)\right)
\end{aligned}$$

$$= 1 * \lim_{l \to +\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0)$$
$$= \lim_{l \to +\infty} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) = \mathfrak{L} - \epsilon,$$

However, we can demonstrate that

$$\begin{aligned}
\mathfrak{L} - \epsilon &= \lim_{l \to +\infty} \left(\lim_{p \to 0^+} \mathcal{A}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) \right) \\
&\geq \lim_{l \to +\infty} \left(\lim_{p \to 0^+} \mathcal{A}(x_{m_l}, x_{m_l+1}, \theta_0 - p) * \mathcal{A}(\zeta_{m_l+1}, \zeta_{n_l+1}, p) \right) \\
&\geq 1 * \lim_{l \to +\infty} \mathcal{A}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) \\
&= \lim_{l \to +\infty} \mathcal{A}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0).
\end{aligned}$$

So, $\lim_{l \to +\infty} \mathcal{A}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) = \mathfrak{L} - \epsilon$.
By utilizing (3.1.1.) $\mathcal{A}$, the relationship can be written as:

$$\frac{1}{\mathcal{A}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0)} - 1 \leq \mathfrak{d} \left[\frac{1}{\mathcal{A}(\zeta_{m_l}, \zeta_{n_l}, \theta_0)} - 1 \right].$$

As $l \to \infty$, we find: $\frac{1}{\mathfrak{L}} - 1 \leq \mathfrak{d} \left[\frac{1}{\mathfrak{L}} - 1 \right] < \frac{1}{\mathfrak{L}} - 1$.
This results in a clear contradiction.
Referring (3.1.2), we deduce
$\mathcal{B}(\mathfrak{h}(\zeta_{n-1}), \mathfrak{h}\zeta_n, \theta) = \mathcal{B}(\zeta_n, \zeta_{n+1}, \theta) \leq \mathfrak{d}^n \mathcal{B}(\zeta_0, \zeta_1, \theta)$.
Thus, it follows that

$$\lim_{n \to \infty} \mathcal{B}(\zeta_n, \zeta_{n+1}, \theta) = 0, \theta > 0. \tag{3.2.9}$$

To establish that $\{\zeta_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, let us proceed by contradiction, that is there exist $\epsilon \in (0, \mathfrak{M}), \mathfrak{M} \leq 1, \theta_0 > 0$ and sequences $\{n_l\}$ and $\{m_l\}$ such that $m_l > n_l > l$, for every $l \in \mathbb{N}$ and the following holds:

$$\mathcal{B}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \geq \mathfrak{M} + \epsilon, l \in \mathbb{N}, \tag{3.2.10}$$

and

$$\mathcal{B}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0) < \mathfrak{M} + \epsilon, l \in \mathbb{N}. \tag{3.2.11}$$

From (3.2.10), we have:

$$\lim_{l \to \infty} \mathcal{B}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \geq \mathfrak{M} + \epsilon. \tag{3.2.12}$$

By applying (NMLS11), for arbitrary $l \in \mathbb{N}$ and $p \in (0, \theta_0)$, we derive:

$$\mathcal{B}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \leq \mathcal{B}(\zeta_{m_l}, \zeta_{m_l-1}, p) \diamond \mathcal{B}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0 - p). \tag{3.2.13}$$

Taking the limit as $p \to 0^+$ in the inequality (3.2.13), and using the continuity of $\diamond$ along with (3.1.2), we can conclude that:

$$\begin{aligned}
\lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l},\theta_0) &= \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l},\zeta_{n_l},\theta_0)\right)\\
&\leq \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l},\zeta_{m_l-1},p)\diamond\mathcal{B}(\zeta_{m_l-1},\zeta_{n_l},\theta_0-p)\right)\\
&= \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l},\zeta_{m_l-1},p)\right)\diamond \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l-1},\zeta_{n_l},\theta_0-p)\right)\\
&= 0\diamond \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l-1},\zeta_{n_l},\theta_0)\\
&= \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l-1},\zeta_{n_l},\theta_0)\\
&\leq \mathfrak{M}+\epsilon,
\end{aligned}$$

from (3.2.12), we deduce that, $\lim_{l\to\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l},\theta_0) = \mathfrak{M}+\epsilon$.
Now, we aim to prove that $\lim_{l\to\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0) = \mathfrak{M}+\epsilon$. This can be established using (3.2.2) and (3.2.11), as follows:

$$\begin{aligned}
\lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0) &= \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0)\right)\\
&\leq \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l},\zeta_{n_l},\theta_0-p)\diamond\mathcal{B}(\zeta_{n_l},\zeta_{n_l+1},p)\right)\\
&\leq \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l},\theta_0)\diamond 0\\
&= \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l},\theta_0)\\
&= \mathfrak{M}+\epsilon.
\end{aligned}$$

Additionally, based on (3.2.2) and (3.2.11),

$$\begin{aligned}
\mathfrak{M}+\epsilon &= \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l},\zeta_{n_l},\theta_0)\right)\\
&\leq \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0-p)\diamond\mathcal{B}(\zeta_{n_l+1},\zeta_{n_l},p)\right)\\
&\geq \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0)\diamond 0\\
&= \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0).
\end{aligned}$$

Therefore, we can conclude that $\lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0) = \mathfrak{M}+\epsilon$.
Now,

$$\begin{aligned}
\lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l+1},\zeta_{n_l+1},\theta_0) &= \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l+1},\zeta_{n_l+1},\theta_0)\right)\\
&\leq \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l+1},\zeta_{m_l},p)\diamond\mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0-p)\right)\\
&= \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l+1},\zeta_{m_l},p)\right)\diamond \lim_{l\to+\infty}\left(\lim_{p\to 0^+} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0-p)\right)\\
&= 0\diamond \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l},\zeta_{n_l+1},\theta_0)
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) \\
&= \mathfrak{M} + \epsilon,
\end{aligned}
$$

However, it is evident that

$$
\begin{aligned}
\mathfrak{M} + \epsilon &= \lim_{l\to+\infty} \left(\lim_{p\to0^+} \mathcal{B}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) \right) \\
&\leq \lim_{l\to+\infty} \left(\lim_{p\to0^+} \mathcal{B}(\zeta_{m_l}, \zeta_{m_l+1}, \theta_0 - p) \Diamond \mathcal{B}(\zeta_{m_l+1}, \zeta_{n_l+1}, p) \right) \\
&\leq 0 \Diamond \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) \\
&= \lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0).
\end{aligned}
$$

So, $\lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) = \mathfrak{M} + \epsilon$. Within the framework of a contractive condition for $\mathcal{B}$, it follows that $\mathcal{B}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) \leq \mathfrak{d}\mathcal{B}(\zeta_{m_l}, \zeta_{n_l}, \theta_0)$. Letting $l \to \infty$, we get $\mathfrak{M} + 1 \leq \mathfrak{d}(1 + \mathfrak{M}) < 1 + \mathfrak{M}$, which clearly leads to a contradiction. By (3.1.3), we have $\mathcal{C}(\mathfrak{h}(\zeta_{n-1}), \mathfrak{h}\zeta_n, \theta) = \mathcal{C}(\zeta_n, \zeta_{n+1}, \theta) \leq \mathfrak{d}^n \mathcal{C}(\zeta_0, \zeta_1, \theta)$.
Therefore, we can conclude that

$$\lim_{n\to\infty} \mathcal{C}(\zeta_n, \zeta_{n+1}, \theta) = 0, \theta > 0. \tag{3.2.14}$$

To show that $\{\zeta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, let us proceed by contradiction. Suppose the opposite, that there exist $\epsilon \in (0, \mathfrak{N}), \mathfrak{N} \leq 1, \theta_0 > 0$ and sequences $\{n_l\}$ and $\{m_l\}$ such that $m_l > n_l > l$, $\forall\ l \in \mathbb{N}$ and the following inequalities hold:

$$\mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \geq \mathfrak{N} + \epsilon, l \in \mathbb{N}, \tag{3.2.15}$$

and

$$\mathcal{C}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0) < \mathfrak{N} + \epsilon, l \in \mathbb{N}. \tag{3.2.16}$$

From (3.2.15), we have:

$$\lim_{l\to\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \geq \mathfrak{N} + \epsilon. \tag{3.2.17}$$

Applying (NMLS16), for any $l \in \mathbb{N}$ and $p \in (0, \theta_0)$, we have:

$$\mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \leq \mathcal{C}(\zeta_{m_l}, \zeta_{m_l-1}, p) \Diamond \mathcal{C}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0 - p). \tag{3.2.18}$$

Taking the limit as $p \to 0^+$ in the inequality (3.2.18) and using continuity of $\Diamond$ along with (3.1.3), we can infer the following:

$$
\begin{aligned}
\lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) &= \lim_{l\to+\infty} \left(\lim_{p\to0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \right) \\
&\leq \lim_{l\to+\infty} \left(\lim_{p\to0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{m_l-1}, p) \Diamond \mathcal{C}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0 - p) \right) \\
&= \lim_{l\to+\infty} \left(\lim_{p\to0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{m_l-1}, p) \right) \Diamond \lim_{l\to+\infty} \left(\lim_{p\to0^+} \mathcal{B}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0 - p) \right) \\
&= 0 \Diamond \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0) \\
&= \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l-1}, \zeta_{n_l}, \theta_0) \\
&\leq \mathfrak{N} + \epsilon,
\end{aligned}
$$

and together with (3.2.17), this implies that $\lim_{l\to\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) = \mathfrak{N} + \epsilon$.
Let us prove that $\lim_{l\to\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) = \mathfrak{N} + \epsilon$. By (3.2.3) and (3.2.16), we can establish that:

$$\begin{aligned}
\lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) &= \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) \right) \\
&\leq \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0 - p) \diamond \mathcal{C}(\zeta_{n_l}, \zeta_{n_l+1}, p) \right) \\
&\leq \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \diamond 0 \\
&= \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \\
&= \mathfrak{M} + \epsilon.
\end{aligned}$$

Additionally, based on (3.2.3) and (3.2.16),

$$\begin{aligned}
\mathfrak{N} + \epsilon &= \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0) \right) \\
&\leq \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0 - p) \diamond \mathcal{C}(\zeta_{n_l+1}, \zeta_{n_l}, p) \right) \\
&\geq \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) \diamond 0 \\
&= \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0).
\end{aligned}$$

Therefore, we can conclude that $\lim_{l\to+\infty} \mathcal{B}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) = \mathfrak{N} + \epsilon$.
Now,

$$\begin{aligned}
\lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) &= \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) \right) \\
&\leq \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l+1}, \zeta_{m_l}, p) \diamond \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0 - p) \right) \\
&= \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l+1}, \zeta_{m_l}, p) \right) \diamond \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0 - p) \right) \\
&= 0 \diamond \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) \\
&= \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) \\
&= \mathfrak{N} + \epsilon,
\end{aligned}$$

However, it is evident that,

$$\begin{aligned}
\mathfrak{N} + \epsilon &= \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{n_l+1}, \theta_0) \right) \\
&\leq \lim_{l\to+\infty} \left(\lim_{p\to 0^+} \mathcal{C}(\zeta_{m_l}, \zeta_{m_l+1}, \theta_0 - p) \diamond \mathcal{C}(\zeta_{m_l+1}, \zeta_{n_l+1}, p) \right) \\
&\leq 0 \diamond \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) \\
&= \lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0).
\end{aligned}$$

So, $\lim_{l\to+\infty} \mathcal{C}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) = \mathfrak{N} + \epsilon$. Under the contractive condition for $\mathcal{C}$, it holds that $\mathcal{C}(\zeta_{m_l+1}, \zeta_{n_l+1}, \theta_0) \leq \mathfrak{d}\mathcal{C}(\zeta_{m_l}, \zeta_{n_l}, \theta_0)$. As $l \to \infty$, this leads to $\mathfrak{N} + 1 \leq \mathfrak{d}(1 + \mathfrak{N}) < 1 + \mathfrak{N}$, results in a clear contradiction.

In all three scenarios, $\{\zeta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, ensuring the existence of $\zeta \in \Xi$ such that $\lim_{n\to\infty} \zeta_n = \zeta$.
As a consequence, the following hold:
$\lim_{n\to\infty} \mathcal{A}(\zeta_n, \zeta, \theta) = \mathcal{A}(\zeta, \zeta, \theta) = \lim_{n,m\to\infty} \mathcal{A}(\zeta_n, \zeta_m, \theta)$
$\lim_{n\to\infty} \mathcal{B}(\zeta_n, \zeta, \theta) = \mathcal{B}(, \zeta, , \zeta, \theta) = \lim_{n,m\to\infty} \mathcal{B}(\zeta_n, \zeta_m, \theta)$ and
$\lim_{n\to\infty} \mathcal{C}(\zeta_n, \zeta, \theta) = \mathcal{C}(, \zeta, \zeta, \theta) = \lim_{n,m\to\infty} \mathcal{C}(\zeta_n, \zeta_m, \theta)$.
Additionally, it follows that $\frac{1}{\mathcal{A}(\zeta_n,\zeta_m,\theta)} - 1 \leq \mathfrak{d}\left[\frac{1}{\mathcal{A}(\zeta_{n-1},\zeta_{m-1},t)} - 1\right]$. As $n, m \to \infty$, this implies
$\frac{1}{\mathcal{A}(\,\zeta,\zeta,\theta)} - 1 \leq \mathfrak{d}\left[\frac{1}{\mathcal{A}(\zeta,\zeta,\theta)} - 1\right]$, which holds only if $\mathcal{A}(, \zeta, , \zeta, \theta) = 1$.
Similarly from $\mathcal{B}(\zeta_m, \zeta_n, \theta) \leq \mathfrak{d}\mathcal{B}(\zeta_{m-1}, \zeta_{n-1}, \theta)$ as $n, m \to \infty$, we derive $\mathcal{B}(\zeta, \zeta, \theta) \leq \mathfrak{d}\mathcal{B}(\zeta, \zeta, \theta)$, which is possible only if $\mathcal{B}(\zeta, \zeta, t) = 0$ and $\mathcal{C}(\zeta_m, \zeta_n, \theta) \leq \mathfrak{d}\mathcal{C}(\zeta_{m-1}, \zeta_{n-1}, \theta)$ as $n, m \to \infty$, we arrive at:
$\mathcal{C}(\zeta, \zeta, t) \leq \mathfrak{d}\mathcal{C}(\zeta, \zeta, t)$, which holds only if $\mathcal{C}(\zeta, \zeta, \theta) = 0$.
Thus, we conclude

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{A}(\zeta_n, \zeta, \theta) = \mathcal{A}(\zeta, \zeta, \theta) = \lim_{n,m\to\infty} \mathcal{A}(\zeta_n, \zeta_m, \theta) = 1,\\
&\lim_{n\to\infty} \mathcal{B}(\zeta_n, \zeta, \theta) = \mathcal{B}(\zeta, \zeta, \theta) = \lim_{n,m\to\infty} \mathcal{B}(\zeta_n, \zeta_m, \theta) = 0 \text{ and}\\
&\lim_{n\to\infty} \mathcal{C}(\zeta_n, \zeta, \theta) = \mathcal{C}(\zeta, \zeta, \theta) = \lim_{n,m\to\infty} \mathcal{C}(\zeta_n, \zeta_m, \theta) = 0.
\end{aligned}$$

We now prove that ζ is a fixed point of $\mathfrak{h}$. To do this, we use Definition(3.1) to establish the following inequalities

$$\frac{1}{\mathcal{A}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta)} - 1 \leq \mathfrak{d}\left[\frac{1}{\mathcal{A}(\zeta, \varphi, \theta)} - 1\right], \mathcal{B}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta) \leq \mathfrak{d}\mathcal{B}(\zeta, \varphi, \theta), \mathcal{C}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta) \leq \mathfrak{d}\mathcal{C}(\zeta, \varphi, \theta). \tag{3.2.19}$$

From these inequalities, we derive

$$\begin{aligned}
\mathcal{A}\left(\zeta, \mathfrak{h}\left(\zeta\right), \theta\right) &\geq \mathcal{A}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) * \mathcal{A}\left(\zeta_{n+1}, \mathfrak{h}\left(\zeta\right), \frac{\theta}{2}\right)\\
&= \mathcal{A}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) * \mathcal{A}\left(\mathfrak{h}\left(\zeta_n\right), \mathfrak{h}\left(\zeta\right), \frac{\theta}{2}\right)\\
&\geq \mathcal{A}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) * \frac{1}{\frac{\mathfrak{d}}{\mathcal{A}\left(\zeta_n, \zeta, \frac{\theta}{2}\right)} + 1 - \mathfrak{d}}\\
\mathcal{B}\left(\zeta, \mathfrak{h}\left(\zeta\right), \theta\right) &\leq \mathcal{B}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) \diamond \mathcal{B}\left(\zeta_{n+1}, \mathfrak{h}\left(\zeta\right), \frac{\theta}{2}\right)\\
&= \mathcal{B}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) \diamond \mathcal{B}\left(\mathfrak{h}\left(\zeta_n\right), \mathfrak{h}\left(\zeta\right), \frac{\theta}{2}\right)\\
&\leq \mathcal{B}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) \diamond \mathfrak{d}\mathcal{B}\left(\zeta_n, \zeta, \frac{\theta}{2}\right) \text{ and}\\
\mathcal{C}\left(\zeta, \mathfrak{h}\left(\zeta\right), \theta\right) &\leq \mathcal{C}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) \diamond \mathcal{C}\left(\zeta_{n+1}, \mathfrak{h}\left(\zeta\right), \frac{\theta}{2}\right)\\
&= \mathcal{C}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) \diamond \mathcal{C}\left(\mathfrak{h}\left(\zeta_n\right), \mathfrak{h}\left(\zeta\right), \frac{\theta}{2}\right)\\
&\leq \mathcal{C}\left(\zeta, \zeta_{n+1}, \frac{\theta}{2}\right) \diamond \mathfrak{d}\mathcal{C}\left(\zeta_n, \zeta, \frac{\theta}{2}\right)
\end{aligned}$$

Letting $n \to \infty$ and applying (3.2.19), (3.2.20) and (3.2.21), we obtain,
$\mathcal{A}(\zeta, \mathfrak{h}(\zeta), \theta) = 1, \mathcal{B}(\zeta, \mathfrak{h}(\zeta), \theta) = 0$ and $\mathcal{C}(\zeta, \mathfrak{h}(\zeta), \theta) = 0$.
Thus $\mathfrak{h}(\zeta) = \zeta$, confirming that ζ is fixed point of $\mathfrak{h}$ and $\mathcal{A}(\zeta, \zeta, \theta) = 1, \mathcal{B}(\zeta, \zeta, \theta) = 0and\mathcal{C}(\zeta, \zeta, \theta) = 0$.
Next, we prove the uniqueness of the fixed point ζ of $\mathfrak{h}$.
Suppose there exists another fixed point $\mathfrak{y}$ of $\mathfrak{h}$ such that
$\mathcal{A}(\zeta, \mathfrak{y}, \theta) < 1, \mathcal{B}(\zeta, \mathfrak{y}, \theta) > 0and\mathcal{C}(\zeta, \mathfrak{y}, \theta) > 0$, for some $\theta > 0$.
From the definition of (3.1), we have the following inequalities

$$\frac{1}{\mathcal{A}(\zeta, \mathfrak{y}, \theta)} - 1 = \frac{1}{\mathcal{A}(\mathfrak{h}(\zeta), \mathfrak{h}(\mathfrak{y}), \theta)} - 1 \leq \mathfrak{d}[\frac{1}{\mathcal{A}(\zeta, \mathfrak{y}, \theta)} - 1] < \frac{1}{\mathcal{A}(\zeta, \mathfrak{y}, \theta)} - 1,$$
$$\mathcal{B}(\zeta, \mathfrak{y}, \theta) = \mathcal{B}(\mathfrak{h}(\zeta), \mathfrak{h}(\mathfrak{y}), \theta) \leq \mathfrak{d}\mathcal{B}(\zeta, \mathfrak{y}, \theta) < \mathcal{B}(\zeta, \mathfrak{y}, \theta) \text{ and}$$
$$\mathcal{C}(\zeta, \mathfrak{y}, \theta) = \mathcal{C}(\mathfrak{h}(\zeta), \mathfrak{h}(\mathfrak{y}), \theta) \leq \mathfrak{d}\mathcal{C}(\zeta, \mathfrak{y}, \theta) < \mathcal{C}(\zeta, \mathfrak{y}, \theta).$$

This leads to a contradiction, as the inequality cannot hold simultaneously. Therefore, we conclude that $\mathcal{A}(\zeta, \mathfrak{y}, \theta) = 1, \mathcal{B}(\zeta, \mathfrak{y}, \theta) = 0$ and $\mathcal{C}(\zeta, \mathfrak{y}, \theta) = 0, \forall \theta > 0$. Thus, ζ is a unique fixed point of $\mathfrak{h}$. □

Example 3.3. Let $\Xi = [0, 2], a * b = ab$ and $a \diamond b = \max\{a, b\}$. Define the neutrosophic sets as follows: $\mathcal{A}(\zeta, \varphi, \theta) = e^{-\frac{\max\{\zeta,\varphi\}}{\theta}}, \mathcal{B}(\zeta, \varphi, \theta) = 1 - e^{-\frac{\max\{\zeta,\varphi\}}{\theta}}$, and $\mathcal{C}(\zeta, \varphi, \theta) = e^{\frac{\max\{\zeta,\varphi\}}{\theta}} - 1$ for all $\zeta, \varphi \in \Xi, \theta > 0$. Then, $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \diamond)$ forms a complete NMLS. Consider the mapping $\mathfrak{h}$ given by
$\mathfrak{h}(\zeta) = \begin{cases} 0, & \zeta = 1 \\ \frac{\zeta}{2}, & \zeta \in [0, 1) \\ \frac{\zeta}{4}, & \zeta \in (1, 2] \end{cases}$.
$\frac{\max\{\zeta/2,\varphi/2\}}{\theta} \leq \frac{\max\{\zeta,\varphi\}}{\theta}$
$\Rightarrow \frac{1}{e^{-\frac{\max\{\zeta/2,\varphi/2\}}{\theta}}} - 1 \leq \frac{1}{e^{-\frac{\max\{\zeta,\varphi\}}{\theta}}} - 1$
$\Rightarrow \frac{1}{\mathcal{A}(\zeta,\mathfrak{y},\theta)} - 1 \leq \mathfrak{d}(\frac{1}{\mathcal{A}(\mathfrak{h}(\zeta),\mathfrak{h}(\mathfrak{y})})$
$\frac{\max\{\zeta/2,\varphi/2\}}{\theta} \leq \frac{\max\{\zeta,\varphi\}}{\theta}$.
$\Rightarrow e^{-\frac{\max\{\zeta/2,\varphi/2\}}{\theta}} \geq e^{-\frac{\max\{\zeta,\varphi\}}{\theta}}$
$\Rightarrow 1 - e^{-\frac{\max\{\zeta/2,\varphi/2\}}{\theta}} \leq 1 - e^{-\frac{\max\{\zeta,\varphi\}}{\theta}}$
$\mathcal{B}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta) \leq \mathfrak{d}\mathcal{B}(\zeta, \varphi, \theta)$.
$\Rightarrow \mathcal{B}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta) \leq \mathfrak{d}\mathcal{B}(\zeta, \varphi, \theta)$
Similarly, $\mathcal{C}(\mathfrak{h}(\zeta), \mathfrak{h}(\varphi), \theta) \leq \mathfrak{d}\mathcal{C}(\zeta, \varphi, \theta)$.
Therefore, $\mathfrak{h}$ is a neutrosophic contractive function for $\frac{1}{2} \leq \mathfrak{d} < 1$. By Theorem (3.2), $\mathfrak{h}$ has a unique fixed point.

4. Application

Let $\Xi = C([\xi, \varrho], \mathbb{R})$ denote the collection of all the real-valued continuous functions defined on $[\xi, \varrho]$. Now, consider the integral equation given below.

$$\delta(\pi) = \eta(e) + \beta \int_{\chi}^{\mu} F(\pi, e)\delta(\pi)\Gamma e \text{ for } \pi, e \in [\chi, \mu], \tag{I}$$

where $\eta(e) \in C([\xi, \varrho], \mathbb{R})$, F $\in \Xi$. Define the functions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as follows:

$$\mathcal{A}(\delta(\pi), \varrho(\pi), \theta) = \sup_{\pi \in [\chi,\mu]} \frac{\theta}{\theta + \max\{\delta(\pi), \varrho(\pi)}},$$
$$\mathcal{B}(\delta(\pi), \varrho(\pi), \theta) = 1 - \sup_{\pi \in [\chi,\mu]} \frac{\theta}{\theta + \max\{\delta(\pi), \varrho(\pi)\}} \text{ and}$$
$$\mathcal{C}(\delta(\pi), \varrho(\pi), \theta) = \inf_{\pi \in [\chi,\mu]} \frac{\theta}{\theta + \max\{\delta(\pi), \varrho(\pi)\}} - 1 \,\forall\, \delta, \varrho \in \Xi \text{ and } \theta > 0.$$

Define $\varsigma * \varphi = \min\{\varsigma.\varphi\}$ and $\varsigma\Diamond\varphi = \max\{\varsigma, \varphi\}$. Then, let $(\Xi, \mathcal{A}, \mathcal{B}, \mathcal{C}, *, \Diamond)$ be a complete NMLS.

Theorem 4.1. Let $\max\{F(\pi, e)\varphi(\pi), F(\pi, e)\varrho(\pi)\}^2 \leq \max\{\varphi(\pi), \varrho(\pi)\}^2 \ \forall \ \varphi, \varrho \in \Xi, \varpi \in (0, 1)$ and $\forall \ \pi, e \in [\chi, \mu]$. Furthermore, assume that $\left(\beta \int_\chi^\mu \Gamma e\right)^2 \leq \varpi < 1$. Then the integral Equation (I) has a unique solution.

Proof
Define $\mathfrak{f} : \Xi \to \Xi$ as $\mathfrak{f}\delta(\pi) = \eta(e) + \beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e$ for all $\pi, e \in [\chi, \mu]$.
For any $\delta, \varrho \in \Xi$, we compute

$$\begin{aligned}
\mathcal{A}(\mathfrak{f}\delta(\pi), \mathfrak{f}\varrho(\pi), \varpi\theta) &= \sup_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\{\mathfrak{f}\delta(\pi), \mathfrak{f}\varrho(\pi)\}} \\
&= \sup_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\left\{\eta(e) + \beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e, \eta(e) + \beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e\right\}} \\
&= \sup_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\left\{\beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e, \beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e\right\}} \\
&= \sup_{\pi\in[\chi,\mu]} \frac{\varpi t}{\varpi\theta + \max\{F(\pi, e)\delta(\pi), F(\pi, e)\varrho(\pi)\} \left(\beta \int_\chi^\mu \Gamma e\right)^2} \\
&\geq \sup_{\pi\in[\chi,\mu]} \frac{\theta}{\theta + \max\{\delta(\pi), \varrho(\pi)\}} \\
&\geq \mathcal{A}(\delta(\pi), \varrho(\pi), \theta). \\
\mathcal{A}(\mathfrak{f}\delta(\pi), \mathfrak{f}\varrho(\pi), \varpi\theta) &\geq \mathcal{A}(\delta(\pi), \varrho(\pi), \theta) \\
\Rightarrow \frac{1}{\mathcal{A}(\mathfrak{f}\delta(\pi), \mathfrak{f}\varrho(\pi), \varpi\theta)} - 1 &\leq \frac{1}{\mathcal{A}(\delta(\pi), \varrho(\pi), \theta)} - 1 \\
\mathcal{B}(\mathfrak{f}\delta(\pi), \mathfrak{f}\varrho(\pi), \varpi\theta) &= 1 - \sup_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\{\mathfrak{f}\delta(\pi), \mathfrak{f}\varrho(\pi)\}} \\
&= 1 - \sup_{\pi\in[\chi,\mu]} \frac{\varpi t}{\varpi\theta + \max\left\{\eta(e) + \beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e, \eta(e) + \beta \int_\chi^\mu F(\pi, e)\varrho(\pi)\Gamma e\right\}} \\
&= 1 - \sup_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\left\{\beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e, \beta \int_\chi^\mu F(\pi, e)\varrho(\pi)\Gamma e\right\}} \\
&= 1 - \sup_{\pi\in[\chi,\mu]} \frac{\delta\theta}{\varpi\theta + \max\{F(\pi, e)\delta(\pi), F(\pi, e)\varrho(\pi)\} \left(\beta \int_\chi^\mu \Gamma e\right)} \\
&\leq 1 - \sup_{\pi\in[\chi,\mu]} \frac{\theta}{\theta + \max\{\delta(\pi), \varrho(\pi)\}} \\
&\leq \mathcal{B}(\delta(\pi), \varrho(\pi), \theta), \\
\mathcal{C}(\mathfrak{f}\delta(\pi), \mathfrak{f}\varrho(\pi), \varpi\theta) &= \inf_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\{\mathfrak{f}\delta(\pi), \mathfrak{f}\varrho(\pi)\}} - 1 \\
&= \inf_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\left\{\eta(e) + \beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e, \eta(e) + \beta \int_\chi^\mu F(\pi, e)\varrho(\pi)\Gamma e\right\}} - 1 \\
&= \inf_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\left\{\beta \int_\chi^\mu F(\pi, e)\delta(\pi)\Gamma e, \beta \int_\chi^\mu F(\pi, e)\varrho(\pi)\Gamma e\right\}} - 1
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\{F(\pi,e)\delta(\pi), F(\pi,e)\varrho(\pi)\}\left(\beta\int_\chi^\mu \Gamma e\right)} - 1 \\
&\leq \inf_{\pi\in[\chi,\mu]} \frac{\varpi\theta}{\varpi\theta + \max\{\delta(\pi), \varrho(\pi)\}} - 1 \\
&\leq \mathcal{C}(\delta(\pi), \varrho(\pi), \theta).
\end{aligned}
$$

As all requirements of Theorem(3.2) are fulfilled. the integral equation admits a unique solution. □

5. conclusion

In this paper, we have successfully introduced the notion of neutrosophic metric-like spaces as a natural extension of both intuitionistic fuzzy metric spaces and classical metric-like spaces. The developed framework provides greater flexibility for handling uncertainty and indeterminacy in mathematical analysis. Fundamental properties such as sequence convergence, G-Cauchy sequences, and contractive mappings have been rigorously explored, leading to new fixed-point theorems under the neutrosophic setting. Illustrative examples confirm the validity of the theoretical results, while potential applications highlight the wider relevance of the proposed structure in advancing fuzzy and neutrosophic mathematical modeling.

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