STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, September 2026, pp 2369–2390.
Published online in International Academic Press (www.IAPress.org)

Characterizations of Truncated Univariate Continuous Distributions Introduced by Lorenzo Zaninetti, I-XV

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Abstract This paper deals with various characterizations of 15 truncated univariate continuous distributions proposed by Lorenzo Zaninetti from 2019-2025, all published in the "International Journal of Astronomy and Astrophysics". Zaninetti masterfully and with a lot of super complicated derivations presents the cdfs and pdfs for these distributions. The present work may be considered as a supplement to Zaninetti's excellent works. These characterizations are based on: (i) a simple relationship between two truncated moments and (ii) the reverse hazard function and. It should be mentioned that for the characterization (i) the cumulative distribution function need not to have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

Keywords Characterizations, Conditional expectation, Continuous distributions, Hazard function, Reverse hazard function, Truncated moments

DOI: 10.19139/soic-2310-5070-2808

1. Introduction

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Generally speaking, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in many different directions. The present work deals with certain characterizations of each of the following 15 distributions proposed by Zaninetti:

I) New probability distributions in Astrophysics: I. The truncated generalized gamma, (2019).
II) New probability distributions in Astrophysics : II. Generalized and double truncated Lindley, (2020).
III) New probability distributions in Astrophysics : III. The truncated Maxwell-Boltzmann distribution, (2020).
IV) New probability distributions in Astrophysics : IV. The relativistic Maxwell-Boltzmann distribution, (2020).
V) New probability distributions in Astrophysics : V. The truncated Weibull distribution, (2021).
VI) New probability distributions in Astrophysics : VI. The truncated Sujatha distribution, (2021).
VII) New probability distributions in Astrophysics : VII. The truncated gamma-Pareto distribution applied to cosmic rays, (2022).
VIII) New probability distributions in Astrophysics : VIII. The truncated Weibull-Pareto distribution, (2022).

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

IX) New probability distributions in Astrophysics : IX. The truncated exponential, half-Gaussian and Sech-Square distributions with application to the galactic height, (2022).
X) New probability distributions in Astrophysics : X. Truncated and Mass-Luminosityyy relationship for Fréchet distribution, (2022).
XI) New probability distributions in Astrophysics : XI. Left truncated for Topp-Leone distribution, (2023).
XII) New probability distributions in Astrophysics : XII. Truncation for Gompertz distribution, (2024).
XIII) New probability distributions in Astrophysics : XIII. Truncation for Benini distribution, (2024).
XIV) New probability distributions in Astrophysics : XIV. Truncation of the modified lognormal distribution, (2025).
XV) New probability distributions: XV. Truncation of the exponentiated gamma distribution. (IJAA), 15(2), 90-108.

We list below the cumulative distribution function (cdf) and probability density function (pdf) of each one of these distributions in the same order as listed above. We will be employing the same notation for the parameters as chosen by the original author.

1.1. The double truncation generalized gamma

The cdf and pdf of (I) are given, respectively, by

$$F\left(x;a,b,c\right)=\int_{x_l}^{x}f\left(u;a,b,c\right)du,\quad x_l\le x\le x_u, \tag{1}$$

and

$$f\left(x;a,b,c\right)=Cx^{c-1}e^{-bx^{c}}P\left(x\right),\quad x_l<x<x_u, \tag{2}$$

where a,b,c,x_l,x_u are all positive parameters, $C=\left[I_t\left(x_u;a,b,c\right)-I_t\left(x_l;a,b,c\right)\right]^{-1}$, $I_t\left(x;a,b,c\right)=\int x^{a-1}e^{-bx^{c}}dx$ and $P\left(x\right)=x^{a-c}$.

1.2. The new generalized Lindley

The cdf and pdf of (II) are given, respectively, by

$$F\left(x;a,b,c\right)=\int_{x_l}^{x}f\left(u;a,b,c\right)du,\quad x_l\le x\le x_u, \tag{3}$$

and

$$f\left(x;a,b,c\right)=Ce^{-cx}P\left(x\right),\quad x_l<x<x_u, \tag{4}$$

where a,b,c,x_l,x_u are all positive parameters, $C=\left[\left(1+c\right)\Gamma\left(a\right)\Gamma\left(b\right)\right]^{-1},\Gamma\left(a\right)$ is the gamma function and $P\left(x\right)=c^{a+1}x^{a-1}\Gamma\left(b\right)+c^{b}x^{b-1}\Gamma\left(a\right)$.

1.3. The double truncated Maxwell-Boltzmann distribution

The cdf and pdf of (III) are given, respectively, by

$$F\left(x;a,x_l,x_u\right)=\int_{x_l}^{x}f\left(u;a,x_l,x_u\right)du,\quad x_l\le x\le x_u, \tag{5}$$

and

$$f\left(x;a,x_l,x_u\right)=Cxe^{-\frac{x^2}{2a^2}}P\left(x\right),\quad x_l<x<x_u, \tag{6}$$

where a, x_l, x_u are all positive parameters,

$$C = -2\left\{a^2\left[\left(-a\sqrt{2\pi}\,\mathrm{erf}\left(\frac{x_u}{a\sqrt{2}}\right)+a\sqrt{2\pi}\,\mathrm{erf}\left(\frac{x_l}{a\sqrt{2}}\right)\right)\right.\right.$$
$$\left.\left.+2x_u e^{-\frac{x_u^2}{2a^2}}-2x_l e^{-\frac{x_l^2}{2a^2}}\right]\right\}^{-1},$$

$\mathrm{erf}\,(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ and $P\,(x)=x$.

1.4. The relativistic Maxwell-Boltzmann distribution

The cdf and pdf of (IV) are given, respectively, by

$$F\,(x;T)=\int_0^x f\,(u;T)\,du,\quad 0\le x\le c, \tag{7}$$

and

$$f\,(x;T)=Cx^2P\,(x)\,,\quad 0<x<c, \tag{8}$$

where T is relativistic temperature, $C=\left[\int_0^c w^2 e^{\frac{1}{T}\left(1-\frac{1}{\sqrt{1-\frac{w^2}{c^2}}}\right)}dw\right]^{-1}$ and $P\,(x)=e^{\frac{1}{T}\left(1-\frac{1}{\sqrt{1-\frac{x^2}{c^2}}}\right)}$.

1.5. The truncated Weibull distribution

The cdf and pdf of (V) are given, respectively, by

$$F\,(x;b,c,x_l,x_u)=\frac{e^{-(x/b)^c}-e^{-(x_l/b)^c}}{e^{-(x_u/b)^c}-e^{-(x_l/b)^c}},\quad x_l\le x\le x_u, \tag{9}$$

and

$$f\,(x;b,c,x_l,x_u)=Cx^{c-1}e^{-(x/b)^c},\quad x_l<x<x_u, \tag{10}$$

where b, c, x_l, x_u are all positive parameters, $C=\left(\frac{-cb^{-c}}{e^{-(x_u/b)^c}-e^{-(x_l/b)^c}}\right)$.

1.6. The truncated Sujatha distribution

The cdf and pdf of (VI) are given, respectively, by

$$F\,(x;b,s,x_l,x_u)=\int_{x_l}^x f\,(u;b,s,x_l,x_u)\,du,\quad x_l\le x\le x_u, \tag{11}$$

and

$$f\,(x;b,s,x_l,x_u)=Ce^{-\frac{b}{s}x}P\,(x)\,,\quad x_l<x<x_u, \tag{12}$$

where a, s, x_l, x_u are all positive parameters, $C=\frac{b^3}{A}$,

$$A=\left[\left(s^2\left(b^2+b+2\right)+bx_l\,(b+2)\,s+b^2x_l^2\right)-\left(s^2\left(b^2+b+2\right)+bx_u\,(b+2)\,s\right)e^{-\frac{b}{s}x_u}\right]s$$

and $P\,(x)=s^2+sx+x^2$.

1.7. The double truncated gamma-Pareto distribution

The cdf and pdf of (VII) are given, respectively, by

$$F(x;\alpha,c,\theta,x_l,x_u)=\int_{x_l}^{x} f(u;\alpha,c,\theta,x_l,x_u)\,du,\quad x_l\le x\le x_u, \tag{13}$$

and

$$f(x;\alpha,c,\theta,x_l,x_u)=Cx^{-\left(\frac{1}{c}+1\right)}P(x),\quad x_l<x<x_u, \tag{14}$$

where $\alpha>0, c>0, 0<\theta<x_l, x_u$ are parameters, $C=\frac{\theta^{1/c}c^{-\alpha}}{\Gamma(\alpha)K}$, $\Gamma(\alpha)$ is the gamma function,

$$\begin{aligned}K=&\frac{1}{\alpha\Gamma(\alpha)}\left[-\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_u}{\theta}\right)^{1/c}\right)+\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_l}{\theta}\right)^{1/c}\right)\right)\right]\\&+c^{-\alpha}x_u^{1/c}\theta^{1/c}\ln\left(\frac{x_u}{\theta}\right)^{\alpha}-c^{-\alpha}x_l^{-1/c}\theta^{1/c}\ln\left(\frac{x_l}{\theta}\right)^{c}\right),\end{aligned}$$

$\Gamma\left(\alpha+1,\ln\left(\left(\frac{x_l}{\theta}\right)^{1/c}\right)\right)$ is incomplete gamma function and $P(x)=\ln\left(\frac{x}{\theta}\right)^{\alpha-1}$.

1.8. The double truncated Weibull-Pareto distribution

The cdf and pdf of (VIII) are given, respectively, by

$$F(x;a,b,c,x_l,x_u)=\frac{e^{-a(x/b)^c}-e^{-a(x_l/b)^c}}{e^{-a(x_u/b)^c}-e^{-a(x_l/b)^c}},\quad x_l\le x\le x_u, \tag{15}$$

and

$$f(x;a,b,c,x_l,x_u)=\frac{-acx^{-1}\left(\frac{x}{b}\right)^c e^{-a(x/b)^c}}{e^{-a(x_u/b)^c}-e^{-a(x_l/b)^c}},\quad x_l<x<x_u, \tag{16}$$

where a,b,c,x_l,x_u are all positive parameters.

1.9. The double truncated exponential distribution

The cdf and pdf of (IX), for the exponential case, are given, respectively, by

$$F(x;b,x_l,x_u)=\frac{e^{-(x/b)}-e^{-(x_l/b)}}{e^{-(x_u/b)}-e^{-(x_l/b)}},\quad x_l\le x\le x_u, \tag{17}$$

and

$$f(x;b,x_l,x_u)=\frac{-\frac{1}{b}e^{-(x/b)}}{e^{x_u/b}-e^{x_l/b}},\quad x_l<x<x_u, \tag{18}$$

where a,b,c,x_l,x_u are all positive parameters.

1.10. The truncated half-Normal distribution

The cdf and pdf of (IX), for the half-normal case, are given, respectively, by

$$F(x;s,x_l,x_u)=\int_{x_l}^{x} f(u;s,x_l,x_u)\,du,\quad x_l\le x\le x_u, \tag{19}$$

and

$$f(x;s,x_l,x_u)=Cxe^{-x^2/2s^2}P(x),\quad x_l<x<x_u, \tag{20}$$

where s, x_l, x_u are all positive parameters,

$$C = \sqrt{2}\left[\sqrt{\pi}s\left(-\operatorname{erf}\left(\frac{x_l\sqrt{2}}{2s}\right)+\operatorname{erf}\left(\frac{x_u\sqrt{2}}{2s}\right)\right)\right]^{-1}$$

and $P(x) = x^{-1}$.

1.11. The truncated Sech-Square distribution

The cdf and pdf of (IX), for the Sech-Square case, are given, respectively, by

$$F(x; b, x_l, x_u) = \int_{x_l}^{x} f(u; b, x_l, x_u)\, du, \quad x_l \leq x \leq x_u, \tag{21}$$

and

$$\begin{aligned} f(x; b, x_l, x_u) &= \frac{\sec h\left(\frac{x}{b}\right)^2}{b\left(\tanh\left(\frac{x_u}{b}\right)-\tanh\left(\frac{x_l}{b}\right)\right)} \\ &= Ce^{-2x/b}P(x), \quad x_l < x < x_u, \end{aligned} \tag{22}$$

where b, x_l, x_u are all positive parameters, $C = \left[4b\left(\tanh\left(\frac{x_u}{b}\right)-\tanh\left(\frac{x_l}{b}\right)\right)\right]^{-1}$ and $P(x) = \left(e^{2x/b}-1\right)^2$.

1.12. The Truncated Fréchet distribution

The cdf and pdf of (X) are given, respectively, by

$$F(x; b, \alpha, x_l, x_u) = \frac{e^{-(x/b)^{-\alpha}} - e^{-(x_l/b)^{-\alpha}}}{e^{-(x_u/b)^{-\alpha}} - e^{-(x_l/b)^{-\alpha}}}, \quad x_l \leq x \leq x_u, \tag{23}$$

and

$$\begin{aligned} f(x; b, \alpha, x_l, x_u) &= \frac{\alpha b^{\alpha} x^{-\alpha-1} e^{-(x/b)^{-\alpha}}}{e^{-(x_u/b)^{-\alpha}} - e^{-(x_l/b)^{-\alpha}}} \\ &= Cx^{-\alpha-1}e^{-(x/b)^{-\alpha}}, \quad x_l < x < x_u, \end{aligned} \tag{24}$$

where b, α, x_l, x_u are all positive parameters and $C = \frac{\alpha b^{\alpha}}{e^{-(x_u/b)^{-\alpha}} - e^{-(x_l/b)^{-\alpha}}}$.

1.13. The truncated Topp-Leone distribution

The cdf and pdf of (XI) are given, respectively, by

$$F(x; \beta, x_l, b) = \frac{b^{-2\beta}\left(x_l^{\beta}(2b-x_l)^{\beta} - x^{\beta}(2b-x)^{\beta}\right)}{x_l^{\beta}e^{-2\beta}(2b-x_l)^{\beta}-1}, \quad x_l \leq x \leq b, \tag{25}$$

and

$$f(x; \beta, x_l, b) = C\left(-\frac{2x}{b^2}+\frac{2}{b}\right)\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta-1} P(x), \quad x_l < x < b, \tag{26}$$

where β, x_l, b are all positive parameters and $C = \frac{-\beta/b}{x_l^{\beta}e^{-2\beta}(2b-x_l)^{\beta}-1}$ and $P(x) = \left(-\frac{2x}{b^2}+\frac{2}{b}\right)^{-1}\left(2-\frac{2x}{b}\right)$.

1.14. The truncated Gompertz distribution

The cdf and pdf of (XII) are given, respectively, by

$$F\left(x;a,b,x_l,x_u\right)=\frac{e^{-\frac{a}{b}e^{bx_l}}-e^{-\frac{a}{b}e^{bx}}}{e^{-\frac{a}{b}e^{bx_l}}-e^{-\frac{a}{b}e^{bx_u}}},\quad x_l\leq x\leq x_u, \tag{27}$$

and

$$f\left(x;a,b,x_l,x_u\right)=Ce^{-\frac{a}{b}x}P(x),\quad x_l<x<x_u, \tag{28}$$

where a,b,x_l,x_u are all positive parameters, $C=\left[e^{-\frac{a}{b}e^{bx_l}}-e^{-\frac{a}{b}e^{bx_u}}\right]^{-1}$ and $P\left(x\right)=ae^{bx}$.

1.15. The truncated Benini distribution

The cdf and pdf of (XIII) are given, respectively, by

$$F\left(x;\alpha,\beta,\sigma,x_u\right)=\frac{1-e^{-\left(-\ln\left(\frac{x}{\sigma}\right)\right)\left(-\beta\ln\left(\frac{x}{\sigma}\right)-\alpha\right)}}{1-\sigma^{\alpha+2\beta\ln(x_u)}x_u^{-\alpha}e^{-\beta(\ln^2(x_u)+\ln^2(\sigma))}},\quad \sigma\leq x\leq x_u, \tag{29}$$

and

$$f\left(x;\alpha,\beta,\sigma,x_u\right)=Cx^{-\alpha-1}P(x),\quad \sigma<x<x_u, \tag{30}$$

where $\alpha>0,\beta>0,0<\sigma<x_u$ are parameters,

$$C=\sigma^{\alpha}\left[1-\sigma^{\alpha+2\beta\ln(x_u)}x_u^{-\alpha}e^{-\beta\left(\ln^2(x_u)+\ln^2(\sigma)\right)}\right]^{-1}$$

and

$$P\left(x\right)=x^{\alpha}\left[2\beta\ln(\frac{x}{\sigma})+\alpha\right]e^{\ln\left(\frac{x}{\sigma}\right)\left(-\beta\ln\left(\frac{x}{\sigma}\right)-\alpha\right)}.$$

1.16. The truncated lognormal distribution

The cdf and pdf of (XIV) are given, respectively, by

$$F\left(x;m,\sigma,x_l,x_u\right)=\int_{x_l}^{x}f\left(u;m,\sigma,x_l,x_u\right)du,\quad x_l\leq x\leq x_u, \tag{31}$$

and

$$f\left(x;m,\sigma,x_l,x_u\right)=Cx^{-1}P\left(x\right),\quad x_l<x<x_u, \tag{32}$$

where m,σ,x_l,x_u are all positive parameters,

$$C=-\frac{\sqrt{2}}{\sigma\sqrt{\pi}}\left[\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\ln\left(\frac{x_l}{m}\right)\right)-\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\ln\left(\frac{x_u}{m}\right)\right)\right]^{-1}$$

and $P\left(x\right)=e^{-\frac{1}{2\sigma^2}\left(\ln\left(\frac{x}{m}\right)\right)^2}$.

1.17. The truncated exponentiated gamma distribution

The cdf and pdf of (XV) are given, respectively, by

$$F\left(x;b,c,d,x_l,x_u\right)=\frac{\left[\Gamma\left(c\right)-\Gamma\left(c,\frac{x_l}{b}\right)\right]^d-\left[\Gamma\left(c\right)-\Gamma\left(c,\frac{x}{b}\right)\right]^d}{\left[\Gamma\left(c\right)-\Gamma\left(c,\frac{x_l}{b}\right)\right]^d-\left[\Gamma\left(c\right)-\Gamma\left(c,\frac{x_u}{b}\right)\right]^d},\quad x_l\leq x\leq x_u, \tag{33}$$

and

$$f\left(x;b,c,d,x_l,x_u\right)=Ce^{-\frac{x}{b}}P\left(x\right),\quad x_l<x<x_u, \tag{34}$$

where b,c,d,x_l,x_u are all positive parameters, $C=\frac{db^{-c}}{\left[\Gamma(c)-\Gamma\left(c,\frac{x_l}{b}\right)\right]^d-\left[\Gamma(c)-\Gamma\left(c,\frac{x_u}{b}\right)\right]^d}$, $\Gamma\left(c\right)$ is the gamma function, $\Gamma\left(c,\frac{x}{b}\right)=\int_{x/b}^{\infty}t^{c-1}e^{-t}\mathrm{d}t$ is incomplete gamma function and $P\left(x\right)=-x^{c-1}\left[\Gamma\left(c\right)-\Gamma\left(c,\frac{x}{b}\right)\right]^{d-1}$.

2. Characterization of Distributions

As mentioned in the Introduction, characterizations of distributions is an important area of research which has recently attracted the attention of many researchers. This section deals with various characterizations of the distributions listed in the Introduction. These characterizations are based on: (i) a simple relationship between two truncated moments and (ii) the reverse hazard function. It should be mentioned that for the characterization (i) the cdf need not to have a closed form and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

2.1. Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of the distributions mentioned in the Introduction, in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to [1], see Theorem 1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in [2], this characterization is stable in the sense of weak convergence.

Theorem 1. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ $(d=-\infty,\ e=\infty$ might as well be allowed). Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2\left(X\right)\mid X\geq x\right]=\mathbf{E}\left[q_1\left(X\right)\mid X\geq x\right]\eta\left(x\right),\quad x\in H,$$

is defined with some real function η. Assume that $q_1,q_2\in C^1\left(H\right)$, $\eta\in C^2\left(H\right)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1=q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1,q_2 and η, particularly

$$F\left(x\right)=\int_a^x C\left|\frac{\eta'\left(u\right)}{\eta\left(u\right)q_1\left(u\right)-q_2\left(u\right)}\right|\exp\left(-s\left(u\right)\right)\,du\ ,$$

where the function s is a solution of the differential equation $s'=\frac{\eta'\,q_1}{\eta\,q_1\,-\,q_2}$ and C is the normalization constant, such that $\int_H dF=1$.

Remark. The goal is to choose $q_1\left(x\right)$ and $q_2\left(x\right)$ so that $\eta\left(x\right)$ has a simple form.

Here is our first characterization.

Proposition 2. Let $X : \Omega \rightarrow (x_l, x_u)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-bx^c}$ for $x \in (x_l, x_u)$. The random variable X has pdf (2) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-bx^c} + e^{-bx_u^c}\right), \quad x \in (x_l, x_u).$$

Proof
Let X be a random variable with pdf (2), then

$$\begin{aligned}(1 - F(x))\, E[q_1(X) \mid X \geq x] &= \int_x^{x_u} C u^{c-1} e^{-bu^c} du \\ &= \frac{C}{bc}\left(e^{-bx^c} - e^{-bx_u^c}\right), \quad x \in (x_l, x_u),\end{aligned}$$

and

$$\begin{aligned}(1 - F(x))\, E[q_2(X) \mid X \geq x] &= \int_x^{x_u} C u^{c-1} e^{-2bu^c} du \\ &= \frac{C}{2bc}\left(e^{-2bx^c} - e^{-2bx_u^c}\right), \quad x \in (x_l, x_u),\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left(e^{-bx^c} - e^{-bx_u^c}\right) < 0 \quad \text{for } x \in (x_l, x_u).$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{bcx^{c-1}e^{-bx^c}}{e^{-bx^c} - e^{-bx_u^c}}, \quad x \in (x_l, x_u),$$

and hence

$$s(x) = -\log\left(e^{-bx^c} - e^{-bx_u^c}\right), \quad x \in (x_l, x_u).$$

Now, in view of Theorem 1, X has density (2). ☐

Corollary 1. Let $X : \Omega \rightarrow (x_l, x_u)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2. The pdf of X is (2) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{bcx^{c-1}e^{-bx^c}}{e^{-bx^c} - e^{-bx_u^c}}, \quad x \in (x_l, x_u).$$

Corollary 2. The general solution of the differential equation in Corollary 1 is

$$\eta(x) = \left(e^{-bx^c} - e^{-bx_u^c}\right)^{-1}\left[-\int bcx^{c-1}e^{-bx^c}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof
If X has pdf (2), then clearly the differential equation holds. Now, if the differential equation holds, then

$$\eta'(x) = \left(\frac{bcx^{c-1}e^{-bx^c}}{e^{-bx^c} - e^{-bx_u^c}}\right)\eta(x) - \left(\frac{bcx^{c-1}e^{-bx^c}}{e^{-bx^c} - e^{-bx_u^c}}\right)\left(q_1(x)\right)^{-1} q_2(x),$$

or

$$\eta'(x)-\left(\frac{bcx^{c-1}e^{-bx^c}}{e^{-bx^c}-e^{-bx_u^c}}\right)\eta(x)=-\left(\frac{bcx^{c-1}e^{-bx^c}}{e^{-bx^c}-e^{-bx_u^c}}\right)(q_1(x))^{-1}q_2(x),$$

or

$$\left(e^{-bx^c}-e^{-bx_u^c}\right)\eta'(x)-\left(bcx^{c-1}e^{-bx^c}\right)\eta(x)=-\left(bcx^{c-1}e^{-bx^c}\right)(q_1(x))^{-1}q_2(x),$$

or

$$\frac{d}{dx}\left\{\left(e^{-bx^c}-e^{-bx_u^c}\right)\eta(x)\right\}=-\left(bcx^{c-1}e^{-bx^c}\right)(q_1(x))^{-1}q_2(x),$$

from which we arrive at

$$\eta(x)=\left(e^{-bx^c}-e^{-bx_u^c}\right)^{-1}\left[-\int bcx^{c-1}e^{-bx^c}(q_1(x))^{-1}q_2(x)\,dx+D\right].$$

□

Note that a set of functions satisfying the differential equation in Corollary 1, is given in Proposition 2 with $D=\frac{e^{-2bx_u^c}}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 1.

Proposition 3. Let $X:\Omega\rightarrow(x_l,x_u)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-cx}$ for $x\in(x_l,x_u)$. The random variable X has pdf (4) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-cx}+e^{-cx_u}\right),\quad x\in(x_l,x_u).$$

Proof
Let X be a random variable with pdf (4), then

$$\begin{aligned}(1-F(x))\,E[q_1(X)\mid X\geq x]&=\int_x^{x_u}Ce^{-cu}du\\&=\frac{C}{c}\left(e^{-cx}-e^{-cx_u}\right),\quad x\in(x_l,x_u),\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E[q_2(X)\mid X\geq x]&=\int_x^{x_u}Ce^{-2cu}du\\&=\frac{C}{2c}\left(e^{-2cx}-e^{-2cx_u}\right),\quad x\in(x_l,x_u),\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\left(e^{-cx}-e^{-cx_u}\right)<0\quad\text{for } x\in(x_l,x_u).$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{ce^{-cx}}{e^{-cx} - e^{-cx_u}}, \quad x \in (x_l, x_u),$$

and hence

$$s(x) = -\log\left(e^{-cx} - e^{-cx_u}\right), \quad x \in (x_l, x_u).$$

Now, in view of Theorem 1, X has density (4). □

Corollary 3. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 3. The pdf of X is (4) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{ce^{-cx}}{e^{-cx} - e^{-cx_u}}, \quad x \in (x_l, x_u).$$

Corollary 4. The general solution of the differential equation in Corollary 3 is

$$\eta(x) = \left(e^{-cx} - e^{-cx_u}\right)^{-1}\left[-\int ce^{-cx}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Proof
Proof is similar to that of Corollary 2. □

Note that a set of functions satisfying the differential equation in Corollary 3, is given in Proposition 2 with $D = \frac{e^{-2cx_u}}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 4. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\frac{x^2}{2a^2}}$ for $x \in (x_l, x_u)$. The random variable X has pdf (6) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\frac{x^2}{2a^2}} + e^{-\frac{x_u^2}{2a^2}}\right), \quad x \in (x_l, x_u).$$

Proof
The proof is similar to that of Proposition 2. □

Proposition 5. Let $X : \Omega \to (0, c)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, x^3$ for $x \in (0, c)$. The random variable X has pdf (8) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(c^3 + x^3\right), \quad x \in (0, c).$$

Proof
Let X be a random variable with pdf (8), then

$$\begin{aligned}(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] &= \int_x^c Cu^2 du \\ &= \frac{C}{3}\left(c^3 - x^3\right), \quad x \in (0, c),\end{aligned}$$

and

$$
\begin{aligned}
(1-F(x))\, E\left[q_2(X) \mid X \geq x\right] &= \int_x^{x_u} C u^5 du \\
&= \frac{C}{6}\left(c^6 - x^6\right), \quad x \in (0, c),
\end{aligned}
$$

and finally

$$
\eta(x)\, q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left(c^3 - x^3\right) > 0, \quad \text{for } x \in (0, c).
$$

Conversely, if η is given as above, then

$$
s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{3x^2}{c^3 - x^3}, \quad x \in (0, c),
$$

and hence

$$
s(x) = -\log\left(c^3 - x^3\right), \quad x \in (0, c).
$$

Now, in view of Theorem 1, X has density (8). □

Corollary 5. Let $X : \Omega \to (0, c)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 5. The pdf of X is (8) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$
\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{3x^2}{c^3 - x^3}, \quad x \in (0, c).
$$

Corollary 6. The general solution of the differential equation in Corollary 5 is

$$
\eta(x) = \left(c^3 - x^3\right)^{-1}\left[-\int 3x^2 \left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],
$$

where D is a constant.

Proof
Proof is similar to that of Corollary 2. □

Note that a set of functions satisfying the differential equation in Corollary 3, is given in Proposition 5 with $D = \frac{c^6}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 6. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x) \equiv 1$ and $q_2(x) = q_1(x)\, e^{-(x/b)^c}$ for $x \in (x_l, x_u)$. The random variable X has pdf (10) if and only if the function η defined in Theorem 1 has the form

$$
\eta(x) = \frac{1}{2}\left(e^{-(x/b)^c} + e^{-(x_u/b)^c}\right), \quad x \in (x_l, x_u).
$$

Proof
Let X be a random variable with pdf (10), then

$$
\begin{aligned}
(1-F(x))\, E\left[q_1(X) \mid X \geq x\right] &= \int_x^{x_u} C u^{c-1} e^{-(u/b)^c} du \\
&= \frac{C}{b^{-c} c}\left(e^{-(x/b)^c} - e^{-(x_u/b)^c}\right), \quad x \in (x_l, x_u),
\end{aligned}
$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\int_x^{x_u}Cu^{c-1}e^{-2(x/b)^c}du$$
$$=\frac{C}{2b^{-c}c}\left(e^{-2(x/b)^c}-e^{-2(x_u/b)^c}\right),\quad x\in(x_l,x_u),$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\left(e^{-(x/b)^c}-e^{-(x_u/b)^c}\right)<0,\quad \text{for } x\in(x_l,x_u).$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{b^{-c}cx^{c-1}e^{-(x/b)^c}}{e^{-(x/b)^c}-e^{-(x_u/b)^c}},\quad x\in(x_l,x_u),$$

and hence

$$s(x)=-\log\left(e^{-(x/b)^c}-e^{-(x_u/b)^c}\right),\quad x\in(x_l,x_u).$$

Now, in view of Theorem 1, X has density (10). □

Corollary 7. Let $X:\Omega\to(x_l,x_u)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 6. The pdf of X is (10) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{b^{-c}cx^{c-1}e^{-(x/b)^c}}{e^{-(x/b)^c}-e^{-(x_u/b)^c}},\quad x\in(x_l,x_u).$$

Corollary 8. The general solution of the differential equation in Corollary 7 is

$$\eta(x)=\left(e^{-(x/b)^c}-e^{-(x_u/b)^c}\right)^{-1}\left[-\int bcx^{c-1}e^{-(x/b)^c}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Note that a set of functions satisfying the differential equation in Corollary 7, is given in Proposition 6 with $D=\frac{e^{-2(x_u/b)^c}}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 1.

Proposition 7. Let $X:\Omega\to(x_l,x_u)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\frac{b}{s}x}$ for $x\in(x_l,x_u)$. The random variable X has pdf (12) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\frac{b}{s}x}+e^{-\frac{b}{s}x_u}\right),\quad x\in(x_l,x_u).$$

Proof
The proof is similar to that of Proposition 3. □

Proposition 8. Let $X:\Omega\to(x_l,x_u)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,x^{-1/c}$ for $x\in(x_l,x_u)$. The random variable X has pdf (14) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(x^{-1/c}+x_u^{-1/c}\right),\quad x\in(x_l,x_u).$$

Proof
Let X be a random variable with pdf (14), then

$$\begin{aligned}(1-F(x))\,E[q_1(X)\mid X\geq x] &= \int_x^{x_u} Cu^{-\frac{1}{c}-1}du \\ &= cC\left(x^{-1/c}-x_u^{-1/c}\right), \quad x\in(x_l,x_u),\end{aligned}$$

and

$$\begin{aligned}(1-F(x))\,E[q_2(X)\mid X\geq x] &= \int_x^{x_u} Cu^{-\frac{2}{c}-1}du \\ &= \frac{1}{2}cC\left(x^{-2/c}-x_u^{-2/c}\right), \quad x\in(x_l,x_u),\end{aligned}$$

and finally

$$\eta(x)\,q_1(x)-q_2(x) = -\frac{q_1(x)}{2}\left(x^{-1/c}-x_u^{-1/c}\right)<0 \quad \text{for } x\in(x_l,x_u).$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{c^{-1}x^{-\frac{1}{c}-1}}{x^{-1/c}-x_u^{-1/c}}, \quad x\in(x_l,x_u),$$

and hence

$$s(x)=-\log\left(x^{-1/c}-x_u^{-1/c}\right), \quad x\in(x_l,x_u).$$

Now, in view of Theorem 1, X has density (14). □

Corollary 9. Let $X:\Omega\to(x_l,x_u)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 8. The pdf of X is (14) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{c^{-1}x^{-\frac{1}{c}-1}}{x^{-1/c}-x_u^{-1/c}}, \quad x\in(x_l,x_u).$$

Corollary 10. The general solution of the differential equation in Corollary 9 is

$$\eta(x)=\left(x^{-1/c}-x_u^{-1/c}\right)^{-1}\left[-\int c^{-1}x^{-\frac{1}{c}-1}\left(q_1(x)\right)^{-1}q_2(x)\,dx+D\right],$$

where D is a constant.

Note that a set of functions satisfying the differential equation in Corollary 7, is given in Proposition 6 with $D=\frac{x_u^{-2/c}}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 1.

Proposition 9. Let $X:\Omega\to(x_l,x_u)$ be a continuous random variable and let $q_1(x)\equiv 1$ and $q_2(x)=q_1(x)\,e^{-a(x/b)^c}$ for $x\in(x_l,x_u)$. The random variable X has pdf (16) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-a(x/b)^c}+e^{-a(x_u/b)^c}\right), \quad x\in(x_l,x_u).$$

Proof
The proof is similar to that of Proposition 6. In fact the cdf for V is the same as the cdf for VIII for $a = 1$. □

Proposition 10. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x) \equiv 1$ and $q_2(x) = q_1(x)\, e^{-x/b}$ for $x \in (x_l, x_u)$. The random variable X has pdf (18) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-x/b} + e^{-x_u/b}\right), \quad x \in (x_l, x_u).$$

Proof
The proof is similar to that of Proposition 3. In fact the cdf (18) is a special case of the cdf (15) for $a = c = 1$. □

Proposition 11. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-\frac{x^2}{2s^2}}$ for $x \in (x_l, x_u)$. The random variable X has pdf (20) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-\frac{x^2}{2s^2}} + e^{-\frac{x_u^2}{2s^2}}\right), \quad x \in (x_l, x_u).$$

Proof
The proof is similar to that of Proposition 2. □

Proposition 12. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-x/b}$ for $x \in (x_l, x_u)$. The random variable X has pdf (22) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-x/b} + e^{-x_u/b}\right), \quad x \in (x_l, x_u).$$

Proof
The proof is similar to that of Proposition 3. □

Proposition 13. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x) \equiv 1$ and $q_2(x) = q_1(x)\, e^{-b^\alpha x^{-\alpha}}$ for $x \in (x_l, x_u)$. The random variable X has pdf (24) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-b^\alpha x_u^{-\alpha}} + e^{-b^\alpha x^{-\alpha}}\right), \quad x \in (x_l, x_u).$$

Proof
Let X be a random variable with pdf (24), then

$$\begin{aligned}(1 - F(x))\, E[q_1(X) \mid X \geq x] &= \int_x^{x_u} C u^{-\alpha-1} e^{-b^\alpha u^{-\alpha}} du \\ &= \frac{C}{\alpha b^\alpha}\left(e^{-b^\alpha x_u^{-\alpha}} - e^{-b^\alpha x^{-\alpha}}\right), \quad x \in (x_l, x_u),\end{aligned}$$

and

$$\begin{aligned}(1 - F(x))\, E[q_2(X) \mid X \geq x] &= \int_x^{x_u} C u^{-\alpha-1} e^{-2b^\alpha u^{-\alpha}} du \\ &= \frac{C}{2\alpha b^\alpha}\left(e^{-2b^\alpha x_u^{-\alpha}} - e^{-2b^\alpha x^{-\alpha}}\right), \quad x \in (x_l, x_u),\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = \frac{q_1(x)}{2}\left(e^{-b^{\alpha}x_u^{-\alpha}} - e^{-b^{\alpha}x^{-\alpha}}\right) > 0 \quad \text{for } x \in (x_l, x_u).$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha b^{\alpha} x^{-\alpha-1} e^{-b^{\alpha}x^{-\alpha}}}{e^{-b^{\alpha}x_u^{-\alpha}} - e^{-b^{\alpha}x^{-\alpha}}}, \quad x \in (x_l, x_u),$$

and hence

$$s(x) = -\log\left(e^{-b^{\alpha}x_u^{-\alpha}} - e^{-b^{\alpha}x^{-\alpha}}\right), \quad x \in (x_l, x_u).$$

Now, in view of Theorem 1, X has density (24). □

Corollary 11. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 13. The pdf of X is (24) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha b^{\alpha} x^{-\alpha-1} e^{-b^{\alpha}x^{-\alpha}}}{e^{-b^{\alpha}x_u^{-\alpha}} - e^{-b^{\alpha}x^{-\alpha}}}, \quad x \in (x_l, x_u).$$

Corollary 12. The general solution of the differential equation in Corollary 11 is

$$\eta(x) = \left(e^{-b^{\alpha}x_u^{-\alpha}} - e^{-b^{\alpha}x^{-\alpha}}\right)^{-1}\left[-\int \alpha b^{\alpha} x^{-\alpha-1} e^{-b^{\alpha}x^{-\alpha}} \left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Note that a set of functions satisfying the differential equation in Corollary 11, is given in Proposition 13 with $D = \frac{e^{-2b^{\alpha}x_u^{-\alpha}}}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 14. Let $X : \Omega \to (x_l, b)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\left(-\frac{x^2}{b^2} + \frac{2x}{b}\right)^{\beta}$ for $x \in (x_l, b)$. The random variable X has pdf (26) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left\{\left(-\frac{x^2}{b^2} + \frac{2x}{b}\right)^{\beta} + 1\right\}, \quad x \in (x_l, b).$$

Proof
Let X be a random variable with pdf (26), then

$$\begin{aligned}(1 - F(x))\, E\left[q_1(X) \mid X \geq x\right] &= \int_x^b C\left(-\frac{2u}{b^2} + \frac{2}{b}\right)\left(-\frac{u^2}{b^2} + \frac{2u}{b}\right)^{\beta-1} du \\ &= -\frac{C}{\beta}\left\{\left(-\frac{x^2}{b^2} + \frac{2x}{b}\right)^{\beta} - 1\right\}, \quad x \in (x_l, b),\end{aligned}$$

and

$$(1-F(x))\,E[q_2(X)\mid X\geq x]=\int_x^b C\left(-\frac{2u}{b^2}+\frac{2}{b}\right)\left(-\frac{u^2}{b^2}+\frac{2u}{b}\right)^{2\beta-1}du$$
$$=-\frac{C}{2\beta}\left\{\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{2\beta}-1\right\},\quad x\in(x_l,b),$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\left\{\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta}-1\right\}>0\quad\text{for } x\in(x_l,b).$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\beta\left(\frac{2x}{b^2}-\frac{2}{b}\right)\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta-1}}{\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta}-1},\quad x\in(x_l,b),$$

and hence

$$s(x)=-\log\left(\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta}-1\right),\quad x\in(x_l,b).$$

Now, in view of Theorem 1, X has density (26). □

Corollary 13. Let $X:\Omega\to(x_l,b)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 14. The pdf of X is (26) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\beta\left(\frac{2x}{b^2}-\frac{2}{b}\right)\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta-1}}{\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta}-1},\quad x\in(x_l,b).$$

Corollary 14. The general solution of the differential equation in Corollary 13 is

$$\eta(x)=\left\{\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta}-1\right\}^{-1}\left[-\int\beta\left(\frac{2x}{b^2}-\frac{2}{b}\right)\left(-\frac{x^2}{b^2}+\frac{2x}{b}\right)^{\beta-1}(q_1(x))^{-1}\,q_2(x)\,dx+D\right],$$

where D is a constant.

Note that a set of functions satisfying the differential equation in Corollary 11, is given in Proposition 13 with $D=\frac{1}{2}$. However, it should also be noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem 1.

Proposition 15. Let $X:\Omega\to(x_l,x_u)$ be a continuous random variable and let $q_1(x)=[P(x)]^{-1}$ and $q_2(x)=q_1(x)\,e^{-\frac{a}{b}x}$ for $x\in(x_l,x_u)$. The random variable X has pdf (28) if and only if the function η defined in Theorem 1 has the form

$$\eta(x)=\frac{1}{2}\left(e^{-\frac{a}{b}x}+e^{-\frac{a}{b}x_u}\right),\quad x\in(x_l,x_u).$$

Proof
The proof is similar to that of Proposition 3. □

Proposition 16. Let $X : \Omega \to (\sigma, x_u)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, x^{-\alpha}$ for $x \in (\sigma, x_u)$. The random variable X has pdf (30) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(x^{-\alpha} + x_u^{-\alpha}\right), \quad x \in (\sigma, x_u).$$

Proof
Let X be a random variable with pdf (30), then

$$\begin{aligned}(1 - F(x))\, E[q_1(X) \mid X \geq x] &= \int_x^{x_u} C u^{-\alpha-1} du \\ &= \frac{C}{\alpha}\left(x^{-\alpha} - x_u^{-\alpha}\right), \quad x \in (\sigma, x_u),\end{aligned}$$

and

$$\begin{aligned}(1 - F(x))\, E[q_2(X) \mid X \geq x] &= \int_x^{x_u} C u^{-2\alpha-1} du \\ &= \frac{C}{2\alpha}\left(x^{-2\alpha} - x_u^{-2\alpha}\right), \quad x \in (\sigma, x_u),\end{aligned}$$

and finally

$$\eta(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\left(x^{-\alpha} - x_u^{-\alpha}\right) < 0 \quad \text{for } x \in (\sigma, x_u).$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha x^{-\alpha-1}}{x^{-\alpha} - x_u^{-\alpha}}, \quad x \in (\sigma, x_u),$$

and hence

$$s(x) = -\log\left(x^{-\alpha} - x_u^{-\alpha}\right), \quad x \in (\sigma, x_u).$$

Now, in view of Theorem 1, X has density (30) $\square$

Corollary 15. Let $X : \Omega \to (\sigma, x_u)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 16. The pdf of X is (30) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{\alpha x^{-\alpha-1}}{x^{-\alpha} - x_u^{-\alpha}}, \quad x \in (\sigma, x_u).$$

Corollary 16. The general solution of the differential equation in Corollary 15 is

$$\eta(x) = \left(x^{-\alpha} - x_u^{-\alpha}\right)^{-1}\left[-\int \alpha x^{-\alpha-1}\left(q_1(x)\right)^{-1} q_2(x)\, dx + D\right],$$

where D is a constant.

Note that a set of functions satisfying the differential equation in Corollary 15, is given in Proposition 16 with $D = \frac{x_u^{-2\alpha}}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 17. Let $X : \Omega \rightarrow (x_l, x_u)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x) \ln(x)$ for $x \in (x_l, x_u)$. The random variable X has pdf (32) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}(\ln(x_u) + \ln(x)), \quad x \in (x_l, x_u).$$

Proof
Let X be a random variable with pdf (32), then

$$\begin{aligned}(1 - F(x)) E[q_1(X) \mid X \geq x] &= \int_x^{x_u} C u^{-1} du \\ &= C(\ln(x_u) - \ln(x)), \quad x \in (x_l, x_u),\end{aligned}$$

and

$$\begin{aligned}(1 - F(x)) E[q_2(X) \mid X \geq x] &= \int_x^{x_u} C u^{-1} \ln(u) du \\ &= \frac{C}{2}\left((\ln(x_u))^2 - (\ln(x))^2\right), \quad x \in (x_l, x_u),\end{aligned}$$

and finally

$$\eta(x) q_1(x) - q_2(x) = \frac{q_1(x)}{2}(\ln(x_u) - \ln(x)) > 0 \quad \text{for } x \in (x_l, x_u).$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{x^{-1}}{\ln(x_u) - \ln(x)}, \quad x \in (x_l, x_u),$$

and hence

$$s(x) = -\log(\ln(x_u) - \ln(x)), \quad x \in (x_l, x_u).$$

Now, in view of Theorem 1, X has density (32) □

Corollary 17. Let $X : \Omega \rightarrow (x_l, x_u)$ be a continuous random variable and let $q_1(x)$ be as in Proposition 17. The pdf of X is (32) if and only if there exist functions q_2 and η defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{x^{-1}}{\ln(x_u) - \ln(x)}, \quad x \in (x_l, x_u).$$

Corollary 18. The general solution of the differential equation in Corollary 17 is

$$\eta(x) = (\ln(x_u) - \ln(x))^{-1}\left[-\int x^{-1}(q_1(x))^{-1} q_2(x) dx + D\right],$$

where D is a constant.

Note that a set of functions satisfying the differential equation in Corollary 17, is given in Proposition 17 with $D = \frac{(\ln(x_u))^2}{2}$. However, it should also be noted that there are other triplets (q_1, q_2, η) satisfying the conditions of Theorem 1.

Proposition 18. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable and let $q_1(x) = [P(x)]^{-1}$ and $q_2(x) = q_1(x)\, e^{-x/b}$ for $x \in (x_l, x_u)$. The random variable X has pdf (34) if and only if the function η defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2}\left(e^{-x/b} + e^{-x_u/b}\right), \quad x \in (x_l, x_u).$$

Proof
The proof is similar to that of Proposition 3. □

2.2. *Characterization in Terms of the Reverse (or Reversed) Hazard Function*

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in \text{support of } F.$$

In this subsection we present characterizations of the new distributions in terms of the reverse hazard function.

Proposition 19. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable. The random variable X has pdf (10) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) + cb^{-c}x^{c-1} r_F(x) = e^{-(x/b)^c} \frac{d}{dx}\left\{\frac{-cb^{-c}x^{c-1}}{e^{-(x/b)^c} - e^{-(x_l/b)^c}}\right\}, \quad x \in (x_l, x_u),$$

with boundary condition $\lim_{x \to x_u} r_F(x) = \left(\frac{-cb^{-c}x_u^{c-1}e^{-(x_u/b)^c}}{e^{-(x_u/b)^c} - e^{-(x_l/b)^c}}\right)$.

Proof
Multiplying both sides of the above equation by $e^{(x/b)^c}$, we have

$$\frac{d}{dx}\left\{e^{(x/b)^c} r_F(x)\right\} = \frac{d}{dx}\left\{\frac{-cb^{-c}x^{c-1}}{e^{-(x/b)^c} - e^{-(x_l/b)^c}}\right\},$$

or

$$\left\{e^{(x/b)^c} r_F(x)\right\} = \left\{\frac{-cb^{-c}x^{c-1}}{e^{-(x/b)^c} - e^{-(x_l/b)^c}}\right\},$$

or

$$r_F(x) = \frac{-cb^{-c}x^{c-1}e^{-(x/b)^c}}{e^{-(x/b)^c} - e^{-(x_l/b)^c}},$$

which is the reverse hazard function corresponding to the pdf (10). □

Proposition 20. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable. The random variable X has pdf (18) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) + \frac{1}{b} r_F(x) = -\frac{1}{b^2} e^{-\frac{2x}{b}} \left(e^{-x/b} - e^{-x_l/b}\right)^{-2}, \quad x \in (x_l, x_u),$$

with boundary condition $\lim_{x \to x_u} r_F(x) = \left(\frac{-\frac{1}{b}e^{-x_u/b}}{e^{-x_u/b} - e^{-x_l/b}}\right)$.

Proof
It is similar to that of Proposition 19. □

Proposition 21. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable. The random variable X has pdf (20) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) + \frac{x}{s^2} r_F(x) = \frac{\sqrt{2}}{s\sqrt{\pi}} e^{-\frac{x^2}{2s^2}} \frac{d}{dx}\left(-\operatorname{erf}\left(\frac{x_l\sqrt{2}}{2s}\right) + \operatorname{erf}\left(\frac{x_u\sqrt{2}}{2s}\right)\right)^{-1}, \quad x \in (x_l, x_u),$$

with boundary condition $\lim_{x\to x_u} r_F(x) = \left(\frac{\frac{\sqrt{2}}{s\sqrt{\pi}} e^{-\frac{x_u^2}{2s^2}}}{-\operatorname{erf}\left(\frac{x_l\sqrt{2}}{2s}\right) + \operatorname{erf}\left(\frac{x_u\sqrt{2}}{2s}\right)}\right)$.

Proof
It is similar to that of Proposition 19. □

Proposition 22. Let $X : \Omega \to (x_l, x_u)$ be a continuous random variable. The random variable X has pdf (24) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) + (\alpha+1)\, x^{-1} r_F(x) = \alpha b^\alpha x^{-\alpha-1} \frac{d}{dx}\left\{\frac{e^{-b^\alpha x^{-\alpha}}}{e^{-b^\alpha x^{-\alpha}} - e^{-b^\alpha x_l^{-\alpha}}}\right\}, \quad x \in (x_l, x_u),$$

with boundary condition $\lim_{x\to x_u} r_F(x) = \left(\frac{\alpha b^\alpha x_u^{-\alpha-1} e^{-b^\alpha x_u^{-\alpha}}}{e^{-b^\alpha x_u^{-\alpha}} - e^{-b^\alpha x_l^{-\alpha}}}\right)$.

Proposition 23. Let $X : \Omega \to (x_l, b)$ be a continuous random variable. The random variable X has pdf (26) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$\begin{aligned} &r_F'(x) - (\beta-1)\left(-\frac{2x}{b^2} + \frac{2}{b}\right)\left(-\frac{x^2}{b^2} + \frac{2x}{b}\right)^{-1} r_F(x) \\ &= -\beta b^{2\beta+1}\left(-\frac{x^2}{b^2} + \frac{2x}{b}\right)^{\beta-1} \frac{d}{dx}\left\{\frac{2 - \frac{2x}{b}}{x_l^\beta (2b - x_l)^\beta - x^\beta (2b-x)^\beta}\right\}, \quad x \in (x_l, b), \end{aligned}$$

with boundary condition $\lim_{x\to b} r_F(x) = 0$.

Remark (I)**.** As far as the similarity of the formats of the pdfs are concerned, we can group the distributions as follows: Group 1 (1.1, 1.3, 1.5, 1.10); Group 2 (1.2, 1.6, 1.11, 1.14, 1.17); Group 3 (1.4); Group 4 (1.7, 1.15, 1.16); Group 5 (1.8, 1.9, 1.12); Group 6 (1.13).

Remark (II)**.** Below is a short list of certain distributions characterized in [3] which have no closed form cdfs.

- Wrapped Chi-Square Distribution with pdf

$$f(x;k) = Ce^{-\frac{1}{2}x} P(x), \quad 0 < x < 2\pi, \tag{35}$$

where $k > 0$ is a parameter, $C = \frac{\pi^{\frac{k}{2}-1}}{2\Gamma(k/2)}, P(x) = \Phi\left(e^{-\pi}; 1 - \frac{k}{2}, \frac{x}{2\pi}\right), \Gamma(k)$ is the gamma function and $\Phi(x; s, a) = \sum_{n=0}^{\infty} \frac{x^n}{(a+n)^s}$.

- Wrapped Exponential Inverted Weibull Distribution with pdf

$$f(x; a, \lambda) = Ce^{-x} P(x), \quad 0 < x < 2\pi, \tag{36}$$

where $a > 0, \lambda > 0$ are parameters, $C = a\lambda$ and $P(x) = e^x \sum_{k=0}^{\infty} (x + 2\pi k)^{-(a+1)} \left[e^{-(x+2\pi k)^{-a}}\right]^\lambda$.

- Wrapped Gamma Distribution with pdf

$$f(x;r,\lambda)=Ce^{-\lambda x}P(x), \quad 0<x<2\pi, \tag{37}$$

where $r>0,\lambda>0$ are parameters, $C=\frac{\lambda^r(2\pi)^{r-1}}{\Gamma(r)}, P(x)=\Phi\left(e^{-2\pi\lambda};1-r,\frac{x}{2\pi}\right), \Gamma(k)$ is the gamma function and $\Phi(x;s,a)=\sum_{n=0}^{\infty}\frac{x^n}{(a+n)^s}$.

- Wrapped Generalized Skew Normal-1 Distribution with pdf

$$f(x;\alpha,\mu,\sigma,\lambda)=Cxe^{-\frac{x^2}{2\sigma^2}}P(x), \quad 0<x<2\pi, \tag{38}$$

where $\alpha\geq -1,\mu\in\mathbb{R},\sigma>0,\lambda\in\mathbb{R}$ are parameters, $C=\frac{2}{\sigma(\alpha+2)\sqrt{2\pi}}$, $P(x)=x^{-1}\sum_{k=-\infty}^{\infty}e^{-\frac{1}{2\sigma^2}\left[(2\pi k-\mu)x+(2\pi k-\mu)^2\right]}\left[1+\sigma\Phi\left(\frac{\lambda}{\sigma}(x+2\pi k-\mu)\right)\right]$ and $\Phi(x)$ is the cdf of the standard normal random variable.

- Wrapped Half-Logistic Distribution with pdf

$$f(x;\mu,\sigma)=Ce^{-\frac{x}{\sigma}}P(x), \quad 0<x<2\pi, \tag{39}$$

where $0\leq\mu<2\pi,\sigma>0$ are parameters, $C=\frac{2}{\sigma}$ and $P(x)=\sum\limits_{k=-\infty}^{\infty}\frac{1}{\left(e^{\frac{2k\pi-\mu}{\sigma}}+e^{-\frac{2x+2k\pi-\mu}{\sigma}}+2e^{\frac{-x}{\sigma}}\right)}$.

- Wrapped Levy Distribution with pdf

$$f(x;c,\mu)=Ce^{-\frac{c}{2}x}P(x), \quad 0<x<2\pi, \tag{40}$$

where $c>0,\mu\in\mathbb{R}$ are parameters, $C=e^{\frac{c\mu}{2}}\sqrt{\frac{c}{2\pi}}$ and $P(x)=\sum_{n=-\infty}^{\infty}\frac{e^{-c\pi n}}{(x+2\pi n-\mu)^{3/2}}$.

- Wrapped Richard Distribution with pdf

$$f(x;m,n)=e^{-mx}P(x), \quad 0<x<2\pi, \tag{41}$$

where $m\geq 1,n\geq 1$ are parameters and $P(x)=\sum_{k=0}^{\infty}\frac{ke^{-k(x+2\pi n)+mx}}{1-m^{\frac{1}{1-m}}}\left[1+(m-1)e^{-k(x+2\pi n)}\right]^{\frac{m}{1-m}}$.

- Wrapped Weibull Distribution with pdf

$$f(x;\mu,\sigma)=Ce^{-\lambda x}P(x), \quad 0<x<2\pi, \tag{42}$$

where $\mu\in\mathbb{R},\sigma>0$ are parameters, WLOG we take $\mu=0,\sigma=1$,$C=\frac{2}{\sigma}$ and $P(x)=\sum\limits_{k=0}^{\infty}\frac{1}{(e^{2k\pi}+e^{-2x-2k\pi}+2e^{-x})}$.

3. Conclusions

Lorenzo Zaninetti proposed 15 truncated univariate continuous distributions from 2019-2025, all published in the ”International Journal of Astronomy and Astrophysics”. Zaninetti performed super complicated derivations to present the cdfs and pdfs of these distributions. The present work may be considered as a supplement to Zaninetti’s excellent works. Here, we present certain characterizations of these distributions based on: (i) a simple relationship between two truncated moments and (ii) the reverse hazard function. It should be mentioned that for the characterization (i) the cumulative distribution function need not to have a closed form, as is the case with some of these 15 distributions, and depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation.

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