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On Properties and Applications of Bimodal Skew Half Normal Distribution

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Abstract In this paper, we develop a modified half-normal distribution capable of accommodating datasets that exhibit bimodal patterns. To demonstrate its usefulness, we investigate several key structural characteristics of the proposed model through graphical techniques. Parameter estimation is carried out via the maximum likelihood approach. We also conduct a simulation study to evaluate the performance of the proposed estimators under various parameter settings. The model is subsequently applied to real-world datasets to assess its practical performance and flexibility. Finally, likelihood ratio tests are employed to compare the new distribution against existing competing distributions.

Keywords Half Normal Distribution, likelihood ratio test, maximum likelihood approach, simulation study, bimodal

AMS 2010 subject classifications 62E15, 62F35

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1. Introduction

The half-normal (HN) distribution is a continuous probability distribution derived by folding the standard normal (N) distribution around zero, considering only the absolute values. It is a special case of the folded N distribution where the mean is zero. The probability density function (PDF) of the HN distribution is given as

$$h(x) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}. \tag{1}$$

The HN distribution is widely used in statistical modeling and applied research due to its ability to describe non-negative, right-skewed data. In quality control, it analyzes absolute deviations from target values, ensuring process consistency. The distribution also serves as a prior in Bayesian statistics, particularly for variance parameters, due to its constrained positive support. Additionally, it appears in reliability engineering to model failure times and in signal processing for noise magnitude analysis. Its mathematical tractability and connection to the N distribution make it a practical choice for scenarios where only positive deviations are relevant, such as measurement errors and financial volatility studies.

Several researchers have continuously worked on the HN distribution, leading to various modifications, extensions, and generalizations. For example, [14] proposed a new extension of the HN distribution, called the slashed HN distribution, and explored its main properties, including higher kurtosis. The study of [9] introduced

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another new HN distribution as a two-parameter lifetime model derived from static fatigue considerations, with a cumulative distribution function (CDF) resembling that of the HN distribution. Again, [15] introduced the beta generalized HN distribution as a flexible extension of the HN and generalized HN models. [13] discussed the discrete HN distribution, which can be obtained as the maximum entropy distribution on non-negative integers under fixed mean and variance, and also appears as a limiting case of a q-hyper-Poisson-I model arising in queueing theory. [19] introduced a new family of positive-support probability distributions called the Modified-HN (MHN) distribution and discussed several of its key structural properties. [10] introduced and studied the Kumaraswamy generalized half-normal (KGHN) distribution for modeling positively skewed data, which includes the HN and generalized HN distributions as special cases. Besides these works, there are other notable contributions in the literature, including [17], [6], [4], [8], [3], [7], [21], [16], [2], [29] among others.

On the other hand, [5] introduced the path-breaking skew normal (SN) distribution which allows fitting data with unimodal asymmetric character. The PDF of the SN distribution is

$$f(x;\lambda) = 2\phi(x)\Phi(\lambda x), \qquad x \in \mathbb{R},\ \lambda \in \mathbb{R}, \tag{2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the PDF and CDF of the standard N distribution, respectively. This distribution has wide-ranging applications across areas such as financial risk analysis, epidemiology, climate and environmental science, medical research, industrial quality control, insurance, and various other applied disciplines. The pioneering concept of the SN distribution [5] inspired the development of numerous skewed distribution families, leading to significant advancements in statistical modeling. These developments demonstrate the ongoing evolution and diversification of skew distributions in statistical theory.

In recent years, researchers have shown growing interest in probability distribution models capable of capturing unimodality, bimodality, and even multimodality. Several authors have proposed new families of distributions to accommodate bimodal data patterns. For instance, [12] introduced a class of bimodal symmetric and asymmetric distributions by extending Azzalini's SN model [5]. [11] proposed another distribution namely alpha-skew-normal (ASN) with the ability to fit uni-bimodal asymmetric data. Their work led to the development of a two-parameter bimodal SN distribution tailored for bimodal data. These advancements provide flexible tools for modeling complex data structures in statistical applications. Some other advanced model towards this study results in a number of bimodal distribution namely bimodal Tanh skew-normal distribution [22] , alpha skew-logistic [28], alpha skew-Laplace [26], and generalized alpha skew-normal [27] distributions, Balakrishnan alpha skew-normal [23], Balakrishnan alpha skew-logistic [24], and Balakrishnan alpha skew-Laplace [25] etc.

Drawing on the theoretical foundation presented in [11], this study formulates a new variant of the HN distribution. The newly proposed distribution differs in important ways from related families such as some bimodal skew-normal distribution and other generalization of HN distributions which are discussed earlier. The bimodal extension of SN distribution extends the SN model by allowing bimodality, but it is defined over the entire real line. In contrast, the newly proposed distribution is supported only on the positive real axis, making it particularly suitable for modeling non-negative data arising in areas such as reliability analysis, survival studies, and environmental sciences. On the other hand, some existing generalization of HN distribution offers additional flexibility in skewness and tail behavior but does not accommodate bimodality. Our newly proposed distribution thus integrates the advantages of both approaches by simultaneously modeling skewness and bimodality on a positive support, thereby filling an important gap in the statistical literature. This paper also studies some important statistical properties of the proposed distribution followed by a characterization result. Besides, parameter estimation of the new distribution is also included. A simulation study is also conducted to observe the behavior of the estimated parameters. Moreover, the flexibility of the proposed model over some other counterparts is demonstrated through applications to real-life datasets.

2. A Bimodal Extension of Half Normal Distribution

Theorem 2.1. A continuous random variable X is said to follow the bimodal skew half-normal (BSHN) distribution when its PDF is defined as

$$f(x) = \frac{\left[1 + (1 - \alpha x)^2\right]}{K(\alpha)} h(x); \quad x > 0, \tag{3}$$

where, $\alpha \in R$ and $h(x)$ is the PDF of half N distribution while normalizing constant $K(\alpha)$ is evaluated as

$$\begin{aligned} K(\alpha) &= \int_0^\infty \left[(1 - \alpha x)^2 + 1\right] h(x)dx \\ &= \int_0^\infty \left[\alpha^2 x^2 - 2\alpha x + 2\right] h(x)dx \\ &= \alpha^2 \int_0^\infty x^2 h(x)dx - 2\alpha \int_0^\infty x h(x)dx + 2\int_0^\infty h(x)dx \\ &= \alpha^2 - 2\sqrt{\frac{2}{\pi}}\alpha + 2. \end{aligned}$$

It is denoted as $X \sim BSHN(\alpha)$.

Theorem 2.2. The corresponding CDF of $BSHN(\alpha)$ distribution is given by

$$F(x) = \frac{1}{K(\alpha)}\left[(2 + \alpha^2)H(x) - \sqrt{2/\pi}\left\{2\alpha(1 - e^{-x^2/2}) + \alpha^2 e^{-x^2/2}\right\}\right], \tag{4}$$

where, $H(\cdot)$ is the CDF of HN distribution.
Proof. Using the PDF provided in equ. (3), CDF can be defined as

$$\begin{aligned} F(x) &= \frac{1}{K(\alpha)} \int_0^x \left[(1 - \alpha x)^2 + 1\right] h(x)dx \\ &= \frac{1}{K(\alpha)}\left[2\int_0^x h(x)dx - 2\alpha \int_0^x x h(x)dx + \alpha^2 \int_0^x x^2 h(x)dx\right] \\ &= \frac{1}{K(\alpha)}\left[(2 + \alpha^2)H(x) - \sqrt{2/\pi}\left\{2\alpha(1 - e^{-x^2/2}) + \alpha^2 e^{-x^2/2}\right\}\right]. \end{aligned}$$

Remark 2.1. Some particular results of the BSHN distribution are given as:

i. If $\alpha = 0$, then BSHN distribution reduces to half N distribution.
ii. If $X \sim BSHN(\alpha)$, then $-X \sim BSHN(-\alpha)$.

Remark 2.2. The PDF of BSHN distribution has atmost two modes.
Proof. Differentiating PDF of BSHN given in eqn. (3), we obtain

$$\begin{aligned} f'(x) &= \frac{\sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} \left(2\alpha + x\left(\alpha^2\left(x^2 - 2\right) - 2\alpha x + 2\right)\right)}{-\alpha^2 + 2\sqrt{\frac{2}{\pi}}\alpha - 2} \\ &= \frac{h(x)}{K(\alpha)}\Big[-\alpha^2 x^3 - 2\alpha + 2\alpha x^2 + 2\alpha^2 x - 2x\Big] \end{aligned}$$

Therefore, $f'(x)$ has at most three zeros, and hence the density function of BSHN has at most two modes.

Remark 2.3. Note on Modality. As shown in Figure 1, the phase diagram and the corresponding density plots clearly states the modality behavior of the BSHN distribution as given below:

- **Unimodal Region ($\alpha \leq \alpha^*$):** For parameter values $\alpha \leq 1.476$, the distribution exhibits unimodal behavior with a single peak located near the origin. As α increases within this range, the distribution becomes more dispersed while maintaining its unimodal character.
- **Critical Threshold ($\alpha = \alpha^*$):** At the critical value $\alpha^* \approx 1.476$, the distribution undergoes a phase transition. The density function at this point displays an inflection characteristic of the unimodal-to-bimodal transition.
- **Bimodal Region ($\alpha > \alpha^*$):** For $\alpha > 1.476$, the distribution manifests clear bimodality. The primary mode remains near the origin, while a secondary mode emerges at higher x-values. As α increases beyond the threshold, the bimodality becomes more pronounced.

This well-defined modality transition underscores the flexibility of the BSHN distribution, enabling it to model both conventional unimodal positive data and more complex bimodal patterns encountered in practical applications.

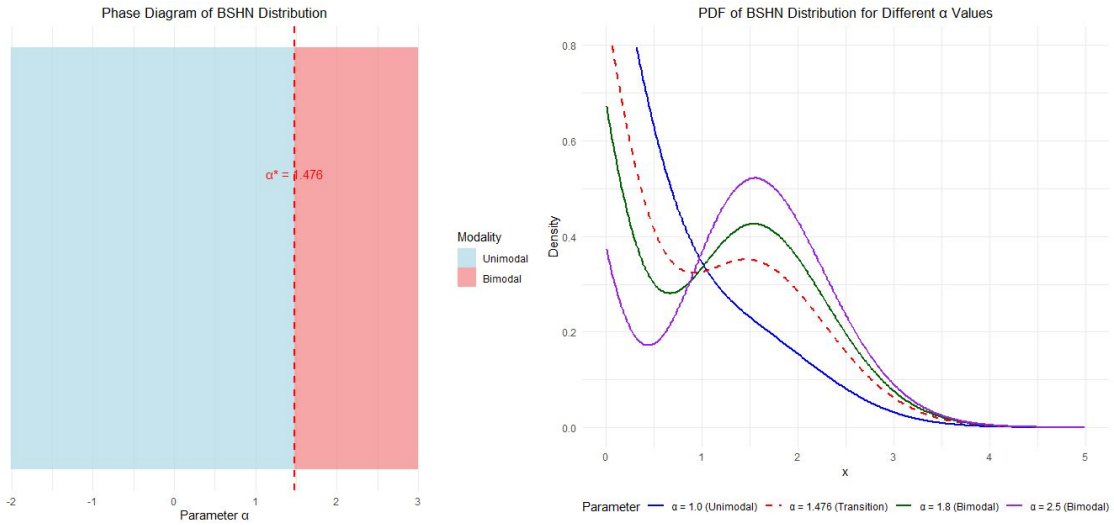


Figure 1. Modality Analysis of BSHN Distribution

Again, Graphical illustrations of the PDF of $BSHN(\alpha)$ distribution for different selection of the parameters are given in Fig. 2. Similarly, graphs of the CDF of $BSHN(\alpha)$ distribution for different parameter values are plotted in Fig. 3. From Fig. 2(a), it can be seen that for larger positive values the distribution exhibits bimodality. As α increases, the second mode becomes more pronounced, and the overall distribution becomes more spread out. Again the transition of the PDF from unimodal to bimodal behavior is observed for some other moderate positive values of α in Fig. 3 (b). Similarly Fig. 3 (c) reflects that for some negative value of α, the distribution remains unimodal but becomes more concentrated near the origin. As α decreases, the peak height increases and the tail shortens, indicating stronger concentration of probability mass near lower values of x. Finally, Fig. 3 (d) shows that the distribution evolves from a unimodal shape to clear bimodality at higher α. Larger positive α generates a stronger second peak and heavier right-tail behavior. Overall, the parameter α plays a crucial role in controlling the modality and skewness of the distribution. Negative α leads to unimodal, highly concentrated densities, while positive α induces bimodality and greater dispersion, making the model flexible for fitting data with diverse shapes.

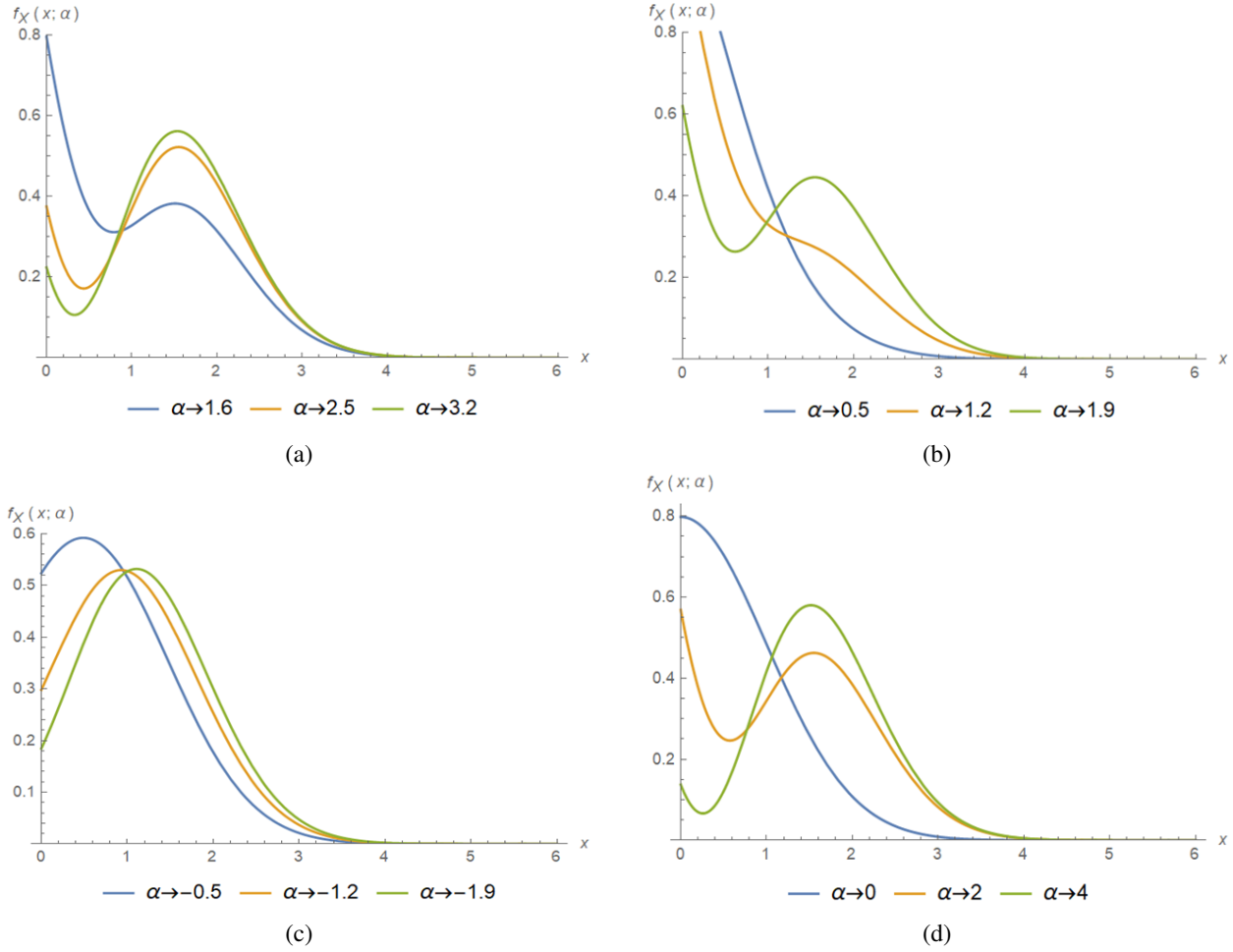


Figure 2. Pictorial view of the density function

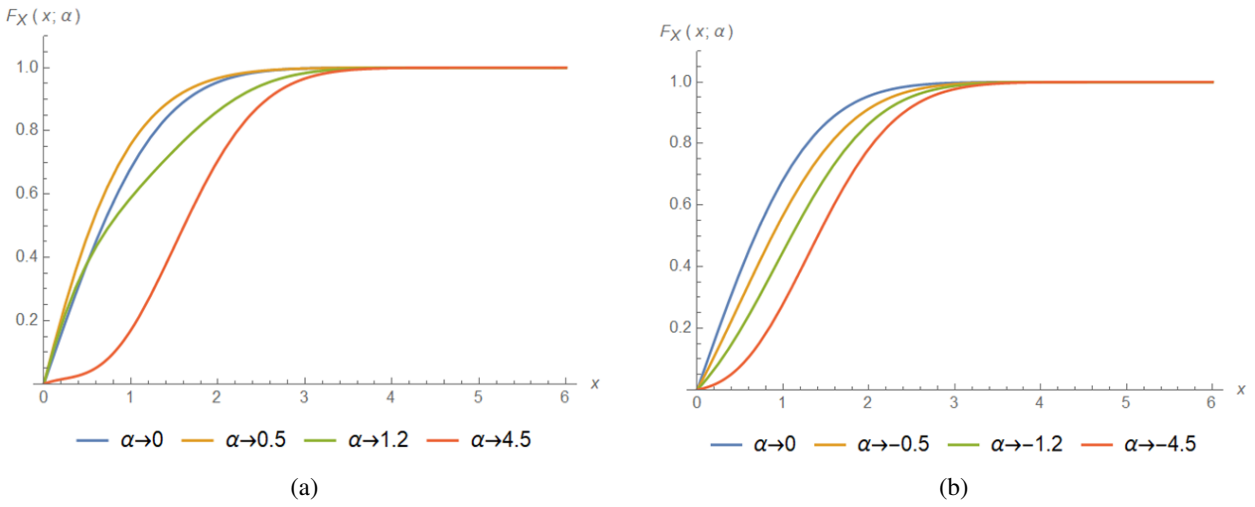


Figure 3. Pictorial View of the CDF.

3. Structural Properties of the New Distribution

This section discusses, with some graphical examples, several significant distributional features of the BSHN distribution.

3.1. Moments and Incomplete Moments

Theorem 3.1.1. The nth order moment of $BSHN(\alpha)$ distribution is calculated as

$$E(X^n) = \frac{1}{\sqrt{\pi}K(\alpha)}\left[2^{\frac{n}{2}+\frac{1}{2}}\left\{-n\alpha\Gamma\left(\frac{n}{2}\right) + \frac{1}{\sqrt{2}}\left(\alpha^2(n+1)+2\right)\Gamma\left(\frac{n+1}{2}\right)\right\}\right]. \tag{5}$$

Proof. Using the PDF provided in equ. (3), n^{th} order moment is defined as

$$\begin{aligned} E(X^n) &= \int_0^\infty x^n f(x)dx \\ &= \frac{1}{K(\alpha)}\left[2\int_0^\infty x^n h(x)dx - 2\alpha\int_0^\infty x^{n+1}h(x)dx + \alpha^2\int_0^\infty x^{n+2}h(x)dx\right] \\ &= \frac{1}{K(\alpha)}\Big[2I_1 - 2\alpha I_2 + \alpha^2 I_3\Big]. \end{aligned} \tag{6}$$

Here, I_1 is the nth order moment of half N distribution while I_2 and I_3 can be calculated using the integrated by parts and results of the same can be obtained as

$$\begin{aligned} I_1 &= \int_0^\infty x^n h(x)dx = \frac{2^{\frac{n-1}{2}+\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}, \\ I_2 &= \int_0^\infty x^{n+1} h(x)dx = \frac{2^{\frac{n}{2}+\frac{1}{2}}\Gamma\left(\frac{n}{2}+1\right)}{\sqrt{\pi}}, \\ I_3 &= \int_0^\infty x^{n+2} h(x)dx = \frac{2^{\frac{n+1}{2}+\frac{1}{2}}\Gamma\left(\frac{n+3}{2}\right)}{\sqrt{\pi}}. \end{aligned}$$

Substituting the value of the integrations in equ. (6), final expression of the BSHN distribution's moment can be expressed as

$$E(X^n) = \frac{1}{\sqrt{\pi}K(\alpha)}\left[2^{\frac{n}{2}+\frac{1}{2}}\left\{-n\alpha\Gamma\left(\frac{n}{2}\right) + \frac{1}{\sqrt{2}}\left(\alpha^2(n+1)+2\right)\Gamma\left(\frac{n+1}{2}\right)\right\}\right].$$

Remark 3.1.1. By substituting $n = 1, 2, 3, 4$ into equation (5), the first four raw moments of the BSHN distribution are obtained as follows

$$\begin{aligned} E(X) &= \frac{1}{K(\alpha)}\left[2\sqrt{\frac{2}{\pi}}\left(\alpha^2+1\right) - 2\alpha\right], \\ E(X^2) &= \frac{1}{K(\alpha)}\left[3\alpha^2 - 4\sqrt{\frac{2}{\pi}}\alpha + 2\right], \\ E(X^3) &= \frac{1}{K(\alpha)}\left[4\sqrt{\frac{2}{\pi}}\left(2\alpha^2+1\right) - 6\alpha\right], \\ E(X^4) &= \frac{1}{K(\alpha)}\left[15\alpha^2 - 16\sqrt{\frac{2}{\pi}}\alpha + 6\right]. \end{aligned}$$

Hence, variance of BSHN distribution can be calculated as

$$Var(X) = \frac{3\sqrt{2\pi}\alpha^2 - 8\alpha + 2\sqrt{2\pi}}{\sqrt{2\pi}\alpha^2 - 4\alpha + 2\sqrt{2\pi}} - \frac{2\left(2\alpha^2 - \sqrt{2\pi}\alpha + 2\right)^2}{\pi K(\alpha)^2}.$$

Similarly, coefficient of skewness (γ_1) and coefficient of kurtosis (γ_2) are calculated as

$$\gamma_1 = \frac{8\left[\,\alpha(s\alpha - 4) + 2s\,\right]\left(L(\alpha)\right)^2}{\left(E(\alpha)\right)^3},$$

$$\gamma_2 = \frac{-12C^4B + 12DC^2\pi A^2 - 8(8\alpha^2 - 3s\alpha + 4)C\pi A^2 B + (15s\alpha^2 - 32\alpha + 6s)\pi^2 A^4}{\left(D\pi A^2 - 2C^2B\right)^2},$$

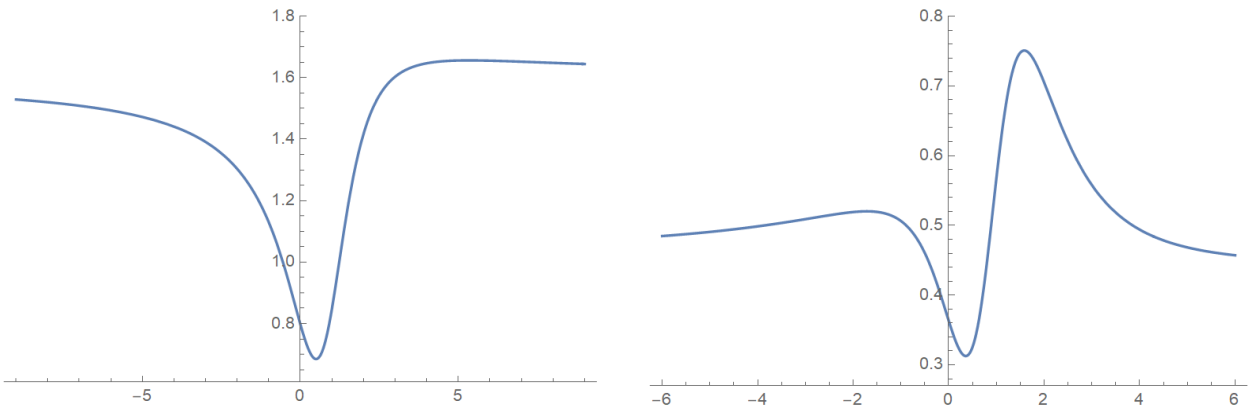
where,

$$s = \sqrt{2\pi}, \quad A = \alpha^2 - 2\sqrt{\tfrac{2}{\pi}}\,\alpha + 2, \quad B = s\alpha^2 - 4\alpha + 2s, \quad C = 2\alpha^2 - s\alpha + 2, \quad D = 3s\alpha^2 - 8\alpha + 2s,$$

$$\begin{aligned} L(\alpha) = {} & \pi s\left((5\alpha^2+1)(\alpha^4+6\alpha^2+4)\alpha^2+8\right) + 24\pi(\alpha^2+2)\alpha^3 \\ & + 64(\alpha^7+2\alpha^5+\alpha^3+\alpha) - 8s(\alpha^2+1)(2\alpha^6+11\alpha^4+4\alpha^2+4) - 2\pi^2(3\alpha^4+8\alpha^2+4)\alpha^3, \end{aligned}$$

$$E(\alpha) = 32(\alpha^5+\alpha) - 16\pi\alpha(\alpha^4+2\alpha^2+2) + \pi s(\alpha^2+2)(3\alpha^4+4\alpha^2+4) - 8s(\alpha^6+\alpha^4-\alpha^2+2),$$

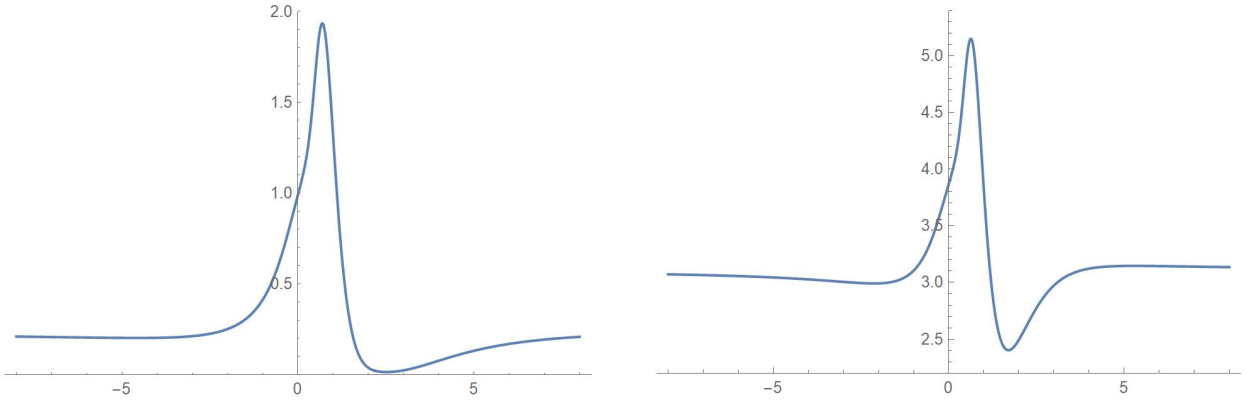
Remark 3.1.2. By optimizing $E(X)$ and $Var(X)$ with respect to the parameter, bounds for the mean and variance for the BSHN distribution can be found out. The resultant value of the bounds for mean and variance are $0.685 \leq E(X) \leq 1.596$ and $0.313 \leq Var(X) \leq 0.751$ respectively. These bounds can be also visualized in the graphs provided in figure (4).



(a) Mean of BSHN (b) Variance of BSHN

Figure 4. Mean and Variance of BSHN distribution.

Remark 3.1.3. By optimizing γ_1 and γ_2 with respect to the parameter, bounds for the skewness and kurtosis for the BSHN distribution can be found out. The resultant value of the bounds for skewness and kurtosis are $0.014 \leq \gamma_1 \leq 1.933$ and $2.402 \leq \gamma_2 \leq 5.152$ respectively. These bounds can be also visualized in the graphs provided in figure (5).



(a) Skewness of BSHN (b) Kurtosis of BSHN

Figure 5. Skewness and kurtosis of BSHN distribution.

Theorem 3.1.2. The nth order incomplete moment of BSHN distribution is calculated as

$$E(X^n|X<y)=\frac{2^{\frac{n}{2}+1}}{y\sqrt{\pi}F(y)K(\alpha)}\left[\sqrt{2}\alpha\sqrt{y^2}\left(\Gamma\left(\frac{n}{2}+1,\frac{y^2}{2}\right)-\Gamma\left(\frac{n}{2}+1\right)\right)\right.$$
$$\left.+y\left\{\alpha^2\left(\Gamma\left(\frac{n+3}{2}\right)-\Gamma\left(\frac{n+3}{2},\frac{y^2}{2}\right)\right)-\Gamma\left(\frac{n+1}{2},\frac{y^2}{2}\right)+\Gamma\left(\frac{n+1}{2}\right)\right\}\right]. \tag{7}$$

Proof.

$$\begin{aligned}E(X^n|X<y)&=\frac{1}{F(y)K(\alpha)}\int_0^y x^n\left[(1-\alpha x)^2+1\right]h(x)dx\\&=\frac{1}{F(y)K(\alpha)}\left[2\int_0^y x^n h(x)dx-2\alpha\int_0^y x^{n+1}h(x)dx+\alpha^2\int_0^y x^{n+2}h(x)dx\right]\\&=\frac{1}{F(y)K(\alpha)}\left[2I_4-2\alpha I_5+\alpha^2 I_6\right].\end{aligned}$$

To calculate the value of the integrations, integration by parts method can be incorporated and those can be obtained as

$$I_4=\int_0^y x^n h(x)dx=\frac{2^{n/2}y^{n+1}\left(y^2\right)^{-\frac{n}{2}-\frac{1}{2}}\left(\Gamma\left(\frac{n+1}{2}\right)-\Gamma\left(\frac{n+1}{2},\frac{y^2}{2}\right)\right)}{\sqrt{\pi}},$$
$$I_5=\int_0^y x^{n+1}h(x)dx=\frac{2^{\frac{n+1}{2}}y^n\left(y^2\right)^{-\frac{n}{2}}\left(\Gamma\left(\frac{n}{2}+1\right)-\Gamma\left(\frac{n}{2}+1,\frac{y^2}{2}\right)\right)}{\sqrt{\pi}},$$
$$I_6=\int_0^y x^{n+2}h(x)dx=\frac{2^{\frac{n}{2}+1}y^{n+1}\left(y^2\right)^{-\frac{n}{2}-\frac{1}{2}}\left(\Gamma\left(\frac{n+3}{2}\right)-\Gamma\left(\frac{n+3}{2},\frac{y^2}{2}\right)\right)}{\sqrt{\pi}}.$$

Substituting these quantities yields the closed-form expression for the incomplete moment as

$$E(X^n|X<y)=\frac{2^{\frac{n}{2}+1}}{y\sqrt{\pi}F(y)K(\alpha)}\left[\sqrt{2}\alpha\sqrt{y^2}\left(\Gamma\left(\frac{n}{2}+1,\frac{y^2}{2}\right)-\Gamma\left(\frac{n}{2}+1\right)\right)\right.$$
$$\left.+y\left\{\alpha^2\left(\Gamma\left(\frac{n+3}{2}\right)-\Gamma\left(\frac{n+3}{2},\frac{y^2}{2}\right)\right)-\Gamma\left(\frac{n+1}{2},\frac{y^2}{2}\right)+\Gamma\left(\frac{n+1}{2}\right)\right\}\right].$$

3.2. Result for Moment Generating Function (MGF)

Theorem 3.2.1. The MGF of the $BSHN(\alpha)$ distribution is given as

$$M(t) = \frac{1}{K(\alpha)}\left[e^{\frac{t^2}{2}}\left(H(t)+1\right)\left(\alpha(\alpha+t(\alpha t-2))+2\right)+\sqrt{\frac{2}{\pi}}\alpha(\alpha t-2)\right]. \tag{8}$$

Proof. Using the PDF provided in equ. (3), MGF is defined as

$$\begin{aligned} M(t) &= \int_0^\infty e^{xt} f(x)dx \\ &= \frac{1}{K(\alpha)}\left[2\int_0^\infty e^{xt}h(x)dx - 2\alpha\int_0^\infty xe^{xt}h(x)dx + \alpha^2\int_0^\infty x^2e^{xt}h(x)dx\right] \\ &= \frac{1}{K(\alpha)}\Big[2I_4 - 2\alpha I_5 + \alpha^2 I_6\Big]. \end{aligned} \tag{9}$$

Here, I_4 is the MGF of half N distribution while I_5 and I_6 can be calculated using the integration by parts method and results of the same can be obtained as

$$\begin{aligned} I_4 &= \int_0^\infty e^{xt}h(x)dx = e^{\frac{t^2}{2}}\left(H(t)+1\right), \\ I_5 &= \int_0^\infty xe^{xt}h(x)dx = e^{\frac{t^2}{2}}t\left(H(t)+1\right)+\sqrt{\frac{2}{\pi}}, \\ I_6 &= \int_0^\infty x^2e^{xt}h(x)dx = e^{\frac{t^2}{2}}\left(t^2+1\right)\left(H(t)+1\right)+\sqrt{\frac{2}{\pi}}t. \end{aligned}$$

Substituting the value of the integrations in equ. (9), final expression of the moment of BSHN distribution can be written as

$$M(t) = \frac{1}{K(\alpha)}\left[e^{\frac{t^2}{2}}\left(H(t)+1\right)\left(\alpha(\alpha+t(\alpha t-2))+2\right)+\sqrt{\frac{2}{\pi}}\alpha(\alpha t-2)\right].$$

Remark 3.2.1. Substituting (it) instead of t in equ. (8), characteristic function of BSHN distribution can be calculated and the expression is given by

$$\Phi(t) = \frac{1}{K(\alpha)}\left[e^{\frac{(it)^2}{2}}\left(H(it)+1\right)\left(\alpha(\alpha+it(\alpha(it)-2))+2\right)+\sqrt{\frac{2}{\pi}}\alpha(\alpha(it)-2)\right]. \tag{10}$$

3.3. Mean Deviation (MD)

The MD of BSHN distribution about mean (μ) is given by

$$\delta_1(z) = 2\mu F(\mu) - 2m_1(\mu)F(\mu). \tag{11}$$

where, $m_1(\mu)$ is the first order incomplete moment defined using the equ. (7) and $F(\mu)$ is obtained from equ. (4).

Proof. The MD about μ and median (M) is calculated as

$$\delta_1(X) = \int_{-\infty}^{\infty}|x-\mu|f(x)dx, \quad \delta_2(X) = \int_{-\infty}^{\infty}|x-M|f(x)dx,$$

respectively, where the median is indicated by M and $\mu = E(X)$.

Now, $\delta_1(X)$ for BSHN distribution may be computed as

$$\begin{aligned}\delta_1(X) &= \int_0^{\infty} |x-\mu| f(x)dx \\ &= -2\int_0^{\mu} xf(x)dx + 2\mu F(\mu) \\ &= -2I_{MD} + 2\mu F(\mu)\end{aligned}$$

where, I_{MD} can be calculated as $\sqrt{\frac{2}{\pi}}\left(1 - e^{-\frac{\mu^2}{2}}\right)$.

Remark 3.3.1. By replacing μ with M in equation (11), the MD about M is obtained.

4. Characterization Result

Characterizations of probability distributions are fundamental in statistical theory as they establish unique identifiers for a model, ensuring that a distribution can be uniquely determined by specific properties or relationships. These results are crucial for verifying model uniqueness, for goodness-of-fit testing, and for providing deeper insight into the structural properties that define a distribution. In this section, characterizations of the BSHN distribution are developed based on relationships involving two truncated moments. Notably, the first characterization is obtained without requiring the cumulative distribution function to be available in a closed-form expression.

4.1. Characterizations based on two truncated moments

This subsection presents several characterizations of the BSHN distribution through a functional relationship involving two truncated moments. The first result is derived by invoking Theorem 4.1.1 of Glanzel (reported in [**?**]), reproduced below for ease of reference. It is worth noting that the statement continues to remain valid even when H is not chosen as a closed interval. Furthermore, the resulting characterization is robust under weak convergence; see [**?**] for additional discussion.

Theorem 4.1.1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and take $H = [d, e]$ with $d < e$ (possibly $d = -\infty$ and $e = +\infty$) as the support of a continuous random variable $X : \Omega \to H$ with distribution function F. Suppose that k and h are real-valued functions on H such that

$$\mathbf{E}\left[k(X) \mid X \geq x\right] = \mathbf{E}\left[h(X) \mid X \geq x\right] \eta(x), \qquad x \in H,$$

where η is a real-valued function. Assume that $k, h \in C^1(H)$, $\eta \in C^2(H)$, and F is twice continuously differentiable and strictly increasing on H. Also assume that the equation $\eta(x)h(x) = k(x)$ has no real solution inside the interior of H. Under these conditions, the distribution F is completely characterized by the functions k, h, and η, and may be written as

$$F(x) = \int_a^x C \left|\frac{\eta'(u)}{\eta(u)h(u) - k(u)}\right| \exp\{-s(u)\}\, du,$$

where s is any solution of the differential equation

$$s'(u) = \frac{\eta'(u)h(u)}{\eta(u)h(u) - k(u)},$$

and $C > 0$ is a normalising constant chosen so that $\int_H dF = 1$.

Proposition 4.1.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and consider the functions

$$h(x) = \frac{x}{(1-\alpha x)^2}, \qquad k(x) = h(x)\, e^{-x^2/2}, \qquad x > 0.$$

Then X follows the distribution with density given in (3) if and only if

$$\eta(x) = \frac{1}{2} e^{-x^2/2}, \qquad x > 0,$$

in the construction above.

Proof. If X has pdf (3), then for $C = \frac{\sqrt{2}}{K(\alpha)\sqrt{\pi}}$, we have

$$\begin{aligned}(1 - F(x))\, E[h(X) \mid X \geq x] &= \int_x^\infty Cue^{-u^2/2} du \\ &= Ce^{-x^2/2}, \quad x > 0,\end{aligned}$$

and

$$\begin{aligned}(1 - F(x))\, E[k(X) \mid X \geq x] &= \int_x^\infty Cue^{-u^2} du \\ &= \frac{1}{2} Ce^{-x^2}, \quad x > 0,\end{aligned}$$

and finally

$$\eta(x)\, h(x) - k(x) = -\frac{1}{2} h(x)\, e^{-x^2/2} < 0 \quad for \;\; x > 0.$$

Conversely, if η has the above form, then

$$s'(x) = \frac{\eta'(x)\, h(x)}{\eta(x)\, h(x) - k(x)} = x,$$

and hence

$$s(x) = \frac{1}{2} x^2, \quad x > 0.$$

In view of Theorem 4.1.1, X has pdf (3) .

Corollary 4.1.1. Let $X : \Omega \to (0, \infty)$ be a continuous random variable and let $h(x)$ be specified as in Proposition 4.1.1. Then X admits density (3) if and only if there exist functions k and η (as defined in Theorem 4.1.1) such that the following first–order differential equation is satisfied:

$$x = \frac{\eta'(x)\, h(x)}{\eta(x)\, h(x) - k(x)}.$$

Corollary 4.1.2. The differential equation presented in Corollary 4.1.1 admits the following general solution

$$\eta(x) = e^{x^2/2} \left[-\int x e^{-x^2/2} (h(x))^{-1} k(x) + D \right],$$

where D is a constant.

Proof. If X has pdf (3), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = x\eta(x) - x\left(h(x)\right)^{-1} k(x),$$

$$\eta'(x) - x\eta(x) = -x\left(h(x)\right)^{-1} k(x),$$

or

$$e^{-x^2/2}\eta'(x) - xe^{-x^2/2}\eta(x) = -xe^{-x^2/2}\left(h(x)\right)^{-1} k(x),$$

or

$$\frac{d}{dx}\left\{e^{-x^2/2}\eta(x)\right\} = -xe^{-x^2/2}\left(h(x)\right)^{-1} k(x),$$

from which we arrive at

$$\eta(x) = e^{x^2/2}\left[-\int xe^{-x^2/2}\left(h(x)\right)^{-1} k(x)\,dx + D\right].$$

A particular class of solutions to this differential equation is given in Proposition 4.1.1 for the case $D = 0$. It should be noted, however, that this is not the only possible choice; in fact, there exist several other triplets (h, k, η) which also fulfil the assumptions of Theorem 4.1.1.

5. Estimation of Parameter

This section deals with the estimation of parameters for the BSHN distribution. To begin with, we introduce a scale-extended version of the BSHN model by incorporating a positive scale parameter $\sigma > 0$. The extension is achieved via the transformation $Y = \sigma X$, where $X \sim BSHN(\alpha)$. Consequently, the PDF of the scaled (extended) BSHN distribution takes the following form:

$$f(y;\sigma,\alpha) = \frac{\left[\left(1 - \frac{\alpha y}{\sigma}\right)^2 + 1\right]}{K(\alpha)} h(y;\sigma), \tag{12}$$

where, $\alpha \in R$ and $\sigma > 0$ and $h(y;\sigma)$ is the PDF of the half-normal distribution with scale parameter σ, given by $h(y;\sigma) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{y^2}{2\sigma^2}\right), \quad y > 0.$. This new extension of the BSHN distribution is denoted as $BSHN(\alpha,\sigma)$.

Let, $y_1, y_2, y_3, \ldots, y_n$ are independently and identically distributed random variable drawn from the $BSHN(\alpha,\sigma)$ distribution. Then the log-likelihood function can be written as

$$l(\theta) = nC - n\log\sigma - n\log\alpha - \sum_{i=1}^{n}\frac{y_i^2}{2\sigma^2} + \sum_{i=1}^{n}\log\left[\left(1 - \frac{\alpha y_i}{\sigma}\right)^2 + 1\right], \tag{13}$$

where, $C = \left(\frac{1}{2}\log 2 - \frac{1}{2}\log\pi - \log K\right)$.

Taking derivatives of equation (13) with respect to the model parameters yields the corresponding likelihood

equations

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{y_i^2}{\sigma^3} + \sum_{i=1}^{n} \left[\frac{1}{A_1} \cdot 2 \left(1 - \frac{\alpha y_i}{\sigma}\right) \cdot \frac{\alpha y_i}{\sigma^2} \right],$$
$$\frac{\partial l}{\partial \alpha} = -\frac{n}{\alpha} - \sum_{i=1}^{n} \left[\frac{1}{A_1} \cdot 2 \left(1 - \frac{\alpha y_i}{\sigma}\right) \cdot \frac{y_i}{\sigma} \right],$$

where, $A_1 = \left(1 - \frac{\alpha y_i}{\sigma}\right)^2 + 1$.

The system of equations stated above may be solved simultaneously to obtain the parameter estimates. As these equations do not possess a closed-form analytical solution, a numerical optimisation routine is required. In this study, we employ the `GenSA` package in `R` to perform the computation.

6. Simulation Study

A simulation experiment is conducted to investigate the sampling behaviour of the proposed estimators using the Metropolis-Hastings (MH) algorithm. The primary reason for choosing the MH algorithm is the analytical unmanageability of the inverse CDF of BSHN distribution. As presented in Eqn. (4), the CDF of the BSHN distribution, does not have a closed form solution that can be easily inverted to generate random variates via the inverse transform method. So, MH algorithm is selected for its robustness and straightforward implementation within the R programming environment. The MH algorithm for generating random samples is given below: A total of 1000 replicated datasets are generated for for three distinct sample sizes:, $n = 100,\ 300$, and 500. For each simulated dataset, the likelihood function is maximised using the `GenSA` routine in `R`. The performance of the estimators is evaluated using the bias and mean squared error (MSE). The numerical findings of the simulation study are summarised in Table 1 to Table 4.

Metropolis–Hastings Algorithm for Random Sample Generation

- Consider the BSHN distribution as the target density, denoted by f.
- Consider the HN distribution as the proposal density, denoted by g.
- Initialize y_0 (an arbitrary positive value).

For each iteration t from 1 to N:

- Generate a candidate value x^* from the proposal distribution g (half-normal).
- Compute the acceptance ratio: $r = \dfrac{f(x^*)}{f(y_t)} \times \dfrac{g(y_t)}{g(x^*)}$.
- Generate a random number $u \sim U(0, 1)$.
- If $u < \min(1, r)$, then accept x^* and set $y_{t+1} = x^*$; otherwise set $y_{t+1} = y_t$.
- Store the accepted sample.
- Discard the first $N - n$ values (burn-in samples) and use the remaining n samples as filtered output.

Table 1. Simulation Results

		$\sigma = 1$			
α	n	Bias (σ)	MSE (σ)	Bias (α)	MSE (α)
	100	0.0692	0.0812	-0.0951	0.0715
-1.5	300	-0.0413	0.0715	0.0814	0.0700
	500	-0.0401	0.0666	0.0810	0.0698
	100	-0.0613	0.0711	-0.0512	0.0933
-1	300	0.0523	0.0618	-0.0415	0.0871
	500	-0.0112	0.0606	-0.0110	0.0837
	100	0.0311	0.0592	-0.1496	0.0859
-0.5	300	0.0213	0.0509	-0.1380	0.0798
	500	0.0122	0.0399	-0.0967	0.0629
	100	-0.0421	0.0515	-0.2891	0.0399
0	300	-0.0198	0.0451	-0.2166	0.0276
	500	0.0538	0.0751	0.0623	0.0222
	100	-0.0498	0.0733	0.0930	0.1869
0.5	300	-0.0371	0.0621	0.0592	0.1594
	500	-0.1582	0.0954	-0.2690	0.2402
	100	-0.1945	0.1741	-0.3162	0.3020
1	300	0.0717	0.0642	0.0981	0.0901
	500	0.0290	0.0458	-0.0492	0.1622
	100	0.0333	0.0471	0.0391	0.1000
1.5	300	0.0281	0.0449	0.0289	0.0718
	500	0.0182	0.0398	0.0190	0.0611

Table 2. Simulation Results

		$\sigma = 2$			
α	n	Bias (σ)	MSE (σ)	Bias (α)	MSE (α)
	100	0.0814	0.0500	0.1690	0.0869
-1.5	300	0.0629	0.0411	-0.0840	0.0723
	500	0.0601	0.0408	-0.0739	0.0549
	100	-0.3619	0.1402	0.1589	0.0794
-1	300	0.3018	0.1138	0.2131	0.0594
	500	0.2879	0.1092	0.1827	0.0531
	100	-0.3888	0.1229	0.1629	0.1349
-0.5	300	0.3701	0.0969	0.1680	0.1107
	500	0.2939	0.0619	0.0658	0.0800
	100	-0.2819	0.0521	0.0155	0.0856
0	300	0.2001	0.0297	0.0222	0.0084
	500	0.1827	0.0270	0.0210	0.0053
	100	0.0988	0.0371	-0.1222	0.0543
0.5	300	0.0834	0.0210	-0.0915	0.0381
	500	0.0560	0.0151	-0.0651	0.0224
	100	-0.3711	0.3918	-0.6271	0.4815
1	300	-0.2699	0.2770	-0.3989	0.3155
	500	0.2460	0.2446	-0.3161	0.2241
	100	-0.1739	0.0370	0.2799	0.0742
1.5	300	0.0937	0.0266	0.2280	0.0692
	500	0.0766	0.0222	0.1952	0.0588

Table 3. Simulation Results

		$\sigma = 3$			
α	n	Bias (σ)	MSE (σ)	Bias (α)	MSE (α)
	100	0.0824	0.0753	-0.0881	0.0612
-1.5	300	-0.0352	0.0674	0.0769	0.0586
	500	-0.0232	0.0598	0.0645	0.0514
	100	-0.0548	0.0682	-0.0475	0.0869
-1	300	0.0459	0.0569	-0.0374	0.0781
	500	-0.0091	0.0532	-0.0082	0.0755
	100	0.0273	0.0551	-0.1317	0.0783
-0.5	300	0.0181	0.0486	-0.1193	0.0704
	500	0.0105	0.0362	-0.0831	0.0572
	100	-0.0384	0.0482	-0.2552	0.0342
0	300	-0.0156	0.0410	-0.1815	0.0259
	500	0.0471	0.0694	0.0576	0.0195
	100	-0.0435	0.0683	0.0842	0.1680
0.5	300	-0.0319	0.0578	0.0527	0.1435
	500	-0.1416	0.0886	-0.2437	0.2172
	100	-0.1746	0.1542	-0.2813	0.2794
1	300	0.0632	0.0589	0.0876	0.0814
	500	0.0254	0.0423	-0.0437	0.1419
	100	0.0305	0.0438	0.0358	0.0911
1.5	300	0.0252	0.0410	0.0256	0.0642
	500	0.0161	0.0354	0.0167	0.0546

Table 4. Simulation Results

		$\sigma = 4$			
α	n	Bias (σ)	MSE (σ)	Bias (α)	MSE (α)
	100	0.0834	0.0752	-0.0791	0.0683
-1.5	300	-0.0252	0.0641	0.0742	0.0601
	500	-0.0207	0.0559	0.0675	0.0530
	100	-0.0523	0.0689	-0.0451	0.0902
-1	300	0.0481	0.0590	-0.0397	0.0797
	500	-0.0089	0.0559	-0.0078	0.0735
	100	0.0298	0.0562	-0.1254	0.0831
-0.5	300	0.0199	0.0489	-0.1085	0.0716
	500	0.0113	0.0388	-0.0799	0.0560
	100	-0.0395	0.0498	-0.2507	0.0362
0	300	-0.0148	0.0432	-0.1755	0.0287
	500	0.0495	0.0704	0.0591	0.0213
	100	-0.0457	0.0712	0.0849	0.1715
0.5	300	-0.0344	0.0595	0.0555	0.1491
	500	-0.1443	0.0916	-0.2330	0.2196
	100	-0.1822	0.1610	-0.2655	0.2891
1	300	0.0608	0.0561	0.0824	0.0770
	500	0.0273	0.0440	-0.0382	0.1374
	100	0.0320	0.0468	0.0332	0.0918
1.5	300	0.0267	0.0433	0.0259	0.0653
	500	0.0146	0.0365	0.0151	0.0539

From the simulation study, several important patterns can be observed. First, as the sample size increases, both the bias and MSE of the estimators decrease consistently across all parameter settings, which empirically

validates the consistency of the maximum likelihood estimators for the proposed distribution. Second, for smaller values of α, the estimators exhibit relatively low bias even at moderate sample sizes. However, as α increases, the magnitude of bias becomes more pronounced, especially for small sample sizes. This indicates that the skewness and bimodality induced by larger values of α introduce additional variability in the estimation, requiring larger samples to achieve stable performance.

Overall, the simulation results confirm the desirable statistical properties of the proposed estimators. The reduction in bias and MSE with increasing sample size and the behavior of bias with respect to α, provide strong evidence for the reliability and efficiency of the maximum likelihood method in the context of the BSHN distribution.

7. Practical Data Applications

This part compares the efficacy and adaptability of the suggested distribution to a number of competing models, including the extended half normal EHN distribution of [29], half normal, HN distribution, Weibull W distribution, Rayleigh R distribution with the help of some real life datasets. Besides, for comparison purposes, model selection criterion like Bayesian information criteria (BIC) and the Akaike information criterion (AIC) are considered.

7.1. Life time dataset of a certain device

This illustration considers the data set considering the lifetime of a certain device (n=30) which was reported by [20]. The Summary statistics for this dataset is given in Table 5. Figure 6 visualizes some non parametric plots for this dataset. The value of AIC, BIC, log-likelihood of the models under consideration along with the value of MLEs are presented in Table 6. As demonstrated in Table 6, the proposed distribution exhibits superior fit and suitability compared to competing models. Additionally, the flexibility of the BSHN distribution is further illustrated in Figure 8. The P-P plot in Figure 7 (a) also shows an almost perfect alignment of points along the diagonal, indicating that the theoretical probabilities from the BSHN model match the empirical probabilities very well. Similarly, the Q-Q plot in Figure 7 (b) demonstrates a strong linear trend with minimal deviation, confirming that the quantiles of the BSHN distribution accurately represent the quantiles of the observed data. This visual evidence strongly supports the BSHN model as an appropriate fit.

Table 5. Summary statistics for the Life time dataset of a certain device

Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum	Variance	Skewness	Kurtosis
0.0094	7.181	9.976	9.039	12.002	13.853	12.401	–0.710	2.741

Table 6. Model fitting results for Life time dataset of a certain device

Distributions	α	σ	μ	$logL$	AIC	BIC
HN	–	0.127	–	-90.27	182.54	183.94
W	–	9.583	1.619	-92.73	189.46	192.26
R	–	6.935	–	-93.55	189.10	190.50
L	–	–	0.203	-91.89	185.78	187.18
EHN	0.223	6.041	–	-82.83	169.66	172.46
$BSHN$	2.827	5.576	—	-78.99	161.98	164.78

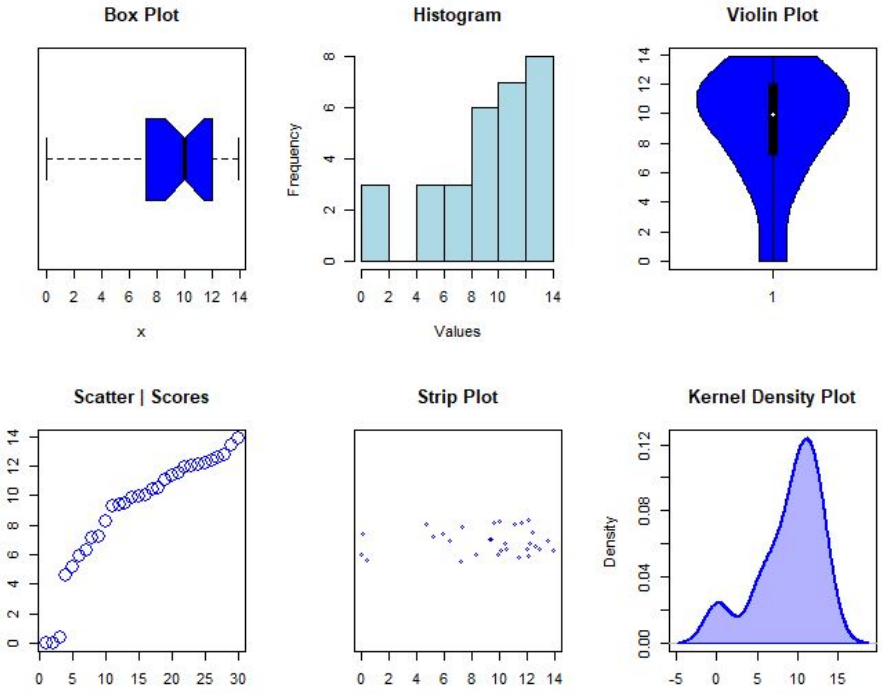
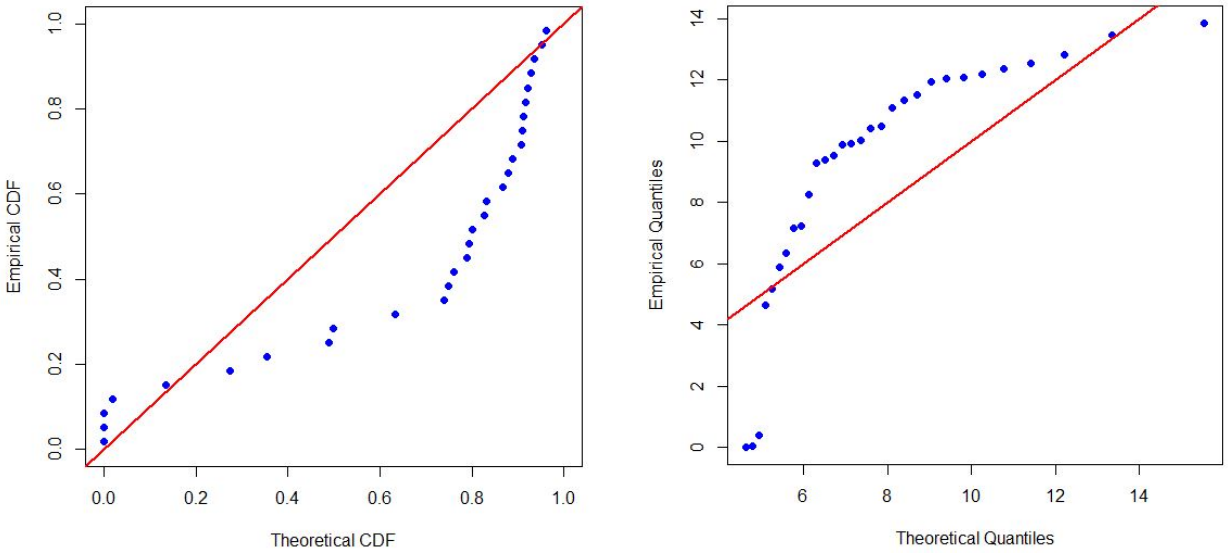


Figure 6. Non-parametric graphical summaries of the Life time dataset of a certain device, including box plot, histogram, violin plot, scatter plot, strip plot, and kernel density estimate.



(a) P-P Plot for BSHN distribution (b) Q-Q Plot for BSHN distribution

Figure 7. P-P and Q-Q Plot for the BSHN distribution

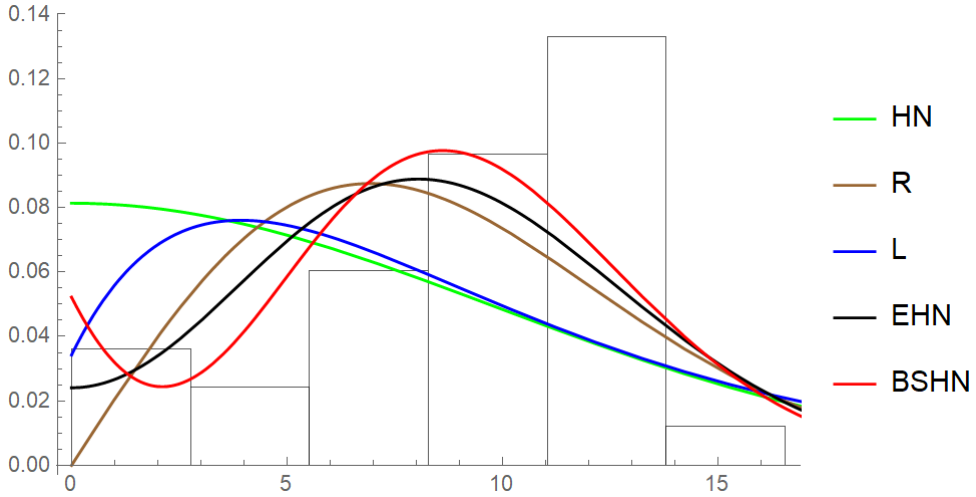


Figure 8. Histogram of the device lifetime dataset with fitted densities from competing models, showing the superior fit of the BSHN distribution.

7.2. Blood Cancer Dataset

A data set concerning 40 patients suffering from blood cancer from one ministry of hospitals in Saudi Arabia is used for this illustration which was previously analyzed by [18]. Summary statistics for this dataset are given in Table 7. Different non parametric plots for the said dataset is depicted in Figure 9. Table 8 reports the value of AIC, BIC, log-likelihood of the models under consideration along with the value of MLEs. As evidenced by the results in Table 8, the proposed distribution demonstrates superior suitability and fit compared to competing models. Furthermore, Figure 11 illustrates the enhanced flexibility of the BSHN distribution in adapting to various data scenarios. The P-P plot in Figure 10 (a) also shows an almost perfect alignment of points along the diagonal, indicating that the theoretical probabilities from the BSHN model match the empirical probabilities very well. Similarly, the Q-Q plot in Figure 10 (b) demonstrates a strong linear trend with minimal deviation, confirming that the quantiles of the BSHN distribution accurately represent the quantiles of the observed data. This visual evidence strongly supports the BSHN model as an appropriate fit.

Table 7. Summary statistics for the lood Cancer Dataset

Minimum	1st Quartile	Median	Mean	3rd Quartile	Maximum	Variance	Skewness	Kurtosis
0.315	2.199	3.348	3.141	4.264	5.381	1.846	–0.430	2.336

Table 8. Model fitting results for blood cancer dataset

Distributions	α	σ	μ	$logL$	AIC	BIC
HN	–	0.366	–	-78.16	158.32	160.01
W	–	2.499	3.518	-69.55	143.10	146.48
R	–	2.415	–	-70.81	143.62	145.31
L	–	–	0.526	-80.25	162.50	164.19
EHN	0.156	2.067	–	-69.27	142.54	145.92
$BSHN$	2.769	1.946	—	-68.75	141.50	144.88

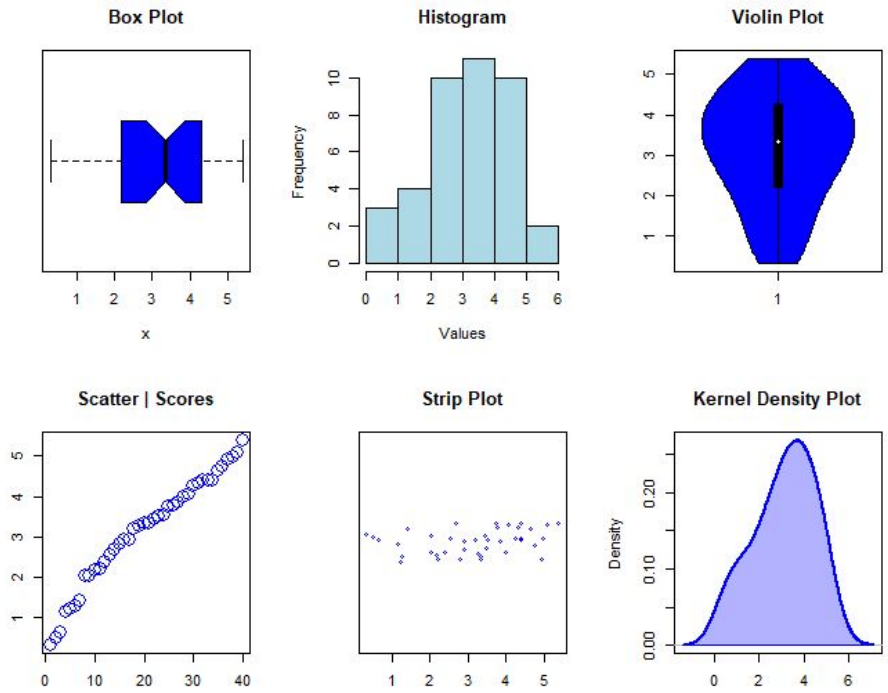
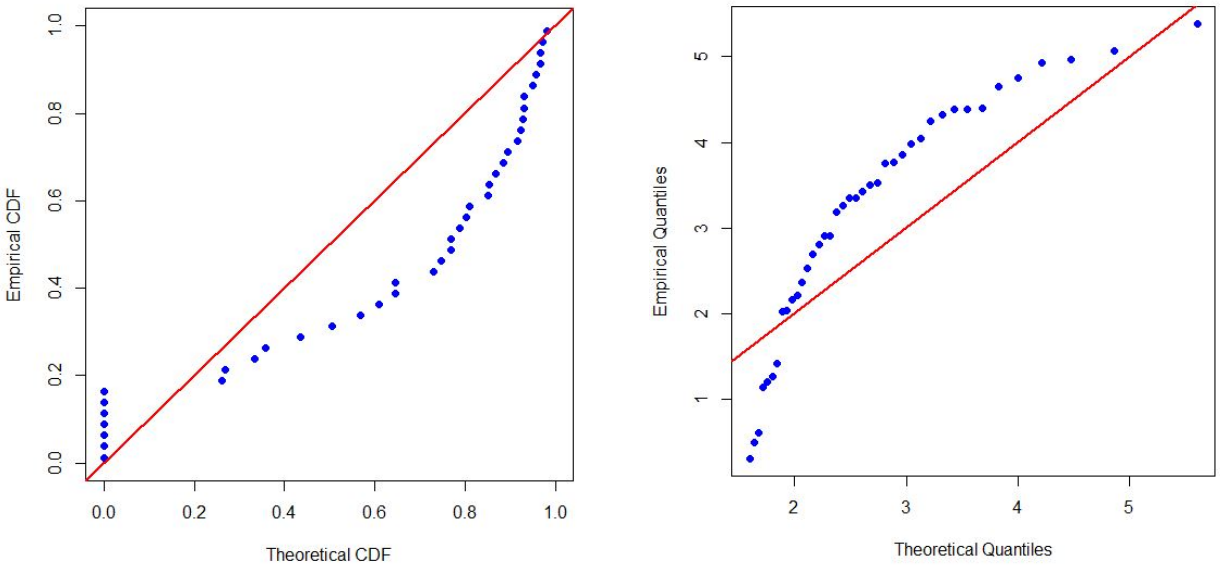


Figure 9. Non-parametric graphical summaries of the Blood Cancer Dataset of a certain device, including box plot, histogram, violin plot, scatter plot, strip plot, and kernel density estimate.



(a) P-P Plot for BSHN distribution

(b) Q-Q Plot for BSHN distribution

Figure 10. P-P and Q-Q Plot for the BSHN distribution

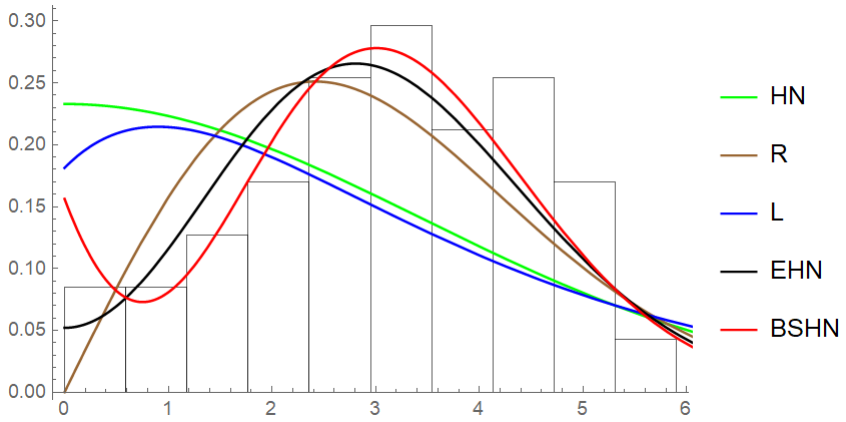


Figure 11. Histogram of the blood cancer dataset with fitted densities from competing models, showing the superior fit of the BSHN distribution.

8. Hypothesis Testing

In order to distinguish the classical (HN) model from its $BSHN$ extension, a likelihood ratio (LR) test is conducted. The null and alternative hypotheses are given by

$$H_0 : \alpha = 0 \quad \text{VS} \quad H_1 : \alpha \neq 0,$$

where α is the additional shape parameter that characterizes the $BSHN$ distribution. The LR test statistic takes the form

$$-2\log(LR) = -2\Big[\log L(\hat{\sigma}_1; \alpha = 0 \mid x) - \log L(\hat{\sigma}_2, \hat{\alpha}_2 \mid x)\Big],$$

where $\hat{\sigma}_1$ denotes the maximum likelihood estimate (MLE) of the scale parameter under H_0 (corresponding to the HN model), and $(\hat{\sigma}_2, \hat{\alpha}_2)$ are the MLEs under H_1 (associated with the $BSHN$ model). Under the null hypothesis, the statistic approximately follows a chi–square distribution with one degree of freedom, which equals the difference in the number of free parameters between the competing models.

Table 9. LRT for different hypotheses for the data set I and II.

Hypothesis	LRT statistic		d.f.	Critical Values at 5 %
	Dataset I	Dataset II		
$H_0 : \alpha = 0$ Vs $H_1 : \alpha \neq 0$	22.56	18.82	1	3.841

The value of the LRT statistics is more than the tabulated critical value at 95% of confidence level for dataset I and II, as can be shown from Table 9. Thus, the data that originated may be said to come from the $BSHN$ distribution.

9. Conclusion

This paper introduced a new flexible probability model, the Bimodal Skew Half-Normal (BSHN) distribution, designed to effectively model data with positive support that exhibits both unimodal and bimodal patterns. A detailed investigation of the theoretical characteristics of the proposed distribution is carried out, including its

structural, probabilistic, and inferential properties. To gain further insight, several characterization results are derived under different mathematical frameworks.

In this paper, we developed the BSHN distribution as a generalization of the HN distribution. A single shape parameter, α, allows the model to transition from a highly concentrated unimodal shape to a clearly bimodal form, providing significant flexibility for fitting diverse data shapes. We also extended the model with a scale parameter and detailed the maximum likelihood method for estimating the parameters. A comprehensive simulation study demonstrated that these estimators perform well and they exhibit low bias as well as MSE, and these measures decrease as the sample size increases, confirming the estimators' consistency and good finite-sample performance.

While the present study evaluates the performance of the proposed estimators in terms of bias and mean squared error, an assessment of coverage probabilities of confidence intervals constitutes an important direction for future research. Such an extension would provide a deeper understanding of the inferential reliability of the estimators under different parameter settings. In addition, future work may consider the development of Bayesian estimation strategies for the BSHN model, regression frameworks incorporating covariates, and multivariate extensions of the distribution to handle correlated data structures. Further, a comprehensive comparative study considering the BSHN distribution against other sophisticated models designed for positive bimodal data also can be conducted in near future. Such a comparison would more fully explain the competitive advantages and potential niches of the proposed model within the methodology of distributions available for modeling complex, bimodal positive data. Exploring these avenues will further enhance the applicability and robustness of the proposed model in diverse practical contexts

Besides, the practical value of the BSHN distribution was confirmed through applications to two real-world datasets. In both cases, the BSHN provided a better fit than several well-known competing models, including the half-normal, Weibull, Rayleigh, and extended half-normal distributions, as evidenced by lower Akaike (AIC) and Bayesian (BIC) information criterion values. A LR test was also conducted to formally compare the BSHN distribution against the standard half-normal model. The test results were statistically significant, providing strong evidence that the new model with its additional shape parameter offers a substantially improved fit to the data.

In summary, the BSHN distribution is a theoretically sound and practically useful tool for statisticians and data analysts. Its ability to model bimodality in positive data, combined with its excellent performance in real applications, makes it a valuable addition to the family of statistical distributions for use in fields like reliability engineering, survival analysis, and environmental science.

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