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A quantile regression model based on the unit- maximum entropy lifetime distribution with application

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Abstract The beta regression is a standard approach for modeling the mean of a bounded response variable in the interval $(0, 1)$ conditional on covariates. However, this well-known model is not an appropriate choice when we are seeking the influence of such covariates on the quantiles of a response variable. As alternatives to the beta distribution, the Kumaraswamy and unit-Weibull distributions have a closed form in their cumulative distribution functions so that quantiles which made them appropriate for modeling quantile of a response versus covariates. In this paper, we introduced the unit-maximum entropy lifetime distribution which is obtained by the transformation of a random variable of maximum entropy lifetime distribution. This model has a closed form in its quantiles which can be used as an alternative to the Kumaraswamy and unit-Weibull distributions in quantile regression models. The appropriateness of the proposed model was evaluated through two simulation studies and application in two real datasets. The results of quantile unit-maximum entropy lifetime regression were compared with the Kumaraswamy and unit-Weibull quantile regression models.

Keywords quantile regression, Kumaraswamy, unit-Weibull, unit-maximum entropy lifetime, likelihood estimation

AMS 2010 subject classifications 62E99; 60E05

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1. Introduction

The Beta regression is a standard approach for modeling response variables such as rates, proportions, or percentages [9, 10, 15, 17, 25] that are defined in the unit interval $(0, 1)$. It assumes that the response variable follows a Beta distribution and the mean of the response is related to several predictors. The predictors are commonly incorporated into the model through a linear model and a link function. The Beta regression model introduced by Cepeda [8] and Ferrari and Cribari-Neto [11] has a markable application in economic and medicine studies [1, 4] due to the model flexibility and simple interpretation of parameters. However, among various applications of random variables with the support of $(0, 1)$, modeling probabilities of phenomena have attracted more attention by researchers. Therefore, the Beta regression can be applied to describe the relationship between the probability of phenomena and predictors in terms of the mean of the model although the description intrinsically is heteroscedastic and asymmetries [10]. Next to the Beta distribution which has a variety of statistical properties and applications, there have been proposed several other distributions on the unit interval in the literature. The Kumaraswamy [22] distribution as a transformation of exponentiated exponential random variable and the unit distributions such as unit-Weibull distribution [19], unit-gamma distribution [13, 29], unit-log logistic [30] and the unit-inverse Gaussian distribution [12] are the alternatives to the Beta distribution for modeling variables on the unit interval. The abovementioned models have been extended to describe the effect of the covariates on the response variable. The Beta regression model [6], the simplex regression model [5], the Kumaraswamy regression

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model [7, 22], the unit-Gamma regression model [24], the unit-logistic regression model [26]; the unit-Lindley regression model [20] and the unit-Weibull regression model [21] have been used by researchers for modeling bounded responses conditional on covariates. Recently a new lifetime distribution called the maximum entropy lifetime (MEL) distribution was proposed by Khosravi et al [16]. This new model is a three-parameter lifetime distribution which was obtained by the maximum entropy principle under the constraints on the mean and a general index. In this paper, we first introduced the unit-MEL distribution by using the transformation. The unit-MEL distribution is defined on the $(0, 1)$ interval with a closed form in the cumulative distribution function. Then based on the above discussion and using the proposed model, we proposed a quantile regression model for modeling quantile of the unit-MEL random variable versus covariates. To incorporate the quantiles into the model for the purpose of implementing quantile regression, we applied a reparameterization which was used by Mazucheli et al [21] to describe the unit-Weibull quantile regression. The rest of the paper is organized as follows. In Section 2 we introduce the unit-MEL distribution and discuss some of its main properties based on the parameterization applied to this model. In Section 3 the unit-MEL quantile regression model and parameters estimation are investigated. In order to evaluate the behavior of maximum likelihood estimators, a simulation study is implemented for the unit-MEL quantile regression and presented in Section 4. In Section 5, the unit-MEL quantile regression is applied to two sets of real data, and the performance was evaluated versus two well-known quantile regression models. Finally, in Section 6, we mention some concluding remarks.

2. The unit-MEL distribution

The MEL distribution which was recently proposed by Khosravi et al [16] is a three parameters distribution with the cumulative distribution function (c.d.f)

$$G(x) = 1 - [1 + \alpha(e^{\beta x} - 1)]^{-\lambda}, \qquad x > 0. \tag{1}$$

In this model which we denote by MEL(α, β, λ), the parameters $\alpha > 0$ and $\lambda > 0$ are the shape and $\beta > 0$ is the scale parameter. For simplicity, we considered one of the shape parameters to be known, that is, MEL$(\alpha, \beta, \lambda = 1)$ which is a special case of the MEL distribution. Using the transformation $Y = e^{-X}$ the cumulative distribution function of unit-MEL is given by

$$F(y) = [1 + \alpha(y^{-\beta} - 1)]^{-1}, \qquad 0 < y < 1, \tag{2}$$

and the probability density function of the unit-MEL can be expressed by

$$f(y) = \alpha\beta y^{-(\beta+1)}[1 + \alpha(y^{-\beta} - 1)]^{-2}, \qquad 0 < y < 1, \tag{3}$$

where $\alpha, \beta > 0$. To define a quantile regression based on the unit-MEL distribution, we first obtain an expression for the quantile of the model. The τth quantile of the unit-MEL distribution which is defined by $F(y) = \tau$ is given by

$$q(\tau|\alpha, \beta) = [1 + \frac{1 - \tau}{\alpha\tau}]^{\frac{-1}{\beta}}, \qquad 0 < \tau < 1. \tag{4}$$

To incorporate the quantile of the unit-MEL into the quantile regression model, we consider re-parameterization $\mu = q(\tau|\alpha, \beta)$ as used by Mazucheli et al [21]. In this case, the parameter α is obtained by

$$\alpha = \frac{1 - \tau}{\tau}[\mu^{-\beta} - 1]^{-1}. \tag{5}$$

Using this parameterization, the new functional forms of the c.d.f and the p.d.f of the unit-MEL can be expressed respectively, as follows:

$$F(y|\mu, \tau, \beta) = [1 + (\frac{1 - \tau}{\tau})(\frac{y^{-\beta} - 1}{\mu^{-\beta} - 1})]^{-1}, \qquad 0 < y < 1 \tag{6}$$

and

$$f(y|\mu,\tau,\beta)=\beta(\frac{1-\tau}{\tau})[\mu^{-\beta}-1]^{-1}y^{-(\beta+1)}[1+(\frac{1-\tau}{\tau})(\frac{y^{-\beta}-1}{\mu^{-\beta}-1})]^{-2}, \qquad 0<y<1 \tag{7}$$

where , $\beta>0, 0<\mu<1$ and the parameter $0<\tau<1$ is assumed to be a known value. Hereafter, we use the notation $Y\sim$ UMEL(μ,β,τ) to denote the unit maximum entropy lifetime distribution considering the new parameters. Figure 1 illustrates the behavior of the p.d.f of the UMEL distribution defined in 7, for some of the selected values of μ,β and τ.

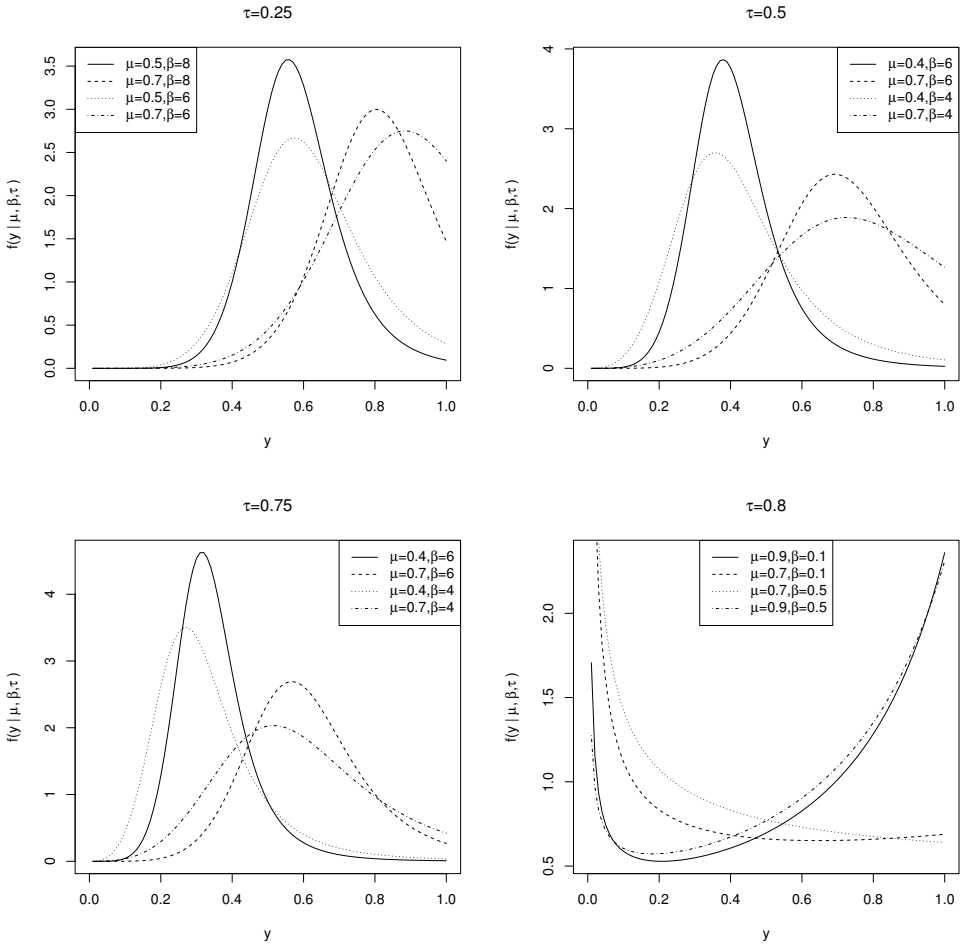


Figure 1. Probability density plot of the UMEL distribution for several selected values of parameters

As illustrated in Figure 1, the p.d.f plot can assume to be positively skewed, negatively skewed, and even symmetric shapes. A detailed examination of the p.d.f plot shows that μ is the location and β is the scale parameters of the UMEL distribution. Therefore, the UMEL distribution can be used as a flexible distribution for modeling various data on the unit interval.

We investigated the behavior of the UMEL p.d.f in terms of μ,β and τ based on the logarithm scale of (7). The derivative of $\log f(y|\mu,\beta,\tau)$ with respect to y is given by

$$\frac{d}{dy}\log f(y|\mu,\tau,\beta)=-\frac{1}{y}\left[\beta+1-\frac{2\beta y^{-\beta}}{C+y^{-\beta}}\right]=-\frac{1}{y}\left[\frac{(\beta+1)C+y^{-\beta}(1-\beta)}{C+y^{-\beta}}\right]=-\frac{A(y)}{y} \tag{8}$$

where $C = \{(\frac{\tau}{1-\tau})(\mu^{-\beta}-1)-1\}$ and $A(y) = [\beta+1-\frac{2\beta y^{-\beta}}{C+y^{-\beta}}]$.

If $\tau > \mu^{\beta}$ and $\beta < 1$ then $f(y|\mu,\tau,\beta)$ is decreasing. **Proof.** The term $A(y)$ in (8) is positive if
1) C¿0 $\Rightarrow (\frac{\tau}{1-\tau})(\mu^{-\beta}-1)-1>0 \Rightarrow \tau > \mu^{\beta}$,
2) $1-\beta>0 \Rightarrow \beta<1$.
Therefore, positive $A(y)$ implies that $\frac{d}{dy}\log f(y|\mu,\tau,\beta)$ is negative, which is concluded that the UMEL p.d.f (7) is decreasing.

If $\tau < \mu^{\beta}$ and $\beta > 1$ then $f(y|\mu,\tau,\beta)$ is increasing. **Proof.** The term $A(y)$ in (8) is negative if
1) C¡0 $\Rightarrow (\frac{\tau}{1-\tau})(\mu^{-\beta}-1)-1>0 \Rightarrow \tau < \mu^{\beta}$,
2) $1-\beta<0 \Rightarrow \beta>1$.
Therefore, negative $A(y)$ implies that $\frac{d}{dy}\log f(y|\mu,\tau,\beta)$ is positive, which is concluded that the UMEL p.d.f (7) is increasing.

If $\tau = \mu^{\beta}$ and $\beta > 1$ ($\beta < 1$) then $f(y|\mu,\tau,\beta)$ is increasing (decreasing). **Proof.** If $C = 0$, it can be concluded that $\tau = \mu^{\beta}$. On the other hand, for $\tau = \mu^{\beta}$ we have,

$$\frac{d}{dy}\log f(y|\mu,\tau,\beta) = -\frac{1-\beta}{y}$$

Therefore, for $\beta > 1$ ($\beta < 1$), $\frac{d}{dy}\log f(y|\mu,\tau,\beta) > 0$ $\left(\frac{d}{dy}\log f(y|\mu,\tau,\beta) < 0\right)$ which implies UMEL p.d.f (7) is increasing (decreasing).

If $\tau < \mu^{\beta}$, $\beta > 1$ and $y > \left(1-(\frac{\tau}{1-\tau})(\mu^{-\beta}-1)\right)^{-\frac{1}{\beta}}$ then $f(y|\mu,\tau,\beta)$ is increasing. **Proof.** The term $A(y)$ is negative if
1) $C < 0 \Rightarrow \tau < \mu^{\beta}$,
2) $1-\beta<0 \Rightarrow \beta>1$,
3) $C+y^{-\beta}>0 \Rightarrow y > \left(1-(\frac{\tau}{1-\tau})(\mu^{-\beta}-1)\right)^{-\frac{1}{\beta}}$.

Therefore, negative $A(y)$ implies that $\frac{d}{dy}\log f(y|\mu,\tau,\beta)$ is positive, that is concluded that the UMEL p.d.f (7) is increasing.
Now, we investigate the situations in which the UMEL p.d.f (7) is unimodal or bathtub. For this aim, we first obtain the second derivative of $\log f(y|\mu,\beta,\tau)$ with respect to y that is given by

$$\frac{d^2}{dy^2}\log f(y|\mu,\tau,\beta) = \frac{1+\beta}{y^2} - 2\beta y^{-(\beta+2)}\left[\frac{C(\beta+1)+y^{-\beta}}{(C+y^{-\beta})^2}\right] = \frac{1+\beta}{y^2} - 2\beta y^{-(\beta+2)}D(y) \tag{9}$$

If $\tau < \mu^{\beta}$ and $y < \left((1+\beta)\left[1-(\frac{\tau}{1-\tau})(\mu^{-\beta}-1)\right]\right)^{-\frac{1}{\beta}}$ then $f(y|\mu,\tau,\beta)$ is bathtub. **Proof.** The second derivative (9) is poitive if
1) $C < 0 \Rightarrow \tau < \mu^{\beta}$,
2) $D(y) < 0 \Rightarrow y^{-\beta} < -C(\beta+1) \Rightarrow y < \left((1+\beta)\left[1-(\frac{\tau}{1-\tau})(\mu^{-\beta}-1)\right]\right)^{-\frac{1}{\beta}}$.

Therfore, under these conditions the second derivative (9) is positive which implies that (7) is bathtub. To interpret the behavior of the probability density function of in practice, we consider the random variable Y to be the loss ratio in actuarial context. A decreasing p.d.f (Result 2) indicates that most policies have low loss ratios, indicating most portfolios are profitable, whereas increasing p.d.f (Result 2) results in policies with high loss ratios. The parameter β controls the skewness toward low or high loss ratios (Result 2), while more complex shapes (Result2) shows a heterogeneous portfolio. Finally, a bathtub-shaped p.d.f (Result 2) reveals distinct groups involving low and high loss ratios.

3. Quantile regression of UMEL distribution

In this section, we define the quantile regression of UMEL distribution based on the parameterization model expressed in (7). The formulation we use here was applied in several studies in the literature [7, 21, 22]. Suppose $Y_1, Y_2, \ldots, Y_n$ is a set of n independent random variables each follows the UMEL distribution with the p.d.f in (7) that is, $Y_i \sim \text{UMEL}(\mu_i, \beta, \tau)$ with unknown parameters μ_i, β, and known parameter $\tau \in (0,1)$ that commonly takes values $0.25, 0.50$, and 0.75 denoting the first, second, and third quantiles. To define the UMEL quantile regression and incorporate the predictors $\mathbf{x}_i = (1, x_{i1}, \ldots, x_{iq})^T$ into the model we use a link function that relates the quantile of the model and the covariates through a linear regression as follows,

$$g(\mu_i) = \delta^T \mathbf{x}_i, \qquad i = 1, 2, \ldots, n, \tag{10}$$

where, $\delta = (\delta_0, \delta_1, \ldots, \delta_q)^T$ is $(q \times 1)$ vector of regression coefficients and $\mathbf{x}_i$ is the vector of the ith observed predictors. Various link functions involving logit, probit, and log-log functions can be used to relate the quantile regression parameters and the predictors. However, the logit function, that is, $\text{logit}(\mu_i) = \mu_i/(1-\mu_i)$ which guaranties the predicted values fall into the $(0,1)$ interval, is the most commonly used link function in the literature. Applying the logit function, the relationship between μ_i and covariates can be expressed by

$$\mu_i = \frac{\exp(\delta_0 + \delta_1 x_{\text{i1}} + \ldots + \delta_q x_{\text{iq}})}{1 + \exp(\delta_0 + \delta_1 x_{\text{i1}} + \ldots + \delta_q x_{\text{iq}})}, \qquad i = 1, 2, \ldots, n \tag{11}$$

In this paper, we used the logit link among the others. This is due to the simple interpretation of the odds ratio and the regression coefficient in describing the quantile parameter μ_i . The coefficients δ_k of UMEL regression quantify covariate impact on the log-odds of the τ-th quantile, analogous to logistic regression but for quantiles rather than means. For instance, $\exp(\delta_k)$ represents the multiplicative change in the odds that Y_i falls below its τ-th conditional quantile per unit increase in x_k. A comprehensive details of odds ratio and interpretation of regression coefficients in logit link function can be found in Agresti [2]. The parameter μ_i in (11) can be the mean of the model as in the Beta regression model or the τ th quantile, $0 < \tau < 1$ as in the UMEL quantile regression model. In both cases the interpretation is straight forward. Here, the β parameter can also be entered in the regression model through a positive link function such as the log link function, that is, $\log(\beta_i) = \omega^T \mathbf{z}_i$ where the vectors $\mathbf{x}_i$ and $\mathbf{z}_i$ may or may not share the same components.

4. Estimation of parameters and inference

In this section we present the estimation approach for fitting the UMEL quantile regression. Suppose $Y_i \sim \text{UMEL}(\mu_i, \beta, \tau), i = 1, 2, \ldots, n$ is a set of random variables where μ_i is the quantile regression equation in (11) and $0 < \tau < 1$ is a known value. Let $\theta = (\delta^T, \beta)^T$ be the vector of regression coefficients and scale parameters to be estimated using the maximum likelihood approach. The log-likelihood function of the UMEL distribution based on the p.d.f (7) is given by

$$\begin{aligned}\ell(\theta|y_1, y_2, \ldots, y_n) = n\log(\beta(\tfrac{1-\tau}{\tau})) - \sum_{i=1}^{n}\log({\mu_i}^{-\beta} - 1) - (\beta+1)\sum_{i=1}^{n}\log(y_i) \\ -2\sum_{i=1}^{n}\log(1 + (\tfrac{1-\tau}{\tau})(\tfrac{{y_i}^{-\beta}-1}{{\mu_i}^{-\beta}-1})).\end{aligned} \tag{12}$$

Therefore, the maximum likelihood estimation of $\theta = (\delta^T, \beta)^T$, that is, $\hat{\theta} = (\hat{\delta}^T, \hat{\beta})^T$ is obtained by maximizing the log-likelihood function (12) through the derivative of this function with respect to each parameter $(\delta_0, \delta_1, \ldots, \delta_q, \beta)$ and setting each equal to zero. It is not possible to obtain a closed form for solutions, so that it must be solved numerically using an optimization approach such as the Newton-Raphson algorithm. As suggested by Ferrary and Cribari-Neto [11], the least square estimation of the parameters, that is $(X^T X)^{-1} X^T y^*$ where $y^* = (\text{logit}(y_1), \ldots, \text{logit}(y_n))^T$ can be used as the starting values for the Newton-Raphson The Newton-Raphson algorithm consistently achieved convergence when using least squares estimates as starting values.

However, in some situations, issues may arise depending on the choice of initial values, or selected algorithm. To avoid convergence problems related to initial values, we recommend using moment estimates as starting points or employing multiple randomized initializations to escape local optima particularly for small samples or near-boundary τ values. If the Newton-Raphson method fails to converge, alternative approaches such as quasi-Newton methods (BFGS), gradient-based algorithms (L-BFGS-B), penalized likelihood, or EM-type algorithms (for boundary estimates) can be applied. These strategies can be implemented using `optim()` or `nlm()` functions in `R` package. Under some regular conditions and for large n , the MLE $\hat{\theta} = (\hat{\delta}^T, \hat{\beta})^T$ is asymptotically multivariate normal, that is, $\hat{\theta} \sim N_{q+2}(\theta, \mathbf{I}^{-1}(\theta))$ where $\theta = (\delta^T, \beta)^T$ and $\mathbf{I}(\theta)$ is the Fisher information matrix which can be partitioned as

$$\mathbf{I}(\theta) = \begin{bmatrix} \mathbf{I}_{\delta\delta} & \mathbf{I}_{\delta\,\beta} \\ \mathbf{I}_{\beta\,\delta} & \mathbf{I}_{\beta\beta} \end{bmatrix} = E[-\psi_\theta(\delta,\beta){\psi_\theta}^T(\delta,\beta)] \tag{13}$$

and $\psi_\theta(\delta,\beta)$ is the score vector of θ which is obtained by

$$\psi_\theta(\delta,\beta) = \left(\frac{\partial \ell(\theta)}{\partial \mu_i}\frac{\partial \mu_i}{\partial \delta_0}, \ldots, \frac{\partial \ell(\theta)}{\partial \mu_i}\frac{\partial \mu_i}{\partial \delta_q}, \frac{\partial \ell(\theta)}{\partial \beta} \right) \tag{14}$$

where $\frac{\partial \mu_i}{\partial \delta_0} = \mu_i(1-\mu_i)$ and $\frac{\partial \mu_i}{\partial \delta_j} = \mu_i(1-\mu_i)x_{ij},\ j = 1, 2, \ldots, q$. The entries of the Fisher information matrix are,

$$\begin{aligned} \mathbf{I}_{\delta\delta} &= -E\left[\frac{\partial^2 \ell(\theta)}{\partial \delta_j \partial \delta_k}\right] = -E\left[\frac{\partial^2 \ell(\theta)}{\partial {\mu_i}^2}{\mu_i}^2(1-\mu_i)^2 x_{ij}x_{ik}\right], \quad j,k = 1,\ldots,q \\ \mathbf{I}_{\delta\,\beta} &= -E\left[\frac{\partial^2 \ell(\theta)}{\partial \delta_j \partial \beta}\right] = -E\left[\frac{\partial^2 \ell(\theta)}{\partial \mu_i \partial \beta}\mu_i(1-\mu_i)x_{ij}\right], \quad j = 1,\ldots,q \\ \mathbf{I}_{\beta\beta} &= -E\left[\frac{\partial^2 \ell(\theta)}{\partial \beta^2}\right] \end{aligned} \tag{15}$$

Since there is no closed form for the Fisher information matrix here, as suggested by Lindsay and Li [18], the estimated observed Fisher information matrix

$$\mathbf{I}(\hat{\theta}) = -\psi_\theta(\delta,\beta){\psi_\theta}^T(\delta,\beta)|_{\theta=\hat{\theta}} \tag{16}$$

is replaced. Using this consistent estimator in replace of the expected Fisher information matrix, the asymptotic $100(1-\eta)\%$ confidence interval for θ_k is given by

$$\theta_k : \hat{\theta}_k \pm z_{\eta/2}\, se\,(\hat{\theta}_k), \quad k = 1,\ldots,q+2 \tag{17}$$

where $z_{\eta/2}$ is the upper tail of the $\eta/2$ quantile of the standard normal distribution and $se(\hat{\theta}_k)$ is the square root of the kth diagonal element of the inversed estimated Fisher information matrix, $\mathbf{I}^{-1}(\hat{\theta})$.

5. Simulation study

In this section, we presented two simulation studies conducted to investigate the properties of maximum likelihood estimators and asymptotic confidence interval of the UMEL quantile regression parameters. For evaluation purposes, we calculated the estimated bias, the estimated root-mean-square error (RMSE), and the estimated coverage probability of a 95% confidence interval that are defined as,

$$\begin{aligned} BIAS(\hat{\theta}) &= \tfrac{1}{B}\sum_{i=1}^{B}(\hat{\theta}_i - \theta) \\ RMSE &= \sqrt{\tfrac{1}{B}\sum_{i=1}^{B}(\hat{\theta}_i - \theta)^2} \\ CP_\theta(n) &= \tfrac{1}{B}\sum_{i=1}^{B} I\left\{\hat{\theta}_i - 1.96\,\hat{s}_{\hat{\theta}_i} < \theta < \hat{\theta}_i + 1.96\,\hat{s}_{\hat{\theta}_i}\right\} \end{aligned} \tag{18}$$

where $I\{.\}$ denotes the indicator function. For assessing the flexibility of the proposed model, we considered two simulation scenarios for generating quantile regression models involving (i) a model with one covariate and (ii) a two covariates model. The simulations are conducted in `R` using the `optim` function to obtain the estimated values.

(i) One predictor quantile regression

In this scenario, we applied a Monte Carlo simulation considering quantile regression equation

$$\text{logit}(\mu_i) = 0.2 - 0.1x_{i1}, \qquad i = 1, 2, \ldots, n$$

where true values are $\delta_0 = 0.2, \delta_1 = -0.1$ and $\beta = 3.5$. The predictor x_1 is generated from $n(0, 1)$. For generating each set, we considered different sample sizes of $n = 50, 100, 200, 300$ such that each Monte Carlo simulation was repeated $B = 10,000$ times. For further investigation, we applied the simulation for important quantiles that is, $\tau = 0.10, 0.25, 0.50, 0.75, 0.90$. The results of the simulation of one covariate quantile regression are given in Figure 2 and Table (1).

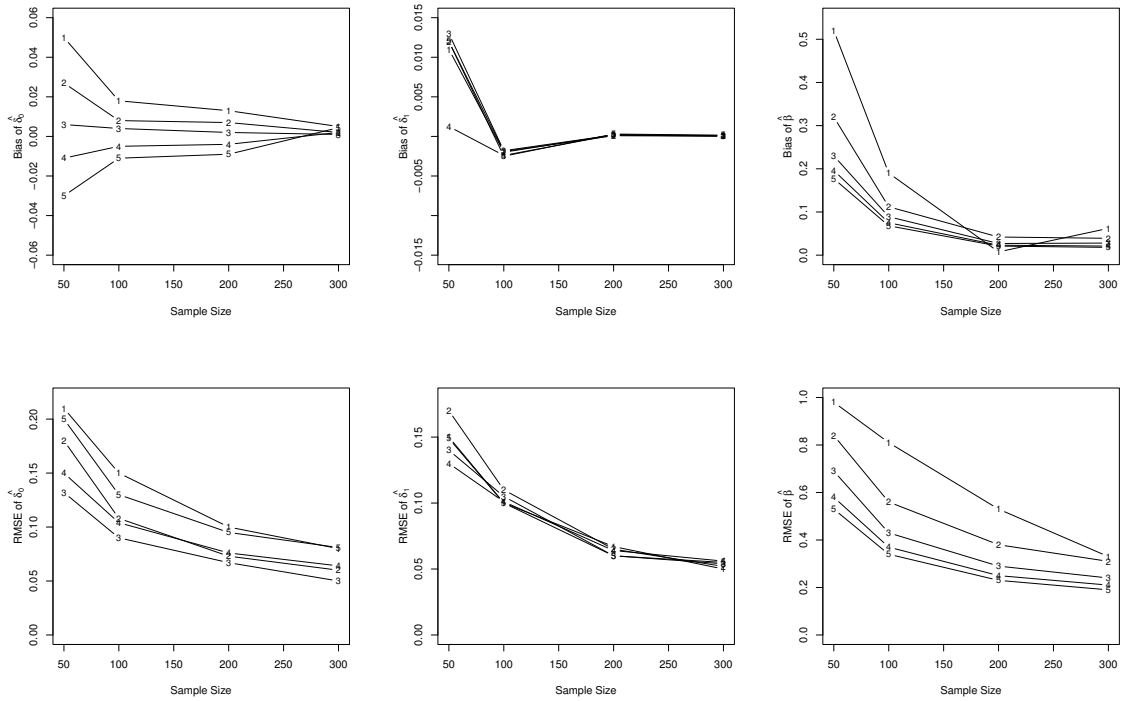


Figure 2. Top: estimated bias of $\hat{\delta}_0$, $\hat{\delta}_1$ and $\hat{\beta}$. Bottom: estimated $RMSE$ of $\hat{\delta}_0$, $\hat{\delta}_1$ and $\hat{\beta}$ for 1: $\tau = 0.10$, 2: $\tau = 0.25$, 3: $\tau = 0.50$, 4: $\tau = 0.75$ and 5: $\tau = 0.90$ values.

(ii) Two predictors quantile regression

In this scenario, we considered a quantile regression with two predictors, that is,

$$\text{logit}(\mu_i) = 0.8 + 1.2x_{i1} - 0.5x_{i2}, \qquad i = 1, 2, \ldots, n$$

where true values are $\delta_0 = 0.8, \delta_1 = 1.2, \delta_2 = -0.5$ and $\beta = 2.5$. The predictors x_1 and x_2 are generated from $n(0, 1)$ and $n(0, 4)$ respectively. Like in the scenario (i), sample sizes $n = 50, 100, 200, 300$ with repeated value $B = 10,000$ for Monte Carlo simulation were considered. Results of the simulation with two predictors are given in Figure (3) and Table (2).

Table 1. Estimated coverage probability for δ_0, δ_1 and β

		n=50			n=100	
τ	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\beta}$	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\beta}$
0.10	94.4	94.8	94.2	95.2	94.4	95.2
0.25	95	95	93.8	94.6	94.8	95.2
0.50	95	95.2	94	95.6	94.8	95.4
0.75	95.4	95.2	93.8	95.4	94.6	95.2
0.90	95	95.1	94.8	95.4	94.8	94.8
		n=200			n=300	
τ	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\beta}$	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\beta}$
0.10	96	95.4	94.6	95	94	94.8
0.25	95	95	95	95.4	94.4	95.2
0.50	94.6	95.2	94.6	95	94.6	95.4
0.75	95.8	95	95.4	95.2	94.6	95.8
0.90	95.4	95.2	94.8	94	94.8	95.8

Table 2. Estimated coverage probability for δ_0, δ_1 , δ_2 and β

		n=50				n=100		
τ	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\delta}_1$	$\hat{\beta}$	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\delta}_1$	$\hat{\beta}$
0.10	94.8	94.2	94.4	95.8	95.6	93.4	94.4	94.8
0.25	95.4	94.4	94	95.2	95.2	94.2	95	96
0.50	94.6	95.4	94.2	94.8	95.2	93.8	95	95.6
0.75	95	94.6	94.2	94.2	95	94.6	94.6	95.2
0.90	95	94.2	94	94.2	95.8	94.2	94.8	95.2
		n=200				n=300		
τ	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\beta}$	$\hat{\delta}_0$	$\hat{\delta}_1$	$\hat{\delta}_2$	$\hat{\beta}$
0.10	95.4	94.6	93	94.4	95	94.2	94.6	93.6
0.25	94.4	95.2	93.6	94.8	94.8	93.8	95.4	95
0.50	94.4	94.8	94.2	94.4	95.4	93	95	95.2
0.75	94.6	94.8	94.4	95	95	93.4	94.8	95.2
0.90	95	94.8	94.6	95	94.8	93.8	94.8	95

The followings are concluded from the simulations study,

a) The highest bias and RMSE of estimated regression parameters and β occur with respect to $\tau = 0.1, 0.9$ values and the lowest with respect to $\tau = 0.5$.

b) The bias and RMSE of estimated parameters decrease as sample size increases.

c) The bias and RMSE of estimated parameters decrease as τ approaches 0.5 , that is the median.

d) The coverage probabilities are quite close to the considered level, 95%.

While the UMEL model performs well in theoretical properties and applications across a range of scenarios, it has potential limitations. For instance, data with highly skewed, heavy-tailed covariates or extreme quantiles (τ near 0 or 1) may present additional challenges for model fitting and interpretation. In such cases, alternative approaches could offer advantages.

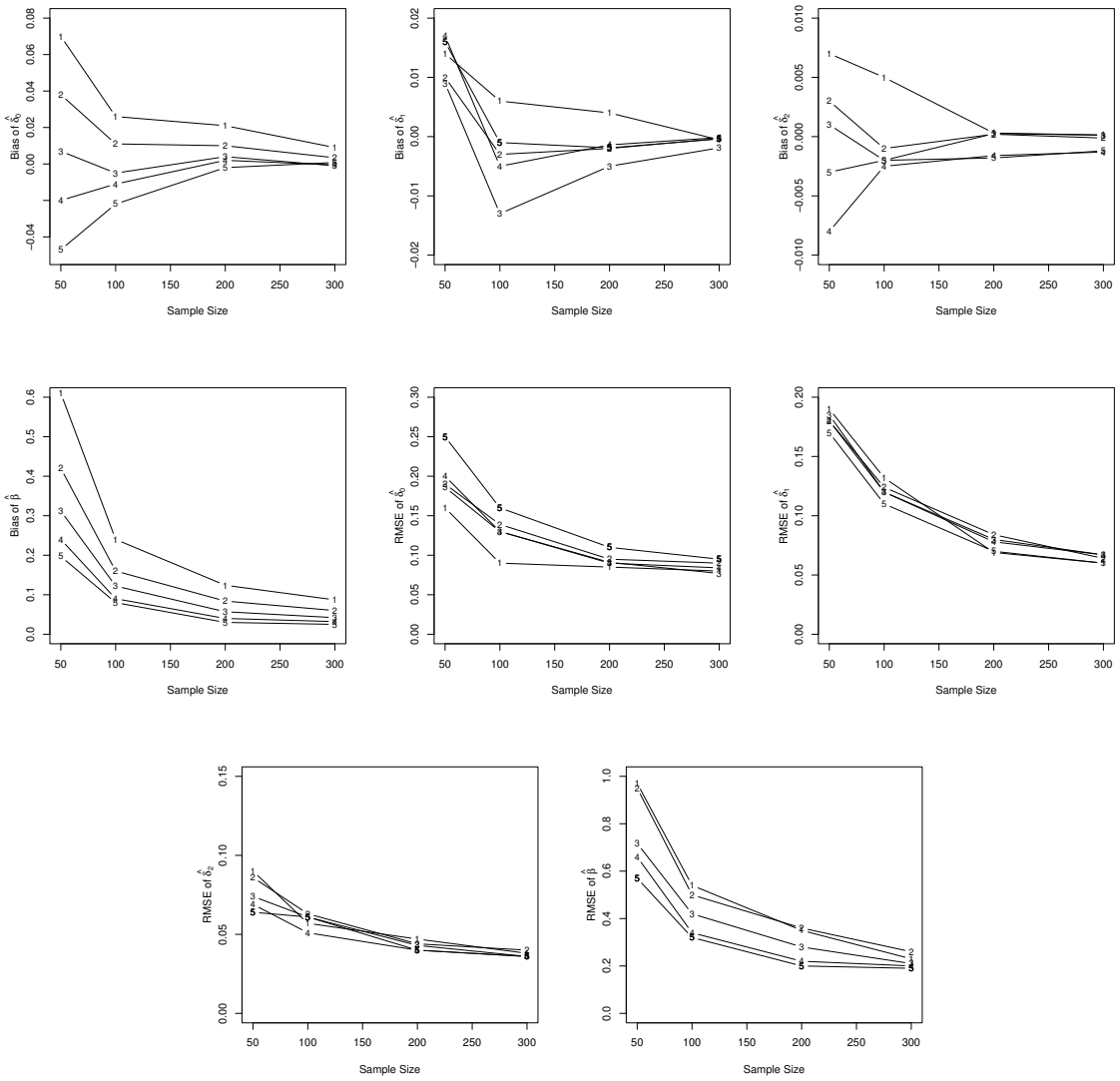


Figure 3. Top: estimated bias of $\hat{\delta}_0$, $\hat{\delta}_1$, $\hat{\delta}_2$and $\hat{\beta}$. Bottom: estimated $RMSE$ of $\hat{\delta}_0$, $\hat{\delta}_1$, $\hat{\delta}_2$ and $\hat{\beta}$ for 1: $\tau = 0.10$, 2: $\tau = 0.25$, 3: $\tau = 0.50$, 4: $\tau = 0.75$ and 5: $\tau = 0.90$ values.

6. Application to real data

In this section, we illustrate the flexibility of UMEL quantile regression through the analysis of two real datasets. To assess the potential of the proposed model we compare the UMEL quantile regression with two alternatives including the Kumaraswamy [22] and unit-Weibull [21] quantile regression models. In order to assess the model adequacy, we applied the Cox-Snell residuals which is defined as,

$$e_i = -\log[1 - F(y_i|\hat{\theta}_i)] = -\log S(y_i|\hat{\theta}_i) \tag{19}$$

where $S(.)$ is the survival function. According to the Cox-Snell residual property, if the regression model fits the data well, then e_i follows a standard exponential distribution. Therefore, the plot of e_i versus $-\log\hat{S}(e_i)$ where $\hat{S}(e_i)$ is the estimated survival function of a standard exponential distribution, should be a straight line with zero

Table 3. Maximum likelihood estimation and 95% confidence interval for quantile regression parameters fitted to PITCH data.

	UMEL		UW		KU	
parameter	MLE	95% CI	MLE	95% CI	MLE	95% CI
δ_0	-4.05	(-5.6, -2.5)	-4.85	(-6.5, -3.2)	-2.60	(-6.59,1.36)
δ_1	0.055	(0.03, 0.07)	0.062	(0.05, .075)	0.37	(-0.003,0.078)
δ_2	0.099	(-1.4, 1.6)	0.70	(-0.87,2.28)	-0.81	(-4.57,2.95)
δ_3	0.44	(0.32, 0.55)	0.45	(0.36, 0.54)	0.43	(0.17,0.69)
δ_4	-0.39	(-1.0, 0.23)	-0.30	(-0.92, .30)	-0.94	(-2.32,0.44)
$beta$	18.71	(13.3, 26.1)	43.2	(31.6, 54.9)	4.82	(3.11,6.53)

Table 4. Likelihood based criteria for fitted quantile regression models .

Criteria	UMEL	UW	KU
AIC	-287.58	-281.74	-254.90
BIC	-278.78	-272.94	-246.10
HQIC	-284.66	-278.82	-251.98

intercept and unit slope. In addition to this, we applied the Akaike (AIC) [3], Bayesian (BIC) [27] and Hannan-Quinn (HQIC) [14] information criteria for selecting the best model. These measures are defined as,

$$AIC = -2\ell(\hat{\theta}) + 2k, \qquad BIC = -2\ell(\hat{\theta}) + k\log(n), \qquad HQIC = -2\ell(\hat{\theta}) + 2k\log(\log(n)).$$

where $\ell(\hat{\theta})$ is the log-likelihood function evaluated at the MLE of parameters, k is the number of parameters and n denotes the sample size.

Heat-Treating Test (PITCH) Data This dataset contains the results of the pitch carbon analysis test which was taken from Montgomery and Runger [23]. The dataset involves 32 observations where pitch is the continuous response variable in the form of a ratio and the explanatory variables are SoakTime (duration of carburizing cycle), SoakPCT (carbon concentration), DiffTime (duration of the diffuse cycle) and DiffPCT (carbon concentration). The response variable and the covariates related to this dataset are

x_1: SoakTime (duration of carburizing cycle)
x_2: SoakPCT (carbon concentration)
x_3: DiffTime (duration of the diffuse cycle)
x_4: DiffPCT (carbon concentration diffused)
y: Pitch (results of the pitch carbon analysis)

The quantile regression model is expressed as

$$\text{logit}(\mu_i) = \delta_0 + \delta_1 x_{i1} + \delta_2 x_{i2} + \delta_3 x_{i3} + \delta_4 x_{i4}, \qquad i = 1, 2, \ldots, 32. \tag{20}$$

Where μ_i represents the median in the UMEL, unit-Weibull (UW) and Kumaraswamy (KU) models. Table (3) illustrates the point estimates and 95% confidence interval of maximum likelihood of parameters for three alternative quantile regression models.
According to the results, the covariates SoakPCT and DiffPCT are not statistically significant in prediction quantile response in all three fitted models since the confidence interval of the parameters contain zero value. As an interpretation of the UMEL quantile regression coefficients, here the coefficient of DiffTime is 0.44 with Odds-Ratio $\exp(0.44) = 1.55$. Therefore, 1-unit increase in DiffTime multiplies the odds of the median PITCH value by 1.55 or equivalently increases odds of median PITCH by 55%

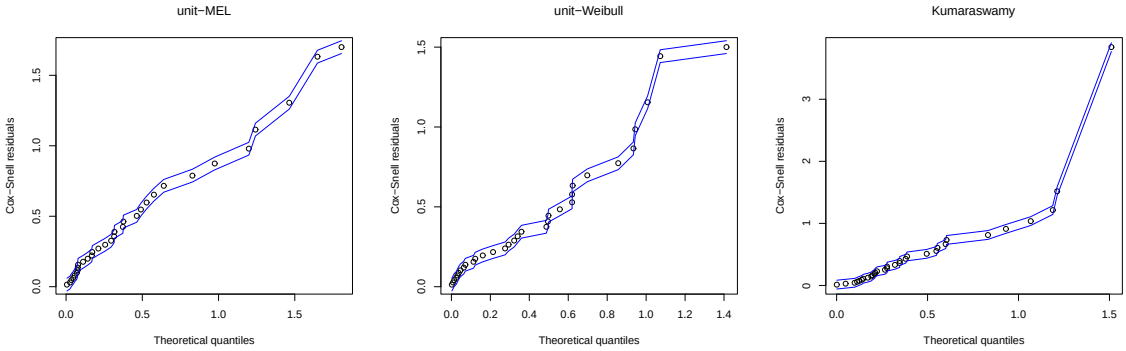


Figure 4. Cox-Snell residual plot for fitted models on PITCH data

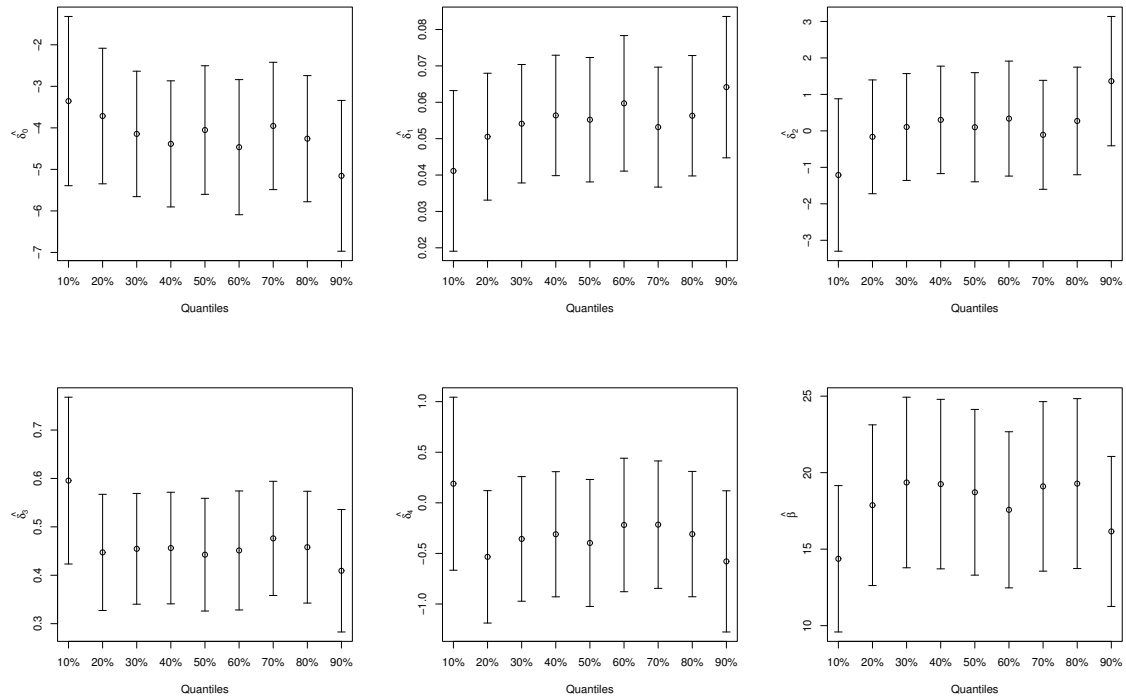


Figure 5. Estimated parameters of UMEL quantile regression with related 95% confidence considering $\tau = 0.1, 0.2, \ldots, 0.9$ for PITCH data

Table (4) displays the AIC, BIC, and HQIC criteria for evaluating the quality of the fitted models. As the result shows, the proposed UMEL quantile regression has a better performance compared to the alternatives in modeling PITCH data due to lower value in all three criteria. For further investigation of the model adequacy, the Cox-Snell residual plots of the fitted models to the PITCH data are illustrated in Figure (4). According to this plot, the Cox-Snell residual of the UMEL model is closer to a straight line which shows a better fit. Figure (5) displays the impact of the various τ 's on regression coefficients parameters. In Figure (6) the effect of different τ values on log-likelihood change of fitted UMEL on PITCH data is investigated. As the plot shows, the Log-likelihood increases by increasing values.

Table 5. Maximum likelihood estimation and 95% confidence interval for quantile regression parameters fitted to reading skills data.

	UMEL		UW		KU	
Parameter	MLE	95% CI	MLE	95% CI	MLE	95% CI
δ_0	1.80	(1.37, 2.23)	1.79	(1.35,2.24)	1.76	(1.36, 2.19)
δ_1	0.98	(0.64, 1.33)	0.76	(0.30, 1.23)	0.84	(0.47, 1.23)
β	5.09	(2.71, 7.48)	0.86	(0.65, 1.07)	3.61	(2.25, 4.96)

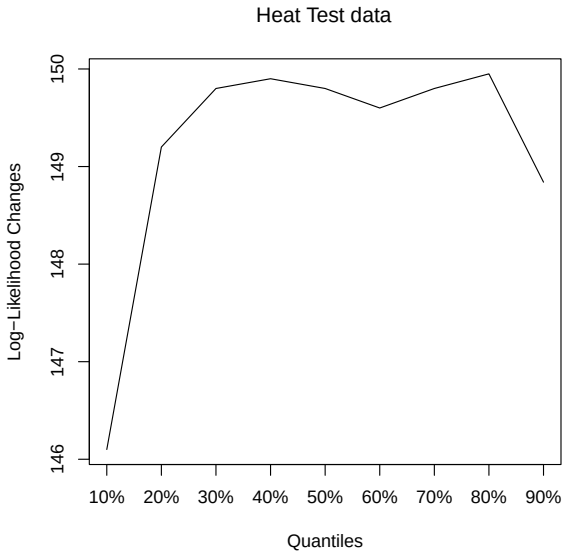


Figure 6. Log-likelihood change for UMEL quantile regression fitted to PITCH data considering $\tau = 0.1, 0.2, \ldots, 0.9$.

Reading Skills Data

In this application, we consider analyzing the reading accuracy data for nondyslexic and dyslexic Australian children [28]. The response variable of interest is accuracy scores on a test of reading taken by 44 children, which is predicted by nonverbal intelligent quotient (iq, converted to z scores). Therefore, the regressor and the response variables are

x_1: iq, converted to z scores
y: reading accuracy

The quantile regression model is expressed as

$$\text{logit}(\mu_i) = \delta_0 + \delta_1 x_{i1}, \qquad i = 1, 2, \ldots, 44. \tag{21}$$

Table (5) shows the point estimation and 95% confidence interval for the estimated parameters. The results of fitted UMEL indicate that all parameters are statistically significant in prediction quantile response.The coefficient of iq is 0.98 with Odds-Ratio $\exp(0.98) = 2.66$. This implies that 1-standard deviation increase in IQ multiplies the odds of median reading accuracy by 2.66. Table (6) illustrates the AIC, BIC, and HQIC criteria of the fitted models. The UMEL regression model has a significantly lower value in all three criteria than the alternatives in modeling Skills data which denotes a better fit than the others.

Table 6. Likelihood based criteria for fitted quantile regression models to Skills data .

Criteria	UMEL	UW	KU
AIC	-69.80	-55.80	-64.60
BIC	-53.09	-39.09	-47.89
HQIC	-67.81	-53.81	-62.61

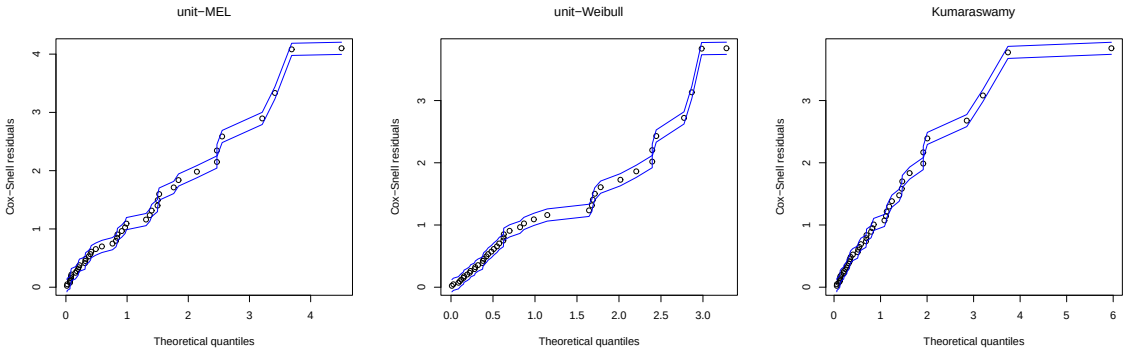


Figure 7. Cox-Snell residual plot for fitted models on Skills data

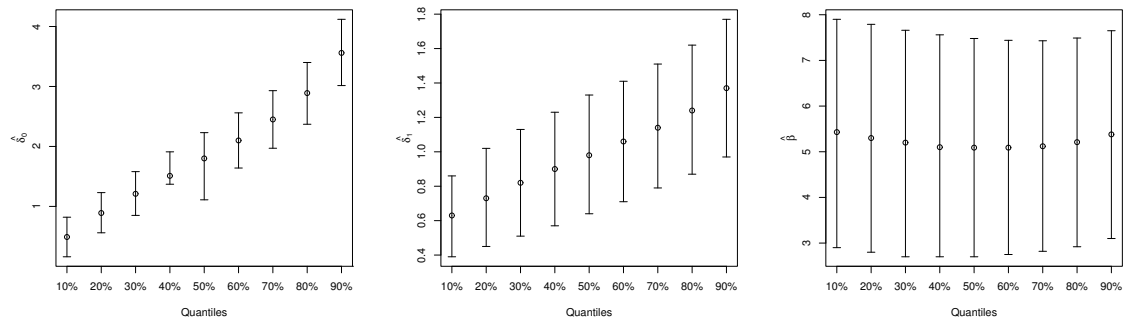


Figure 8. Estimated parameters of UMEL quantile regression with related 95% confidence considering $\tau = 0.1, 0.2, \ldots, 0.9$ for Skills data

For investigating model adequacy in fitting Skills data, the Cox-Snell residual plots of the fitted models are illustrated in Figure (7). According to this plot, the Cox-Snell residual of the fitted UMEL to the Skills data is closer to a straight line than the others. Figure (8) displays the impact of the various τ 's on β and regression coefficients parameters. In Figure (9) the effect of different τ values on log-likelihood change of fitted UMEL on Skills data is investigated. As the plot shows, the Log-likelihood increases by increasing τ values.

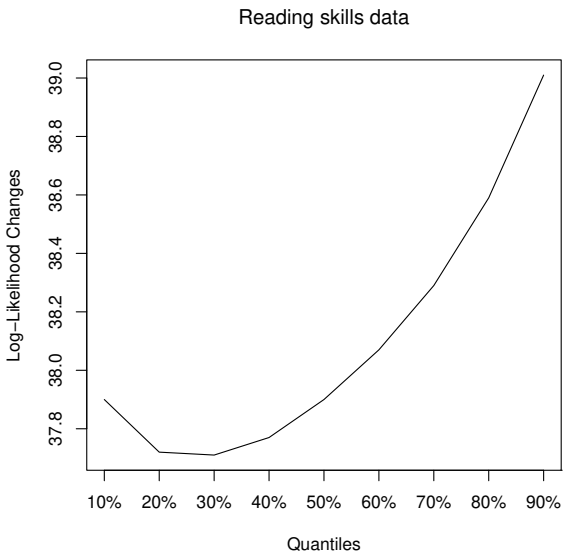


Figure 9. Log-likelihood change for UMEL quantile regression fitted to Skills data considering $\tau = 0.1, 0.2, \ldots, 0.9$.

As shown here, the real data applications illustrate the strengths of the UMEL model in capturing conditional quantiles of bounded response variables. Beyond illustrating superiority over the alternatives, the model fitting and interpretation depends on the complexity of covariate relationships and data heterogeneity. In some cases, small sample sizes or highly skewed covariates may require careful model checking and consideration of potential estimation variability. Moreover, the interpretation coefficients in the quantile regression involves understanding their influence on the specific conditional quantile rather than the mean, which may be less intuitive.

7. Conclusion and remarks

In this paper, we introduced the unit-maximum entropy lifetime distribution which the random variable is defined on the unit interval $(0, 1)$. This model has a closed form in its c.d.f and quantiles which make it an appropriate model for fitting quantile regression. The appropriateness of the proposed model was evaluated through two simulation studies. The performance of the quantile unit-maximum entropy lifetime regression was compared with the Kumaraswamy and unit-Weibull quantile regression through two real datasets. For these real datasets, the UMEL quantile regression has better fitting than the Kumaraswamy and unit-Weibull quantile regression models based on the three information criteria and Cox-Snell residual plot. Generally, this study indicated that the proposed model can be used as an appropriate model to the alternative models in fitting bounded data and application in quantile regression.

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