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Goodness-of-fit tests for Lindley distribution based on varentropy estimators

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Abstract The Lindley distribution is used in a variety of statistical applications, including survival analysis, reliability modeling, and Bayesian inference. Recently, Alizadeh and Shafaei (2023) introduced some estimators for varentropy of a continuous random variable. The present article applies these estimators and construct some tests of fit for Lindley distribution. Percentage points and type I error of the new tests are obtained and then power values of the proposed tests against various alternatives are computed. The results of a simulation study show that the tests have a good performance in terms of power. Finally, a real data set is used to illustrate the application of the proposed tests.

Keywords Varentropy estimator, Lindley distribution, Maximum likelihood estimates, Monte Carlo simulation, Percentage points, Test power

AMS 2010 subject classifications 62G10, 94A15

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1. Introduction

The Lindley distribution is a probability distribution that has found numerous applications in statistical modeling. It was first introduced by Dennis Lindley in 1958 as a distribution for modeling the time until a rare event occurs. Since then, it has been used in a variety of statistical applications, including survival analysis, reliability modeling, and Bayesian inference. One of the advantages of the Lindley distribution is that it can be used to model data that are non-negative and skewed, which are common in many fields such as biology, economics, and engineering. Therefore, constructing goodness of fit tests for the Lindley distribution is an important issue. In this article, we develop some goodness of tests for the Lindley distribution using different varentropy estimators.

The Lindley distribution has been discussed by many authors in different practical cases, such as Bayesian estimation Ali et al. (2013), loading-sharing system mode Singh and Gupta (2012) and stress-strength reliability model Al-Mutairi el al. (2013). It deserves mentioning that the Lindley distribution provides a flexible shape to model the lifetime data. Moreover, Ghitany et al. (2008) presented a comprehensive study about its important mathematical and statistical properties, estimation of parameter and application showing the superiority of Lindley distribution over of the bank customers.

If the density function of the random variable X be as follows, then we say that X has a Lindley distribution.

$$f_0(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \qquad x>0,\ \ \theta>0.$$

Lindley distribution was proposed by Lindley (1958) in the context of Bayesian statistics, as a counter example of fiducial statistics. The cumulative distribution function of the Lindley distribution is as

$$F_0(x;\theta) = 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}.$$

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The mean and variance of the distribution are

$$\mu = E(X) = \frac{\theta + 2}{\theta(\theta + 1)},$$

and

$$\sigma^2 = Var(X) = \frac{\theta^2 + 4\theta + 2}{\theta^2(\theta + 1)^2}.$$

The Lindley distribution has been used in a variety of statistical applications, including survival analysis, reliability modeling, and Bayesian inference. One of the advantages of the Lindley distribution is that it can be used to model data that are non-negative and skewed, which are common in many fields such as biology, economics, and engineering. In Bayesian inference, the Lindley distribution is often used as a prior distribution for a parameter of interest. The choice of the Lindley prior is motivated by its simplicity and its ability to accommodate a wide range of prior beliefs about the parameter.

Goodness of fit tests are statistical methods used to evaluate whether a set of data is consistent with a particular probability distribution. These tests are widely used in many fields of science and engineering, and have numerous applications in statistical modeling. Goodness of fit tests help researchers to determine whether a particular distribution is a good fit for their data, allowing them to make more accurate predictions and inferences. Goodness of fit tests have important applications in many fields, including biology, engineering, finance, physics, and psychology. For example, in biology, goodness of fit tests can be used to test whether the distribution of a particular trait in a population follows a particular genetic model. In finance, goodness of fit tests can be used to test whether stock returns follow a particular distribution, such as the normal or lognormal distribution. In psychology, goodness of fit tests can be used to test whether a set of responses to a questionnaire follows a particular response model, such as the Rasch model.

Overall, goodness of fit tests are powerful tools for evaluating the fit of a particular distribution to a set of data, and can be used to make important inferences and predictions in a wide range of fields. They are essential for ensuring the validity and accuracy of statistical models and predictions.

The present paper begins with some estimators for varentropy of a continuous random variable and new test statistics for testing a hypothesis that the sample comes from an Lindley distribution based on the varentropy estimators. Then, the results of our simulation studies are described. The percentage points and type I error of our test statistics are obtained for different sample sizes based on 100,000 sample values generated by a Monte Carlo experiment. Finally, power values of the proposed tests against various alternatives are computed. Simulation results indicate that the varentropy-based tests perform well against different alternatives. Finally, a real data set is used to illustrate the application of the proposed tests. Some concluding remarks are contained in Section 6.

2. Varentropy and its estimators

Shannon entropy (1948) of a continuous random variable is a well-known information measure that represents the expectation of the information content of an absolutely continuous random variable. The corresponding variance is termed varentropy and is used in various applications of information theory. Recent contributions on the varentropy can be found in various articles by Bobkov and Madiman (2011), Kontoyiannis and Verdu (2013, 2014), Arikan (2016), Buono and Longobardi (2020), Di Crescenzo et al. (2021), Di Crescenzo and Paolillo (2021) and Maadani et al. (2021).

Suppose that a random variable X has distribution function $F(x)$ with continuous density function $f(x)$. The random variable

$$IC(X) = -\log f(X)\,,$$

is often referred as the information content of $f(x)$. The expectation of the information content of X, is termed differential entropy and given by

$$H(X) = E\left[IC(X)\right] = -E\left[\log f(X)\right] = -\int_{-\infty}^{\infty} f(x)\log f(x)dx\,.$$

Intuitively, $H(X)$ measures the expected uncertainty contained in $f(x)$ about the predictability of an outcome of X.
The variance of the information content of X is defined as

$$\begin{aligned} V(X) &= Var\left[IC(X)\right] = Var\left[\log f(X)\right] = E\left[\left(IC(X)\right)^2\right] - E^2\left[IC(X)\right] \\ &= E\left[\left(IC(X)\right)^2\right] - \left[H(X)\right]^2 \\ &= \int_{-\infty}^{\infty} f(x)[\log f(x)]^2 dx - \left[\int_{-\infty}^{\infty} f(x)\log f(x)dx\right]^2. \end{aligned}$$

The varentropy thus measures the variability in the information content of . This measure has been investigated by some authors. Fradelizi et al. (2016) used it to find an optimal varentropy bound for log-concave distributions. Also, a sharp uniform bound on varentropy for log-concave distributions is presented by Madiman (2014). An alternative way to calculate a bound for varentropy is discussed in Goodarzi et al. (2017). Application of the varentropy in reliability theory is conducted by Di Crescenzo and Paolillo (2021). Also, the properties of the varentropy of order statistics are studied by Maadani et al. (2021).
Recently, the problem of estimating the varentropy of a continuous random variable is investigated by Alizadeh and Shafaei (2023). They suggested some varentropy estimators and stated their properties. Also, they used a Monte Carlo simulation to compute the root mean square error (RMSE) of the estimators and then a comparison between the proposed estimators in terms of RMSE is performed. Their estimators are described as follows.
Suppose $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ are order statistics of a random sample of size n from an unknown continuous distribution F with a probability density function f. We interest to estimate of varentropy $V(X)$ of an unknown continuous probability density function f.

2.1. The first estimator

The first estimator is based on the fact that $V(X)$ can be expressed as

$$V(X) = \int_0^1 \log^2\left\{\frac{d}{dp}F^{-1}(p)\right\}^{-1} dp - \left[\int_{-\infty}^{\infty} \log\left\{\frac{d}{dp}F^{-1}(p)\right\}^{-1} dp\right]^2.$$

We replace the distribution function F by the empirical distribution function F_n, and use a difference operator instead of the differential operator and then estimate the derivative of $F^{-1}(p)$ by a function of the order statistics. Assuming that $X_1, \ldots, X_n$ is a random sample, the first estimator is given by

$$\begin{aligned} VV_{mn} &= \tfrac{1}{n}\sum_{i=1}^{n} \log^2\left(\tfrac{2m/n}{X_{(i+m)}-X_{(i-m)}}\right) - \left[\tfrac{1}{n}\sum_{i=1}^{n} \log\left(\tfrac{2m/n}{X_{(i+m)}-X_{(i-m)}}\right)\right]^2 \\ &= \tfrac{1}{n}\sum_{i=1}^{n} \log^2\left(X_{(i+m)} - X_{(i-m)}\right) - \left[\tfrac{1}{n}\sum_{i=1}^{n} \log\left(X_{(i+m)} - X_{(i-m)}\right)\right]^2. \end{aligned}$$

where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. According to Alizadeh and Shafaei (2023), $VV_{mn} \xrightarrow{P.} V(X)$ as $n \to \infty, m \to \infty, m/n \to 0$.

2.2. The second estimator

Alizadeh and Shafaei (2023) adjusted the weights of the first estimator, in order to take into account the fact that the differences are truncated around the smallest and the largest data points. (i.e., $X_{(i+m)} - X_{(i-m)}$ is replaced by $X_{(i+m)} - X_{(1)}$ when $i \leq m$ and $X_{(i+m)} - X_{(i-m)}$ is replaced by $X_{(n)} - X_{(i-m)}$ when $i \geq n - m + 1$). The second estimator is given by

$$VE_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log^2\left(\frac{c_i m/n}{X_{(i+m)} - X_{(i-m)}}\right) - \left[\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{c_i m/n}{X_{(i+m)} - X_{(i-m)}}\right)\right]^2,$$

where

$$c_i = \begin{cases} 1 + \frac{i-1}{m}, & 1 \leq i \leq m, \\ 2, & m+1 \leq i \leq n-m, \\ 1 + \frac{n-i}{m}, & n-m+1 \leq i \leq n. \end{cases}$$

It is easy to show that $VE_{mn} \xrightarrow{P.} V(X)$ as $n \to \infty\,, m \to \infty$, $m/n \to 0$.

2.3. The third estimator

The third estimator is as

$$VD = \int_{-\infty}^{\infty} \hat{f}(x) \left[\log \hat{f}(x)\right]^2 dx - \left[\int_{-\infty}^{\infty} \hat{f}(x) \log \hat{f}(x) dx\right]^2,$$

where $\hat{f}$ is the kernel density function estimation of f and is defined by

$$\hat{f}(x) = \frac{1}{nh} \sum_{j=1}^{n} k\left(\frac{x - X_j}{h}\right),$$

where h is a bandwidth and k is a kernel function which satisfies

$$\int_{-\infty}^{\infty} k(x) dx = 1\,.$$

Usually, k will be a symmetric probability density function.

2.4. The fourth estimator

Since

$$V(X) = E\left[(\log f(X))^2\right] - E^2\left[\log f(X)\right]\,,$$

a simple estimator of varentropy can be obtained as

$$\frac{1}{n} \sum_{i=1}^{n} (\log f(X_i))^2 - \left[\frac{1}{n} \sum_{i=1}^{n} \log f(X_i)\right]^2.$$

Therefore, Alizadeh and Shafaei (2023) proposed the following estimator.

$$VB_n = \frac{1}{n} \sum_{i=1}^{n} \left(\log \hat{f}(X_i)\right)^2 - \left[\frac{1}{n} \sum_{i=1}^{n} \log \hat{f}(X_i)\right]^2,$$

where

$$\hat{f}(X_i) = \frac{1}{nh} \sum_{j=1}^{n} k(\frac{X_i - X_j}{h})\,,$$

and the kernel function is chosen to be the standard normal density function and the bandwidth h is chosen to be the normal optimal smoothing formula, $h = 1.06 s n^{-\frac{1}{5}}$, where s is the sample standard deviation. They showed that $VB_n \xrightarrow{P.} V(X)\,,\ as\ n \to \infty\,.$

2.5. The fifth estimator

Based on a local linear model, Alizadeh and Shafaei (2023) obtained an estimator of varentropy as

$$VC_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log^2\left(\frac{\sum\limits_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})(j-i)}{n\sum\limits_{j=i-m}^{i+m}\left(X_{(j)}-\bar{X}_{(i)}\right)^2}\right) - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\sum\limits_{j=i-m}^{i+m}(X_{(j)}-\bar{X}_{(i)})(j-i)}{n\sum\limits_{j=i-m}^{i+m}\left(X_{(j)}-\bar{X}_{(i)}\right)^2}\right)\right]^2,$$

where

$$\bar{X}_{(i)} = \frac{1}{2m+1}\sum_{j=i-m}^{i+m} X_{(j)}\,.$$

This estimator is consistent, $VC_{mn} \xrightarrow{P.} V(X)$ as $n \to \infty\,, m \to \infty$, $m/n \to 0$.

2.6. The sixth estimator

Alizadeh and Shafaei (2023) proposed to estimate the varentropy $V(X)$ of an unknown continuous probability density function f by

$$\begin{aligned} VA_{mn} &= \tfrac{1}{n}\sum_{i=1}^{n}\log^2\left\{\tfrac{\hat{f}(X_{(i+m)})+\hat{f}(X_{(i-m)})}{2}\right\} - \left[\tfrac{1}{n}\sum_{i=1}^{n}\log\left\{\tfrac{\hat{f}(X_{(i+m)})+\hat{f}(X_{(i-m)})}{2}\right\}\right]^2 \\ &= \tfrac{1}{n}\sum_{i=1}^{n}\log^2\left\{\hat{f}(X_{(i+m)})+\hat{f}(X_{(i-m)})\right\} - \left[\tfrac{1}{n}\sum_{i=1}^{n}\log\left\{\hat{f}(X_{(i+m)})+\hat{f}(X_{(i-m)})\right\}\right]^2, \end{aligned}$$

where

$$\hat{f}(X_i) = \frac{1}{nh}\sum_{j=1}^{n} k(\frac{X_i - X_j}{h})\,,$$

and the kernel function is chosen to be the standard normal density function and the bandwidth h is chosen to be the normal optimal smoothing formula, $h = 1.06sn^{-\frac{1}{5}}$, where s is the sample standard deviation. They proved that $VA_{mn} \xrightarrow{P.} V(X)$ as $n \to \infty\,, m \to \infty$, $m/n \to 0$.

2.7. Comparison of the estimators

Alizadeh and Shafaei (2023) performed a simulation study to analyze the behaviors of the proposed estimators of varentropy. They for different sample sizes 100,000 samples were generated and then obtained their root of mean squared errors (RMSEs). They used the normal, exponential and uniform distributions and reported the RMSEs of the estimators. In varentropy estimation it is necessary to determine the value of m for given n. Based on the recommendation of Alizadeh and Shafaei (2023), we use the following heuristic formula:

$$m = \left[\sqrt{n} + 0.5\right].$$

The RMSE values and standard deviation of the six varentropy estimators at different sample size for normal, exponential and uniform distributions are reported by Alizadeh and Shafaei (2023). They observed that these estimators perform well under different distributions. Under normal distribution, the estimator VD has the best performance in terms of RMSE, and also the estimator VA has a better performance than the other estimators in terms of standard deviation. For the exponential population, again the estimators VD and VA have the least RMSE and SD than the competitors, respectively. Under uniform distribution, the estimator VA has a good performance in compared to the other estimators. Generally, they concluded that the estimators VD and VA behave better than the other estimators.

3. The proposed tests for Lindley distribution

Given a random sample $X_1, \ldots, X_n$ from a continuous probability distribution F with a density function $f(x)$, the hypothesis of interest is

$$H_0: \ f(x) = f_0(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\,, \qquad for\ some\ \theta \in \Theta,$$

where θ is specified or unspecified and $\Theta = \mathbb{R}^+$. The alternative to H_0 is

$$H_1: \ f(x) \neq f_0(x;\theta), \qquad for\ any\ \theta\,.$$

Without loss of any generality, one can reduce the above problem of goodness-of-fit, to testing the hypothesis of uniformity on the unit interval, by means of the probability integral transformation $U = F_0(X)$. Therefore if $U_i = F_0(X_i)\ ,\ i = 1, 2, ..., n$ be the transformed sample, the problem becomes the following testing uniformity.

$$H_0: \ f(u) = 1, \quad 0 < u < 1,$$

against

$$H_1: \ f(u) \neq 1, \quad 0 < u < 1.$$

Now, we use the test proposed by Alizadeh and Shafaei (2023) for uniformity. Therefore, the proposed test statistics can be constructed as follows.
Let $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ denote the order statistics of the sample. Alizadeh and Shafaei (2023) showed that for an f concentrated on $[0, 1]$ one always has

$$V(X) \geq 0\,,$$

with the minimum value of $V(X)$, zero, being uniquely attained by the $U(0, 1)$ density. Based on this property, we construct our test of H_0. A consistent test of the hypothesis of uniformity is then given by

$$T_n = \hat{V}(X),$$

where $\hat{V}(X)$ is the sample estimate of $V(X)$. In the previous section, we investigated some estimators for varentropy $V(X)$, and now we use them here.
Since the mentioned estimators, $\hat{V}(X)$, converges in probability to $V(X)$ as $n \to \infty$, under the null hypothesis H_0, T_n converges in probability to 0 as $n \to \infty$ and under an alternative distribution on [0,1] with absolutely continuous density f, T_n converges in probability to a number larger than zero as $n \to \infty$.
Based on different varentropy estimators, we propose the following test statistics for test of fit for Lindley distribution.

$$TV = \ VV_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log^2\left(U_{(i+m)} - U_{(i-m)}\right) - \left[\frac{1}{n}\sum_{i=1}^{n} \log\left(U_{(i+m)} - U_{(i-m)}\right)\right]^2;$$

$$TE = \ VE_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log^2\left(\frac{c_i m/n}{U_{(i+m)} - U_{(i-m)}}\right) - \left[\frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{c_i m/n}{U_{(i+m)} - U_{(i-m)}}\right)\right]^2;$$

$$TD = \ VD = \int_{-\infty}^{\infty} \hat{f}(u)\Big[\log \hat{f}(u)\Big]^2 du - \left[\int_{-\infty}^{\infty} \hat{f}(u)\log \hat{f}(u) du\right]^2;$$

$$TB = \ VB_n = \frac{1}{n}\sum_{i=1}^{n} \Big(\log \hat{f}(U_i)\Big)^2 - \left[\frac{1}{n}\sum_{i=1}^{n} \log \hat{f}(U_i)\right]^2;$$

$$TC = VC_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log^2\left(\frac{\sum_{j=i-m}^{i+m}(U_{(j)}-\bar{U}_{(i)})(j-i)}{n\sum_{j=i-m}^{i+m}(U_{(j)}-\bar{U}_{(i)})^2}\right) - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{\sum_{j=i-m}^{i+m}(U_{(j)}-\bar{U}_{(i)})(j-i)}{n\sum_{j=i-m}^{i+m}(U_{(j)}-\bar{U}_{(i)})^2}\right)\right]^2;$$

$$TA = VA_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log^2\left\{\hat{f}(U_{(i+m)})+\hat{f}(U_{(i-m)})\right\} - \left[\frac{1}{n}\sum_{i=1}^{n}\log\left\{\hat{f}(U_{(i+m)})+\hat{f}(U_{(i-m)})\right\}\right]^2.$$

where $U_i = F_0(X_i;\hat{\theta})$, $i = 1, 2, ..., n$, denote the transformed sample. Also, $\hat{\theta}$ is the maximum likelihood estimate of the parameter θ,

$$\hat{\theta} = \frac{-\left(\bar{X}-1\right)+\sqrt{\left(\bar{X}-1\right)^2+8\bar{X}}}{2\bar{X}}, \qquad \bar{X} > 0.$$

Since the empirical varentropy converges to the varentropy of X, i.e., $\hat{V}(X) \to V(X)$, the proposed tests are consistent. Under the null hypothesis H_0, the proposed test statistics converge in probability to 0 as $n \to \infty$ and under an alternative distribution H_1, they converge in probability to a number larger than zero as $n \to \infty$.
Guided by these properties, given any significance level α, and any finite sample size n, our test procedure is then defined by the critical region

$$T_n = \hat{V}(X) \geq C_{1-\alpha}$$

where $C_{1-\alpha}$ is set so that the test has the desired level α for given n. For specific α and n, the $C_{1-\alpha}$ can be obtained by Monte Carlo methods. For the proposed tests, we determine the $C_{1-\alpha}$ based on Monte Carlo simulation.

4. Percentage points and power study

In this section, we obtain the percentage points of the proposed test statistics by Monte Carlo methods and then power values of the tests are computed.

4.1. Critical values and the actual sizes

Distribution of the test statistics TV, TE, TD, TB, TC and TA under the null hypothesis cannot be evaluated analytically. Therefore, the critical values of the test statistics are computed by the Monte Carlo method.
For selected values of the sample size n, 100,000 samples of size n from Lindley distribution are generated. For each sample, the test statistics are computed. For level α, the percentage point $C_{1-\alpha}$ of the distribution of T_n is estimated by the $1-\alpha$ percentile of the empirical distribution function of T_n based on the observed 100,000 samples. These estimates are presented in Table 1.

Moreover, Figure 1 shows the empirical density of the proposed test statistics for $n = 50$, with 100,000 simulated random samples. We can see that the test statistics TA and TD have smaller variance than the other test statistics.
In Table 2 the estimated type I error control using the 0.05 percentiles of the proposed tests are evaluated and reported ($\alpha = 0.05$). According to the results of Table 2, the value of the type I error is close to the nominal value and therefore the actual sizes of the tests are acceptable. Generally, we see that the actual sizes of the proposed tests are acceptable and therefore we can use these tests in practice.

4.2. Power study

We compute the power values of the proposed tests by Monte Carlo simulations. In power comparison, we consider the following alternatives.

Table 1. Percentage points of the proposed test statistics at level $\alpha = 0.05$

n	TV	TE	TD	TB	TC	TA
10	0.4991	0.3184	0.1125	0.2004	0.4573	0.0244
20	0.3593	0.2172	0.0960	0.1591	0.3146	0.0409
30	0.2739	0.1668	0.0859	0.1356	0.2404	0.0495
40	0.2242	0.1359	0.0780	0.1203	0.1960	0.0520
50	0.1911	0.1137	0.0716	0.1088	0.1664	0.0521
75	0.1482	0.0887	0.0622	0.0920	0.1280	0.0523
100	0.1265	0.0789	0.0553	0.0806	0.1114	0.0527

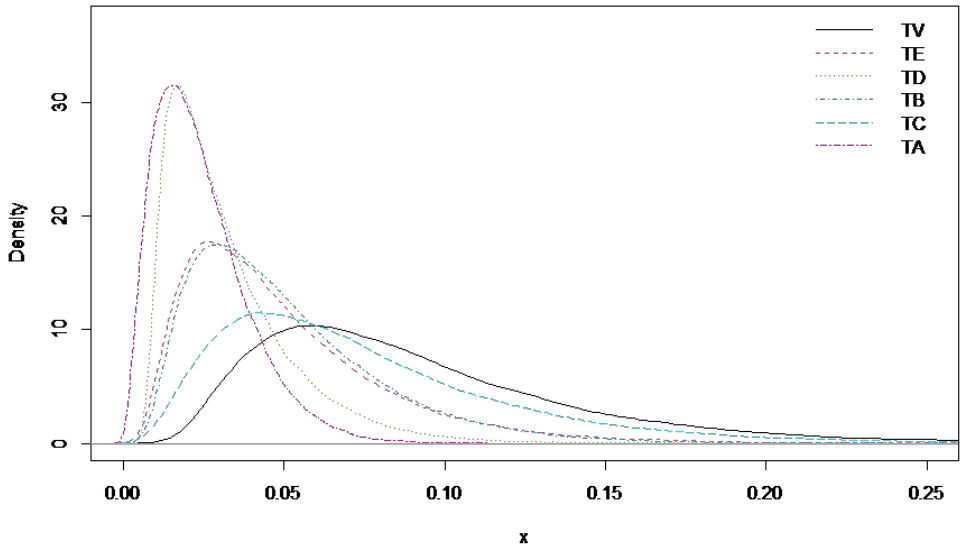

Figure 1. Estimated empirical densities of the proposed test statistics generated with 100,000 samples of size $n = 50$ from the Lindley distribution.

- the exponential distribution $Exp(\theta)$ with density $\theta \exp(-\theta x)$;
- the Weibull distribution with density $\theta \exp(-\theta x)$, denoted by $W(\theta)$;
- the gamma distribution with density $\Gamma(\theta)^{-1} x^{\theta-1} \exp(-x)$, denoted by $\Gamma(\theta)$;
- the half-normal HN distribution with density $\sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^2}{2}\right)$;
- the lognormal law $LN(\theta)$ with density $\frac{1}{\theta x\sqrt{2\pi}} \exp\left(-\frac{(\log x)^2}{2\theta^2}\right)$;
- the Pareto distribution $Pa(\theta)$ with density $\frac{\theta}{x^{\theta+1}}$;
- the uniform distribution U with density 1, $0 \leq x \leq 1$;
- the Beta distribution $Beta(\alpha, \beta)$ with density $\frac{x^{\alpha-1}(1-x)^{\beta-1}}{Beta(\alpha,\beta)}$, $0 \leq x \leq 1$;
- Chen's [7] distribution $CH(\theta)$ with distribution function $\frac{x^{\alpha-1}(1-x)^{\beta-1}}{Beta(\alpha,\beta)}$, $0 \leq x \leq 1$;
- the linear increasing failure rate law $LF(\theta)$ with density $(1+\theta x) \exp\left(-x - \frac{\theta x^2}{2}\right)$;
- the modified extreme value $EV(\theta)$ with distribution function $1 - \exp\left(\theta^{-1}(1-e^x)\right)$;
- Dhillon's [8] law $DL(\theta)$ with distribution function $1 - \exp\left(-(\log(x+1))^{\theta+1}\right)$.

Table 2. Type I error control of the tests for the nominal significance level $\alpha = 0.05$.

$L(\theta)$	n	TV	TE	TD	TB	TC	TA
$L(0.5)$	10	0.0502	0.0490	0.0495	0.0497	0.0509	0.0479
	20	0.0487	0.0484	0.0514	0.0513	0.0487	0.0483
	30	0.0503	0.0499	0.0486	0.0494	0.0496	0.0472
	50	0.0492	0.0505	0.0498	0.0498	0.0490	0.0488
$L(2)$	10	0.0511	0.0517	0.0514	0.0506	0.0520	0.0510
	20	0.0510	0.0522	0.0525	0.0518	0.0517	0.0545
	30	0.0530	0.0534	0.0513	0.0514	0.0527	0.0510
	50	0.0506	0.0525	0.0499	0.0495	0.0503	0.0499
$L(4)$	10	0.0517	0.0526	0.0528	0.0514	0.0523	0.0534
	20	0.0523	0.0533	0.0536	0.0518	0.0530	0.0561
	30	0.0528	0.0532	0.0505	0.0508	0.0527	0.0527
	50	0.0506	0.0544	0.0533	0.0522	0.0505	0.0542
$L(8)$	10	0.0529	0.0543	0.0533	0.0520	0.0541	0.0554
	20	0.0533	0.0548	0.0548	0.0533	0.0532	0.0578
	30	0.0497	0.0533	0.0510	0.0502	0.0509	0.0519
	50	0.0517	0.0552	0.0520	0.0504	0.0520	0.0523

These alternatives include densities f with decreasing failure rates (DFR), increasing failure rates (IFR) as well as models with unimodal failure rate (UFR) functions and bathtub failure rate (BFR) functions.
We compute the power values of the tests under the above alternatives by Monte Carlo simulations as follows. Under each alternative 100,000 samples of size 10, 20, 30 and 50 are generated and the test statistics are calculated. Then power of the corresponding test is computed by the frequency of the event "the statistic is in the critical region". Tables 3 and 4 display and compare the power values of the tests at the significance level $\alpha = 0.05$.

From Tables 3-4, we observe that no single test can be said to perform the best against all alternatives. However, for almost alternatives the tests based on TD, TE, TV and TA statistics have the most power against different alternatives. Two tests TD and TA have a good performance against IFR alternatives. For UFR alternatives, the tests TE and TD have the most power and the test TV has a higher power against DFR and BFR alternatives. Therefore, although there is no uniformly most powerful test against all alternatives, the tests based on TD, TE, TV and TA statistics can be recommended. The powerful tests against different alternatives are presented in Table 5.
In other hand, from Table 2, we found that the actual sizes of the proposed tests based on varentropy estimators were acceptable. Consequently, we can confidently recommend the proposed tests in practice.

5. An Illustrative Example

Through an example, we illustrate how the proposed tests can be applied to test the goodness of fit for the Lindley distribution when the observations are available.

Example 1. We consider the data set discussed by Ghitany et al. (2008). The data set consists waiting times (in minutes) before service of 100 bank customers. The waiting times (in minutes) are as follows:

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

Histogram of these data and a fitted Lindley density function are displayed in Figure 2.

Table 3. Empirical powers of the tests against IFR alternatives at significance level 5%.

Alternative	n	TV	TE	TD	TB	TC	TA
$W(1.4)$	10	0.0405	0.0494	0.1005	0.0868	0.0413	0.0748
	20	0.0236	0.0630	0.1565	0.1437	0.0296	0.1395
	30	0.0169	0.0884	0.2137	0.1934	0.0280	0.2015
	50	0.0112	0.1587	0.3464	0.2998	0.0337	0.3194
$\Gamma(2)$	10	0.0365	0.0448	0.0957	0.0844	0.0371	0.0672
	20	0.0187	0.0481	0.1462	0.1376	0.0228	0.1123
	30	0.0107	0.0677	0.1931	0.1758	0.0187	0.1633
	50	0.0067	0.1219	0.3169	0.2769	0.0224	0.2696
HN	10	0.0511	0.0589	0.0846	0.0756	0.0532	0.0761
	20	0.0433	0.0754	0.1181	0.1058	0.0491	0.1321
	30	0.0406	0.0961	0.1473	0.1309	0.0514	0.1732
	50	0.0379	0.1370	0.2042	0.1780	0.0560	0.2313
U	10	0.2217	0.2865	0.2502	0.1860	0.2337	0.3347
	20	0.4635	0.6353	0.4348	0.3658	0.5095	0.6490
	30	0.6947	0.8442	0.5828	0.5146	0.7354	0.7771
	50	0.9421	0.9868	0.8198	0.7651	0.9567	0.9198
CH(1)	10	0.0540	0.0601	0.0812	0.0716	0.0555	0.0760
	20	0.0513	0.0809	0.1099	0.0976	0.0577	0.1302
	30	0.0487	0.0939	0.1288	0.1145	0.0579	0.1595
	50	0.0489	0.1379	0.1816	0.1558	0.0666	0.2113
CH(1.5)	10	0.0667	0.1077	0.2466	0.1686	0.0739	0.1805
	20	0.0714	0.2501	0.4375	0.3620	0.1093	0.4409
	30	0.0950	0.4139	0.6064	0.5278	0.1733	0.6256
	50	0.1948	0.7224	0.8478	0.7776	0.3681	0.8418
LF(2)	10	0.0509	0.0657	0.1114	0.0955	0.0534	0.0963
	20	0.0422	0.1000	0.1794	0.1593	0.0525	0.1944
	30	0.0395	0.1419	0.2377	0.2098	0.0584	0.2669
	50	0.0383	0.2398	0.3635	0.3203	0.0797	0.3927
LF(4)	10	0.0537	0.0768	0.1517	0.1235	0.0577	0.1217
	20	0.0448	0.1428	0.2679	0.2338	0.0638	0.2818
	30	0.0459	0.2231	0.3712	0.3277	0.0834	0.4049
	50	0.0618	0.4054	0.5704	0.5061	0.1373	0.5947
EV(0.5)	10	0.0545	0.0624	0.0818	0.0721	0.0560	0.0790
	20	0.0505	0.0794	0.1120	0.0992	0.0558	0.1316
	30	0.0481	0.0940	0.1335	0.1171	0.0566	0.1617
	50	0.0478	0.1374	0.1816	0.1558	0.0649	0.2116
EV(1.5)	10	0.0683	0.0922	0.1435	0.1136	0.0726	0.1346
	20	0.0736	0.1765	0.2407	0.2045	0.0965	0.3037
	30	0.0901	0.2627	0.3245	0.2789	0.1277	0.4111
	50	0.1292	0.4491	0.4991	0.4362	0.2097	0.5802

Krishna and Kumar (2011) considered four reliability models, namely exponential, Lindley, gamma, and lognormal. According to Bayesian information criterion (BIC), they found that the Lindley model is the best fit for these data. Thus, Lindley distribution is fitting the above data quite satisfactorily. The main advantage of using Lindley distribution over gamma and lognormal distributions is that it involves only one parameter. Hence, maximum likelihood and other inferential procedures become simple to deal with, especially from computational point of view.

Table 4. Empirical powers of the tests against UFR, DFR and BFR alternatives at level 5%.

Alternative	n	TV	TE	TD	TB	TC	TA
$LN(0.8)$	10	0.0573	0.0786	0.1282	0.1103	0.0617	0.1129
	20	0.0444	0.1102	0.1999	0.1785	0.0582	0.2056
	30	0.0402	0.1470	0.2643	0.2401	0.0628	0.2872
	50	0.0435	0.2396	0.4137	0.3691	0.0865	0.4276
$LN(1.5)$	10	0.4131	0.4480	0.2776	0.2199	0.4219	0.3938
	20	0.6971	0.7453	0.4855	0.4093	0.7145	0.6688
	30	0.8567	0.8818	0.6396	0.5728	0.8657	0.7860
	50	0.9722	0.9786	0.8429	0.8008	0.9742	0.9144
$DL(1)$	10	0.0469	0.0592	0.0923	0.0852	0.0486	0.0814
	20	0.0313	0.0656	0.1285	0.1217	0.0374	0.1225
	30	0.0223	0.0764	0.1605	0.1510	0.0332	0.1592
	50	0.0177	0.1148	0.2432	0.2235	0.0356	0.2378
$DL(1.5)$	10	0.0348	0.0464	0.1338	0.1066	0.0359	0.0791
	20	0.0146	0.0633	0.2211	0.1994	0.0214	0.1610
	30	0.0082	0.1046	0.3145	0.2836	0.0225	0.2590
	50	0.0075	0.2249	0.5243	0.4588	0.0400	0.4458
$W(0.8)$	10	0.1421	0.1444	0.0603	0.0466	0.1431	0.1018
	20	0.2754	0.2765	0.0642	0.0418	0.2764	0.1530
	30	0.4034	0.3818	0.0695	0.0455	0.3932	0.1561
	50	0.6165	0.5746	0.0932	0.0584	0.5934	0.1680
$\Gamma(0.4)$	10	0.3933	0.3893	0.1206	0.0713	0.3831	0.2360
	20	0.7557	0.7478	0.1788	0.1035	0.7391	0.3914
	30	0.9167	0.9048	0.2427	0.1500	0.9023	0.4443
	50	0.9929	0.9903	0.4006	0.2734	0.9898	0.5590
$CH(0.5)$	10	0.2881	0.2814	0.0790	0.0479	0.2815	0.1605
	20	0.6038	0.5870	0.1033	0.0558	0.5875	0.2596
	30	0.8010	0.7736	0.1283	0.0706	0.7798	0.2793
	50	0.9594	0.9444	0.2050	0.1200	0.9474	0.3342

Table 5. Powerful tests against different alternatives

IFR	UFR	DFR-BFR
TD & TA	TD & TE	TV

The proposed procedures can be used to investigate whether the data come from a Lindley distribution. The ML estimator of $theta$ is computed as:

$$\hat{\theta} = 0.1866\,.$$

Then, the value of each test statistic is computed and also the critical value of each test at the significance level 0.05 is obtained by Monte Carlo simulation. Results are summarized in Table 6.

Because the value of each test statistic is smaller than the corresponding critical value, the Lindley hypothesis is not rejected for these data at the significance level of 0.05. Therefore, we can conclude that the underlying distribution of these data is a Lindley distribution.

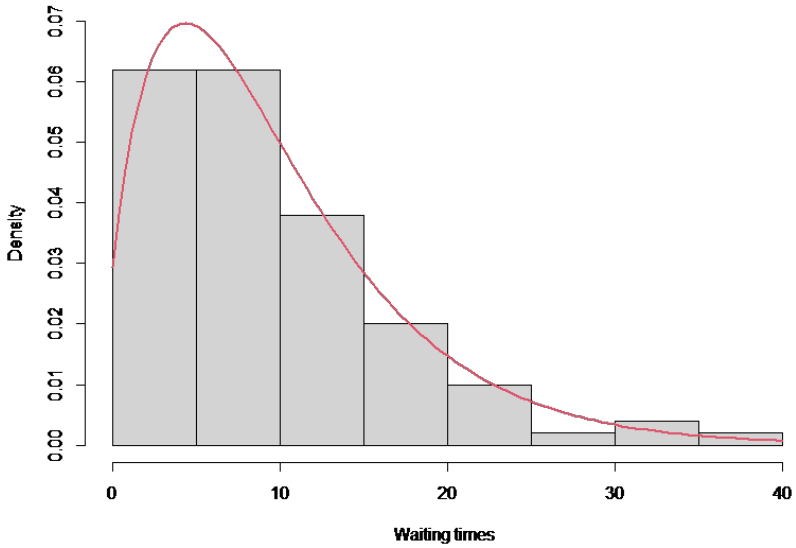

Figure 2. Histogram of data in Example 1 and a fitted Lindley density function.

Table 6. The value of the test statistics and critical values at 5% level.

Test	Value of the test statistic	Critical value	Decision
TV	0.02819	0.12651	Not reject H_0
TE	0.01809	0.07887	Not reject H_0
TD	0.03165	0.05529	Not reject H_0
TB	0.04927	0.08064	Not reject H_0
TC	0.01842	0.11140	Not reject H_0
TA	0.02956	0.05265	Not reject H_0

6. Conclusions

In this paper, we first discussed about the varentropy estimators of a continuous random variable. We then constructed some new goodness of fit tests for Lindley distribution based on estimators of varentropy. The proposed test statistics are easy to compute and consistency and the other properties of the proposed test statistics have presented.
We obtained the percentage points and type I error of the suggested tests. Power values of the proposed tests against various alternatives for different sample sizes were reported. Power simulations of the new tests based on the varentropy estimators showed that the varentropy-based tests are viable for testing the hypothesis of Lindley distribution. In fact, our power studies indicated that generally the proposed tests based on TD, TE, TV and TA statistics have the most power against different alternatives.

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