STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **x**, Month 202x, pp 0–45.
Published online in International Academic Press (www.IAPress.org)

A Unified Framework for Characterizing Wrapped Distributions Using Truncated Moments and Reverse Hazard Function

G. G. Hamedani [1,*], Amin Roshani [2], Nadeem S. Butt [3]

[1] *Department of Mathematical and Statistical Sciences, Marquette University, Milwaukee, USA*
[2] *Department of Statistics, Lorestan University, Khorramabad, Iran*
[3] *Department of Family and Community Medicine, King Abdulaziz University, Jeddah, Saudi Arabia*

Abstract Bell and Nadarajah [1] provided an exceptional review of the existing "Wrapped Distributions" in an alphabetic order. We suggest every interested reader carefully read the "Introduction" given in their paper. The current paper deals with various characterizations of these distributions in two main sections, one for the continuous case and the other for the discrete case. The characterizations for the continuous wrapped case are based on a simple relationship between two truncated moments. It should be mentioned that for this characterization the cumulative distribution function need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation. The fact that our characterizations in the continuous case does not require that the distribution function have closed form, set them apart from other existing characterizations results. We like to point out that our paper is a theoretical research in a challenging and mathematically elegant field, called the "Characterizations of Distributions".
As we mentioned in our previous works, sometimes in real life cases, it is very difficult to obtain samples from a continuous distribution. The observed values are generally discrete due to the fact that they are not measured in continuum. In some cases, it may be possible to measure the observations via a continuous scale, however, they may be recorded in a manner in which a discrete model seems more suitable. Consequently, the discrete models are appearing quite frequently in applied fields and have attracted the attention of many researchers. The characterizations for the discrete wrapped case will be based on certain function of the random variable whose form depends on the nature of the distribution of the random variable.

Keywords Characterizations, Conditional expectation, Continuous distributions, Discrete distributions, Reverse hazard function

AMS 2010 subject classifications 62E10

DOI: 10.19139/soic-2310-5070-2867

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. Our characterizations are in the directions mentioned in the Abstract, which is completely different from the ones reported in the literature.

To illustrate the practical usefulness of such results, consider two classical applications. A well–known result due to Cramér states that if X and Y are independent random variables such that their sum $X + Y$ is normally distributed, then both X and Y must themselves be normally distributed. This property uniquely characterizes the

*Correspondence to: G. G. Hamedani (Email: gholamhoss.hamedani@marquette.edu). Department of Mathematical and Statistical Sciences, Marquette University, Milwaukee, WI 53201-1881, USA.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

normal law: the stability of the normal distribution under addition. As a technical example, consider independent measurement errors arising from two different instruments, say X from a thermometer and Y from a pressure gauge. If the total error $X+Y$ in a derived physical quantity (for instance, an equation of state involving both measurements) is empirically found to be normally distributed, then Cramér's theorem ensures that each of the component errors must themselves follow a normal law. This provides a rigorous justification for modeling experimental errors by Gaussian distributions, a cornerstone assumption in many areas of physics, engineering, and signal processing.

Another example arises from the exponential distribution, which is the only continuous distribution possessing the memoryless property. This identity provides a complete characterization of the exponential law and is widely used in applied probability. For instance, in reliability analysis, if the remaining lifetime of a component does not depend on how long it has already lasted, then modeling the lifetime by an exponential distribution is well justified. This principle underlies the exponential failure-time model commonly used in survival analysis, queuing theory, and reliability engineering.

We will employ the same notations for the parameters used by the original authors of the proposed wrapped distributions.

We list below first the cumulative distribution functions (cdf) and probability density functions (pdf) of the wrapped continuous distributions mentioned in [1], 1.1-1.45. Then we list the cdfs and probability mass functions (pmfs) of the discrete wrapped distributions mentioned in [1], 1.46-1.55.

1.1. Wrapped Akash (WAk) Distribution

$$F(x;\lambda)=\int_0^x f(u;\lambda)\,du,\quad 0\le x\le 2\pi \tag{1}$$

and

$$f(x;\lambda)=Ce^{-\lambda x}P(x),\quad 0<x<2\pi, \tag{2}$$

where $\lambda>0$ is a parameter, $C=\frac{\lambda^3}{(\lambda^2+2)(1-e^{-2\pi\lambda})^3}$ and $P(x)=\left(1+x^2\right)\left(1-e^{-2\pi\lambda}\right)^2+4\pi e^{-2\pi\lambda}\left(x+\pi+\pi e^{-2\pi\lambda}-xe^{-2\pi\lambda}\right)$.

1.2. Wrapped Aradhana (WAr) Distribution

$$F(x;\lambda)=\int_0^x f(u;\lambda)\,du,\quad 0\le x\le 2\pi, \tag{3}$$

and

$$f(x;\lambda)=Ce^{-\lambda x}P(x),\quad 0<x<2\pi, \tag{4}$$

where $\lambda>0$ is a parameter, $C=\frac{\lambda^3}{\lambda^2+2\lambda+2}$ and $P(x)=\frac{(1+x)^2}{(1-e^{-2\pi\lambda})}+\frac{4(1+x)\pi e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}+\frac{4\pi^2e^{-2\pi\lambda}\left(1+e^{-2\pi\lambda}\right)}{(1-e^{-2\pi\lambda})^3}$.

1.3. Wrapped Binormal (WBN) Distribution, WLOG for $\mu=0,\sigma_1=\sigma_2=1$

$$F(x)=\int_0^x f(u)\,du\quad 0\le x\le 2\pi, \tag{5}$$

and

$$f(x)=Cxe^{-x^2/2}P(x),\quad 0<x<2\pi, \tag{6}$$

where $C=\frac{1}{\sqrt{2\pi}}$ and $P(x)=x^{-1}\sum_{k=-\infty}^{\infty}e^{-2k\pi(x+k\pi)}$.

1.4. Wrapped Birnbaum-Saunders (WB-S) Distribution

$$F(x;\mu,\delta)=\int_0^x f(u;\mu,\delta)\,du, \quad 0\le x\le 2\pi, \tag{7}$$

and

$$f(x;\mu,\delta)=Ce^{-\left(\frac{\delta+1}{4\mu}\right)x}P(x), \quad 0<x<2\pi, \tag{8}$$

where $0\le\mu<2\pi,\delta>0$ are parameters, $C=\frac{e^{\delta/2}\sqrt{\delta+1}}{4\sqrt{\pi\mu}}e^{-\frac{k\pi(\delta+1)}{2\mu}}$ and $P(x)=\sum_{k=0}^{\infty}\left(\frac{x+2k\pi+\frac{\delta\mu}{\delta+1}}{(x+2k\pi)^{3/2}}\right)e^{-\frac{\delta^2\mu}{4(x+2k\pi)(\delta+1)}}$.

1.5. Wrapped Cauchy (WC) Distribution

$$F(x;\gamma,\mu)=\int_0^x f(u;\gamma,\mu)\,du, \quad 0\le x\le 2\pi, \tag{9}$$

and

$$f(x;\gamma,\mu)=\frac{C}{\cosh\gamma-\cos(x-\mu)}, \quad 0<x<2\pi, \tag{10}$$

where $\gamma>0,\mu\in\mathbb{R},\delta>0$ are parameters and $C=\frac{\sinh\gamma}{2\pi}$.

1.6. Wrapped Chi-Square (WC-S)Distribution

$$F(x;k)=\int_0^x f(u;k)\,du, \quad 0\le x\le 2\pi, \tag{11}$$

and

$$f(x;k)=Ce^{-\frac{1}{2}x}P(x), \quad 0<x<2\pi, \tag{12}$$

where $k>0$ is a parameter, $C=\frac{\pi^{\frac{k}{2}-1}}{2\Gamma(k/2)}, P(x)=\Phi\left(e^{-\pi};1-\frac{k}{2},\frac{x}{2\pi}\right), \Gamma(k)$ is the gamma function and $\Phi(x;s,a)=\sum_{n=0}^{\infty}\frac{x^n}{(a+n)^s}$.

1.7. Wrapped Exponential (WE) Distribution

$$F(x;\lambda)=\frac{1-e^{-\lambda x}}{1-e^{-2\pi\lambda}}, \quad 0\le x\le 2\pi, \tag{13}$$

and

$$f(x;k)=Ce^{-\lambda x}, \quad 0<x<2\pi, \tag{14}$$

where $\lambda>0$ is a parameter and $C=\frac{\lambda}{1-e^{-2\pi\lambda}}$.

1.8. Wrapped Exponential Inverted Weibull (WEIW) Distribution

$$F(x;a,\lambda)=\int_0^x f(u;a,\lambda)\,du, \quad 0\le x\le 2\pi, \tag{15}$$

and

$$f(x;a,\lambda)=Ce^{-x}P(x), \quad 0<x<2\pi, \tag{16}$$

where $a>0,\lambda>0$ are parameters, $C=a\lambda$ and $P(x)=e^x\sum_{k=0}^{\infty}(x+2\pi k)^{-(a+1)}\left[e^{-(x+2\pi k)^{-a}}\right]^{\lambda}$.

1.9. Wrapped Gamma (WG) Distribution

$$F(x;r,\lambda)=\int_0^x f(u;r,\lambda)\,du, \quad 0\le x\le 2\pi, \tag{17}$$

and

$$f(x;r,\lambda)=Ce^{-\lambda x}P(x), \quad 0<x<2\pi, \tag{18}$$

where $r>0,\lambda>0$ are parameters, $C=\frac{\lambda^r(2\pi)^{r-1}}{\Gamma(r)}$, $P(x)=\Phi\left(e^{-2\pi\lambda};1-r,\frac{x}{2\pi}\right)$, $\Gamma(k)$ is the gamma function and $\Phi(x;s,a)=\sum_{n=0}^{\infty}\frac{x^n}{(a+n)^s}$.

1.10. Wrapped Generalized Geometric Stable (WGGS) Distribution

$$F(x)=\frac{1}{2\pi}\left\{1+2\sum_{k=1}^{\infty}\left[\frac{\alpha_k}{k}\sin(kx)+\frac{\beta_k}{k}-\frac{\beta_k}{k}\cos(kx)\right]\right\}, \quad 0\le x\le 2\pi, \tag{19}$$

and

$$f(x)=C\cos(\pi-x)P(x), \quad 0<x<2\pi, \tag{20}$$

where $C=\frac{1}{2\pi}$, and $P(x)=\frac{1}{\cos(\pi-x)}\left(1+2\sum_{k=1}^{\infty}\left[\alpha_k\cos(kx)+\beta_k\sin(kx)\right]\right)$.

1.11. Wrapped Generalized Gompertz (WGG) Distribution

$$F(x;a,b,c)=\frac{1}{\Gamma(c)}\sum_{k=-\infty}^{\infty}\left[\Gamma\left(c,e^{\frac{2\pi k-a}{b}}\right)-\Gamma\left(c,e^{\frac{x+2\pi k-a}{b}}\right)\right], \quad 0\le x\le 2\pi, \tag{21}$$

and

$$f(x;a,b,c)=Ce^{\frac{c}{b}x}P(x), \quad 0<x<2\pi, \tag{22}$$

where $0\le a<2\pi, b>0, c>0$ are parameters, $x+2\pi k>a, C=\frac{1}{b\Gamma(c)}$, $\Gamma(c)$ is the gamma function, $P(x)=\sum_{k=-\infty}^{\infty}\exp\left\{\frac{c}{b}(2\pi k-a)-e^{\frac{x+2\pi k-a}{b}}\right\}$ and $\Gamma(c,x)$ is the incomplete gamma function.

1.12. Wrapped Generalized Normal Laplace (WGNL) Distribution

$$F(x)=\int_0^x f(u)\,du, \quad 0\le x\le 2\pi, \tag{23}$$

and

$$f(x)=\frac{1}{2\pi}\left\{1+2\sum_{k=1}^{\infty}\left[\alpha_k\cos(kx)+\beta_k\sin(kx)\right]\right\}, \quad 0<x<2\pi. \tag{24}$$

1.13. Wrapped Generalized Skew Normal-1 (WGSN-1) Distribution

$$F(x;\alpha,\mu,\sigma,\lambda)=\int_0^x f(u;\alpha,\mu,\sigma,\lambda)\,du, \quad 0\le x\le 2\pi, \tag{25}$$

and

$$f(x;\alpha,\mu,\sigma,\lambda)=Cxe^{-\frac{x^2}{2\sigma^2}}P(x), \quad 0<x<2\pi, \tag{26}$$

where $\alpha\ge -1,\mu\in\mathbb{R},\sigma>0,\lambda\in\mathbb{R}$ are parameters, $C=\frac{2}{\sigma(\alpha+2)\sqrt{2\pi}}$, $P(x)=x^{-1}\sum_{k=-\infty}^{\infty}e^{-\frac{1}{2\sigma^2}\left[(2\pi k-\mu)x+(2\pi k-\mu)^2\right]}\left[1+\sigma\Phi\left(\frac{\lambda}{\sigma}(x+2\pi k-\mu)\right)\right]$ and $\Phi(x)$ is the cdf of the standard normal random variable.

1.14. Wrapped Generalized Skew Normal-2 (WGSN-2) Distribution

$$F\left(x;\alpha,\beta,\mu,\omega\right)=\int_0^x f\left(u;\alpha,\beta,\mu,\omega\right)du,\quad 0\leq x\leq 2\pi, \tag{27}$$

and

$$f\left(x;\alpha,\beta,\mu,\omega\right)=Cxe^{-\frac{x^2}{2w^2}}P\left(x\right),\quad 0<x<2\pi, \tag{28}$$

where $\alpha\in\mathbb{R},\beta\in\mathbb{R},\mu\in\mathbb{R},\omega>0$ are parameters, $C=\frac{2}{\omega\sqrt{2\pi}}$,

$$P\left(x\right)=x^{-1}\sum_{k=-\infty}^{\infty}e^{-\frac{1}{2\omega^2}\left[(2\pi k-\mu)x+(2\pi k-\mu)^2\right]}\Phi\left(\alpha\left(\frac{x+2\pi k-\mu}{\omega}\right)+\beta\left(\frac{x+2\pi k-\mu}{\omega}\right)^3\right),$$

and $\Phi\left(x\right)$ is the cdf of the standard normal random variable.

1.15. Wrapped Half-Logistic (WHL) Distribution

$$F\left(x;\mu,\sigma\right)=\int_0^x f\left(u;\mu,\sigma\right)du,\quad 0\leq x\leq 2\pi, \tag{29}$$

and

$$f\left(x;\mu,\sigma\right)=Ce^{-\frac{x}{\sigma}}P\left(x\right),\quad 0<x<2\pi, \tag{30}$$

where $0\leq\mu<2\pi,\sigma>0$ are parameters, $C=\frac{2}{\sigma}$ and $P(x)=\sum\limits_{k=-\infty}^{\infty}\frac{1}{\left(e^{\frac{2k\pi-\mu}{\sigma}}+e^{-\frac{2x+2k\pi-\mu}{\sigma}}+2e^{\frac{-x}{\sigma}}\right)}$.

1.16. Wrapped Half-Normal (WHN) Distribution

$$F\left(x;\mu,\sigma\right)=\int_0^x f\left(u;\mu,\sigma\right)du,\quad 0\leq x\leq 2\pi, \tag{31}$$

and

$$f\left(x;\mu,\sigma\right)=Ce^{-\frac{x}{\sigma}}P\left(x\right),\quad 0<x<2\pi, \tag{32}$$

where $0\leq\mu<2\pi,\sigma>0$ are parameters, $C=\frac{2}{\sigma}$ and $P(x)=\sum\limits_{k=0}^{\infty}\frac{1}{\left(e^{\frac{2k\pi-\mu}{\sigma}}+e^{-\frac{2x+2k\pi-\mu}{\sigma}}+2e^{\frac{-x}{\sigma}}\right)}$.

1.17. Wrapped [12]'s Skew Laplace (W[12]'sSL) Distribution

$$F(x;\beta,c,p)=\frac{pe^{c(\beta+2\pi-2\beta\pi)}\left(1-e^{-cx}\right)+\left(1-p\right)e^{(2\pi-1)c\beta}\left(e^{c\beta}-1\right)}{e^{2\pi c}-1},\quad 0\leq x\leq 2\pi, \tag{33}$$

and

$$f\left(x;\beta,c,p\right)=Ce^{-cx}P\left(x\right),\quad 0<x<2\pi, \tag{34}$$

where $\beta\in\mathbb{R},c.0,p\in(0,1)$ are parameters, $C=\frac{1}{e^{2\pi c}-1}$ and $P\left(x\right)=pce^{c(\beta+2\pi-2\beta\pi)}+\left(1-p\right)e^{c(x+(1+(2\pi-1))\beta)}$.

1.18. Wrapped Hyperexponential (WHE) Distribution

$$F(x;\lambda_1,\lambda_2)=\int_0^x f(u;\lambda_1,\lambda_2)du,\quad 0\leq x\leq 2\pi, \tag{35}$$

and

$$f\left(x;\lambda_1,\lambda_2\right)=Ce^{-\lambda_1 x}P\left(x\right),\quad 0<x<2\pi, \tag{36}$$

where $\lambda_1>0,\lambda_2>0$ are parameters, $C=\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}$ and $P\left(x\right)=\sum_{k=0}^{\infty}e^{-2k\pi\lambda_1}-e^{-(\lambda_2-\lambda_1)x}\sum_{k=0}^{\infty}e^{-2k\pi\lambda_2}$.

1.19. Wrapped Ishita (WIshita) Distribution

$$F(x;\lambda)=\int_0^x f(u;\lambda)\,du,\quad 0\le x\le 2\pi, \tag{37}$$

and

$$f(x;\lambda)=Ce^{-\lambda x}P(x),\quad 0<x<2\pi, \tag{38}$$

where $\lambda>0$ is a parameter, $C=\frac{\lambda^3}{\lambda+2}$ and $P(x)=\frac{\lambda+x^2}{1-e^{-2\pi\lambda}}+\frac{4\pi x e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^2}+\frac{4\pi^2 e^{-2\pi\lambda}\left(1+e^{-2\pi\lambda}\right)}{(1-e^{-2\pi\lambda})^3}$.

1.20. Wrapped Laplace (WL) Distribution

$$F(x;\kappa,\lambda)=\int_0^x f(u;\kappa,\lambda)\,du,\quad 0\le x\le 2\pi, \tag{39}$$

and

$$f(x;\kappa,\lambda)=Ce^{-\lambda\kappa x}P(x),\quad 0<x<2\pi, \tag{40}$$

where $\lambda>0$ is a parameter, $C=\frac{\lambda\kappa}{1+\kappa^2}$ and $P(x)=\frac{1}{1-e^{-2\pi\lambda\kappa}}+\frac{e^{\left(\lambda\kappa+\frac{\lambda}{\kappa}\right)x}}{e^{\pi\lambda/\kappa}-1}$.

1.21. Wrapped Length-Biased Weighted Exponential (WLBWE) Distribution

$$F(x;\alpha,\lambda)=\int_0^x f(u;\alpha,\lambda)\,du,\quad 0\le x\le 2\pi, \tag{41}$$

and

$$f(x;\alpha,\lambda)=Ce^{-\lambda x}P(x),\quad 0<x<2\pi, \tag{42}$$

where $\alpha>0,\lambda>0$ are parameters, $C=\frac{[\lambda(\alpha+1)]^2}{\alpha(\alpha+2)}$ and $P(x)=\frac{1}{1-e^{-2\pi\lambda}}\left(x+\frac{2\pi e^{-2\pi\lambda}}{1-e^{-2\pi\lambda}}\right)-\frac{e^{-\alpha\lambda x}}{1-e^{-2\pi\lambda(1+\alpha)}}\left[x+\frac{2\pi e^{-2\pi\lambda(1+\alpha)}}{1-e^{-2\pi\lambda(1+\alpha)}}\right]$.

1.22. Wrapped Levy (WLevy) Distribution

$$F(x;c,\mu)=\int_0^x f(u;c,\mu)\,du,\quad 0\le x\le 2\pi, \tag{43}$$

and

$$f(x;c,\mu)=Ce^{-\frac{c}{2}x}P(x),\quad 0<x<2\pi, \tag{44}$$

where $c>0,\mu\in\mathbb{R}$ are parameters, $C=e^{\frac{c\mu}{2}}\sqrt{\frac{c}{2\pi}}$ and $P(x)=\sum_{n=-\infty}^{\infty}\frac{e^{-c\pi n}}{(x+2\pi n-\mu)^{3/2}}$.

1.23. Wrapped Linnik (WLinnik) Distribution

$$F(x;\lambda)=\int_0^x f(u;\lambda)\,du,\quad 0\le x\le 2\pi, \tag{45}$$

and

$$f(x;\lambda)=Ce^{-\lambda x}P(x),\quad 0<x<2\pi, \tag{46}$$

where $\lambda>0$ is a parameter, $C=\frac{\lambda^2}{(1+\lambda)(1-e^{-2\pi\lambda})^2}$ and $P(x)=(1+x)\left(1-e^{-2\pi\lambda}\right)+2\pi e^{-2\pi\lambda}$.

1.24. Wrapped Lindley (WLindley) Distribution

$$F\left(x;\sigma,\alpha\right)=\frac{1}{2\pi}\left[x+2\sum_{k=1}^{\infty}\frac{\sin\left(kx\right)}{k\left(1+\sigma k^{\alpha}\right)}\right],\quad 0\leq x\leq 2\pi, \tag{47}$$

and

$$f\left(x;\sigma,\alpha\right)=C(\cos x)P\left(x\right),\quad 0<x<2\pi, \tag{48}$$

where $\sigma>0, 0<\alpha\leq 2$ are parameter, $C=\frac{1}{2\pi}$ and $P\left(x\right)=\left(\cos x\right)^{-1}\left[1+2\sum_{k=1}^{\infty}\frac{\cos(kx)}{1+\sigma k^{\alpha}}\right]$.

1.25. Wrapped Lomax (WLomax) Distribution

$$F\left(x;\sigma,\alpha\right)=\sum_{k=0}^{\infty}\left(1+\frac{2k\pi}{\sigma}\right)^{-\alpha}-\sum_{k=0}^{\infty}\left(1+\frac{x+2k\pi}{\sigma}\right)^{-\alpha},\quad 0\leq x\leq 2\pi, \tag{49}$$

and

$$f\left(x;\sigma,\alpha\right)=Cx^{-(\alpha+1)}P\left(x\right),\quad 0<x<2\pi, \tag{50}$$

where $\sigma>0,\alpha>0$ are parameters, $C=\frac{\alpha}{\sigma}$ and $P\left(x\right)=\sum_{k=0}^{\infty}\left(x^{-1}+\frac{1+2k\pi x^{-1}}{\sigma}\right)^{-\alpha-1}$.

1.26. Wrapped Modified Lindley (WML) Distribution

$$F\left(x;\lambda\right)=\int_{0}^{x}f\left(u;\lambda\right)du,\quad 0\leq x\leq 2\pi, \tag{51}$$

and

$$f\left(x;\lambda\right)=Ce^{-\lambda x}P\left(x\right),\quad 0<x<2\pi, \tag{52}$$

where $\lambda>0$ is a parameter, $C=\frac{\lambda}{1-e^{-4\pi\lambda}}$ and $P\left(x\right)=1+e^{-2\pi\lambda}+\frac{e^{-\lambda x}}{1+\lambda}\left[2\lambda x-1+\frac{4\lambda\pi e^{-4\lambda\pi}}{1-e^{-4\lambda\pi}}\right]$.

1.27. Wrapped Weibull Pareto (WWP) Distribution

$$F\left(x;\mu,\sigma\right)=\int_{0}^{x}f\left(u;\mu,\sigma\right)du,\quad 0\leq x\leq 2\pi, \tag{53}$$

and

$$f\left(x;\mu,\sigma\right)=Ce^{-x}P\left(x\right),\quad 0<x<2\pi, \tag{54}$$

where $\mu\geq 0,\sigma>0$ are parameters, $x>\mu$, WLOG we take $\mu=0,\sigma=1$, $C=1$ and $P\left(x\right)=\sum\limits_{k=-\infty}^{\infty}\frac{1}{e^{2k\pi}+e^{-2x-2k\pi}+2e^{-x}}$.

1.28. Wrapped Normal (WN-1) Distribution

$$F\left(x;\mu,\sigma\right)=\sum_{k=-\infty}^{\infty}\left[\Phi\left(x+2\pi k\right)-\Phi\left(2\pi k\right)\right],\quad 0\leq x\leq 2\pi, \tag{55}$$

and

$$f\left(x;\mu,\sigma\right)=Cxe^{-x^{2}/2}P\left(x\right),\quad 0<x<2\pi, \tag{56}$$

where $\mu\in\mathbb{R},\sigma>0$ are parameters, WLOG we take $\mu=0,\sigma=1$, $C=\frac{1}{\sqrt{2\pi}}$ and $P\left(x\right)=x^{-1}\sum_{k=-\infty}^{\infty}e^{-\pi kx-2(\pi k)^{2}}$.

1.29. Wrapped Pareto (WP) Distribution

$$F\left(x;\lambda,\alpha\right)=\int_{0}^{x}f\left(u;\lambda,\alpha\right)du,\quad 0\leq x\leq 2\pi, \tag{57}$$

and

$$f\left(x;\lambda,\alpha\right)=Ce^{-\lambda x}P\left(x\right),\quad 0<x<2\pi, \tag{58}$$

where $\lambda>0,\alpha>0$ are parameters, $C=\frac{\alpha^{-1/\alpha}}{\Gamma(1/\alpha)},\Gamma\left(\alpha\right)$ is the gamma function and $P\left(x\right)=\int_{0}^{\infty}\frac{\lambda se^{\lambda x(1-s)}}{1-e^{-2\pi\lambda s}}s^{\frac{1}{\alpha}-1}e^{-\frac{s}{\alpha}}ds.$

1.30. Wrapped Quasi-Lindley (WQL) Distribution

$$F\left(x;a,b\right)=\int_{0}^{x}f\left(u;a,b\right)du,\quad 0\leq x\leq 2\pi, \tag{59}$$

and

$$f\left(x;a,b\right)=Ce^{-bx}P\left(x\right),\quad 0<x<2\pi, \tag{60}$$

where $a>0,b>0$ are parameters, $C=\frac{b}{a+1}$ and $P\left(x\right)=\frac{a+bx}{1-e^{-2\pi b}}+\frac{2\pi be^{-2\pi b}}{(1-e^{-2\pi b})^{2}}.$

1.31. Wrapped Rama (WRama) Distribution

$$F\left(x;\lambda\right)=\int_{0}^{x}f\left(u;\lambda\right)du,\quad 0\leq x\leq 2\pi, \tag{61}$$

and

$$f\left(x;\lambda\right)=Ce^{-\lambda x}P\left(x\right),\quad 0<x<2\pi, \tag{62}$$

where $\lambda>0$ is a parameter, $C=\frac{\lambda^{4}}{(\lambda^{3}+6)(1-e^{-2\lambda\pi})}$ and $P\left(x\right)=x^{3}+\left(\frac{6\pi e^{-2\lambda\pi}}{1-e^{-2\lambda\pi}}\right)x^{2}+\left(\frac{2\pi\left(e^{-2\lambda\pi}+1\right)}{1-e^{-2\lambda\pi}}\right)\left(\frac{6\pi e^{-2\lambda\pi}}{1-e^{-2\lambda\pi}}\right)x+1+\frac{8\pi^{2}e^{-2\lambda\pi}\left(e^{-4\lambda\pi}+4e^{-2\lambda\pi}+1\right)}{(1-e^{-2\lambda\pi})^{3}}.$

1.32. Wrapped Richard (WRichard) Distribution

$$F\left(x;m,n\right)=\int_{0}^{x}f\left(u;m,n\right)du,\quad 0\leq x\leq 2\pi, \tag{63}$$

and

$$f\left(x;m,n\right)=e^{-mx}P\left(x\right),\quad 0<x<2\pi, \tag{64}$$

where $m\geq 1,n\geq 1$ are parameters and $P\left(x\right)=\sum_{k=0}^{\infty}\frac{ke^{-k(x+2\pi n)+mx}}{1-m^{\frac{1}{1-m}}}\left[1+\left(m-1\right)e^{-k(x+2\pi n)}\right]^{\frac{m}{1-m}}.$

1.33. Wrapped Shanker (WShanker) Distribution

$$F\left(x;\lambda\right)=\frac{1}{1-e^{-2\pi\lambda}}\left(1-e^{-\lambda x}-\frac{\lambda xe^{-\lambda x}}{\lambda^{2}+1}\right)+\frac{2\pi\lambda\left(1-e^{-\lambda x}\right)e^{-2\pi\lambda}}{\left(\lambda^{2}+1\right)\left(1-e^{-2\pi\lambda}\right)^{2}},\quad 0\leq x\leq 2\pi, \tag{65}$$

and

$$f\left(x;\lambda\right)=Ce^{-\lambda x}P\left(x\right),\quad 0<x<2\pi, \tag{66}$$

where $\lambda>0$ is a parameter, $C=\frac{\lambda^{2}}{\lambda^{2}+1}$ and $P\left(x\right)=\frac{\lambda+x}{1-e^{-2\pi\lambda}}+\frac{2\pi e^{-2\pi\lambda}}{(1-e^{-2\pi\lambda})^{2}}.$

1.34. Wrapped Skew Laplace (WSL) Distribution

$$F(x;\lambda_1,\lambda_2,\lambda_3,\mu,\sigma)=\int_0^x f(u;\lambda_1,\lambda_2,\lambda_3,\mu,\sigma)\,du,\quad 0\le x\le 2\pi, \tag{67}$$

and

$$f(x;\lambda_1,\lambda_2,\lambda_3,\mu,\sigma)=Ce^{-x}P(x)\,,\quad 0<x<2\pi, \tag{68}$$

where $\lambda_1\in\mathbb{R},\lambda_2>0,\lambda_3\in\mathbb{R},\ \mu\in\mathbb{R},\sigma>0$ are parameters, WLOG we take $\mu=0,\sigma=1,\ C=\frac{1}{2},\ P(x)=1+\frac{e^{2x}+1}{e^{2\pi}-1}+e^{x}A(x;\lambda_1,\lambda_2,\lambda_3)\,,\ A(x;\lambda_1,\lambda_2,\lambda_3)=\sum_{k=-\infty}^{\infty}e^{-|x+2\pi k|}\left[1-e^{-|g(x+2\pi k)|}\right]\operatorname{sign}\left[g(x+2\pi k)\right]$ and $g(x)=\frac{\lambda_1 x+\lambda_3 x^3}{\sqrt{1+\lambda_2 x^2}}$.

1.35. Wrapped Normal (WN-2) Distribution

$$F(x;\xi,\zeta,\lambda)=\int_0^x f(u;\xi,\zeta,\lambda)\,du,\quad 0\le x\le 2\pi, \tag{69}$$

and

$$f(x;\xi,\zeta,\lambda)=Cxe^{-x^2/2}P(x)\,,\quad 0<x<2\pi, \tag{70}$$

where $\xi>0,\zeta\in\mathbb{R},\lambda\in\mathbb{R}$ are parameters, WLOG we take $\zeta=0,\xi=1,\ C=\frac{2}{\sqrt{2\pi}},\ P(x)=x^{-1}\sum_{k=-\infty}^{\infty}e^{-2\pi kx-2(\pi k)^2}\Phi(\lambda(x+2\pi k))$ and $\Phi(x)$ is the cdf of the standard normal random variable.

1.36. Wrapped Stable (WS) Distribution

cdf not analytically expressible,

and

pdf not analytically expressible,

1.37. Wrapped Student's t (WSt) Distribution

$$F(x;\mu,\lambda,\nu)=\int_0^x f(u;\mu,\lambda,\nu)\,du,\quad 0\le x\le 2\pi, \tag{71}$$

and

$$f(x;\mu,\lambda,\nu)=Cx^{-\frac{\nu+1}{2}-1}P(x)\,,\quad 0<x<2\pi, \tag{72}$$

where $0\le\mu<2\pi,\lambda>0,\nu>0$ are parameters, $C=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\lambda\Gamma(\nu/2)\sqrt{\pi\nu}},\Gamma(\nu)$ is the gamma function and $P(x)=\sum_{k=-\infty}^{\infty}x^{\frac{\nu+1}{2}+1}\left[1+\frac{(x+2\pi k-\mu)^2}{\lambda^2\nu}\right]^{-\frac{\nu+1}{2}}$.

1.38. Wrapped Transmuted Exponential (WTE) Distribution

$$F(x;\lambda,\Lambda)=\frac{\left(e^{-\lambda x}\right)\left[(e^{-2\pi\lambda}-1)+\Lambda\left(1+(e^{-2\pi\lambda}-1)\right)-e^{-\lambda x}\right]}{(e^{-2\pi\lambda}-1)^2},\quad 0\le x\le 2\pi, \tag{73}$$

and

$$f(x;\lambda,\Lambda)=Ce^{-\lambda x}P(x)\,,\quad 0<x<2\pi, \tag{74}$$

where $\lambda>0,|\Lambda|\le 1$ are parameters, $C=\frac{2\lambda\Lambda}{(e^{-2\pi\lambda}-1)^2}$ and $P(x)=(e^{-\lambda x}-1)-\lambda(\Lambda+1)(e^{-2\pi\lambda}-1)$.

1.39. Wrapped Two-Parameter Lindley (WTPL) Distribution

$$
\begin{aligned}
F(x;\zeta,\alpha) = & \frac{1}{1-e^{-2\pi\zeta}}\left(1-e^{-\zeta x}-\frac{\alpha\zeta x}{\alpha+\zeta}\right) \\
& + \frac{2\pi\alpha\zeta}{\alpha+\zeta}\left(1-e^{-\zeta x}\right)\frac{e^{-2\pi\zeta}}{\left(1-e^{-2\pi\zeta}\right)^2}, \quad 0 \le x \le 2\pi,
\end{aligned} \tag{75}
$$

and

$$
f(x;\zeta,\alpha) = Ce^{-\zeta x}P(x), \quad 0 < x < 2\pi, \tag{76}
$$

where $\zeta > 0, \alpha > -\zeta$ a parameters, $C = \frac{\zeta^2}{(\zeta+\alpha)(1-e^{-2\pi\zeta})^2}$ and $P(x) = (1+\alpha x)\left(1-e^{-2\pi\zeta}\right) + 2\pi\alpha e^{-2\pi\zeta}$.

1.40. Wrapped Two-Sided Lindley (WTSL) Distribution

$$
F(x;\alpha,\beta) = \int_0^x f(u;\alpha,\beta)\,du, \quad 0 \le x \le 2\pi, \tag{77}
$$

and

$$
f(x;\alpha,\beta) = Ce^{-\alpha x}P(x), \quad 0 < x < 2\pi, \tag{78}
$$

where $\alpha > 0, \beta > 0$ a parameters, $C = \frac{1}{2(\alpha+1)(e^{-2\pi\alpha}-1)^2(\beta+1)(e^{-2\pi\beta}-1)^2}$ and

$$
\begin{aligned}
P(x) = & \alpha^2 e^{2\pi\alpha}(\beta+1)\left(e^{-2\pi\beta}-1\right)^2\left[(x+1)e^{2\pi\alpha}-x+2\pi-1\right] \\
& + \beta^2 e^{x(\alpha+\beta)}(\alpha+1)\left(e^{-2\pi\alpha}-1\right)^2\left[(2\pi+1-x)e^{2\pi\beta}+x-1\right].
\end{aligned}
$$

1.41. Wrapped Variance Gamma (WVG) Distribution

Lacks a closed-form pdf or cdf.

1.42. Wrapped Weighted Exponential (WWE) Distribution

$$
F(x;\alpha,\lambda) = \int_0^x f(u;\alpha,\lambda)\,du, \quad 0 \le x \le 2\pi, \tag{79}
$$

and

$$
f(x;\alpha,\lambda) = Ce^{-\lambda x}P(x), \quad 0 < x < 2\pi, \tag{80}
$$

where $\alpha > 0, \lambda > 0$ are parameters, $C = \frac{\lambda(\alpha+1)}{\alpha}$ and $P(x) = \sum_{k=0}^{\infty} e^{-2k\pi\lambda}\left[1-e^{-\alpha\lambda(x+2k\pi)}\right]$.

1.43. Wrapped Weibull (WWeibull) Distribution

$$
F(x;\mu,\sigma) = \int_0^x f(u;\mu,\sigma)\,du, \quad 0 \le x \le 2\pi, \tag{81}
$$

and

$$
f(x;\mu,\sigma) = Ce^{-\lambda x}P(x), \quad 0 < x < 2\pi, \tag{82}
$$

where $\mu \in \mathbb{R}, \sigma > 0$ are parameters, WLOG we take $\mu = 0, \sigma = 1$,$C = \frac{2}{\sigma}$ and $P(x) = \sum\limits_{k=0}^{\infty} \frac{1}{(e^{2k\pi}+e^{-2x-2k\pi}+2e^{-x})}$.

1.44. Wrapped XGamma (WXG) Distribution

$$F(x;\lambda) = \int_0^x f(u;\lambda)\,du, \quad 0 \le x \le 2\pi, \tag{83}$$

and

$$f(x;\lambda) = Ce^{-\lambda x} P(x), \quad 0 < x < 2\pi, \tag{84}$$

where $\lambda > 0$ is a parameter, $C = \frac{\lambda^2}{(\lambda+1)(1-e^{-2\pi\lambda})}$ and

$$P(x) = 1 + \frac{\lambda x^2}{2} + 2\pi\lambda\left[(\pi - x)e^{-2\pi\lambda} + (x+\pi)\right]\frac{e^{-2\pi\lambda}}{\left(1-e^{-2\pi\lambda}\right)^2}.$$

1.45. Wrapped XLindley (WXL) Distribution

$$F(x;\lambda) = \int_0^x f(u;\lambda)\,du, \quad 0 \le x \le 2\pi, \tag{85}$$

and

$$f(x;\lambda) = Ce^{-\lambda x} P(x), \quad 0 < x < 2\pi, \tag{86}$$

where $\lambda > 0$ is a parameter, $C = \frac{\lambda^2}{(\lambda+1)^2(1-e^{-2\pi\lambda})^2}$ and $P(x) = \left(1-e^{-2\pi\lambda}\right)(\lambda + x + 2) + 2\pi e^{-2\pi\lambda}$.

1.46. Wrapped Binomial (WBinomial) Distribution

$$F(x;r,m,n,p) = \sum_{u=0}^{r} \lambda^u P(u), \quad x = \frac{2\pi r}{m}, r = 0,1,...,m-1, \tag{87}$$

and

$$f(x;r,m,n,p) = \lambda^r P(r), \quad x = \frac{2\pi r}{m}, r = 0,1,...,m-1, \tag{88}$$

where $r = 0,1,...,m-1, m \ge 1, n \ge m-1, 0 < p < 1$ are parameters, $\lambda = \left(\frac{p}{1-p}\right)$ and $P(r) = \sum_{k=0}^{\left[\frac{n}{m}-\frac{x}{2\pi}\right]} \binom{n}{\frac{mx}{2\pi}+km} p^{km}(1-p)^{n-km}$.

1.47. Wrapped Discrete Cauchy (WDC) Distribution

There is no r on the RHS of the formula given for the pmf.

1.48. Wrapped Discrete Exponential (WDE) Distribution

$$F(x;m,\lambda) = \frac{1 - e^{-\frac{2\pi\lambda(x+1)}{m}}}{1-e^{-2\pi\lambda}}, \quad x \in I, \tag{89}$$

and

$$f(x;m,\lambda) = Ce^{-\lambda x}, \quad x \in I, \tag{90}$$

where $\lambda > 0$ are parameters, $C = \frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}$ and $I = \{0,1,...,m-1\}$.

1.49. Wrapped Discrete Mittag-Leffler (WDML) Distribution

No closed form for cdf or pmf.

1.50. Wrapped Discrete Skew Laplace (WDSL) Distribution

$$F\left(x;r,m,p,q\right)=C\sum_{r=0}^{x}q^{u}P\left(u\right),\quad x=\frac{2\pi r}{m},\quad r\in I, \tag{91}$$

and

$$f\left(x;r,m,p,q\right)=Cq^{r}P\left(r\right),\quad x=\frac{2\pi r}{m},\quad r\in I, \tag{92}$$

where $r=0,1,...,m-1,m\in\mathbb{N},p\in(0,1),q\in(0,1)$ are parameters, $C=\frac{(1-p)(1-q)}{(1-pq)(1-p^{m})(1-q^{m})}$ and $P\left(r\right)=q^{m-2r}\left(1-p^{m}\right)+\left(p/q\right)^{r}\left(1-q^{m}\right)$.

1.51. Wrapped Geometric (WG) Distribution

$$F\left(x;r,m,\delta\right)=\sum_{r=0}^{x}\frac{\delta\left(1-\delta\right)^{r}}{1-\left(1-\delta\right)^{m}},\quad x\in I, \tag{93}$$

and

$$f\left(x;r,m,\delta\right)=\frac{\delta\left(1-\delta\right)^{x}}{1-\left(1-\delta\right)^{m}},\quad x\in I, \tag{94}$$

where $x=0,1,...,m-1,m\in\mathbb{N},\delta>0$ are parameters.

1.52. Wrapped Negative Binomial (WNB) Distribution

$$F\left(x;r,m,n,p\right)=\sum_{u=0}^{r}\left(1-p\right)^{u}P\left(u\right),\quad x=\frac{2\pi r}{m},\quad r\in I, \tag{95}$$

and

$$f\left(x;r,m,n,p\right)=\left(1-p\right)^{r}P\left(r\right),\quad x=\frac{2\pi r}{m},\quad r\in I, \tag{96}$$

where $r=0,1,...,m-1,m\in\mathbb{N},n\in\mathbb{Z}^{+},p\in[0,1]$ are parameters and $P\left(r\right)=\sum_{k=0}^{\infty}\binom{r+km+n-1}{n-1}p^{n}\left(1-p\right)^{km}$.

1.53. Wrapped Poisson (WP) Distribution

$$F\left(x;r,m,\lambda\right)=\sum_{u=0}^{r}\lambda^{r}P\left(r\right),\quad x=\frac{2\pi r}{m},\quad r\in I, \tag{97}$$

and

$$f\left(x;r,m,\lambda\right)=\lambda^{r}P\left(r\right),\quad x=\frac{2\pi r}{m},\quad r\in I, \tag{98}$$

where $r=0,1,...,m-1,m\in\mathbb{N},\lambda>0$ are parameters and $P\left(r\right)=\sum_{k=0}^{\infty}\frac{e^{-\lambda}\lambda^{km}}{(r+km)!}$.

1.54. Wrapped Poisson-Lindley (WPL) Distribution

$$F\left(x;r,m,\theta\right)=C\sum_{u=0}^{r}\lambda^{u}P\left(u\right),\quad x=\frac{2\pi r}{m},\quad r\in I, \tag{99}$$

and

$$f\left(x;r,m,\theta\right)=C\lambda^{r}P\left(r\right),\quad x=\frac{2\pi r}{m},\quad r\in I, \tag{100}$$

where $r = 0, 1, ..., m-1, m \in \mathbb{N}, \theta > 0$ are parameters, $C = \frac{\theta^2(1+\theta)^{m-3}}{[(1+\theta)^m - 1]^2}, \lambda = \frac{1}{1+\theta}$ and $P(r) = (r + \theta + 2)[(1+\theta)^m - 1] + m$.

1.55. Wrapped Zero-Inflated Poisson (WZIP) Distribution

$$F(x; r, m, \lambda, \omega) = \begin{cases} \omega + (1-\omega) e^{-\lambda}, & x = 0 \\ C \sum_{u=0}^{x} \lambda^u P(u), & x \in \mathbb{N} \end{cases} \tag{101}$$

and

$$f(x; r, m, \lambda, \omega) = \begin{cases} \omega + (1-\omega) e^{-\lambda}, & x = 0 \\ C \lambda^x P(x), & x \in \mathbb{N} \end{cases} \tag{102}$$

where $r = 0, 1, ...m-1, m \in \mathbb{N}, \lambda > 0, \omega \in [0,1], C = (1-\omega) e^{-\lambda}$ and $P(x) = \frac{1}{x!}$.

2. Wrapped Continuous Distributions

As mentioned in the Introduction, characterizations of distributions is an important area of research which has recently attracted the attention of many researchers. This section deals with various characterizations of the continuous distributions listed in the Introduction. These characterizations are based on a simple relationship between two truncated moments. It should be mentioned that for this characterization the cdf need not have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

2.1. Characterizations Based on Two Truncated Moments

Our first characterization result employs a theorem due to [52], see Theorem 1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in [53], this characterization is stable in the sense of weak convergence.

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ $(d = -\infty,\ e = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}[q_2(X) \mid X \geq x] = \mathbf{E}[q_1(X) \mid X \geq x]\, \eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η, particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u) q_1(u) - q_2(u)} \right| \exp(-s(u))\, du,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\, q_1}{\eta\, q_1 - q_2}$ and C is the normalization constant, such that $\int_H dF = 1$.

Remark. The goal is to have as simple the function $\eta(x)$ as possible, so it is important to choose appropriate functions $q_1(x)$ and $q_2(x)$ to achieve the goal.

We have grouped the distributions which have the same format and provided just one characterization proof for the Proposition 1 below for the continuous case. Similarly, for the discrete case.

Proposition 1. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x) e^{-\lambda x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (2) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\lambda x} + e^{-2\pi\lambda}\right), \quad x \in (0, 2\pi).$$

Proof
Let X be a random variable with pdf (2), then

$$(1 - F(x)) E[q_1(X) \mid X \geq x] = \int_x^{2\pi} Ce^{-\lambda u} du = \frac{C}{\lambda}\left(e^{-\lambda x} - e^{-2\pi\lambda}\right), \quad x \in (0, 2\pi),$$

and

$$(1 - F(x)) E[q_2(X) \mid X \geq x] = \int_x^{2\pi} Ce^{-2\lambda u} du = \frac{C}{2\lambda}\left(e^{-2\lambda x} - e^{-4\pi\lambda}\right), \quad x \in (0, 2\pi),$$

and finally

$$\eta(x) q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left(e^{-\lambda x} - e^{-2\pi\lambda}\right) < 0 \text{ for } x \in (0, 2\pi).$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}, \quad x \in (0, 2\pi),$$

and hence

$$s(x) = -\log\left(e^{-\lambda x} - e^{-2\pi\lambda}\right), \quad x \in (0, 2\pi).$$

Now, in view of Theorem 1, X has density (2). □

Corollary 1. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 1. The pdf of X is (2) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}, \quad x \in (0, 2\pi).$$

Corollary 2. The general solution of the differential equation in Corollary 1 is

$$\eta(x) = \left(e^{-\lambda x} - e^{-2\pi\lambda}\right)^{-1}\left[-\int \lambda e^{-\lambda x}(q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof
If X has pdf (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \left(\frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}\right)\eta(x) - \left(\frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}\right)(q_1(x))^{-1} q_2(x),$$

or

$$\eta'(x) - \left(\frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}\right)\eta(x) = -\left(\frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}\right)(q_1(x))^{-1} q_2(x),$$

or

$$\left(e^{-\lambda x} - e^{-2\pi\lambda}\right)\eta'(x) - \left(\lambda e^{-\lambda x}\right)\eta(x) = -\left(\lambda e^{-\lambda x}\right)(q_1(x))^{-1} q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(e^{-\lambda x} - e^{-2\pi\lambda}\right)\eta(x)\right\} = -\left(\lambda e^{-\lambda x}\right)(q_1(x))^{-1} q_2(x),$$

from which we arrive at

$$\eta(x) = \left(e^{-\lambda x} - e^{-2\pi\lambda}\right)^{-1}\left[-\int \lambda e^{-\lambda x}(q_1(x))^{-1} q_2(x)\,dx + D\right].$$

□

Note that a set of functions satisfying the differential equation in Corollary 1, is given in Proposition 1 with $D = \frac{e^{-4\pi\lambda}}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 2. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\lambda x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (4) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\lambda x} + e^{-2\pi\lambda}\right), \quad x \in (0, 2\pi).$$

Corollary 3. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2. The pdf of X is (4) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}, \quad x \in (0, 2\pi).$$

Corollary 4. The general solution of the differential equation in Corollary 3 is

$$\eta(x) = \left(e^{-\lambda x} - e^{-2\pi\lambda}\right)^{-1}\left[-\int \lambda e^{-\lambda x}(q_1(x))^{-1} q_2(x)\,dx + D\right],$$

where D is a constant.

Proposition 3. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\left(\frac{\delta+1}{4\mu}\right)x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (8) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\left(\frac{\delta+1}{4\mu}\right)x} + e^{-\left(\frac{\delta+1}{4\mu}\right)(2\pi)}\right), \quad x \in (0, 2\pi).$$

Corollary 5. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3. The pdf of X is (8) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\left(\frac{\delta+1}{4\mu}\right) e^{-\left(\frac{\delta+1}{4\mu}\right)x}}{e^{-\left(\frac{\delta+1}{4\mu}\right)x} - e^{-\left(\frac{\delta+1}{4\mu}\right)(2\pi)}}, \quad x \in (0, 2\pi).$$

Corollary 6. The general solution of the differential equation in Corollary 5 is

$$\eta(x)=\left(e^{-\left(\frac{\delta+1}{4\mu}\right)x}-e^{-\left(\frac{\delta+1}{4\mu}\right)(2\pi)}\right)^{-1}\left[-\int\left(\frac{\delta+1}{4\mu}\right)e^{-\left(\frac{\delta+1}{4\mu}\right)x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 4. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\frac{1}{2}x}$ for $x\in(0,2\pi)$. The random variable X has pdf (12) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\frac{1}{2}x}+e^{-\pi}\right),\quad x\in(0,2\pi).$$

Corollary 7. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4. The pdf of X is (12) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\frac{1}{2}e^{-\frac{1}{2}x}}{e^{-\frac{1}{2}x}-e^{-\pi}},\quad x\in(0,2\pi).$$

Corollary 8. The general solution of the differential equation in Corollary 7 is

$$\eta(x)=\left(e^{-\frac{1}{2}x}-e^{-\pi}\right)^{-1}\left[-\int\frac{1}{2}e^{-\frac{1}{2}x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 5. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)\equiv 1$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (14) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 9. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 5. The pdf of X is (14) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 10. The general solution of the differential equation in Corollary 9 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 6. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-x}$ for $x\in(0,2\pi)$. The random variable X has pdf (16) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-x}+e^{-2\pi}\right),\quad x\in(0,2\pi).$$

Corollary 11. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 6. The pdf of X is (16) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{e^{-x}}{e^{-x}-e^{-2\pi}},\quad x\in(0,2\pi).$$

Corollary 12. The general solution of the differential equation in Corollary 11 is

$$\eta(x)=\left(e^{-x}-e^{-2\pi}\right)^{-1}\left[-\int e^{-x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 7. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (18) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 13. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 7. The pdf of X is (18) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 14. The general solution of the differential equation in Corollary 13 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int \lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 8. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\frac{1}{\sigma}x}$ for $x\in(0,2\pi)$. The random variable X has pdf (30) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\frac{1}{\sigma}(2\pi)}+e^{-\frac{1}{\sigma}x}\right),\quad x\in(0,2\pi).$$

Corollary 15. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 8. The pdf of X is (30) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\frac{1}{\sigma}e^{-\frac{1}{\sigma}x}}{e^{-\frac{1}{\sigma}x}-e^{-\frac{1}{\sigma}(2\pi)}},\quad x\in(0,2\pi).$$

Corollary 16. The general solution of the differential equation in Corollary 15 is

$$\eta(x)=\left(e^{-\frac{1}{\sigma}x}-e^{-\frac{1}{\sigma}(2\pi)}\right)^{-1}\left[-\int \frac{1}{\sigma}e^{-\frac{1}{\sigma}x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 9. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\frac{1}{\sigma}x}$ for $x\in(0,2\pi)$. The random variable X has pdf (32) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\frac{1}{\sigma}(2\pi)}+e^{-\frac{1}{\sigma}x}\right),\quad x\in(0,2\pi).$$

Corollary 17. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 9. The pdf of X is (32) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\frac{1}{\sigma} e^{-\frac{1}{\sigma}x}}{e^{-\frac{1}{\sigma}x} - e^{-\frac{1}{\sigma}(2\pi)}}, \quad x \in (0, 2\pi).$$

Corollary 18. The general solution of the differential equation in Corollary 17 is

$$\eta(x) = \left(e^{-\frac{1}{\sigma}x} - e^{-\frac{1}{\sigma}(2\pi)}\right)^{-1} \left[-\int \frac{1}{\sigma} e^{-\frac{1}{\sigma}x} (q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 10. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-cx}$ for $x \in (0, 2\pi)$. The random variable X has pdf (34) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-cx} + e^{-2\pi c}\right), \quad x \in (0, 2\pi).$$

Corollary 19. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 10. The pdf of X is (34) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{ce^{-cx}}{e^{-cx} - e^{-2\pi c}}, \quad x \in (0, 2\pi).$$

Corollary 20. The general solution of the differential equation in Corollary 19 is

$$\eta(x) = \left(e^{-cx} - e^{-2\pi c}\right)^{-1} \left[-\int ce^{-cx} (q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 11. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\lambda_1 x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (36) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\lambda_1 x} + e^{-2\pi\lambda_1}\right), \quad x \in (0, 2\pi).$$

Corollary 21. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 11. The pdf of X is (36) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\lambda_1 e^{-\lambda_1 x}}{e^{-\lambda_1 x} - e^{-2\pi\lambda_1}}, \quad x \in (0, 2\pi).$$

Corollary 22. The general solution of the differential equation in Corollary 21 is

$$\eta(x) = \left(e^{-\lambda_1 x} - e^{-2\pi\lambda_1}\right)^{-1} \left[-\int \lambda_1 e^{-\lambda_1 x} (q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 12. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (38) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 23. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 12. The pdf of X is (38) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 24. The general solution of the differential equation in Corollary 23 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 13. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda\kappa x}$ for $x\in(0,2\pi)$. The random variable X has pdf (40) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda\kappa x}+e^{-2\pi\lambda\kappa}\right),\quad x\in(0,2\pi).$$

Corollary 25. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 13. The pdf of X is (40) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda\kappa e^{-\lambda\kappa x}}{e^{-\lambda\kappa x}-e^{-2\pi\lambda\kappa}},\quad x\in(0,2\pi).$$

Corollary 26. The general solution of the differential equation in Corollary 25 is

$$\eta(x)=\left(e^{-\lambda\kappa x}-e^{-2\pi\lambda\kappa}\right)^{-1}\left[-\int\lambda\kappa e^{-\lambda\kappa x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 14. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (42) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 27. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 14. The pdf of X is (42) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 28. The general solution of the differential equation in Corollary 27 is

$$\eta(x) = \left(e^{-\lambda x} - e^{-2\pi\lambda}\right)^{-1}\left[-\int \lambda e^{-\lambda x}\left(q_1(x)\right)^{-1} q_2(x)\,dx + D\right],$$

where D is a constant.

Proposition 15. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\,e^{-\frac{c}{2}x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (44) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\frac{c}{2}x} + e^{-2\pi\frac{c}{2}}\right), \quad x \in (0, 2\pi).$$

Corollary 29. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 15. The pdf of X is (44) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \frac{\frac{c}{2}e^{-\frac{c}{2}x}}{e^{-\frac{c}{2}x} - e^{-2\pi\frac{c}{2}}}, \quad x \in (0, 2\pi).$$

Corollary 30. The general solution of the differential equation in Corollary 29 is

$$\eta(x) = \left(e^{-\frac{c}{2}x} - e^{-2\pi\frac{c}{2}}\right)^{-1}\left[-\int \frac{c}{2}e^{-\frac{c}{2}x}\left(q_1(x)\right)^{-1} q_2(x)\,dx + D\right],$$

where D is a constant.

Proposition 16. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\,e^{-\lambda x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (46) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\lambda x} + e^{-2\pi\lambda}\right), \quad x \in (0, 2\pi).$$

Corollary 31. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 16. The pdf of X is (46) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}, \quad x \in (0, 2\pi).$$

Corollary 32. The general solution of the differential equation in Corollary 31 is

$$\eta(x) = \left(e^{-\lambda x} - e^{-2\pi\lambda}\right)^{-1}\left[-\int \lambda e^{-\lambda x}\left(q_1(x)\right)^{-1} q_2(x)\,dx + D\right],$$

where D is a constant.

Proposition 17. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\,e^{-\lambda x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (52) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\lambda x} + e^{-2\pi\lambda}\right), \quad x \in (0, 2\pi).$$

Corollary 33. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 17. The pdf of X is (52) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 34. The general solution of the differential equation in Corollary 33 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 18. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-x}$ for $x\in(0,2\pi)$. The random variable X has pdf (54) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-x}+e^{-2\pi}\right),\quad x\in(0,2\pi).$$

Corollary 35. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 18. The pdf of X is (54) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{e^{-x}}{e^{-x}-e^{-2\pi}},\quad x\in(0,2\pi).$$

Corollary 36. The general solution of the differential equation in Corollary 35 is

$$\eta(x)=\left(e^{-x}-e^{-2\pi}\right)^{-1}\left[-\int e^{-x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 19. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (58) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 37. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 19. The pdf of X is (58) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 38. The general solution of the differential equation in Corollary 37 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 20. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-bx}$ for $x\in(0,2\pi)$. The random variable X has pdf (60) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-bx}+e^{-2\pi b}\right),\quad x\in(0,2\pi).$$

Corollary 39. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 20. The pdf of X is (60) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{be^{-bx}}{e^{-bx}-e^{-2\pi b}},\quad x\in(0,2\pi).$$

Corollary 40. The general solution of the differential equation in Corollary 39 is

$$\eta(x)=\left(e^{-bx}-e^{-2\pi b}\right)^{-1}\left[-\int be^{-bx}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 21. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (62) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 41. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 21. The pdf of X is (62) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 42. The general solution of the differential equation in Corollary 41 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 22. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-mx}$ for $x\in(0,2\pi)$. The random variable X has pdf (64) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-mx}+e^{-2\pi m}\right),\quad x\in(0,2\pi).$$

Corollary 43. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 22. The pdf of (64) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{me^{-mx}}{e^{-mx}-e^{-2\pi m}},\quad x\in(0,2\pi).$$

Corollary 44. The general solution of the differential equation in Corollary 43 is

$$\eta(x)=\left(e^{-mx}-e^{-2\pi m}\right)^{-1}\left[-\int me^{-mx}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 23. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (66) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 45. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 23. The pdf of X is (66) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 46. The general solution of the differential equation in Corollary 45 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 24. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-x}$ for $x\in(0,2\pi)$. The random variable X has pdf (68) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-x}+e^{-2\pi}\right),\quad x\in(0,2\pi).$$

Corollary 47. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 24. The pdf of X is (68) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{e^{-x}}{e^{-x}-e^{-2\pi}},\quad x\in(0,2\pi).$$

Corollary 48. The general solution of the differential equation in Corollary 47 is

$$\eta(x)=\left(e^{-x}-e^{-2\pi}\right)^{-1}\left[-\int e^{-x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 25. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (74) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 49. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 25. The pdf of X is (74) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 50. The general solution of the differential equation in Corollary 49 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 26. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\zeta x}$ for $x\in(0,2\pi)$. The random variable X has pdf (76) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\zeta x}+e^{-2\pi\zeta}\right),\quad x\in(0,2\pi).$$

Corollary 51. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 26. The pdf of X is (76) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\zeta e^{-\zeta x}}{e^{-\zeta x} - e^{-2\pi\zeta}}, \quad x \in (0, 2\pi).$$

Corollary 52. The general solution of the differential equation in Corollary 51 is

$$\eta(x) = \left(e^{-\zeta x} - e^{-2\pi\zeta}\right)^{-1} \left[-\int \zeta e^{-\zeta x} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 27. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\alpha x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (78) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\alpha x} + e^{-2\pi\alpha}\right), \quad x \in (0, 2\pi).$$

Corollary 53. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 27. The pdf of X is (78) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha e^{-\alpha x}}{e^{-\alpha x} - e^{-2\pi\alpha}}, \quad x \in (0, 2\pi).$$

Corollary 54. The general solution of the differential equation in Corollary 53 is

$$\eta(x) = \left(e^{-\alpha x} - e^{-2\pi\alpha}\right)^{-1} \left[-\int \alpha e^{-\alpha x} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 28. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\lambda x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (80) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\lambda x} + e^{-2\pi\lambda}\right), \quad x \in (0, 2\pi).$$

Corollary 55. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 28. The pdf of X is (80) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x} - e^{-2\pi\lambda}}, \quad x \in (0, 2\pi).$$

Corollary 56. The general solution of the differential equation in Corollary 55 is

$$\eta(x) = \left(e^{-\lambda x} - e^{-2\pi\lambda}\right)^{-1} \left[-\int \lambda e^{-\lambda x} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 29. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\lambda x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (82) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\lambda x} + e^{-2\pi\lambda}\right), \quad x \in (0, 2\pi).$$

Corollary 57. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 29. The pdf of X is (82) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 58. The general solution of the differential equation in Corollary 57 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 30. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (84) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 59. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 30. The pdf of X is (84) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 60. The general solution of the differential equation in Corollary 59 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 31. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\lambda x}$ for $x\in(0,2\pi)$. The random variable X has pdf (86) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\lambda x}+e^{-2\pi\lambda}\right),\quad x\in(0,2\pi).$$

Corollary 61. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 31. The pdf of X is (86) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\lambda e^{-\lambda x}}{e^{-\lambda x}-e^{-2\pi\lambda}},\quad x\in(0,2\pi).$$

Corollary 62. The general solution of the differential equation in Corollary 61 is

$$\eta(x)=\left(e^{-\lambda x}-e^{-2\pi\lambda}\right)^{-1}\left[-\int\lambda e^{-\lambda x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proposition 32. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-x^2/2}$ for $x\in(0,2\pi)$. The random variable X has pdf (6) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-x^2/2}+e^{-2\pi^2}\right),\quad x\in(0,2\pi).$$

Proof
Let X be a random variable with pdf (6), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\int_x^{2\pi}Cue^{-u^2/2}du=C\left(e^{-x^2/2}-e^{-2\pi^2}\right),\quad x\in(0,2\pi),$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^{2\pi}Cue^{-u^2}du=\frac{C}{2}\left(e^{-x^2}-e^{-4\pi^2}\right),\quad x\in(0,2\pi),$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\left(e^{-x^2/2}-e^{-2\pi^2}\right)<0 \text{ for } x\in(0,2\pi).$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{xe^{-x^2/2}}{e^{-x^2/2}-e^{-2\pi^2}},\quad x\in(0,2\pi),$$

and hence

$$s(x)=-\log\left(e^{-x^2/2}-e^{-2\pi^2}\right),\quad x\in(0,2\pi).$$

Now, in view of Theorem 1, X has density (6). □

Corollary 63. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 32. The pdf of X is (6) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{xe^{-x^2/2}}{e^{-x^2/2}-e^{-2\pi^2}},\quad x\in(0,2\pi).$$

Corollary 64. The general solution of the differential equation in Corollary 63 is

$$\eta(x)=\left(e^{-x^2/2}-e^{-2\pi^2}\right)^{-1}\left[-\int xe^{-x^2/2}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof. If X has pdf (6), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x)=\left(\frac{xe^{-x^2/2}}{e^{-x^2/2}-e^{-2\pi^2}}\right)\eta(x)-\left(\frac{xe^{-x^2/2}}{e^{-x^2/2}-e^{-2\pi^2}}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\eta'(x)-\left(\frac{xe^{-x^2/2}}{e^{-x^2/2}-e^{-2\pi^2}}\right)\eta(x)=-\left(\frac{xe^{-x^2/2}}{e^{-x^2/2}-e^{-2\pi^2}}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\left(e^{-x^2/2}-e^{-2\pi^2}\right)\eta'(x)-\left(xe^{-x^2/2}\right)\eta(x)=-\left(xe^{-x^2/2}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(e^{-x^2/2}-e^{-2\pi^2}\right)\eta\left(x\right)\right\}=-\left(xe^{-x^2/2}\right)\left(q_1\left(x\right)\right)^{-1}q_2\left(x\right),$$

from which we arrive at

$$\eta\left(x\right)=\left(e^{-x^2/2}-e^{-2\pi^2}\right)^{-1}\left[-\int xe^{-x^2/2}\left(q_1\left(x\right)\right)^{-1}q_2\left(x\right)dx+D\right].$$

Note that a set of functions satisfying the differential equation in Corollary 63, is given in Proposition 32 with $D=\frac{e^{-4\pi^2}}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 33. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1\left(x\right)=\left[P\left(x\right)\right]^{-1}$ and $q_2\left(x\right)=q_1\left(x\right)e^{-x^2/2}$ for $x\in(0,2\pi)$. The random variable X has pdf (56) if and only if the function η defined in Theorem 1 has the form

$$\eta\left(x\right)=\frac{1}{2}\left(e^{-x^2/2}+e^{-2\pi^2}\right),\quad x\in(0,2\pi).$$

Corollary 65. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1\left(x\right)$ be as in Proposition 33. The pdf of X is (56) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'\left(x\right)q_1\left(x\right)}{\eta\left(x\right)q_1\left(x\right)-q_2\left(x\right)}=\frac{xe^{-x^2/2}}{e^{-x^2/2}-e^{-2\pi^2}},\quad x\in(0,2\pi).$$

Corollary 66. The general solution of the differential equation in Corollary 65 is

$$\eta\left(x\right)=\left(e^{-x^2/2}-e^{-2\pi^2}\right)^{-1}\left[-\int xe^{-x^2/2}\left(q_1\left(x\right)\right)^{-1}q_2\left(x\right)dx+D\right],$$

where D is a constant.

Proposition 34. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1\left(x\right)=\left[P\left(x\right)\right]^{-1}$ and $q_2\left(x\right)=q_1\left(x\right)e^{-x^2/2}$ for $x\in(0,2\pi)$. The random variable X has pdf (70) if and only if the function η defined in Theorem 1 has the form

$$\eta\left(x\right)=\frac{1}{2}\left(e^{-x^2/2}+e^{-2\pi^2}\right),\quad x\in(0,2\pi).$$

Corollary 67. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1\left(x\right)$ be as in Proposition 34. The pdf of X is (70) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'\left(x\right)q_1\left(x\right)}{\eta\left(x\right)q_1\left(x\right)-q_2\left(x\right)}=\frac{xe^{-x^2/2}}{e^{-x^2/2}-e^{-2\pi^2}},\quad x\in(0,2\pi).$$

Corollary 68. The general solution of the differential equation in Corollary 67 is

$$\eta\left(x\right)=\left(e^{-x^2/2}-e^{-2\pi^2}\right)^{-1}\left[-\int xe^{-x^2/2}\left(q_1\left(x\right)\right)^{-1}q_2\left(x\right)dx+D\right],$$

where D is a constant.

Proposition 35. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x) e^{-x^2/2\sigma^2}$ for $x \in (0, 2\pi)$. The random variable X has pdf (26) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-x^2/2\sigma^2} + e^{-4\pi^2/2\sigma^2}\right), \quad x \in (0, 2\pi).$$

Corollary 69. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 35. The pdf of X is (26) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\frac{x}{\sigma^2} e^{-x^2/2\sigma^2}}{e^{-x^2/2\sigma^2} - e^{-4\pi^2/2\sigma^2}}, \quad x \in (0, 2\pi).$$

Corollary 70. The general solution of the differential equation in Corollary 69 is

$$\eta(x) = \left(e^{-x^2/2\sigma^2} - e^{-4\pi^2/2\sigma^2}\right)^{-1} \left[-\int \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} (q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 36. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x) e^{-x^2/2\omega^2}$ for $x \in (0, 2\pi)$. The random variable X has pdf (28) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-x^2/2\omega^2} + e^{-4\pi^2/2\omega^2}\right), \quad x \in (0, 2\pi).$$

Corollary 71. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 36. The pdf of X is (28) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{\frac{x}{\omega^2} e^{-x^2/2\omega^2}}{e^{-x^2/2\omega^2} - e^{-4\pi^2/2\omega^2}}, \quad x \in (0, 2\pi).$$

Corollary 72. The general solution of the differential equation in Corollary 71 is

$$\eta(x) = \left(e^{-x^2/2\omega^2} - e^{-4\pi^2/2\omega^2}\right)^{-1} \left[-\int \frac{x}{\omega^2} e^{-x^2/2\omega^2} (q_1(x))^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 37. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = \sin(x - \mu)$ and $q_2(x) = q_1(x) \log(\cosh\gamma - \cos(x - \mu))$ for $x \in (0, 2\pi)$. The random variable X has pdf (10) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\{\log(\cosh\gamma - \cos(2\pi - \mu)) + \log(\cosh\gamma - \cos(x - \mu))\}, \quad x \in (0, 2\pi).$$

Proof
Let X be a random variable with pdf (10), then

$$\begin{aligned}(1 - F(x)) E[q_1(X) \mid X \geq x] &= \int_x^{2\pi} C\left(\frac{\sin(u - \mu)}{\cosh\gamma - \cos(u - \mu)}\right) du \\ &= C\{\log(\cosh\gamma - \cos(2\pi - \mu)) - \log(\cosh\gamma - \cos(x - \mu))\}, \quad x \in (0, 2\pi),\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E\left[q_2(X)\mid X\geq x\right] &= \int_x^{2\pi} C\left(\frac{\sin(u-\mu)}{\cosh\gamma-\cos(u-\mu)}\right)\left(\log\left(\cosh\gamma-\cos(u-\mu)\right)\right)du \\ &= \frac{C}{2}\left\{\left(\log\left(\cosh\gamma-\cos(2\pi-\mu)\right)\right)^2-\left(\log\left(\cosh\gamma-\cos(x-\mu)\right)\right)^2\right\}, \quad x\in(0,2\pi),\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\left(\log\left(\cosh\gamma-\cos(2\pi-\mu)\right)-\log\left(\cosh\gamma-\cos(x-\mu)\right)\right)<0 \text{ for } x\in(0,2\pi).$$

Conversely, if η is given as above, then

$$\begin{aligned}s'(x) &= \frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)} \\ &= \frac{\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}}{\log\left(\cosh\gamma-\cos(2\pi-\mu)\right)-\log\left(\cosh\gamma-\cos(x-\mu)\right)}, \quad x\in(0,2\pi),\end{aligned}$$

and hence

$$s(x)=-\log\left(\log\left(\cosh\gamma-\cos(2\pi-\mu)\right)-\log\left(\cosh\gamma-\cos(x-\mu)\right)\right), \quad x\in(0,2\pi).$$

Now, in view of Theorem 1, X has density (10). $\square$

Corollary 73. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 37. The pdf of X is (10) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}}{\log\left(\cosh\gamma-\cos(2\pi-\mu)\right)-\log\left(\cosh\gamma-\cos(x-\mu)\right)}, \quad x\in(0,2\pi).$$

Corollary 74. The general solution of the differential equation in Corollary 73 is

$$\begin{aligned}\eta(x) = {} & \left(\log\left(\cosh\gamma-\cos(2\pi-\mu)\right)-\log\left(\cosh\gamma-\cos(x-\mu)\right)\right)^{-1}\times \\ & \left[-\int\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],\end{aligned}$$

where D is a constant.

Proof
If X has pdf (10), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\begin{aligned}\eta'(x) = {} & \left(\frac{\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}}{\log\left(\cosh\gamma-\cos(2\pi-\mu)\right)-\log\left(\cosh\gamma-\cos(x-\mu)\right)}\right)\eta(x) \\ & -\left(\frac{\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}}{\log\left(\cosh\gamma-\cos(2\pi-\mu)\right)-\log\left(\cosh\gamma-\cos(x-\mu)\right)}\right)\left(q_1(x)\right)^{-1}q_2(x),\end{aligned}$$

or

$$\eta'(x)-\left(\frac{\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}}{\log(\cosh\gamma-\cos(2\pi-\mu))-\log(\cosh\gamma-\cos(x-\mu))}\right)\eta(x)$$
$$=-\left(\frac{\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}}{\log(\cosh\gamma-\cos(2\pi-\mu))-\log(\cosh\gamma-\cos(x-\mu))}\right)(q_1(x))^{-1}q_2(x),$$

or

$$(\log(\cosh\gamma-\cos(2\pi-\mu))-\log(\cosh\gamma-\cos(x-\mu)))\,\eta'(x)-\left(\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}\right)\eta(x)$$
$$=-\left(\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}\right)(q_1(x))^{-1}q_2(x),$$

or

$$\frac{d}{dx}\{(\log(\cosh\gamma-\cos(2\pi-\mu))-\log(\cosh\gamma-\cos(x-\mu)))\,\eta(x)\}$$
$$=-\left(\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}\right)(q_1(x))^{-1}q_2(x),$$

from which we arrive at

$$\eta(x)=(\log(\cosh\gamma-\cos(2\pi-\mu))-\log(\cosh\gamma-\cos(x-\mu)))^{-1}\times$$
$$\left[-\int\frac{\sin(x-\mu)}{(\cosh\gamma-\cos(x-\mu))}(q_1(x))^{-1}q_2(x)\,dx+D\right].$$

$\square$

Note that a set of functions satisfying the differential equation in Corollary 73, is given in Proposition 37 with $D=\frac{(\log(\cosh\gamma-\cos(2\pi-\mu)))^2}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 1.

Proposition 38. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}\sin(\pi-x)$ and $q_2(x)=q_1(x)\sin^2(\pi-x)$ for $x\in(0,2\pi)$. The random variable X has pdf (20) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\sin^2(\pi-x),\quad x\in(0,2\pi).$$

Proof
Let X be a random variable with pdf (20), then

$$(1-F(x))\,E[q_1(X)\mid X\geq x]=\int_x^{2\pi}C\cos(\pi-u)\sin(\pi-x)\,du$$
$$=\frac{C}{2}\sin^2(\pi-x),\quad x\in(0,2\pi),$$

and

$$\begin{aligned}(1-F(x))\,E[q_2(X)\mid X\geq x] &= \int_x^{2\pi} C\cos(\pi-u)\sin^3(\pi-u)\,du\\ &= \frac{C}{4}\sin^4(\pi-x),\quad x\in(0,2\pi),\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x) = -\frac{q_1(x)}{2}\sin^2(\pi-x)<0 \ \text{ for } \ x\in(0,2\pi).$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{2\cos(\pi-x)}{\sin(\pi-x)},\quad x\in(0,2\pi),$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{2\cos(\pi-x)\sin(\pi-x)}{\sin^2(\pi-x)},\quad x\in(0,2\pi),$$

and hence

$$s(x)=-\log\left(\sin^2(\pi-x)\right),\quad x\in(0,2\pi).$$

Now, in view of Theorem 1, X has density (20). □

Corollary 75. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 38. The pdf of X is (20) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{2\cos(\pi-x)\sin(\pi-x)}{\sin^2(\pi-x)},\quad x\in(0,2\pi).$$

Corollary 76. The general solution of the differential equation in Corollary 75 is

$$\eta(x)=\left(\sin^2(\pi-x)\right)^{-1}\left[-\int 2\cos(\pi-x)\sin(\pi-x)\,(q_1(x))^{-1}\,q_2(x)\,dx+D\right],$$

where D is a constant.

Proof
If X has pdf (20), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x)=\left(\frac{2\cos(\pi-x)\sin(\pi-x)}{\sin^2(\pi-x)}\right)\eta(x)-\left(\frac{2\cos(\pi-x)\sin(\pi-x)}{\sin^2(\pi-x)}\right)(q_1(x))^{-1}\,q_2(x),$$

or

$$\eta'(x)-\left(\frac{2\cos(\pi-x)\sin(\pi-x)}{\sin^2(\pi-x)}\right)\eta(x)=-\left(\frac{2\cos(\pi-x)\sin(\pi-x)}{\sin^2(\pi-x)}\right)(q_1(x))^{-1}\,q_2(x),$$

or

$$\left(\sin^2(\pi - x)\right)\eta'(x) - (2\cos(\pi - x)\sin(\pi - x))\,\eta(x) = -\,(2\cos(\pi - x)\sin(\pi - x))\,(q_1(x))^{-1}\,q_2(x)\,,$$

or

$$\frac{d}{dx}\left\{\left(\sin^2(\pi - x)\right)\eta(x)\right\} = -\,(2\cos(\pi - x)\sin(\pi - x))\,(q_1(x))^{-1}\,q_2(x)\,,$$

from which we arrive at

$$\eta(x) = \left(\sin^2(\pi - x)\right)^{-1}\left[-\int 2\cos(\pi - x)\sin(\pi - x)\,(q_1(x))^{-1}\,q_2(x)\,dx + D\right].$$

□

Note that a set of functions satisfying the differential equation in Corollary 75, is given in Proposition 38 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem1.1.

Proposition 39. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}\sin(\pi - x)$ and $q_2(x) = q_1(x)\sin^2(\pi - x)$ for $x \in (0, 2\pi)$. The random variable X has pdf (24) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\sin^2(\pi - x)\,,\quad x \in (0, 2\pi)\,.$$

Corollary 77. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 39. The pdf of X is (24) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x) - q_2(x)} = \frac{2\cos(\pi - x)\sin(\pi - x)}{\sin^2(\pi - x)},\quad x \in (0, 2\pi)\,.$$

Corollary 78. The general solution of the differential equation in Corollary 77 is

$$\eta(x) = \left(\sin^2(\pi - x)\right)^{-1}\left[-\int 2\cos(\pi - x)\sin(\pi - x)\,(q_1(x))^{-1}\,q_2(x)\,dx + D\right],$$

where D is a constant.

Proposition 40. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\,e^{\frac{c}{b}x}$ for $x \in (0, 2\pi)$. The random variable X has pdf (22) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{\frac{c}{b}(2\pi)} + e^{\frac{c}{b}x}\right),\quad x \in (0, 2\pi)\,.$$

Proof
Let X be a random variable with pdf (22), then

$$(1 - F(x))\,E[q_1(X)\mid X \ge x] = \int_x^{2\pi} Ce^{\frac{c}{b}u}du = \frac{b}{c}C\left(e^{\frac{c}{b}(2\pi)} - e^{\frac{c}{b}x}\right),\quad x \in (0, 2\pi)\,,$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^{2\pi}Ce^{\frac{c}{b}u}du=\frac{b}{2c}C\left(e^{2\frac{c}{b}(2\pi)}-e^{2\frac{c}{b}x}\right),\ x\in(0,2\pi),$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=\frac{q_1(x)}{2}\left(e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}\right)>0 \text{ for } x\in(0,2\pi).$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\frac{c}{b}e^{\frac{c}{b}x}}{e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}},\quad x\in(0,2\pi),$$

and hence

$$s(x)=-\log\left(e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}\right),\quad x\in(0,2\pi).$$

Now, in view of Theorem 1, X has density (22). □

Corollary 79. Let $X:\Omega\to(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 40. The pdf of X is (22) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\frac{c}{b}e^{\frac{c}{b}x}}{e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}},\quad x\in(0,2\pi).$$

Corollary 80. The general solution of the differential equation in Corollary 79 is

$$\eta(x)=\left(e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}\right)^{-1}\left[-\int\frac{c}{b}e^{\frac{c}{b}x}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Proof
If X has pdf (22), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x)=\left(\frac{\frac{c}{b}e^{\frac{c}{b}x}}{e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}}\right)\eta(x)-\left(\frac{\frac{c}{b}e^{\frac{c}{b}x}}{e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\eta'(x)-\left(\frac{\frac{c}{b}e^{\frac{c}{b}x}}{e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}}\right)\eta(x)=-\left(\frac{\frac{c}{b}e^{\frac{c}{b}x}}{e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\left(e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}\right)\eta'(x)-\left(\frac{c}{b}e^{\frac{c}{b}x}\right)\eta(x)=-\left(\frac{c}{b}e^{\frac{c}{b}x}\right)\left(q_1(x)\right)^{-1}q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}\right)\eta\left(x\right)\right\}=-\left(\frac{c}{b}e^{\frac{c}{b}x}\right)\left(q_{1}\left(x\right)\right)^{-1}q_{2}\left(x\right),$$

from which we arrive at

$$\eta\left(x\right)=\left(e^{\frac{c}{b}(2\pi)}-e^{\frac{c}{b}x}\right)^{-1}\left[-\int\left(\frac{c}{b}e^{\frac{c}{b}x}\right)\left(q_{1}\left(x\right)\right)^{-1}q_{2}\left(x\right)dx+D\right].$$

□

Note that a set of functions satisfying the differential equation in Corollary 79, is given in Proposition 40 with $D=\frac{e^{2\frac{c}{b}(2\pi)}}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 1.

Proposition 41. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_{1}\left(x\right)=\sin^{2}\left(x\right)\left[P\left(x\right)\right]^{-1}$ and $q_{2}\left(x\right)=q_{1}\left(x\right)\sin^{2}\left(x\right)$ for $x\in(0,2\pi)$. The random variable X has pdf (48) if and only if the function η defined in Theorem 1 has the form

$$\eta\left(x\right)=\frac{4}{5}\sin^{2}\left(x\right),\quad x\in(0,2\pi).$$

Proof
Let X be a random variable with pdf (48), then

$$\left(1-F\left(x\right)\right)E\left[q_{1}\left(X\right)\mid X\geq x\right]=\int_{x}^{2\pi}C\sin^{2}\left(u\right)\cos\left(u\right)du=-\frac{C}{3}\sin^{3}\left(x\right),\quad x\in(0,2\pi),$$

and

$$\left(1-F\left(x\right)\right)E\left[q_{2}\left(X\right)\mid X\geq x\right]=\int_{x}^{2\pi}C\sin^{4}\left(u\right)\cos\left(u\right)du=-\frac{C}{5}\sin^{5}\left(x\right),\quad x\in(0,2\pi),$$

and finally

$$\eta\left(x\right)q_{1}\left(x\right)-q_{2}\left(x\right)=-\frac{q_{1}\left(x\right)}{5}\sin^{2}\left(x\right)<0\ \text{ for }\ x\in(0,2\pi).$$

Conversely, if η is given as above, then

$$s'\left(x\right)=\frac{\eta'\left(x\right)q_{1}\left(x\right)}{\eta\left(x\right)q_{1}\left(x\right)-q_{2}\left(x\right)}=-8\left(\frac{\cos\left(x\right)}{\sin\left(x\right)}\right),\quad x\in(0,2\pi),$$

and hence

$$s\left(x\right)=-8\log\left(\sin\left(x\right)\right),\quad x\in(0,2\pi).$$

Now, in view of Theorem 1, X has density (48). □

Corollary 81. Let $X:\Omega\rightarrow(0,2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 41. The pdf of X is (48) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'\left(x\right)q_{1}\left(x\right)}{\eta\left(x\right)q_{1}\left(x\right)-q_{2}\left(x\right)}=-8\left(\frac{\cos\left(x\right)}{\sin\left(x\right)}\right),\quad x\in(0,2\pi).$$

Corollary 82. The general solution of the differential equation in Corollary 81 is

$$\eta(x) = \left(\sin^2(x)\right)^{-1}\left[\int 8\cos(x)\sin(x)\left(q_1(x)\right)^{-1} q_2(x)\,dx + D\right],$$

where D is a constant.

Proof
If X has pdf (48), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \left(-8\left(\frac{\cos(x)}{\sin(x)}\right)\right)\eta(x) - \left(-8\left(\frac{\cos(x)}{\sin(x)}\right)\right)\left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\eta'(x) - \left(-8\left(\frac{\cos(x)}{\sin(x)}\right)\right)\eta(x) = -\left(-8\left(\frac{\cos(x)}{\sin(x)}\right)\right)\left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\left(\sin^2(x)\right)\eta'(x) - \left(-8\left(\cos(x)\sin(x)\right)\right)\eta(x) = -\left(-8\left(\cos(x)\sin(x)\right)\right)\left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(\sin^2(x)\right)\eta(x)\right\} = -\left(-8\left(\cos(x)\sin(x)\right)\right)\left(q_1(x)\right)^{-1} q_2(x),$$

from which we arrive at

$$\eta(x) = \left(\sin^2(x)\right)^{-1}\left[\int 8\left(\cos(x)\sin(x)\right)\left(q_1(x)\right)^{-1} q_2(x)\,dx + D\right].$$

□

Note that a set of functions satisfying the differential equation in Corollary 81, is given in Proposition 41 with $D = 0$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 42. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\,x^{-\alpha}$ for $x \in (0, 2\pi)$. The random variable X has pdf (50) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(x^{-\alpha} + (2\pi)^{-\alpha}\right), \quad x \in (0, 2\pi).$$

Proof
Let X be a random variable with pdf (50), then

$$(1 - F(x))\,E\left[q_1(X) \mid X \geq x\right] = \int_x^{2\pi} C u^{-\alpha-1} du = \frac{C}{\alpha}\left(x^{-\alpha} - (2\pi)^{-\alpha}\right), \quad x \in (0, 2\pi),$$

and

$$(1 - F(x))\,E\left[q_2(X) \mid X \geq x\right] = \int_x^{2\pi} C u^{-2\alpha-1} du = \frac{C}{2\alpha}\left(x^{-2\alpha} - (2\pi)^{-2\alpha}\right), \quad x \in (0, 2\pi),$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left(x^{-\alpha} - (2\pi)^{-\alpha}\right) < 0 \text{ for } x \in (0, 2\pi).$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha x^{-\alpha-1}}{x^{-\alpha} - (2\pi)^{-\alpha}}, \quad x \in (0, 2\pi),$$

and hence

$$s(x) = -\log\left(x^{-\alpha} - (2\pi)^{-\alpha}\right), \quad x \in (0, 2\pi).$$

Now, in view of Theorem 1, X has density (50). $\square$

Corollary 83. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 42. The pdf of X is (50) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha x^{-\alpha-1}}{x^{-\alpha} - (2\pi)^{-\alpha}}, \quad x \in (0, 2\pi).$$

Corollary 84. The general solution of the differential equation in Corollary 83 is

$$\eta(x) = \left(x^{-\alpha} - (2\pi)^{-\alpha}\right)^{-1}\left[-\int \alpha x^{-\alpha-1}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof
If X has pdf (50), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \left(\frac{\alpha x^{-\alpha-1}}{x^{-\alpha} - (2\pi)^{-\alpha}}\right)\eta(x) - \left(\frac{\alpha x^{-\alpha-1}}{x^{-\alpha} - (2\pi)^{-\alpha}}\right)\left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\eta'(x) - \left(\frac{\alpha x^{-\alpha-1}}{x^{-\alpha} - (2\pi)^{-\alpha}}\right)\eta(x) = -\left(\frac{\alpha x^{-\alpha-1}}{x^{-\alpha} - (2\pi)^{-\alpha}}\right)\left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\left(x^{-\alpha} - (2\pi)^{-\alpha}\right)\eta'(x) - \left(\alpha x^{-\alpha-1}\right)\eta(x) = -\left(\alpha x^{-\alpha-1}\right)\left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(x^{-\alpha} - (2\pi)^{-\alpha}\right)\eta(x)\right\} = -\left(\alpha x^{-\alpha-1}\right)\left(q_1(x)\right)^{-1} q_2(x),$$

from which we arrive at

$$\eta(x) = \left(x^{-\alpha} - (2\pi)^{-\alpha}\right)^{-1}\left[-\int \alpha x^{-\alpha-1}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right].$$

$\square$

Note that a set of functions satisfying the differential equation in Corollary 83, is given in Proposition 42 with $D = \frac{(2\pi)^{-2\alpha}}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 43. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, x^{-\frac{\nu+1}{2}}$ for $x \in (0, 2\pi)$. The random variable X has pdf (72) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(x^{-\frac{\nu+1}{2}} + (2\pi)^{-\frac{\nu+1}{2}}\right), \quad x \in (0, 2\pi).$$

Corollary 85. Let $X : \Omega \to (0, 2\pi)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 43. The pdf of X is (72) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\left(\frac{\nu+1}{2}\right) x^{-\frac{\nu+1}{2}-1}}{x^{-\frac{\nu+1}{2}} - (2\pi)^{-\frac{\nu+1}{2}}}, \quad x \in (0, 2\pi).$$

Corollary 86. The general solution of the differential equation in Corollary 85 is

$$\eta(x) = \left(x^{-\frac{\nu+1}{2}} - (2\pi)^{-\frac{\nu+1}{2}}\right)^{-1}\left[-\int \left(\frac{\nu+1}{2}\right) x^{-\frac{\nu+1}{2}-1} (q_1(x))^{-1}\, q_2(x)\, dx + D\right],$$

where D is a constant.

Proposition 44. For the WS the cdf and pdf are not analytically expressible. No characterizations via our Propositions are possible.

Proposition 45. WVG lacks a closed-form pdf or cdf. No characterizations via our Propositions are possible.

3. Wrapped Discrete Distributions

Characterizations of discrete distributions are important to many researchers in the applied fields. An investigator will be vitally interested to know if their model fits the requirements of a particular discrete distribution. To this end, one will depend on the characterizations of this distribution which provide conditions under which the underlying distribution is indeed that particular discrete distribution.

As we mentioned in the Abstract, the characterizations for the discrete wrapped case will be based on certain function of the random variable whose form depends on the nature of the distribution of the random variable.

Proposition 46. Let $R : \Omega \to I = \{0, 1, ..., m-1\}$ be a random variable. The pmf of R is (88) if and only if

$$E\left\{[P(R)]^{-1} \mid R \le k\right\} = \frac{1}{(\lambda - 1)}\left(\frac{\lambda^{k+1} - 1}{\sum_{r=0}^{k} \lambda^r P(r)}\right). \tag{103}$$

Proof
If X has pmf (88), then for $k \in I$, the left-hand side of (103), using finite geometric sum formula, will be

$$\begin{aligned}(F(k))^{-1} \sum_{r=0}^{k} \{\lambda^r\} &= \left(\frac{1}{\sum_{r=0}^{k} \lambda^r P(r)}\right)\left(\frac{\lambda^{k+1} - 1}{\lambda - 1}\right) \\ &= \frac{1}{(\lambda - 1)}\left(\frac{\lambda^{k+1} - 1}{\sum_{r=0}^{k} \lambda^r P(r)}\right).\end{aligned}$$

Conversely, if (103) holds, then

$$\sum_{r=0}^{k}\left\{[P(x)]^{-1} f(x)\right\} = F(k)\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k+1}-1}{\sum_{r=0}^{k}\lambda^{r}P(r)}\right). \tag{104}$$

From (104), we also have

$$\begin{aligned}\sum_{r=0}^{k-1}\left\{[P(x)]^{-1} f(x)\right\} &= F(k-1)\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k}-1}{\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right)\\ &= (F(k)-f(k))\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k}-1}{\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right).\end{aligned} \tag{105}$$

Now, subtracting (105) from (104), yields

$$\begin{aligned}\left\{[P(k)]^{-1}\right\} f(k) =&F(k)\left\{\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k+1}-1}{\sum_{r=0}^{k}\lambda^{r}P(r)}\right)-\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k}-1}{\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right)\right\}+\\ &f(k)\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k}-1}{\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right).\end{aligned}$$

From the above equality, after some algebra, we have

$$\begin{aligned}\frac{f(k)}{F(k)} &= \frac{\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k+1}-1}{\sum_{r=0}^{k}\lambda^{r}P(r)}\right)-\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k}-1}{\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right)}{[P(k)]^{-1}-\frac{1}{(\lambda-1)}\left(\frac{\lambda^{k}-1}{\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right)}\\ &= \frac{\lambda^{k}P(k)}{\sum_{r=0}^{k}\lambda^{r}P(r)},\end{aligned}$$

which is the reverse hazard function corresponding to the pmf (88), so X has pmf (88). □

Proposition 47. Discrete case number 1.47, WDC.

There is no r on the RHS of the formula given for the pmf.

Proposition 48. Let $X:\Omega\to I$ be a random variable. The pmf of X is (90) if and only if

$$E\left\{[P(X)]^{-1} \mid X\leq k\right\} = \left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda(k+1)}{m}}}\right)\left(\frac{e^{-\lambda(k+1)}-1}{e^{-\lambda}-1}\right). \tag{106}$$

Proof
If X has pmf (90), then for $k\in\mathbb{N}^{*}$, the left-hand side of (106), using finite geometric sum formula, will be

$$\begin{aligned}(F(k))^{-1}\sum_{x=0}^{k}\left\{Ce^{-\lambda x}\right\} &= C\left(\frac{1-e^{-2\pi\lambda}}{1-e^{-\frac{2\pi\lambda(k+1)}{m}}}\right)\left(\frac{e^{-\lambda(k+1)}-1}{e^{-\lambda}-1}\right)\\ &= \left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-2\pi\lambda}}\right)\left(\frac{1-e^{-2\pi\lambda}}{1-e^{-\frac{2\pi\lambda(k+1)}{m}}}\right)\left(\frac{e^{-\lambda(k+1)}-1}{e^{-\lambda}-1}\right)\\ &= \left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda(k+1)}{m}}}\right)\left(\frac{e^{-\lambda(k+1)}-1}{e^{-\lambda}-1}\right).\end{aligned}$$

Conversely, if (106) holds, then

$$\sum_{x=0}^{k}\left\{[P(x)]^{-1}f(x)\right\} = F(k)\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda(k+1)}{m}}}\right)\left(\frac{e^{-\lambda(k+1)}-1}{e^{-\lambda}-1}\right). \tag{107}$$

From (107), we also have

$$\begin{aligned}\sum_{x=0}^{k-1}\left\{[P(x)]^{-1}f(x)\right\} &= F(k-1)\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda k}{m}}}\right)\left(\frac{e^{-\lambda k}-1}{e^{-\lambda}-1}\right)\\ &= (F(k)-f(k))\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda k}{m}}}\right)\left(\frac{e^{-\lambda k}-1}{e^{-\lambda}-1}\right).\end{aligned} \tag{108}$$

Now, subtracting (108) from (107), yields

$$\begin{aligned}\left\{[P(k)]^{-1}\right\}f(k) =& F(k)\left\{\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda(k+1)}{m}}}\right)\left(\frac{e^{-\lambda(k+1)}-1}{e^{-\lambda}-1}\right)-\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda k}{m}}}\right)\left(\frac{e^{-\lambda k}-1}{e^{-\lambda}-1}\right)\right\}+\\ & f(k)\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda k}{m}}}\right)\left(\frac{e^{-\lambda k}-1}{e^{-\lambda}-1}\right).\end{aligned}$$

From the above equality, after some algebra, we have

$$\frac{f(k)}{F(k)} = \frac{\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda(k+1)}{m}}}\right)\left(\frac{e^{-\lambda(k+1)}-1}{e^{-\lambda}-1}\right)-\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda k}{m}}}\right)\left(\frac{e^{-\lambda k}-1}{e^{-\lambda}-1}\right)}{[P(k)]^{-1}-\left(\frac{1-e^{-\frac{2\pi\lambda}{m}}}{1-e^{-\frac{2\pi\lambda k}{m}}}\right)\left(\frac{e^{-\lambda k}-1}{e^{-\lambda}-1}\right)} = \frac{\left(1-e^{-\frac{2\pi\lambda}{m}}\right)e^{-\lambda k}}{1-e^{-\frac{2\pi\lambda(k+1)}{m}}},$$

which is the reverse hazard function corresponding to the pmf (90), so X has pmf (90). □

Proposition 49. Discrete case number 1.49, WDML.

No closed form for cdf or pmf.

Proposition 50. Let $R:\Omega\to I$ be a random variable. The pmf of R is (92) if and only if

$$E\left\{[P(R)]^{-1}\mid R\le k\right\} = \frac{1-q^{(k+1)}}{(1-q)\sum_{r=0}^{k}q^{r}P(r)}. \tag{109}$$

Proof
If R has pmf (92), then for $k \in I$, the left-hand side of (109), using finite geometric sum formula, will be

$$\begin{aligned}(F(k))^{-1}\sum_{r=0}^{k}\{Cq^r\} &= C\left(\frac{1}{C\sum_{r=0}^{k}q^rP(r)}\right)\left(\frac{q^{(k+1)}-1}{q-1}\right)\\ &= \frac{1-q^{(k+1)}}{(1-q)\sum_{r=0}^{k}q^rP(r)}.\end{aligned}$$

Conversely, if (109) holds, then

$$\sum_{r=0}^{k}\left\{[P(r)]^{-1}f(r)\right\} = F(k)\left(\frac{1-q^{(k+1)}}{(1-q)\sum_{r=0}^{k}q^rP(r)}\right). \tag{110}$$

From (110), we also have

$$\begin{aligned}\sum_{r=0}^{k-1}\left\{[P(r)]^{-1}f(r)\right\} &= F(k-1)\left(\frac{1-q^k}{(1-q)\sum_{r=0}^{k-1}q^rP(r)}\right)\\ &= (F(k)-f(k))\left(\frac{1-q^k}{(1-q)\sum_{r=0}^{k-1}q^rP(r)}\right).\end{aligned} \tag{111}$$

Now, subtracting (111) from (110), yields

$$\begin{aligned}\left\{[P(k)]^{-1}\right\}f(k) =& F(k)\left\{\left(\left(\frac{1-q^{k+1}}{(1-q)\sum_{r=0}^{k}q^rP(r)}\right)-\left(\frac{1-q^k}{(1-q)\sum_{r=0}^{k-1}q^rP(r)}\right)\right)\right\}+\\ & f(k)\left(\frac{1-q^k}{(1-q)\sum_{r=0}^{k-1}q^rP(r)}\right).\end{aligned}$$

From the above equality, after some algebra, we have

$$\begin{aligned}\frac{f(k)}{F(k)} &= \frac{\left(\frac{1-q^{k+1}}{(1-q)\sum_{r=0}^{k}q^rP(r)}\right)-\left(\frac{1-q^k}{(1-q)\sum_{r=0}^{k-1}q^rP(r)}\right)}{[P(k)]^{-1}-\left(\frac{1-q^k}{(1-q)\sum_{r=0}^{k-1}q^rP(r)}\right)}\\ &= \frac{q^kP(k)}{\sum_{r=0}^{k}q^rP(r)},\end{aligned}$$

which is the reverse hazard function corresponding to the pmf (92), so X has pmf (92). □

Proposition 51. Let $X:\Omega\to\{0,1,...,m-1\}$ be a random variable. The pmf of X is (94) if and only if

$$E\left\{\left[\left(\frac{1-(1-\delta)^m}{\delta}\right)(1-\delta)^X\right]\mid X\le k\right\} = \left(\frac{1-(1-\delta)^m}{1-(1-\delta)^2}\right)\left(1+(1-\delta)^{(k+1)}\right). \tag{112}$$

Proof
If X has pmf (94), then for $k\in\{0,1,...,m-1\}$, the left-hand side of (112), using finite geometric sum formula, will be

$$
\begin{aligned}
(F(k))^{-1}\sum_{x=0}^{k}(1-\delta)^{2x} &= \left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{k+1}}\right)\sum_{x=0}^{k}(1-\delta)^{2x}\\
&= \left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{k+1}}\right)\left(\frac{1-(1-\delta)^{2(k+1)}}{1-(1-\delta)^{2}}\right)\\
&= \left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{2}}\right)\left(1+(1-\delta)^{(k+1)}\right).
\end{aligned}
$$

Conversely, if (112) holds, then

$$
\sum_{x=0}^{k}\left\{\left(\frac{1-(1-\delta)^{m}}{\delta}\right)(1-\delta)^{x}f(x)\right\} = F(k)\left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{2}}\right)\left(1+(1-\delta)^{(k+1)}\right). \tag{113}
$$

From (113), we also have

$$
\begin{aligned}
\sum_{x=0}^{k-1}\left\{\left(\frac{1-(1-\delta)^{m}}{\delta}\right)(1-\delta)^{x}f(x)\right\} &= F(k-1)\left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{2}}\right)\left(1+(1-\delta)^{k}\right)\\
&= (F(k)-f(k))\left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{2}}\right)\left(1+(1-\delta)^{k}\right).
\end{aligned} \tag{114}
$$

Now, subtracting (114) from (113), yields

$$
\begin{aligned}
\left\{\left(\frac{1-(1-\delta)^{m}}{\delta}\right)(1-\delta)^{k}\right\}f(k) = F(k)\left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{2}}\right)\left\{\left(1+(1-\delta)^{(k+1)}\right)-\left(1+(1-\delta)^{k}\right)\right\}+\\
f(k)\left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{2}}\right)\left(1+(1-\delta)^{k}\right).
\end{aligned}
$$

From the above equality, after some algebra, we have

$$
\begin{aligned}
\frac{f(k)}{F(k)} &= \frac{\left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{2}}\right)\left\{\left(1+(1-\delta)^{(k+1)}\right)-\left(1+(1-\delta)^{k}\right)\right\}}{\left(\frac{1-(1-\delta)^{m}}{\delta}\right)(1-\delta)^{k}-\left(\frac{1-(1-\delta)^{m}}{1-(1-\delta)^{2}}\right)\left(1+(1-\delta)^{k}\right)}\\
&= \frac{\delta(1-\delta)^{k}}{1-(1-\delta)^{(k+1)}},
\end{aligned}
$$

which is the reverse hazard function corresponding to the pmf (94), so X has pmf (94) □

Proposition 52. Let $R:\Omega\to I$ be a random variable. The pmf of R is (96) if and only if

$$
E\left\{[P(R)]^{-1}\mid R\le k\right\} = \frac{1-(1-p)^{(k+1)}}{p\sum_{r=0}^{k}q^{r}P(r)}. \tag{115}
$$

Proof
If R has pmf (96), then for $k\in\mathbb{N}^{*}$, the left-hand side of (115), using finite geometric sum formula, will be

$$(F(k))^{-1}\sum_{r=0}^{k}\{(1-p)^r\}=\left(\frac{1}{\sum_{r=0}^{k}q^rP(r)}\right)\left(\frac{(1-p)^{(k+1)}-1}{(1-p)-1}\right)$$
$$=\frac{1-(1-p)^{(k+1)}}{p\sum_{r=0}^{k}q^rP(r)}.$$

Conversely, if (115) holds, then

$$\sum_{r=0}^{k}\left\{[P(r)]^{-1}f(r)\right\}=F(k)\left(\frac{1-(1-p)^{(k+1)}}{p\sum_{r=0}^{k}q^rP(r)}\right). \tag{116}$$

From (116), we also have

$$\sum_{r=0}^{k-1}\left\{[P(r)]^{-1}f(r)\right\}=F(k-1)\left(\frac{1-(1-p)^k}{p\sum_{r=0}^{k-1}q^rP(r)}\right)$$
$$=(F(k)-f(k))\left(\frac{1-q^k}{p\sum_{r=0}^{k-1}q^rP(r)}\right). \tag{117}$$

Now, subtracting (117) from (116), yields

$$\left\{[P(k)]^{-1}\right\}f(k)=F(k)\left\{\left(\left(\frac{1-q^{k+1}}{p\sum_{r=0}^{k}q^rP(r)}\right)-\left(\frac{1-q^k}{p\sum_{r=0}^{k-1}q^rP(r)}\right)\right)\right\}+$$
$$f(k)\left(\frac{1-q^k}{p\sum_{r=0}^{k-1}q^rP(r)}\right).$$

From the above equality, after some algebra, we have

$$\frac{f(k)}{F(k)}=\frac{\left(\frac{1-q^{k+1}}{p\sum_{r=0}^{k}q^rP(r)}\right)-\left(\frac{1-q^k}{p\sum_{r=0}^{k-1}q^rP(r)}\right)}{[P(k)]^{-1}-\left(\frac{1-q^k}{p\sum_{r=0}^{k-1}q^rP(r)}\right)}=\frac{(1-p)^kP(k)}{\sum_{r=0}^{k}(1-p)^rP(r)},$$

which is the reverse hazard function corresponding to the pmf (96), so X has pmf (96). □

Proposition 53. Let $R:\Omega\to\{0,1,...,m-1\}$ be a random variable. The pmf of R is (98) if and only if

$$E\left\{[P(R)]^{-1}\mid R\le k\right\}=\frac{\lambda^{(k+1)}-1}{(\lambda-1)\sum_{r=0}^{k}\lambda^rP(r)}. \tag{118}$$

Proof
If R has pmf (98), then for $k\in\{0,1,...,m-1\}$, the left-hand side of (118), using finite geometric sum formula, will be

$$(F(k))^{-1}\sum_{r=0}^{k}\{\lambda^r\}=\left(\frac{1}{\sum_{r=0}^{k}\lambda^rP(r)}\right)\left(\frac{\lambda^{(k+1)}-1}{\lambda-1}\right)$$
$$=\frac{\lambda^{(k+1)}-1}{(\lambda-1)\sum_{r=0}^{k}\lambda^rP(r)}.$$

Conversely, if (118) holds, then

$$\sum_{r=0}^{k}\left\{[P(r)]^{-1} f(r)\right\} = F(k)\left(\frac{\lambda^{(k+1)}-1}{(\lambda-1)\sum_{x=0}^{k}\lambda^{r}P(r)}\right). \tag{119}$$

From (119), we also have

$$\begin{aligned}\sum_{r=0}^{k-1}\left\{[P(r)]^{-1} f(r)\right\} &= F(k-1)\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{x=0}^{k-1}\lambda^{r}P(r)}\right)\\ &= (F(k)-f(k))\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right).\end{aligned} \tag{120}$$

Now, subtracting (120) from (119), yields

$$\begin{aligned}\left\{[P(k)]^{-1}\right\} f(k) =& F(k)\left\{\left(\frac{\lambda^{k+1}-1}{(\lambda-1)\sum_{r=0}^{k}\lambda^{r}P(r)}\right)-\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right)\right\}+\\ & f(k)\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{r=0}^{k}\lambda^{r}P(r)}\right).\end{aligned}$$

From the above equality, after some algebra, we have

$$\begin{aligned}\frac{f(k)}{F(k)} &= \frac{\left(\frac{\lambda^{k+1}-1}{(\lambda-1)\sum_{r=0}^{k}\lambda^{r}P(r)}\right)-\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right)}{[P(k)]^{-1}-\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{r=0}^{k-1}\lambda^{r}P(r)}\right)}\\ &= \frac{\lambda^{k}P(k)}{\sum_{r=0}^{k}\lambda^{r}P(r)},\end{aligned}$$

which is the reverse hazard function corresponding to the pmf (98), so X has pmf (98). □

Proposition 54. Let $R:\Omega\to\{0,1,...,m-1\}$ be a random variable. The pmf of R is (98) if and only if

$$E\left\{[P(R)]^{-1} \mid R\le k\right\} = \frac{(1+\theta)^{(k+1)}-1}{\theta\sum_{r=0}^{k}(1+\theta)^{r}P(r)}. \tag{121}$$

Proof
Proof is the same as that of Proposition 53. □

Proposition 55. Let $X:\Omega\to\mathbb{N}^{*}$ be a random variable. The pmf of X is (100) if and only if

$$E\left\{[P(X)]^{-1} \mid X\le k\right\} = \frac{\lambda^{(k+1)}-1}{(\lambda-1)\sum_{r=0}^{k}\lambda^{r}P(r)}. \tag{122}$$

Proof
If X has pmf (100), then for $k\in\mathbb{N}$, the left-hand side of (122), using finite geometric sum formula, will be

$$(F(k))^{-1}\sum_{x=1}^{k} C\{\lambda^x\} = \left(\frac{1}{C\sum_{x=1}^{k}\lambda^x P(x)}\right) C\left(\frac{\lambda^{(k+1)}-1}{\lambda-1}\right)$$
$$= \frac{\lambda^{(k+1)}-1}{(\lambda-1)\sum_{x=1}^{k}\lambda^x P(x)}.$$

Conversely, if (122) holds, then

$$\sum_{x=1}^{k}\left\{[P(x)]^{-1} f(x)\right\} = F(k)\left(\frac{\lambda^{(k+1)}-1}{(\lambda-1)\sum_{x=1}^{k}\lambda^x P(x)}\right). \tag{123}$$

From (123), we also have

$$\sum_{x=1}^{k-1}\left\{[P(x)]^{-1} f(x)\right\} = F(k-1)\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{x=1}^{k-1}\lambda^x P(x)}\right)$$
$$= (F(k)-f(k))\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{x=1}^{k-1}\lambda^x P(x)}\right). \tag{124}$$

Now, subtracting (124) from (123), yields

$$\left\{[P(k)]^{-1}\right\} f(k) = F(k)\left\{\left(\frac{\lambda^{k+1}-1}{(\lambda-1)\sum_{x=1}^{k}\lambda^x P(x)}\right) - \left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{x=1}^{k-1}\lambda^x P(x)}\right)\right\} +$$
$$f(k)\left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{x=1}^{k-1}\lambda^x P(x)}\right).$$

From the above equality, after some algebra, we have

$$\frac{f(k)}{F(k)} = \frac{\left(\frac{\lambda^{k+1}-1}{(\lambda-1)\sum_{x=1}^{k}\lambda^x P(x)}\right) - \left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{x=1}^{k-1}\lambda^x P(x)}\right)}{[P(k)]^{-1} - \left(\frac{\lambda^{k}-1}{(\lambda-1)\sum_{x=1}^{k-1}\lambda^x P(x)}\right)} = \frac{\lambda^k P(k)}{\sum_{x=0}^{k}\lambda^x P(x)},$$

which is the reverse hazard function corresponding to the pmf (100), so X has pmf (100). □

Proposition 56. Let $X:\Omega\to\mathbb{N}^*$ be a random variable. The pmf of X is (102) if and only if

$$E\left\{[P(X)]^{-1} \mid X\le k\right\} = \frac{\lambda^{(k+1)}-1}{(\lambda-1)\sum_{r=0}^{k}\lambda^r P(r)}. \tag{125}$$

Proof
Proof is the same as that of Proposition 55. □

Acknowledgements

The authors are grateful to the Editor and two Reviewers for their comments and suggestions which has improved greatly the presentation of the material.

REFERENCES

1. W. Bell and S. Nadarajah, *A Review of Wrapped Distributions for Circular Data*, Mathematics, vol. 12, no. 16, p. 2440, 2024.
2. A. Al-Khazaleh and T. R. Al-Meanazel, *Wrapped Akash distribution*, Electronic Journal of Applied Statistical Analysis, vol. 14, pp. 305–317, 2021.
3. R. Ganaie, V. Rajagopalan, and A. Rather, *Weighted Aradhana distribution: Properties and applications*, Journal of Information Technology Research, vol. IX, pp. 392–406, 2019.
4. I. Ramabhadra Sarma, A. V. Dattatreya Rao, and S. V. S. Girija, *On characteristic functions of wrapped half logistic and binormal distributions*, International Journal of Statistical Systems, vol. 4, pp. 33–45, 2009.
5. M. A. Saraiva and F. J. de Azevedo Cysneiros, *Wrapped Birnbaum-Saunders distribution: Definition and estimation*, Nexus Mathematica, vol. 5, p. 200010, 2022.
6. P. Lévy, *L'addition des variables aléatoires définies sur une circonférence*, Bulletin de la Société Mathématique de France, vol. 67, pp. 1–41, 1939.
7. S. Roy and M. A. S. Adnan, *Wrapped Chi-Square Distribution*, Available online: https://www.researchgate.net/publication/263080292_Wrapped_Chi-square_distribution, Accessed 3 August 2024, 2024.
8. J. S. Rao and T. J. Kozubowski, *A wrapped exponential circular model*, Proceedings of the Academy of Sciences, vol. 5, pp. 43–56, 2001.
9. P. S. Subrahmanyam, A. V. Dattatreya Rao, and S. V. S. Girija, *On wrapping of exponentiated inverted Weibull distribution*, International Journal of Innovative Research in Science and Technology, vol. 3, pp. 18–25, 2017.
10. C. A. Coelho, *The wrapped gamma distribution and wrapped sums and linear combinations of independent gamma and Laplace distributions*, Journal of Statistical Theory and Practice, vol. 1, pp. 1–29, 2007.
11. K. Jayakumar and T. Sajayan, *Wrapped generalized geometric stable distributions with an application to wind direction*, Far East Journal of Theoretical Statistics, vol. 66, pp. 147–166, 2022.
12. S. Roy and M. A. S. Adnan, *Wrapped generalized Gompertz distribution: An application to ornithology*, Journal of Biometrics & Biostatistics, vol. 3, pp. 153–156, 2012.
13. W. J. Reed and A. Pewsey, *Two nested families of skew-symmetric circular distributions*, Test, vol. 18, pp. 516–528, 2009.
14. C. S. Kumar and R. Alex, *Wrapped generalized skew normal distribution*, Stochastic Modeling and Applications, vol. 25, pp. 1–7, 2021.
15. E. Hernández-Sánchez and B. Scarpa, *A wrapped flexible generalized skew-normal model for a bimodal circular distribution of wind directions*, Chilean Journal of Statistics, vol. 3, pp. 129–141, 2012.
16. J. Varghese and K. K. Jose, *Wrapped hb-skewed laplace distribution and its application in meteorology*, Biometrics & Biostatistics International Journal, vol. 7, pp. 525–530, 2018.
17. M. A. S. Adnan and S. Roy, *Wrapped hypo-exponential distribution*, Journal of Statistical Management Systems, vol. 16, pp. 1–11, 2013.
18. T. R. Al-Meanazel and A. Al-Khazaleh, *Wrapped Ishita distribution*, Journal of Statistical Applications & Probability, vol. 10, pp. 293–299, 2021.
19. S. R. Jammalamadaka and T. Kozubowski, *A new family of circular models: The wrapped Laplace distributions*, Advances in Applied Statistics, vol. 3, pp. 77–103, 2003.
20. S. Bhattacharjee and D. Borah, *Wrapped length biased weighted exponential distribution*, Thai Journal of Statistics, vol. 17, pp. 223–234, 2019.
21. N. I. Fisher, *Statistical Analysis of Circular Data*, Cambridge University Press, Cambridge, UK, 1996.
22. S. Joshi and K. K. Jose, *Wrapped Lindley distribution*, Communications in Statistics — Theory and Methods, vol. 47, pp. 1013–1021, 2018.
23. S. Jacob, *Study on Circular Distributions*, Ph.D. dissertation, University of Calicut, India, 2012.
24. R. Subba Rao, V. Ravindranath, A. V. Dattatreya Rao, G. Prasad, and P. Ravi Kishore, *Wrapped Lomax distribution: A new circular probability model*, International Journal of Engineering and Technology, vol. 7, pp. 150–152, 2018.
25. C. Chesneau, L. Tomy, and M. Jose, *Wrapped modified Lindley distribution*, Journal of Statistical Management Systems, vol. 24, pp. 1025–1040, 2021.
26. P. S. Subrahmanyam, A. V. Dattatreya Rao, and S. V. S. Girija, *On wrapping of new Weibull Pareto distribution*, International Journal of Advanced Research and Review, vol. 2, pp. 10–20, 2017.
27. K. V. Mardia, *Statistics of Directional Data*, Academic Press, London, UK, 1972.
28. S. R. Jammalamadaka and T. J. Kozubowski, *A general approach for obtaining wrapped circular distributions via mixtures*, Sankhyā A, vol. 79, pp. 133–157, 2017.
29. S. Bhattacharjee, *Exploration of some distributional properties, parameter estimates and applications of the wrapped quasi Lindley distribution*, Journal of Mathematics and Computer Science, vol. 10, pp. 2390–2407, 2020.
30. W. Bell and S. Nadarajah, *The wrapped Rama distribution*, Scientific Reports, vol. 14, no. 1, p. 31936, 2024.
31. A. E. Demirelli and M. Gürcan, *New wrapped distribution via Richard link function*, AIP Conference Proceedings, vol. 1863, p. 350002, 2017.
32. T. R. Al-Meanazel and A. Al-Khazaleh, *Wrapped Shanker distribution*, Italian Journal of Pure and Applied Mathematics, vol. 46, pp. 184–194, 2021.
33. A. Yilmaz, *Wrapped flexible skew Laplace distribution*, Journal of the Turkish Statistical Association, vol. 11, pp. 53–64, 2018.
34. A. Pewsey, *The wrapped skew-normal distribution on the circle*, Communications in Statistics — Theory and Methods, vol. 29, pp. 2459–2472, 2000.
35. A. Pewsey, *The wrapped stable family of distributions as a flexible model for circular data*, Computational Statistics & Data Analysis, vol. 52, pp. 1516–1523, 2008.
36. A. Pewsey, T. Lewis, and M. Jones, *The wrapped t family of circular distributions*, Australian & New Zealand Journal of Statistics, vol. 49, pp. 79–91, 2007.

37. A. Yilmaz and C. Biçer, *A new wrapped exponential distribution*, Mathematical Sciences, vol. 12, pp. 285–293, 2018.
38. S. Bhattacharjee, I. Ahmed, and K. K. Das, *Wrapped two-parameter Lindley distribution for modelling circular data*, Thai Journal of Statistics, vol. 19, pp. 81–94, 2020.
39. C. Biçer and A. Yilmaz, *Wrapped two sided Lindley distribution*, in Research and Reviews in Science and Mathematics, Gece Academy Publishing, Ankara, Turkey, pp. 1–13, 2022.
40. M. A. S. Adnan and S. Roy, *Wrapped variance gamma distribution with an application to wind direction*, Journal of Environmental Statistics, vol. 6, pp. 1–10, 2024.
41. S. Roy and M. A. S. Adnan, *Wrapped weighted exponential distributions*, Statistics & Probability Letters, vol. 82, pp. 77–83, 2012.
42. H. Al-Mofleh and S. Sen, *The wrapped Xgamma distribution for modeling circular data appearing in geological context*, arXiv, vol. abs/1903.00177, 2019.
43. E. Zinhom, M. M. Nassar, S. S. Radwan, and A. Elmasry, *The wrapped XLindley distribution*, Environmental and Ecological Statistics, vol. 30, pp. 669–686, 2023.
44. S. V. S. Girija, A. V. Dattatreya Rao, and V. L. N. Srihari, *On wrapped binomial model characteristics*, Mathematics & Statistics, vol. 2, pp. 231–234, 2014.
45. S. V. S. Girija, G. V. L. N. Srihari, and R. Srinivas, *On discrete wrapped Cauchy model*, Mathematics Theory and Modeling, vol. 9, pp. 15–23, 2019.
46. R. Srinivas, G. V. L. N. Srihari, and S. V. S. Girija, *On construction of discrete wrapped Lindley distribution*, i-Manager's Journal on Mathematics, vol. 7, pp. 10–19, 2008.
47. S. Jacob and K. Jayakumar, *Wrapped geometric distribution: A new probability model for circular data*, Journal of Statistical Theory and Applications, vol. 12, pp. 348–355, 2013.
48. K. Jayakumar and S. Jacob, *Wrapped skew Laplace distribution on integers: A new probability model for circular data*, Open Journal of Statistics, vol. 2, pp. 106–114, 2012.
49. G. V. L. N. Srihari, S. V. S. Girija, and A. V. D. Rao, *On wrapped negative binomial model*, International Journal of Mathematical Archive, vol. 9, pp. 148–153, 2018.
50. N. I. Fisher, T. Lewis, and B. J. J. Embleton, *Statistical Analysis of Spherical Data*, Cambridge University Press, New York, NY, USA, 1987.
51. A. Thavaneswaran, S. Mandal, and D. Pathmanathan, *Estimation for wrapped zero inflated Poisson and wrapped Poisson distributions*, International Journal of Statistics and Probability, vol. 5, pp. 1–8, 2016.
52. W. Glänzel, *A characterization theorem based on truncated moments and its application to some distribution families*, in Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, pp. 75–84, 1987.
53. W. Glänzel, *Some consequences of a characterization theorem based on truncated moments*, Statistics: A Journal of Theoretical and Applied Statistics, vol. 21, no. 4, pp. 613–618, 1990.