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Enhancing Parameter Estimation for Fuzzy Robust Regression in the Presence of Outliers

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Abstract This study presents an enhanced algorithm for parameter estimation in fuzzy robust regression (FRR), aimed at improving the reliability of estimates in the presence of outliers. The standard approach of using ordinary least squares (OLS) struggles when dealing with both outlier effects and the uncertainty inherent in data. By combining traditional FRR analysis with the Huber loss function, this research addresses these challenges effectively. The performance of the algorithm is evaluated using real-world datasets and a simulation study, demonstrating its ability to minimize the impact of outliers. Furthermore, the algorithm not only outperforms OLS but also serves as a robust alternative to traditional methods, including Huber, Hampel, Tukey, Andrews, MM-estimates and existing FRR approaches.

Keywords Outliers, Robust Regression, Membership Function, Fuzzy Robust Regression, Parameter Estimation

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1. Introduction

The OLS approach is very popular when used in conventional regression analysis because it is simple and efficiently establishes the relationship between dependent and independent variables. However, OLS is highly sensitive to outliers or deviations from assumptions about normality and more generally, homoscedasticity of the data. Outliers can have a significant impact on parameter estimates, leading to an incorrect interpretation of our model, especially when using real-world data. As such, robust regression (RR) methods have been specifically designed to address the impact of outliers. According to Huber's M-estimation approach, RR has become the main methodology for handling this issue [1]. Typically, the accuracy of this method exceeds that of least-squares techniques by incorporating a weighting factor to reduce the impact of influential observations.

Zadeh introduced the concept of fuzziness and the theory of fuzzy sets in 1965 [2]. Since then, fuzzy information has been incorporated into regression analysis across various scientific fields. Fuzzy regression analysis expands the classical regression concept into a fuzzy environment to model uncertainty in the data. The fuzzy regression model analysis is categorized into two main types: the first involves the Linear Programming (LP) approach of Tanaka et al., and the second involves the fuzzy least squares (FLS) approach of Diamond [3]. Tanaka et al. [4] introduced a fuzzy model for linear regression, which led to extensive studies in fuzzy regression analysis. However, their method may lead to misinterpretations when outliers are present. Hung and Yang [5] developed an omission approach to solve this issue by applying Tanaka's method to the dataset after removing certain observations. Bardossy [6] defined the general forms of regression equations and explained how fuzzy regression should be formulated as a mathematical programming problem. In the context of fuzzy regression, an observation with a larger residual value than the others is referred to as an outlier [5].

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FRR unifies the advantages of robust statistics and fuzzy regression to address data that includes both fuzziness and outliers. In cases where data uncertainty and outliers coexist, this hybrid approach has received attention in many applications. This method uniquely adjusts the regression model so that outliers in the data have less influence, while considering the fuzziness of the data, leading to more accurate and reliable estimates. Yen et al. [7] introduced asymmetric triangular fuzzy numbers (TFNs) to fuzzy regression analysis. Chang and Lee [8] introduced an expanded approach to weighted fuzzy least squares (WFLS) regression. Their model involved an interactive decision-making process and relied on non-fuzzy input and output data. The weighting function in their method is based on the degree of membership and the WFLS is constructed using a weight matrix [9]. Buckley [10] proposed a hypothesis testing approach for fuzzy linear regression parameters based on confidence intervals. Subsequently, Nasrabadi et al. [11] introduced a LP method to detect outliers within the fuzzy linear regression framework. Varga [12] developed robust estimation techniques applicable to both fuzzy and classical regression models. Taheri and Arefi [13] advanced fuzzy hypothesis testing by introducing a method based on calculating the areas of fuzzy numbers. Celikyilmaz and Turksen [14] developed the fuzzy functions approach to construct fuzzy system models for regression and classification problems, introducing a novel framework for modeling uncertainty. Yang and Lin [15] proposed a FLS regression analysis for fuzzy input and output data. To address heterogeneous data and detect outliers, they integrated clusterwise regression and a noise cluster approach into their modeling framework. Khammar et al. [16] presented the robust least squares fuzzy regression model with a kernel function. This model enhanced the flexibility of fuzzy regression, especially in high-dimensional spaces, and was able to retain more accuracy in characterizing the non-linearity of data. The authors later developed a fuzzy varying coefficient regression model dependent on the Huber loss function that improved outlier resistance and enhanced estimation accuracy of model parameters [17]. Bas [18] offered a fuzzy regression functions approach that combines fuzzy set theory and multiple regression analysis. A RR technique is used, allowing forecasts to be made robustly, even in the presence of outliers and it outperforms methods from the literature. Kong and Song [19] established a FRR model using exponential-type kernel functions to enhance the forecasting performance of fuzzy regression models. Additionally, the gh-transformation was used to prevent negative spreads in the predictions. Later, Kula et al. [20] proposed the FRR method, which transforms explanatory and response variables into TFNs and estimates the parameters as crisp values.

This study proposes a modified fuzzy robust regression (MFRR) model, in which both dependent and independent variables are expressed as TFNs, while parameter estimates are obtained as crisp values. This method builds on the model introduced by Kula et al. [20] and introduces modifications to improve robustness and estimation accuracy. The structure of the article is as follows: In Section 2, parameter estimation in multiple linear regression is discussed. Section 3 addresses the concepts of RR Section 4 explains multiple fuzzy regression models. Section 5 explains the FRR model and algorithm. Section 6 presents the proposed MFRR model and algorithm. Section 7 presents the results of the application on real-world data and a simulation study, using tables and graphs to assess the performance of the proposed method. Finally, conclusions and discussions are revealed in Section 8.

2. Multiple Linear Regression Model(MLRM)

Regression analysis was initially introduced by Galton in the 19th century. He formulated a mathematical model to describe the statistical relationships between variables. In this model, regression illustrates how a dependent variable (Y) is related to independent variables $(X_1, X_2, \text{etc.})$. The basic regression equation is represented as [21]:

$$Y_i = \beta_0 + \sum_{j=1}^{p} \beta_j X_{ij} + \varepsilon_i, \quad i = 1, 2, ..., n. \tag{1}$$

The general model, as presented in Eq.(1), is expressed in matrix form as follows:

$$Y = X\beta + \varepsilon,$$

where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix}, \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}, \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix},$$

β signifies the vector of regression coefficients and ε indicates the vector of errors (residuals). A standard assumption in this framework is as follows:

$$E(\varepsilon) = 0, \tag{2}$$

$$E(\varepsilon^T \varepsilon) = \begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2 I_n, \tag{3}$$

where I_n represents the identity matrix of order n and σ^2 is the variance of errors. The residual sum of squares can be expressed as:

$$\varepsilon^T \varepsilon = (Y - X\hat{\beta})^T (Y - X\hat{\beta}), \tag{4}$$

$$= Y^T Y - 2\hat{\beta} X^T Y + \hat{\beta}^T X^T X \hat{\beta}, \tag{5}$$

where $\hat{\beta}$ is the least squares estimator of β, $\hat{\beta}$ can be solved by minimizing the residual sum of squares using the first-order condition:

$$\frac{\partial}{\partial \hat{\beta}} (\varepsilon^T \varepsilon) = 0, \tag{6}$$

$$-2X^T Y + 2X^T X \hat{\beta} = 0, \tag{7}$$

$$X^T X \hat{\beta} = X^T Y, \tag{8}$$

$$\hat{\beta} = (X^T X)^{-1} (X^T Y). \tag{9}$$

OLS estimated regression models are very sensitive to outliers and not robust in their presence. An outlier is normally defined as an observation that deviates from the main trend of the data. This is not a problem if the outlier is simply an extreme value at the tail of a normal distribution. But when an outlier results from non-normal measurement errors or other violations of the fundamental OLS assumptions, the validity of the regression results is compromised, particularly when estimation techniques are used that fail to sufficiently account for such irregularities.

3. Robust Regression Model (RRM)

RR is a statistical technique especially suited to cases where residuals do not follow a normal distribution, or when outliers make an important contribution to the model. This is a very useful approach when the datasets you are looking at have outliers, because it makes the produced models less sensitive to their effects. Conventional methods often produce biased predictions when the assumptions of regression analysis are violated, and transformations fail to eliminate the impact of outliers. For such situations RR is the most convenient way to go as it is much less sensitive to outliers and thus results in more accurate and reliable results. RR reduces outlier effects through M-estimation and MM-estimation implementations that maintain efficiency along with resilience [22–24]. However, in the RR method, the results depend on selecting a loss function (ρ). Compared to traditional 0LS, the calculation method imposes a greater computational burden. An analysis of robust estimation methods, specifically M-estimators and MM-estimators, is presented below.

3.1. M-Estimation

M-estimation is one of the common techniques of estimation in RR wherein 'M' means that estimation is based on the maximum likelihood. Minimizing the effect of outliers is widely seen as the advantage of M-estimation. Among the best known M-estimation methods are Huber, Hampel, Andrews and Tukey. These approaches use different functions for reduction of outliers' influence resulting in the more stable and reliable parameter estimates while the data contains the anomalies.

M-estimation can be regarded as both an extension of the maximum likelihood estimation method and a robust estimation technique [25]. Using this technique, it is possible to remove some data points [26]. However, not all of the time is this the best option, as by excluding data, important information can be missed. The fundamental principle of M-estimation is to minimize the residual function ρ, which is designed to accommodate the peculiarities of the dataset being analyzed. This can be written this way:

$$\min_{\beta}\sum_{i=1}^{n}\rho(\nu_i)=\min_{\beta}\sum_{i=1}^{n}\rho(\frac{\varepsilon_i}{\hat{\sigma}_i})=\min_{\beta}\sum_{i=1}^{n}\rho(\frac{y_i-\sum\limits_{j=0}^{p}x_{ij}\beta_j}{\hat{\sigma}_i}), \tag{10}$$

where $\hat{\sigma}_i=\frac{MAD}{0.6745}=\frac{\text{median}|\varepsilon_i-\text{median}(\varepsilon_i)|}{0.6745}$. Analyzing the first partial derivative of Eq.(10) allows for the determination of optimal regression coefficients effectively.

$$\sum_{i=1}^{n}\psi(\frac{y_i-\sum\limits_{j=0}^{p}x_{ij}\beta_j}{\hat{\sigma}_i})x_{ij}=0, \qquad j=0,1,2,...,p, \tag{11}$$

where $\psi(\nu_i)=\rho'(\nu_i),\ x_{ij}$ corresponds to the $i-th$ observation for the $j-th$ independent variable, and $x_{i0}=1$. Draper and Smith [24] present a solution for Eq.(11) by defining a weighted function:

$$W(\varepsilon_i)=\frac{\psi(\frac{y_i-\sum\limits_{j=0}^{p}x_{ij}\beta_j}{\hat{\sigma}_i})}{(\frac{y_i-\sum\limits_{j=0}^{p}x_{ij}\beta_j}{\hat{\sigma}_i})}. \tag{12}$$

In terms of the weighted function, the estimated equations for the model parameters (11) can be expressed as follows:

$$\sum_{i=1}^{n}W_i(\frac{y_i-\sum\limits_{j=0}^{p}x_{ij}\beta_j}{\hat{\sigma}_i})x_{ij}=0. \tag{13}$$

In matrix notation, Eq.(13) can be represented as follows:

$$X^TWX\beta=X^TWY, \tag{14}$$

where W is an $n\times n$ matrix whose diagonal elements represent the weights. Eq. (14) is referred to as the weighted least squares (WLS) equation. The solution to this equation provides an estimator for β; specifically:

$$\hat{\beta}=(X^TWX)^{-1}(X^TWY). \tag{15}$$

The following four objective functions are employed in the paper. The ρ and W functions of Huber's are as follows:

$$\rho(\nu)=\begin{cases}\frac{\nu^2}{2} & ,|\nu|\leq k\\ k|\nu|-\frac{k^2}{2} & ,|\nu|>k\end{cases} \quad \text{and} \quad W(\nu)=\begin{cases}1 & ,|\nu|\leq k\\ \frac{k}{|\nu|} & ,|\nu|>k\end{cases} \tag{16}$$

The term tuning constant in common usage in robust statistics is usually a cutoff point. When the assumptions of a normal error structure are satisfied, such as in cases where least squares estimates are deemed optimal, applying the Huber method may incur a certain degree of efficiency loss. This loss in efficiency is quantified as the premium paid to safeguard against unreliable least squares estimates in scenarios characterized by non-normal distributions. The tuning constant k for this premium is generally set at 1.5 if this premium is established at 5%. [27]. The selection of the tuning constant aims to maintain a reasonably high level of efficiency under normal conditions. Specifically, the Huber method is designed to achieve a certain percentage of efficiency when errors follow a normal distribution while simultaneously providing a robust defense against outliers.
The ρ and W functions of Hampel's are as follows:

$$\rho(\nu) = \begin{cases} \frac{\nu^2}{2} & , 0 < |\nu| \le a \\ a|\nu| - \frac{a^2}{2} & , a < |\nu| \le b \\ -\dfrac{a(c-\nu)}{2(c-b)} + \dfrac{a(b+c-a)}{2} & , b < |\nu| \le c \\ \dfrac{a(b+c-a)}{2} & , c < |\nu| \end{cases} \text{ and } W(\nu) = \begin{cases} 1 & , 0 < |\nu| \le a \\ \dfrac{a}{\nu}\,\mathrm{sgn}(\nu) & , a < |\nu| \le b \\ \dfrac{a}{\nu}\left(\dfrac{c-|\nu|}{c-b}\right)\mathrm{sgn}(\nu) & , b < |\nu| \le c \\ 0 & , c < |\nu| \end{cases} \tag{17}$$

where the constants $a = 1.7$, $b = 3.4$ and $c = 8.5$ are designated as the cutoff points for the Hampel estimator. These values are considered appropriate choices for the constants, which enhances the robustness of the estimator against outliers [27].
The ρ and W functions of Andrew's are as follows:

$$\rho(\nu) = \begin{cases} k^2\left(1 - \cos\left(\dfrac{|\nu|}{k}\right)\right) & , |\nu| \le k\pi \\ 2k^2 & , |\nu| > k\pi \end{cases} \quad \text{and} \quad W(\nu) = \begin{cases} \dfrac{1}{\nu}\sin\left(\dfrac{\nu}{k}\right) & , |\nu| \le k\pi \\ 0 & , |\nu| > k\pi \end{cases} \tag{18}$$

where k is referred to as the cutoff point for the Andrews estimator. When the scale is known, a value of $k = 1.339$ corresponds to a premium of 5%. However, if the scale is not specified, it can be set to either $k = 1.5$ or $k = 2.1$ as alternative options [27].
The ρ and W functions of Tukey's bisquare (biweight estimator) are as follows:

$$\rho(\nu) = \begin{cases} \dfrac{1}{6}\left(1 - \left(1 - \left(\dfrac{\nu}{k}\right)^2\right)^3\right) & , |\nu| \le k \\ \dfrac{1}{6} & , |\nu| > k \end{cases} \quad \text{and} \quad W(\nu) = \begin{cases} \left(1 - \left(\dfrac{\nu}{k}\right)^2\right)^2 & , |\nu| \le k \\ 0 & , |\nu| > k \end{cases} \tag{19}$$

where k is designated as the cutoff point for the biweight estimator. When the scale is known, a value of $k = 4.685$ corresponds to a premium of 5%. In cases where the scale is not established, alternative values of $k = 5.0$ or $k = 6.0$ may be utilized [27]. A comprehensive overview of M-estimation using weights from Huber, Hampel, Andrews, and Tukey is provided. The M-estimator procedure is summarized as follows:
Step 1: Estimate the regression coefficient for the data using OLS method.
Step 2: Conduct classical assumption testing for the regression model.
Step 3: Identify outliers in the dataset.
Step 4: Calculate the estimates $\hat{\beta}$ using OLS.
Step 5: Calculate the value of $\hat{y}_i$.
Step 6: Calculate the residual value ε_i.
Step 7: Calculate the value of $\hat{\sigma}_i$.
Step 8: Calculate the value of $\nu_i = \frac{\varepsilon_i}{\hat{\sigma}_i}$.
Step 9: Calculate the weighted value of W_i by employing one of the weights derived from Huber, Hampel, Andrews or Tukey.
Step 10: Calculate $\hat{\beta}_M$ using the WLS method as defined in Eq.(15) with the weighted matrix W_i.

Step 11: In the iterative estimation of RRM coefficients, the process is terminated if the absolute difference between the coefficients in iterations $k+1$ and k is less than a specified tolerance level (ϵ) . If this condition is not satisfied, the procedure continues to Step 5, where k denotes the iteration number, and $\epsilon > 0$ represents a small positive constant.

Figure 1 presents a graphical comparison of both the loss functions ρ and the corresponding weight functions associated with four widely used robust estimators: Huber, Hampel, Tukey, and Andrews. The left panel displays the ρ curves, illustrating how each function penalizes residuals differently depending on their magnitude. The right panel shows the weight functions, indicating the influence assigned to each observation as a function of residual size.

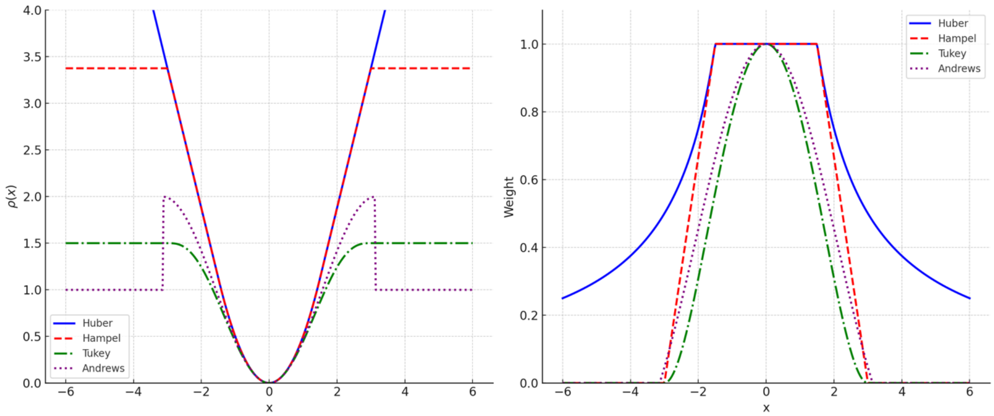

Figure 1. The loss and weight functions of Huber, Hampel, Tukey and Andrews.

3.2. *MM-Estimation*

The MM estimation procedure involves estimating the regression parameter using S-estimation, which minimizes the residual scale from M-estimation, and then continues with M-estimation. The goal of MM-estimation is to produce estimates with a high breakdown value and increased efficiency. The breakdown value is a widely used metric indicating the percentage of outliers that can be handled before they impact the model [23]. MM-estimator is the solution of

$$\sum_{i=1}^{n} \psi\left(\frac{y_i - \sum_{j=0}^{p} x_{ij}\beta_j}{\hat{\sigma}_{MM}}\right) x_{ij} = 0, \tag{20}$$

where $\hat{\sigma}_{MM}$ represents the standard deviation derived from the residuals of the S-estimation. The MM-estimates procedure is summarized as follows [22]:

Step 1: Calculate the initial estimates of the coefficients $\hat{\beta}^{(1)}$ and the corresponding residuals $\varepsilon_i^{(1)}$, $i = 1, 2, ..., n$ by employing a high breakdown point estimator such as S-estimators with Huber or bisquare weight function.

Step 2: Calculate the M-estimation of the scale of residuals $\hat{\sigma}_\varepsilon$ using the results from Step 1.

Step 3: The residuals (from Step 1) and the scale (from Step 2) are employed in the first iteration of WLS to find the M-estimates of the regression parameters,

$$\sum_{i=1}^{n} W_i\left(\frac{\varepsilon_i^{(1)}}{\hat{\sigma}_\varepsilon}\right) x_{ij} = 0, \tag{21}$$

where W_i can be chosen as Huber or bisquare weights.

Step 4: Calculate new weights $W_i^{(2)}$ by utilizing the residuals from Step 3.

Step 5: The $\hat{\sigma}_\varepsilon$ is kept fixed from Step 2, Steps 3 and 4 are reiterated until convergence.

4. Multiple Fuzzy Regression Model(MFRM)

The MFRM extends the concept of fuzzy linear models by capturing the relationships between multiple independent variables and a dependent variable, all represented as TFNs, while keeping the parameters crisp. In this framework, each independent variable X_i and the dependent variable Y are expressed as TFNs in the form $X = (x, x_l, x_r)$ and $Y = (y, y_l, y_r)$, where the central (modal) values (x, y), the left spreads(x_l, y_l) and the right spreads (x_r, y_r) are clearly defined. The regression equation is formulated as $Y_i = a + bX_i$, where the TFNs X_i and Y_i are used for all observations $i = 1, 2, \ldots, n$, and the parameters a and b are crisp values. When the parameters are non-fuzzy, the FLS optimization problem is defined as follows [20]:

$$\min\ r(a,b) = \sum d(a + bX_i, Y_i)^2, \tag{22}$$

where

$$d(a + bX_i, Y_i)^2 = (a + bx_i - y_i - (bx_{l_i} - y_{l_i}))^2 + (a + bx_i - y_i - (bx_{r_i} - y_{r_i}))^2 + (a + bx_i - y_i)^2 \tag{23}$$

This optimization problem aims to analyze the relationship between the dependent variable Y and the independent variable X within a fuzzy framework. This approach is employed to more effectively manage the uncertainties and outliers commonly encountered in traditional regression analysis. The FLS method takes account of the uncertainties in the parameters and therefore makes more reliable prediction. As an extended multiple regression model, Eq.(22) expresses the FLS model. The optimization problem is defined under this context as follows:

$$\min\ r(a, b_1, ..., b_p) = \sum d(a + b_1X_{i1} + b_2X_{i2} + ... + b_pX_{ip}, Y_i)^2. \tag{24}$$

This leads to the determination of the parameter estimates as follows:

$$\hat{\beta} = (X^TX + X_L^TX_L + X_R^TX_R)^{-1}(X^TY + X_L^TY_L + X_R^TY_R), \tag{25}$$

where

$$X = \begin{bmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, X_L = \begin{bmatrix} 1 & x_{11} - x_{l_{11}} & \cdots & x_{1p} - x_{l_{1p}} \\ 1 & x_{21} - x_{l_{21}} & \cdots & x_{2p} - x_{l_{2p}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} - x_{l_{n1}} & \cdots & x_{np} - x_{l_{np}} \end{bmatrix},$$

$$X_R = \begin{bmatrix} 1 & x_{11} + x_{r_{11}} & \cdots & x_{1p} + x_{r_{1p}} \\ 1 & x_{21} + x_{r_{21}} & \cdots & x_{2p} + x_{r_{2p}} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} + x_{r_{n1}} & \cdots & x_{np} + x_{r_{np}} \end{bmatrix}, Y_L = \begin{bmatrix} y_1 - y_{l_1} \\ y_2 - y_{l_2} \\ \vdots \\ y_n - y_{l_n} \end{bmatrix}, Y_R = \begin{bmatrix} y_1 + y_{r_1} \\ y_2 + y_{r_2} \\ \vdots \\ y_n + y_{r_n} \end{bmatrix}.$$

The observed index is $i = 1, 2, 3, ..., n$, and the number of variables is $j = 1, 2, 3, ..., p$. To obtain parameter estimates, the inverse of $(X^TX + X_L^TX_L + X_R^TX_R)$ must be calculated. In ordinary regression, when the X matrix satisfies the necessary condition, the inverse of $(X^TX + X_L^TX_L + X_R^TX_R)$ will be taken [9].

5. Fuzzy Robust Regression Model (FRRM)

FRR represents a key research field in statistical modeling because it addresses imprecise data situations and outliers. This approach merges two critical statistical approaches: The methodology combines fuzzy regression together with RR to address uncertainty through fuzzy sets and reduce outlier effects [28]. The methodology enables automatic outlier evaluation by generating membership values near zero or significantly below standard data point thresholds [20].

The goal of the FRR is to build the model with all available data. In real-world conditions, however, irregular data will always exist. The observed values can range between these irregularities, from errors in the observed values to deficiencies or inaccuracies in the observation methods, incomplete measured values, or outliers. Thus, when applying the traditional regression method in building a model using such data, the model may lose its validity. The objective of this method focuses on diminishing irregular data effects which occur while building predictive models [29]. As previously referenced, the generalized WFLS regression model introduced by Chang and Lee [8] forms the basis of this approach. In this model, the weighted function is determined by the degree of membership and the WFLS is constructed through a weight matrix. A membership function is given, membership value is found, and the weight matrix is formed on these values. The weight matrix is expressed as a diagonal matrix ($W_i = \text{diag}(\mu(\varepsilon_1), \mu(\varepsilon_2), ..., \mu(\varepsilon_n))$, where $\mu(\varepsilon_i)$ are the elements reflecting the degrees of membership). Consequently, the WFLS function can be computed as follows:

$$\hat{\beta} = \left(X^T W X + X_L^T W X_L + X_R^T W X_R\right)^{-1} \left(X^T W Y + X_L^T W Y_L + X_R^T W Y_R\right) \tag{26}$$

Provided that $(X^T X)^{-1}$ exists and W is a non-zero matrix. The FRR procedure is summarized as follows [20]:
Step 1: The estimation of the regression parameters for the TFNs $X_i = (x_i, x_{l_i}, x_{r_i})$ and $Y_i = (y_i, y_{l_i}, y_{r_i})$ is obtained from Eq.(25).
Step 2: Calculate the value of $\hat{y}_i$.
Step 3: Calculate the residual value ε_i.
Step 4: The median is calculated based on the absolute residual values, and distances are found using the formula:

$$D_i = \| \text{abs}(\varepsilon_i) - \text{median}(\text{abs}(\varepsilon_i)) \|, \quad i = 1, 2, ..., n, \tag{27}$$

where $\| \cdot \|$ represents Euclidean distance.
Step 5: Define the membership function:

$$\mu(\varepsilon_i) = \begin{cases} 1 & , |\varepsilon_i| \leq a \\ \dfrac{b - |\varepsilon_i|}{b - a} & , a < |\varepsilon_i| < b \\ 0 & , \text{otherwise} \end{cases} \tag{28}$$

where $a = \text{median}(D_i)$ and $b = \max(D_i) + \text{mad}(\varepsilon_i)/0.6745$.
Step 6: Calculate the weighted value W_i.
Step 7: Calculate $\hat{\beta}_{FRR}$ using the WFLS method as defined in Eq. (26) with the weighted matrix W_i.
Step 8: In the iterative estimation of FRRM coefficients, the process is terminated if the absolute difference between the coefficients in iterations $k + 1$ and k is less than a specified tolerance level (ϵ) . If this condition is not satisfied, the procedure continues to Step 2, where k denotes the iteration number, and $\epsilon > 0$ represents a small positive constant.

6. Modified Fuzzy Robust Regression (MFRR)

MFRR effectively extends traditional FRR analysis to datasets with both uncertainty and outliers by combining the traditional FRR analysis with the Huber loss function. Our method differs from that proposed by Kula et al. [20]. In their approach, they handle uncertain data and maintain robustness with respect to outliers, but the MFRR improves on this further by incorporating the Huber loss function, which is specifically devised to cope with the impact of outliers. Hence, the framework of MFRR solves these two problems simultaneously, i.e., uncertainty and outliers, providing a more reliable mechanism for analysis. In this approach, we model the inherent uncertainty in the data using TFNs, along with the use of the Huber loss function in order to deal with the influence of outliers. After fitting the model, the TFNs ($X = (x_{ij}, x_{lij}, x_{rij})$ and $Y = (y_i, y_{li}, y_{ri})$) are transformed into crisp values ($X_c = x_{ijc}$ and $Y_c = y_{ic}$) for computing the predicted values $\hat{y}_i$ and residuals ε_i . The matrix representations of these crisp values are given as follows:

$$X_c = \begin{bmatrix} 1 & x_{11c} & \cdots & x_{1pc} \\ 1 & x_{21c} & \cdots & x_{2pc} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1c} & \cdots & x_{npc} \end{bmatrix} \text{ and } Y_c = \begin{bmatrix} y_{1c} \\ y_{2c} \\ \vdots \\ y_{nc} \end{bmatrix}.$$

Studies have developed multiple methods for transforming fuzzy data into crisp data, yet among these methods the centroid technique stands out as the most widespread solution [30]. The fuzzy data x_{ij}, y_i are transformed into crisp values x_{ijc}, y_{ic}, which are usually referred to as the centroids of x_{ij} and y_i, using the following formula:

$$x_{ijc} = \frac{1}{3}(x_{ij} + x_{l_{ij}} + x_{r_{ij}}) \quad \text{and} \quad y_{ic} = \frac{1}{3}(y_i + y_{l_i} + y_{r_i}). \tag{29}$$

Clearly, if the observed data consists of symmetric TFNs, the centroid coincides with the center of symmetry of each TFN. The MFRR procedure is summarized as follows:
Step 1: The estimation of the regression parameters for the TFNs $X_i = (x_i, x_{l_i}, x_{r_i})$ and $Y_i = (y_i, y_{l_i}, y_{r_i})$ is obtained from Eq. (25).
Step 2: Calculate the value of $\hat{y}_i$.
Step 3: Calculate the residual value ε_i.
Step 4: Calculation of the Huber median: To compute a robust measure of central tendency for the residuals, the Huber loss function was employed. This function reduces the influence of outliers, providing a more reliable estimate of the central location. The calculations were performed in R using the MASS package. The Huber median, based on the Huber loss function, was computed as follows: First, the absolute values of the residuals were taken, and the Huber function was applied to these values.
Step 5: Calculate the distances using the Huber median and absolute residual values:

$$D_i = \| p_i - q_i \|, \; i = 1, 2, ..., n, \tag{30}$$

where $\| \cdot \|$ represents Euclidean distance, $p_i = \text{abs}(\varepsilon_i)$ and $q_i =$Huber median.
Step 6: Define the membership function:

$$\mu(\varepsilon_i) = \begin{cases} 1 & , |\varepsilon_i| \leq a \\ \dfrac{b - |\varepsilon_i|}{b - a} & , a < |\varepsilon_i| < b \\ 0 & , \text{otherwise} \end{cases} \tag{31}$$

where $a = \text{median}(D_i)$ and $b = \max(D_i) + \text{mad}(\varepsilon_i)/0.6745$.
Step 7: Calculate the weighted value W_i.
Step 8: Calculate $\hat{\beta}_{MFRR}$ using the WFLS method as defined in Eq. (26) with the weighted matrix W_i.
Step 9: In the iterative estimation of MFRR model coefficients, the process is terminated if the absolute difference between the coefficients in iterations $k+1$ and k is less than a specified tolerance level (ϵ) . If this condition is not satisfied, the procedure continues to Step 2, where k denotes the iteration number, and $\epsilon > 0$ represents a small positive constant.

7. Applications

Under this section a real-world data and a simulation study are used to evaluate MFRR's performance.

7.1. Real-World Data Application: Insurance Data Set

This study evaluates the proposed method (MFRR) using the insurance dataset presented in Kula et al. [20]. This dataset, as shown in Table 1, contains two independent variables and a dependent variable, where X_1 and X_2 represent the number of months and the number of claims in the corresponding month, respectively, while

Y represents the payments in the corresponding months. For both FRR and MFRR methods, crisp variables are fuzzified. The independent variable values are fuzzified as follows: the center(x_i), left spread $x_{l_i} = x_i/8$, right spread $x_{r_i} = x_i/7$ and the dependent variable values: center (y_i), left spread $y_{l_i} = y_i/8$ and right spread $y_{r_i} = y_i/7$. The analysis was conducted using R code, which included the OLS, M-estimator, MM-estimator, FRR and MFRR methods.

Table 1. Data set (Kula et al. 2012)

X_1	X_2	$Y * 10^4$	X_1	X_2	$Y * 10^4$
1	1270	125	7	3169	631
2	2630	387	8	3448	545
3	3653	589	9	3163	583
4	3045	591	10	3096	606
5	3232	609	11	3765	753
6	3681	654	12	4481	898

The outcomes of the residual analysis, as shown in Figure 2, reveal that the eighth observation is identified as an outlier. This conclusion is drawn from the standardized residuals, calculated as the ratio of the residuals to their standard deviation. The standardized residual for the eighth observation exceeded 2, confirming its classification as an outlier. This threshold is based on the assumption of normal distribution, where approximately 95% of observations fall within the range of -2 to $+2$. Consequently, standardized residuals outside this range are deemed outliers. Utilizing this threshold enhances the reliability of the analysis and ensures the validity of the model by mitigating the influence of extreme observations.

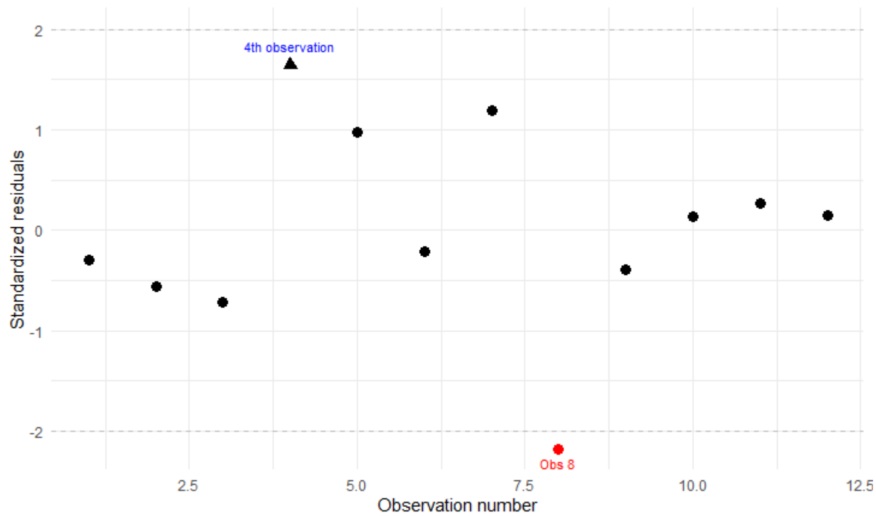

Figure 2. Standardized residuals plot.

Table 2 presents the parameter estimates for the regression models, illustrating the results obtained through various methodologies. As shown, the parameter estimates exhibit consistent signs and are nearly equivalent in magnitude when compared to those derived from robust and FRR methods. This consistency arises because the weight matrix is generated through the membership function. The degree of membership from each observation allows for model estimation in the regression framework while reducing risks from outlier effects. The MFRR demonstrates resilience against outliers, resulting in superior performance compared to classical methods.

Table 2. Regression parameter estimates for the insurance dataset.

Estimations\Methods	OLS	Huber	Hampel	Tukey	Andrews	MM	FRR	MFRR
$\hat{\beta}_0$	-118.4504	-123.0439	-121.3489	-130.7774	-120.6027	-121.4522	-103.9568	-103.3980
$\hat{\beta}_1$	12.2628	14.1149	13.8057	16.5405	13.2381	13.7049	15.8823	15.6325
$\hat{\beta}_2$	0.1925	0.1900	0.1908	0.1847	0.1916	0.1911	0.1804	0.1811

Table 3 includes residuals derived from the OLS, M, MM, FRR, and MFRR techniques, while Table 4 contains observation-weight values based on the analyzed methods. The weight matrix obtains its values from the membership function to establish the observation-influencing capabilities in the regression model. This approach therefore minimizes the possibility for adverse effects from outliers, thus resulting in more robust estimations of parameters. As shown in Table 3, the eighth observation is classified as an outlier due to its significantly large residual values across all methods. In particular, its residuals are very high in OLS (-98.3076), Hampel (-101.8568), MM (-102.1921), FRR (-100.0739), and MFRR (-102.4170), suggesting a marked deviation from the trend expected.

The weight analysis in Table 4 further confirms this classification, as the eighth observation receives the lowest weights in the Tukey (0.0000), FRR (0.0124), and MFRR (0.0000) methods. These results show that these methods reduce the influence of the outlier by scaling it down and thus help reduce the impact of the outlier on parameter estimation and model robustness.

The fourth observation exhibits relatively high residual values, particularly in the OLS (74.3113), MM (75.6516), FRR (82.1510), and MFRR (80.1889) methods. As shown in Figure 2, its standardized residual is around 2. Therefore, this observation is not an extreme outlier but should still be treated with care in RR analysis.

As shown in Table 4 and Figure 3, the weight assigned to this observation is notably lower in the FRR (0.2146) and MFRR (0.2765) methods. These reduced weights indicate that the fuzzy robust models down-weight this observation, limiting its effect on the estimated regression coefficients.

When examining the mean absolute error (MAE) values in the analysis, the differences in performance between the methods appear to be minimal. The Tukey method achieves the lowest MAE of 29.3637, demonstrating its efficiency in managing outliers. Both FRR and MFRR show strong performance with MAE values of 30.9301 and 31.5131, respectively, highlighting their effectiveness in minimizing the impact of outliers. In contrast, OLS produces the highest MAE at 32.7741, indicating its lower robustness to outliers. The MAE of the MM-estimate (31.1631) is closer to that of MFRR, reinforcing the idea that the MFRR method is comparable to traditional robust approaches. Degrees of membership come up in FRR and MFRR, which express how each observation affects the regression model. Non-outlier observations remain with weights close to 1, while the outliers have much smaller degrees of membership, and thus they have much less impact.

Table 3. Residuals for the OLS, M, MM, FRR and MFRR methods.

Observations \Methods	OLS	Huber	Hampel	Tukey	Andrews	MM	FRR	MFRR
1	-13.2563	-7.3226	-9.7292	4.7034	-11.0125	-9.9837	-16.0162	-17.9883
2	-25.2858	-17.7856	-20.9763	-0.9911	-22.8748	-21.6210	-15.2240	-17.9578
3	-32.4510	-24.2315	-27.9353	-4.4511	-30.1560	-28.8486	-13.6415	-16.8802
4	74.3113	79.1504	76.2446	93.2890	75.1203	75.6516	82.1510	80.1889
5	44.0555	47.5126	44.7657	60.2149	44.0464	44.2060	50.5365	48.4976
6	-9.6288	-6.8952	-9.6938	5.7567	-10.2360	-10.3148	-1.3393	-3.7717
7	53.6559	53.2504	51.1726	60.7683	51.6433	50.8372	52.1362	50.6566
8	-98.3076	-99.8639	-101.8568	-93.2957	-101.0611	-102.1921	-100.0739	-102.4170
9	-17.7148	-21.8397	-23.2942	-19.2046	-21.6830	-23.4259	-26.5461	-27.9869
10	5.9182	-0.2271	-1.3186	-0.3721	0.9185	-1.3253	-7.3426	-8.3676
11	11.8893	5.5735	4.2534	6.5418	6.4763	4.1060	3.0966	1.8844
12	6.8141	0.4459	-1.1405	2.7762	1.0272	-1.4457	3.0577	1.5607
Mean Absolute Error	32.7741	30.3415	31.0317	29.3637	31.3546	31.1631	30.9301	31.5131

Table 4. Weights for the OLS, M, MM, FRR and MFRR methods.

Observations\ Methods	OLS	Huber	Hampel	Tukey	Andrews	MM	FRR	MFRR
1	1.0000	1.0000	1.0000	0.9808	0.4730	0.9919	0.9608	1.0000
2	1.0000	1.0000	1.0000	0.9991	0.4625	0.9622	0.9698	1.0000
3	1.0000	1.0000	1.0000	0.9828	0.4526	0.9332	0.9876	1.0000
4	1.0000	0.4566	0.5497	0.0000	0.3407	0.5872	0.2146	0.2765
5	1.0000	0.7607	0.9362	0.0000	0.4267	0.8468	0.5713	0.6730
6	1.0000	1.0000	1.0000	0.9713	0.4734	0.9913	1.0000	1.0000
7	1.0000	0.6787	0.8190	0.0000	0.4089	0.8001	0.5533	0.6460
8	1.0000	0.3619	0.3525	0.0000	0.2490	0.3290	0.0124	0.0000
9	1.0000	1.0000	1.0000	0.7043	0.4639	0.9557	0.8420	0.9296
10	1.0000	1.0000	1.0000	0.9999	0.4762	1.0000	1.0000	1.0000
11	1.0000	1.0000	1.0000	0.9630	0.4751	0.9986	1.0000	1.0000
12	1.0000	1.0000	1.0000	0.9933	0.4762	1.0000	1.0000	1.0000

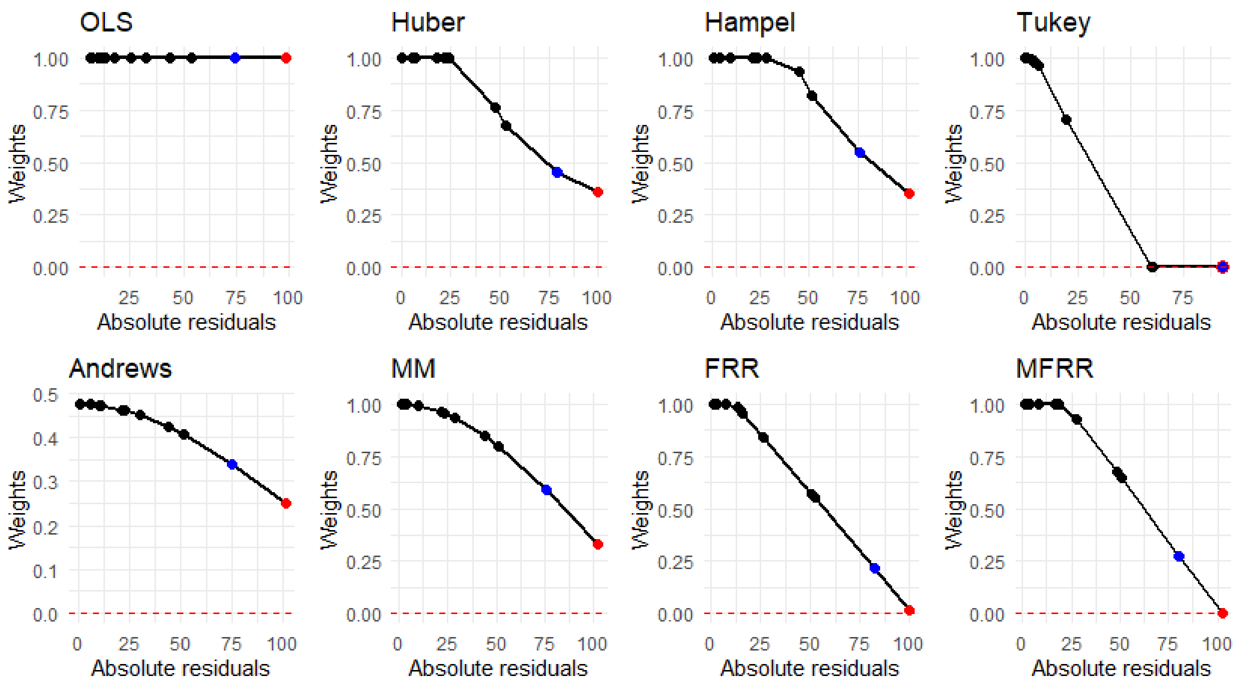

Figure 3. Weight functions for the OLS, M, MM, FRR and MFRR methods.

7.2. Simulation Study

In this part too, eight regression problems involving one dependent variable and three independent variables were analyzed to compare the methods and evaluate the results. The number of observations for each regression problem was set at 35, 55, 95, 150, 200, and 300. The artificial data for these regression problems were generated from normal distributions for the independent variables: X_1, X_2 and X_3 are generated from $N(\mu = 0, \sigma = 1)$. The dependent variable Y was defined based on the following linear relationship:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \varepsilon_i, \quad i = 1, 2, \dots, n. \tag{32}$$

where $\beta_j = 1$ for $j = 0, 1, 2, 3$ and the error term ε_i follows a normal distribution with a mean of zero and a standard deviation of one . To assess the robustness of the model, outliers were introduced in 5%, 10%, and 15% of the observations for X_1, X_2, X_3 and Y. These outliers were generated from a uniform distribution ranging from one to ten. The independent variable values are fuzzified as follows: the center(x_i), left spread $x_{l_i} = x_i/1.5$, right spread $x_{r_i} = x_i/2$ and the dependent variable values: center (y_i), left spread $y_{l_i} = y_i/1.5$ and right spread $y_{r_i} = y_i/2$.

In Tables 5, 6 and 7, the regression model estimates for the OLS, Huber, Hampel, Tukey, Andrews, MM and FRR methods, alongside the estimates obtained using the proposed method, are presented. A careful examination of these tables reveals that the parameter estimates derived from the proposed method are consistent in sign and exhibit magnitudes closely aligned with those obtained via classical robust and fuzzy robust methods. These findings are particularly noteworthy given the presence of outliers affecting 5%, 10% and 15% of the observations in X_1, X_2, X_3 and Y.

Table 5. Regression parameter estimates in the presence of 5% outliers.

Estimations\ Methods	OLS	Huber	Hampel	Tukey	Andrews	MM	FRR	MFRR
N=35								
$\hat{\beta}_0$	0.6912	0.6940	0.6912	0.6894	0.6905	0.6887	0.4881	0.5956
$\hat{\beta}_1$	0.8085	0.8073	0.8085	0.8110	0.8096	0.8119	0.9176	0.8482
$\hat{\beta}_2$	0.9545	0.9542	0.9545	0.9514	0.9531	0.9505	0.9386	0.9913
$\hat{\beta}_3$	0.8922	0.8929	0.8922	0.8905	0.8915	0.8900	0.7732	0.8103
N=55								
$\hat{\beta}_0$	0.5810	0.6659	0.6948	0.7085	0.7106	0.7642	0.6558	0.6509
$\hat{\beta}_1$	0.5212	0.4625	0.4211	0.4283	0.4290	0.6122	0.4648	0.4716
$\hat{\beta}_2$	0.7628	0.8117	0.7746	0.7927	0.7883	1.0603	0.7966	0.7969
$\hat{\beta}_3$	0.4663	0.7169	0.8601	0.8421	0.8507	0.9036	0.7995	0.7707
N=95								
$\hat{\beta}_0$	1.2357	1.1653	1.0517	1.0138	1.0504	1.0148	1.0857	1.1056
$\hat{\beta}_1$	0.7593	1.0172	1.1659	1.0376	1.1826	1.0439	1.0143	1.0186
$\hat{\beta}_2$	1.4383	1.2256	0.9094	0.8250	0.9312	0.8193	1.2016	1.2420
$\hat{\beta}_3$	1.4235	1.2052	0.9778	0.8955	1.0012	0.8912	1.2289	1.2452
N=150								
$\hat{\beta}_0$	1.1467	1.0375	1.0148	1.0171	1.0283	1.0144	1.0170	1.0218
$\hat{\beta}_1$	1.4585	1.2422	1.1933	1.2082	1.1994	1.2103	1.2633	1.2704
$\hat{\beta}_2$	1.0991	1.0925	1.0659	1.0737	1.0712	1.0745	1.1488	1.1489
$\hat{\beta}_3$	0.6292	0.8362	0.8324	0.8628	0.8559	0.8647	0.9548	0.9477
N=200								
$\hat{\beta}_0$	1.0814	1.0042	0.9993	0.9921	0.9997	0.9913	0.9563	0.9610
$\hat{\beta}_1$	1.2185	1.1045	1.0966	1.0802	1.0588	1.1071	1.1001	1.1060
$\hat{\beta}_2$	0.8290	0.9738	1.0213	1.0166	0.9834	1.0344	0.8775	0.8825
$\hat{\beta}_3$	0.8740	0.9819	1.0497	1.0376	1.0444	1.0334	0.8870	0.8926
N=300								
$\hat{\beta}_0$	1.0876	1.0802	1.0421	1.0437	1.0467	1.0429	1.0253	1.0275
$\hat{\beta}_1$	0.8963	1.0121	0.8926	0.9003	0.8960	0.9008	1.0171	1.0178
$\hat{\beta}_2$	1.3507	1.1215	0.9797	0.9588	0.9888	0.9502	1.2363	1.2417
$\hat{\beta}_3$	1.3177	1.1538	0.9727	0.9499	0.9592	0.9463	1.2539	1.2581

Table 6. Regression parameter estimates in the presence of 10% outliers.

Estimations\ Methods	OLS	Huber	Hampel	Tukey	Andrews	MM	FRR	MFRR
N=35								
$\hat{\beta}_0$	0.9671	0.7515	0.6689	0.6421	0.6886	0.6409	0.6503	0.6753
$\hat{\beta}_1$	0.8037	0.8971	0.8983	0.9488	0.8226	0.9490	0.8059	0.8152
$\hat{\beta}_2$	0.5541	0.8417	0.9227	1.0343	0.7590	1.0350	0.7538	0.7516
$\hat{\beta}_3$	2.4893	1.6540	1.0483	1.0440	1.0476	1.0448	1.1345	1.2877
N=55								
$\hat{\beta}_0$	0.9438	0.7761	0.8011	0.7979	0.8083	0.8254	0.6488	0.6525
$\hat{\beta}_1$	0.7702	0.6921	0.6634	0.7136	0.6603	0.7293	0.6274	0.6302
$\hat{\beta}_2$	1.4585	1.2108	1.1618	1.2141	1.1567	1.2702	1.1181	1.1233
$\hat{\beta}_3$	0.6365	0.8714	1.1884	0.8189	1.1868	1.0655	0.7894	0.7889
N=95								
$\hat{\beta}_0$	1.0494	1.0553	1.0418	1.0377	1.0391	1.0379	0.9923	0.9947
$\hat{\beta}_1$	1.0484	1.2447	1.2711	1.2816	1.2704	1.2805	1.2944	1.2896
$\hat{\beta}_2$	1.0810	0.9364	0.8993	0.8930	0.9049	0.8943	0.9887	0.9912
$\hat{\beta}_3$	0.9778	0.9676	0.9693	0.9438	0.9857	0.9471	0.9778	0.9781
N=150								
$\hat{\beta}_0$	1.2515	1.1000	1.0399	1.0604	1.0263	1.0605	1.0497	1.0562
$\hat{\beta}_1$	1.2016	1.1757	1.1518	1.1713	1.1453	1.1687	1.1654	1.1682
$\hat{\beta}_2$	1.4532	1.1291	0.9181	1.0212	0.8605	1.0308	1.1365	1.1507
$\hat{\beta}_3$	0.7307	0.7734	0.7868	0.7714	0.8067	0.7762	0.7771	0.7814
N=200								
$\hat{\beta}_0$	1.0716	0.9937	0.9724	0.9838	0.9677	0.9822	0.9390	0.9429
$\hat{\beta}_1$	1.1458	1.0379	0.9428	1.0062	0.8860	1.0192	1.0710	1.0777
$\hat{\beta}_2$	0.8969	1.0774	1.0643	1.0842	1.0227	1.0829	0.9714	0.9695
$\hat{\beta}_3$	1.5116	1.2088	1.1211	1.0730	1.1211	1.0642	1.3749	1.3825
N=300								
$\hat{\beta}_0$	1.1085	1.0747	1.0618	1.0676	1.0648	1.0681	1.0032	1.0077
$\hat{\beta}_1$	0.9407	0.9169	0.9226	0.9489	0.9432	0.9487	0.8693	0.8732
$\hat{\beta}_2$	0.5500	0.8196	0.9080	0.9324	0.9197	0.9320	0.8086	0.8029
$\hat{\beta}_3$	1.4048	1.1026	1.0178	0.9897	1.0068	0.9902	1.2170	1.2284

Table 7. Regression parameter estimates in the presence of 15% outliers.

Estimations\ Methods	OLS	Huber	Hampel	Tukey	Andrews	MM	FRR	MFRR
N=35								
$\hat{\beta}_0$	0.4323	0.6235	0.6414	0.6879	0.6658	0.6879	0.5940	0.5859
$\hat{\beta}_1$	0.6695	0.7374	0.7387	0.7564	0.7082	0.7557	0.7397	0.7559
$\hat{\beta}_2$	1.5477	1.1018	1.0547	0.9448	0.9739	0.9430	1.0361	1.0802
$\hat{\beta}_3$	0.5796	0.8907	0.9253	1.0029	0.9994	1.0053	0.9545	0.9134
N=55								
$\hat{\beta}_0$	0.7982	0.8945	0.9180	0.9331	0.9093	0.7747	0.8814	0.8797
$\hat{\beta}_1$	1.1818	0.9487	0.8775	0.8418	0.8467	0.6147	1.0163	1.0211
$\hat{\beta}_2$	1.6013	1.5380	1.5667	1.5465	1.5537	0.8040	1.4526	1.4565
$\hat{\beta}_3$	0.4702	0.9576	1.1183	1.1276	1.1110	0.9610	1.0126	0.9990
N=95								
$\hat{\beta}_0$	1.2220	1.2174	1.1957	1.2029	1.1996	0.9668	1.1731	1.1751
$\hat{\beta}_1$	0.2648	0.8093	0.9544	1.0989	1.0281	1.0062	0.3935	0.3995
$\hat{\beta}_2$	1.5274	1.4467	1.4214	1.3562	1.4458	0.6915	1.6516	1.6466
$\hat{\beta}_3$	1.7402	1.0788	0.9102	0.9436	0.8914	0.7716	1.2621	1.2782
N=150								
$\hat{\beta}_0$	0.8900	0.9467	0.9645	0.9907	0.9407	0.9891	0.8296	0.8316
$\hat{\beta}_1$	1.3912	1.0780	1.0449	1.1234	0.8775	1.1243	1.1598	1.1689
$\hat{\beta}_2$	0.5594	0.7618	0.8581	0.9261	0.8679	0.9285	0.6412	0.6397
$\hat{\beta}_3$	0.6150	0.6347	0.7355	0.7735	0.6651	0.7743	0.5253	0.5328
N=200								
$\hat{\beta}_0$	0.9310	0.9571	0.9766	0.9663	0.9866	0.9617	0.9108	0.9114
$\hat{\beta}_1$	0.8826	1.0485	1.1160	1.1384	1.1123	1.1253	0.9655	0.9627
$\hat{\beta}_2$	1.0669	1.0526	1.0743	1.0373	1.1260	1.0374	1.0538	1.0562
$\hat{\beta}_3$	1.1221	1.0362	1.0375	0.9737	1.0467	0.9718	1.1494	1.1517
N=300								
$\hat{\beta}_0$	1.0190	1.0522	1.0747	1.0646	1.0896	1.0627	0.9787	0.9796
$\hat{\beta}_1$	0.4558	0.9386	1.0782	1.0326	1.1123	1.0249	0.8645	0.8587
$\hat{\beta}_2$	1.5054	1.1449	1.0056	0.9194	0.9934	0.9110	1.1840	1.1950
$\hat{\beta}_3$	1.3110	1.1412	1.0814	1.0622	1.0914	1.0608	1.2089	1.2134

Table 8 presents the comparative MAE results for various regression methods, including OLS, RR methods (Huber, Hampel, Tukey, Andrews), MM, FRR and MFRR. The analysis is conducted under three levels of contamination (5%, 10% and 15% outliers) across varying sample sizes (N).

For datasets containing 5% outliers, the MFRR method consistently delivered the lowest MAE values, demonstrating superior robustness across all sample sizes. At N = 35 , the MAE of MFRR (0.8638) was substantially lower than that of OLS (0.9158) and Huber (0.9158). Similarly, at N = 300 , MFRR achieved a MAE of 1.1589, surpassing both FRR (1.2277) and MM (1.2304). Traditional robust methods, while effective to some extent, showed marginally higher MAE values compared to MFRR.

As the proportion of outliers increased to 10%, the differences in performance between the methods became more pronounced. The MFRR method continued to outperform all alternatives, with a notable advantage in both small and large sample sizes. At N = 35, MFRR achieved a MAE of 1.3867, outperforming OLS (1.5761), Huber (1.4065), and FRR (1.4978). At N = 300 , the MFRR method maintained its edge with a MAE of 1.3663, while FRR and MM method lagged slightly behind with MAE values of 1.4456 and 1.4359, respectively.

The robustness of the MFRR method became most evident under the most challenging scenario, with 15% contamination. Across all sample sizes, MFRR consistently achieved the lowest MAE values, showcasing its effectiveness in managing high contamination levels. For instance, at N = 300 , MFRR recorded a MAE of 1.9519, compared to 2.0673 for FRR, 2.0615 for MM method, and 2.1653 for OLS. Even at smaller sample sizes (N = 35), MFRR excelled with a MAE of 1.2804, significantly lower than FRR (1.3521) and traditional methods such as Tukey (1.3414) and Huber (1.3561).

The MFRR method consistently outperforms others, achieving the lowest MAE values across all levels of contamination and sample sizes. Traditional robust methods such as Huber, Hampel, Tukey and Andrews exhibit higher sensitivity to contamination, especially under moderate and high contamination levels. Meanwhile, OLS remains the least effective, consistently producing the highest MAE values due to its vulnerability to outliers. As contamination increases, the performance gap between MFRR and other methods becomes more pronounced, underscoring its robustness.

Table 8. MAE values for the OLS, M, MM, FRR and MFRR methods.

N \ Methods	OLS	Huber	Hampel	Tukey	Andrews	MM	FRR	MFRR
5% Outliers								
35	0.9158	0.9158	0.9158	0.9155	0.9156	0.9154	0.9198	0.8638
55	0.9265	0.8556	0.8822	0.8771	0.8793	0.9257	0.8718	0.8158
95	1.0780	1.0306	1.0197	1.0455	1.0187	1.0459	1.0379	0.9752
150	1.2970	1.2605	1.2658	1.2616	1.2627	1.2613	1.2608	1.1905
200	1.1821	1.1590	1.1612	1.1616	1.1636	1.1597	1.1796	1.1105
300	1.2576	1.2095	1.2251	1.2291	1.2253	1.2304	1.2277	1.1589
10% Outliers								
35	1.5761	1.4065	1.4726	1.4468	1.5207	1.4464	1.4978	1.3867
55	1.4456	1.4085	1.4871	1.4011	1.4860	1.4680	1.4371	1.3497
95	0.9947	0.9389	0.9361	0.9344	0.9371	0.9346	0.9545	0.8965
150	1.6474	1.5993	1.6331	1.6070	1.6511	1.6054	1.5997	1.5098
200	1.6402	1.5598	1.5707	1.5671	1.5853	1.5670	1.5903	1.5017
300	1.4974	1.4291	1.4326	1.4359	1.4339	1.4359	1.4456	1.3663
15% Outliers								
35	1.4938	1.3561	1.3488	1.3414	1.3505	1.3413	1.3521	1.2804
55	1.6633	1.5991	1.6192	1.6236	1.6248	1.8028	1.6160	1.5220
95	1.8483	1.6694	1.6656	1.6650	1.6670	1.8013	1.7484	1.6446
150	1.9028	1.8378	1.8472	1.8709	1.8453	1.8719	1.8917	1.7818
200	1.8096	1.7924	1.7899	1.7905	1.8011	1.7909	1.7999	1.6996
300	2.1653	2.0538	2.0522	2.0598	2.0577	2.0615	2.0673	1.9519

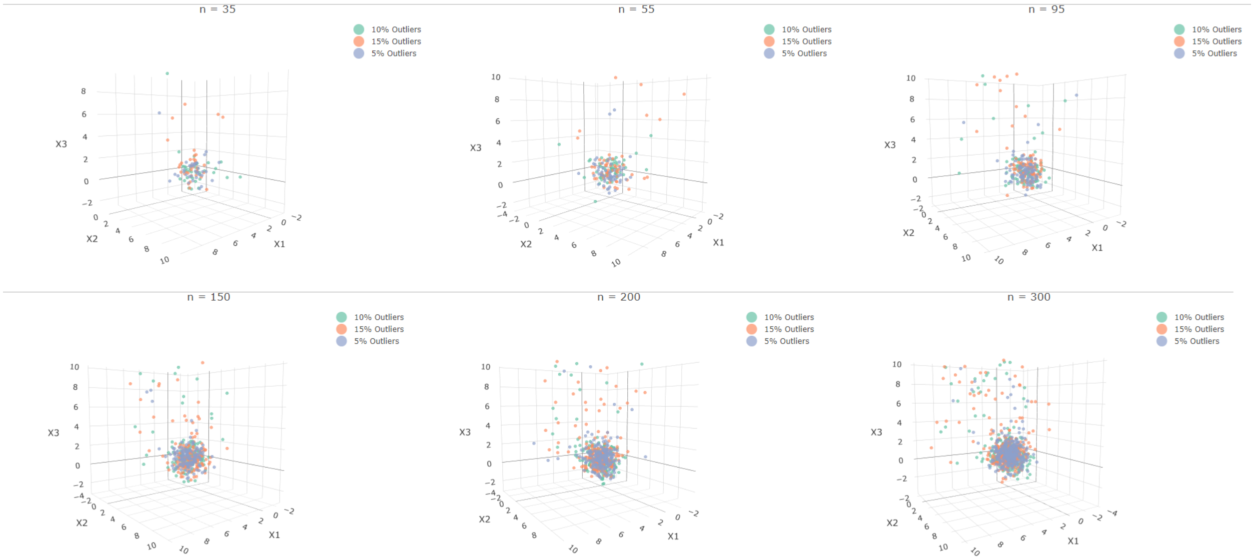

Figure 4. Scatter plot for the generated datasets (N = 35, 55, 95, 150, 200, 300) with 5%, 10% and 15% outliers.

8. Conclusion

This study presents an enhanced method based on Kula et al. [16], offering an effective solution for regression models dealing with datasets that include outliers and fuzzy data. The proposed MFRR method minimizes the impact of outliers and captures the fuzziness of the data in a correct way. The new approach proved superior to OLS and other published models for producing lower MAE results when faced with high levels of contamination. The MFRR model shows the capability to detect outliers and assess how other observations affect the model due to its membership function. This reduces subjectivity and effectively limits the negative effects of outliers on the regression parameter estimation accuracy and reliability. Future research may focus on exploring alternative loss functions, integrating the MFRR method with other robust techniques, or extending its application to different types of regression problems, such as nonlinear regression.

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