STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **16**, August 2026, pp 1618–1639.
Published online in International Academic Press (www.IAPress.org)

Estimation in generalized Lindley distribution with randomly censored data

Shahdie Marganpoor, and Vahid Ranjbar*

Department of Statistics, Golsetan University, Iran

Abstract In this article, we obtained the point and interval estimations for a generalized Lindley distribution (GLD) based on randomly censored data. The maximum likelihood (ML) and Bayes estimation method are used to estimate the unknown parameters of the GLD. Furthermore, approximate confidence intervals (ACIs) for the unknown parameters were constructed. Markov chain Monte Carlo (MCMC) method applied to find the Bayes estimation. Also, highest posterior density (HPD) credible intervals (CRIs) were obtained for the parameters. Gibbs within Metropolis-Hasting samplers used to generate samples from the posterior density functions. A real data set is discussed to illustrate the proposed methods. we performed a Monte Carlo simulation study.

Keywords Generalized Lindley distribution, Random censoring; Maximum likelihood estimation, Bayes estimation, MCMC method, HPD credible interval.

AMS 2010 subject classifications 62N01, 62N05, 62F10, 62F15.

DOI: 10.19139/soic-2310-5070-1183

1. Introduction

There are many distributions for modeling lifetime data. Among the known parametric models, the most popular are the gamma, lognormal, exponentiated exponential and the Weibull distributions. However, the above mentioned four distributions suffer from a number of drawbacks. The survival functions of the gamma and lognormal distributions cannot be expressed in closed forms and one needs numerical integration. moreover, none of them exhibit bathtub shapes for their hazard rate functions. The four above mentioned distributions exhibit only monotonically increasing, monotonically decreasing or constant hazard rates. This is a major weakness because most real-life systems exhibit bath tub shapes for their hazard rate functions and at least three of the four distributions exhibit constant hazard rates. This is a very unrealistic feature because there are hardly any real-life systems that have constant hazard rates. Nadarajah et.al ([34]) introduce a generalized Lindley (GL) distribution which overcomes these mentioned drawbacks. It is most conveniently specified in terms of the cumulative distribution function:

$$F(x) = \left[1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha}, \quad x > 0 \tag{1}$$

$$f(x) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)e^{-\lambda x}\left[1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha-1}, \quad x > 0 \tag{2}$$

*Correspondence to: Vahid Ranjbar (Email: v.ranjbar@gu.ac.ir, vahidranjbar@gmail.com). Department of Statistics, Golestan University. Gorgan, Iran.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

and

$$h(x) = \frac{\frac{\alpha\lambda^2}{1+\lambda}(1+x)e^{-\lambda x}\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha-1}}{1-\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha}}, \quad x>0 \tag{3}$$

where $\lambda > 0$ and $\alpha > 0$ are the shape and scale parameters, respectively. The plot of the hazard rate function for various values of the shape parameter (λ) and scale parameter (α) is shown in Figure 1.

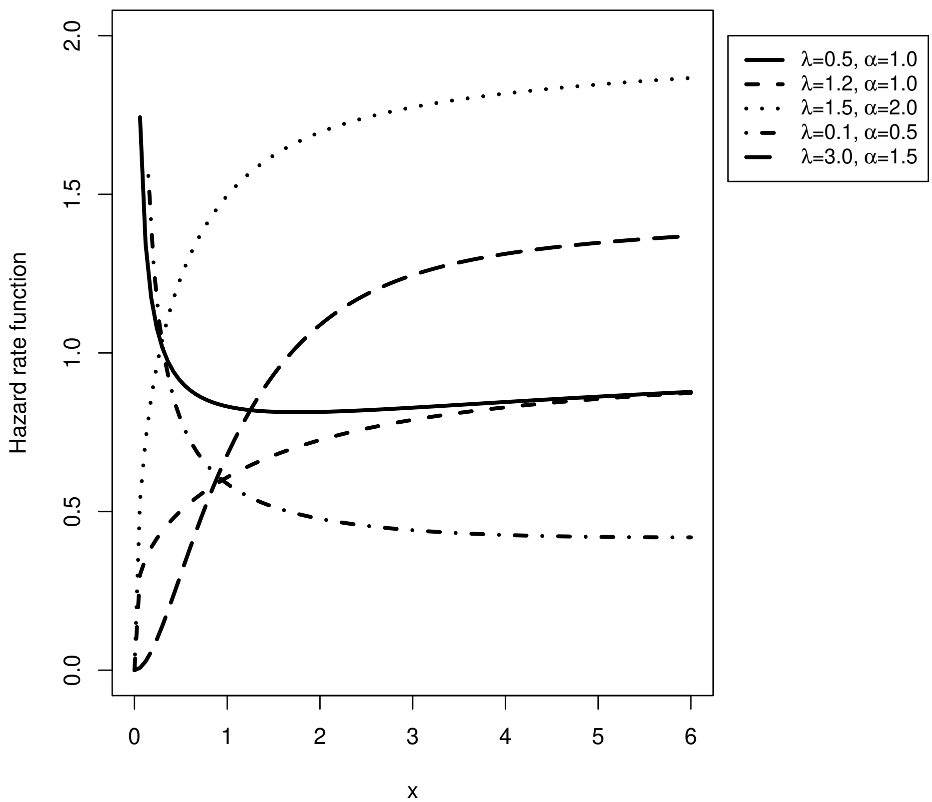

Figure 1. Hazard functions of the GL model for selected α and λ.

For $\alpha = 1$, Eq. (1) reduces to the Lindley distribution (Lindley [32]). GL density function has the attractive feature of allowing for monotonically decreasing, monotonically increasing and bath tub shaped hazard rate functions while not allowing for constant hazard rate functions (see Nadarajah et.al [34]).

Another motivation for the GL distribution can be described as follows. Consider the one parameter Lindley distribution (Lindley [32]). This distribution is becoming increasing popular for modeling lifetime data. For example, Ghitany et al. [19] developed different distributional properties, reliability characteristics and some inferential procedures for the Lindley distribution in complete sample case. Krishna and Kumar [28] discussed reliability estimation in Lindley distribution with progressively type II right censored sample. Mazucheli and Achcar [33] described Lindley distribution applied to competing risk lifetime data. Ali et al. [2] studied the effect of the loss function on Bayes estimate, posterior risk and hazard function for Lindley distribution. Kumar et al. [26] discussed estimation of $P(Y < X)$ in Lindley distribution using progressively first failure censoring. Dube et al. [11] described maximum likelihood and Bayes estimates of parameters and reliability characteristics of Lindley distribution under progressive first failure censoring. Garg et. al [17] Analyzed the estimation of parameters and reliability characteristics in Lindley distribution using randomly censored data. Now, suppose $X_1, X_2, ..., X_\alpha$ are independent random variables with Lindley distribution and represent the failure times of the components of a series system, assumed to be independent. Then the probability that the system will fail before time x is given by

$$P(\max(X_1, ..., X_\alpha) \leq x) = \left[1 - (1 + \frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^\alpha$$

So, Eq. (1) gives the distribution of the failure of a series system with independent components.

In life testing experiments, the data are frequently censored. Censoring arises in a life testing experiment, when exact lifetimes are known only for a portion of test items and remainder of the lifetimes are known only to exceed certain values under a life test. There are several types of censoring schemes which are used in life testing experiments. In literature, the two most popular censoring schemes are conventional Type I and Type II . These censoring schemes do not allow units to be removed from the test at points other than the final termination point. Such intermittent removals are studied in progressive censoring Balakrishnan and Aggarwala [3], Kumar et al. [27] and Chaturvedi et al. [5]. Another type of censoring called random censoring occurs when the item under study is lost or removed from the experiment before its failure. Random censoring is an important censoring in which the time of censoring is not fixed but taken as random.

The random censoring was introduced in literature by Gilbert [22]. Many authors studied this type of censoring such as Breslow and Crowley [4], Koziol and Green [25] and Csorgo and Horvath [7]. Kim [24] implemented chi-square goodness of fit tests for randomly censored data. Ghitany [20] analyzed Rayleigh survival model and its application to randomly censored data. Ghitany and Al-Awadhi [21] studied maximum likelihood estimation of Burr-XII distribution parameters under random censoring. Friesl and Hurt [15] considered exponential distribution under random censorship. Saleem and Aslam [36] discussed the Bayesian analysis of the Rayleigh survival time assuming the random censor time. Saleem and Raza [37] studied the Bayesian analysis of the exponential survival time assuming the exponential censor time. Danish and Aslam [8, 9] discussed the Bayesian estimation for generalized exponential and Weibull distributions under randomly censored, respectively. Krishna et al. [31] studied the estimation in Maxwell distribution with randomly censored data and Garg et al. [16] discussed the generalized inverted exponential distribution with randomly censored data. Recently, Krishna and Goel [30] dealt with maximum likelihood and Bayes estimation in randomly censored geometric distribution. Danish et al. [10] dealt with Bayesian inference for the Burr-XII distribution under randomly censored data and Krishna and Goel [31] studied the classical and Bayesian inference in two parameters exponential distribution with randomly censored data. Garg et al. [17] Estimate the parameters and reliability characteristics in Lindley distribution using randomly censored data. EL-Sagheer et al. [14] dealt with bayesian inference for the randomly censored Burr XII distribution. Kumar and Kumar [26] studied the estimation in inverse Weibull distribution based on randomly censored data.

Mathematically, the random censoring can be described as follows: suppose n identical items are put on test with their lifetimes as $X_1, X_2, ..., X_n$ which are independent and identically distributed (iid) random variables with probability density function (pdf) $f_X(x)$ and cumulative distribution function (cdf) $F_X(x)$. Also, let $T_1, T_2, ..., T_n$ be the random censoring times of these items. Suppose pdf and cdf of T_i s are $f_T(t)$ and $F_T(t)$, respectively. Further, let X_is and T_is be mutually independent. Note that, between X_is and T_is, only one will be observed. The actual observed time be $Y_i = \min(X_i, T_i); i = 1, 2, ..., n$. Also, define the indicator variable D_i as $D_i = (1$ if $X_i \leq T_i$ otherwise $0)$. Note that D_i is a random variable with Bernoulli probability mass function given by $P[D_i = j] = p^j(1-p)^{1-j}; j = 0, 1$ and $p = P[X_i \leq T_i]$. Since X_is and T_is are independent, so will be Y_i and D_i, $\forall i = 1, 2, ..., n$. Now, it is simple to show that the joint pdf of Y and D is given by

$$f_{Y,D}(y, d) = \left[f_X(y)\bar{F}_T(y)\right]^d \left[f_T(y)\bar{F}_X(y)\right]^{1-d} \quad ; y > 0, d = 0, 1. \tag{4}$$

where, $\bar{F}_T(y) = 1 - F_T(y)$ and $\bar{F}_X(y) = 1 - F_X(y)$. Also, the probability of failure of an item before its censoring is given by

$$\begin{aligned} p &= P[An\ item\ fails] = P(D=1) = P(X \leq T) \\ &= \int_0^\infty F_X(t)dF_T(t) = \int_0^\infty F_X(t)f_T(t)dt \end{aligned} \quad (5)$$

Now, let the failure time X and censoring time T follow $GL(\theta, \alpha)$ and $GL(\lambda, \alpha)$, respectively. Then using equations (4), (1) and (2) we get the joint pdf of randomly censored variables (Y,D) as

$$\begin{aligned} f(y,d,\alpha,\theta,\lambda) = \alpha(\frac{\theta^2}{1+\theta})^d(\frac{\lambda^2}{1+\lambda})^{1-d}(1+y)e^{-y[d\theta+(1-d)\lambda]} \\ \times \left[1-(1+\frac{\theta y}{1+\theta})e^{-\theta y}\right]^{d(\alpha-1)} \left[1-(1+\frac{\lambda y}{1+\lambda})e^{-\lambda y}\right]^{(1-d)(\alpha-1)} \\ \times \left[1-\left(1-(1+\frac{\lambda y}{1+\lambda})e^{-\lambda y}\right)^\alpha\right]^d \left[1-\left(1-(1+\frac{\theta y}{1+\theta})e^{-\theta y}\right)^\alpha\right]^{1-d} \end{aligned} \quad (6)$$

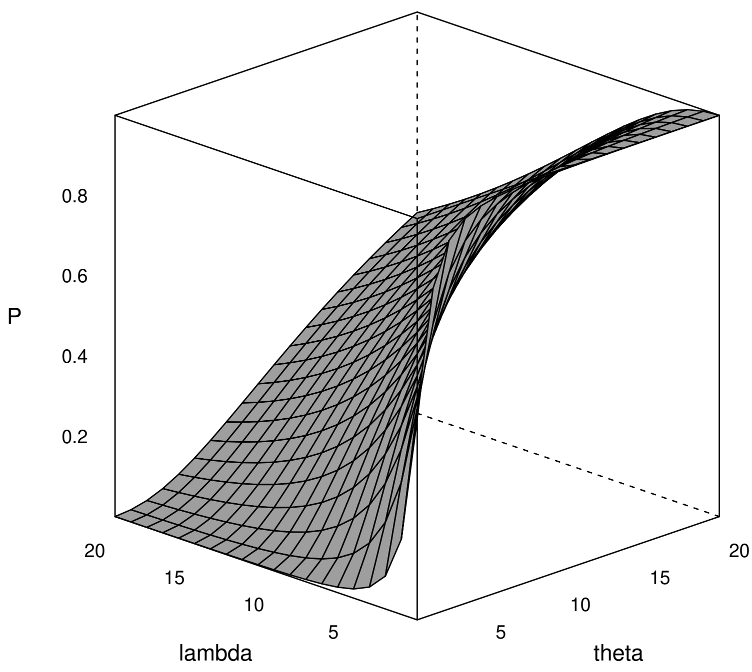

Figure 2. Probability of failure of the GL model for selected θ and λ when $\alpha = 2$.

The plot of the probability of failure for various values of the λ and θ when $\alpha = 2$ is shown in Figure 2. As can be seen, increasing the value of θ as well as decreasing the value of λ will increase the value of the probability of failure.

In this article, our main objective is to develop the classical and Bayesian estimation procedures in GL distribution based on the randomly censored data. Rest of the paper is organized as follows: In Section 2, we derive maximum likelihood estimators of the parameters as well as asymptotic confidence intervals and coverage probabilities of the unknown parameters are constructed based on expected Fisher information matrix. Section 3 deals with the bootstrap p and bootstrap t confidence intervals of the parameters. Expected test time of the experiment based on randomly censored data from GL distribution is discussed in section 4. In Section 5, Bayes estimators of the parameters under squared error loss function (SELF) with gamma informative and non-informative priors are obtained using Lindley's approximation method and Markov chain Monte Carlo (MCMC) techniques. Highest posterior density (HPD) credible intervals for the parameters based on MCMC techniques are developed. Section 6 deals with a simulation study to compare the performance of the developed estimators. In Section 7, findings are illustrated by a randomly censored real data set. Finally, conclusions of this article are given in section 8.

2. Maximum likelihood estimation

The likelihood function for GL based on randomly censored sample data $(y, d) = (y_1, d_1), ..., (y_n, d_n)$ of size n as shown in Section 1, from Equation (6), is given by

$$\begin{aligned} L(y,d,\alpha,\theta,\lambda) &= \alpha^n(\frac{\theta^2}{1+\theta})^m(\frac{\lambda^2}{1+\lambda})^{(n-m)}\prod_{i=1}^{n}(1+y_i)e^{-[\sum_{i=1}^{n}\theta d_i y_i+\lambda(1-d_i)y_i]} \\ &\times\prod_{i=1}^{n}\left[1-(1+\frac{\theta y_i}{1+\theta})e^{-\theta y_i}\right]^{d_i(\alpha-1)}\left[1-(1+\frac{\lambda y_i}{1+\lambda})e^{-\lambda y_i}\right]^{(1-d_i)(\alpha-1)} \\ &\times\prod_{i=1}^{n}\left[1-\left(1-(1+\frac{\lambda y_i}{1+\lambda})e^{-\lambda y_i}\right)^{\alpha}\right]^{d_i}\left[1-\left(1-(1+\frac{\theta y_i}{1+\theta})e^{-\theta y_i}\right)^{\alpha}\right]^{1-d_i} \end{aligned} \tag{7}$$

where, $m = \sum_{i=1}^{n} d_i$. The log-likelihood function for the GL, corresponding to Equation (7) is

$$\begin{aligned} &l(y,d,\alpha,\theta,\lambda) = n\ln\alpha + m(2\ln\theta - \ln(1+\theta)) + (n-m)(2\ln\lambda - \ln(1+\lambda)) \\ &+\sum_{i=1}^{n}\ln(1+y_i) - (\sum_{i=1}^{n}\theta d_i y_i + \lambda(1-d_i)y_i) + (\alpha-1)\sum_{i=1}^{n}d_i\ln\left(1-(1+\frac{\theta y_i}{1+\theta})e^{-\theta y_i}\right) \\ &+(\alpha-1)\sum_{i=1}^{n}(1-d_i)\ln\left(1-(1+\frac{\lambda y_i}{1+\lambda})e^{-\lambda y_i}\right) + \sum_{i=1}^{n}(1-d_i)\ln\left(1-(1-(1+\frac{\theta y_i}{1+\theta})e^{-\theta y_i})^{\alpha}\right) \\ &+\sum_{i=1}^{n}(d_i)\ln\left(1-(1-(1+\frac{\lambda y_i}{1+\lambda})e^{-\lambda y_i})^{\alpha}\right) \end{aligned} \tag{8}$$

Taking the first derivative of Equation (8) with respect to α, θ and λ and setting each of these derivatives equal to zero, we obtain the likelihood equations for the parameters α, θ and λ as follows:

$$\frac{n}{\alpha} + \sum_{i=1}^{n} d_i \ln F_L(y_i,\theta)(1 - F_L^{-\alpha}(y_i,\theta)) + \sum_{i=1}^{n}(1-d_i)\ln F_L(y_i,\theta)(1-F_L^{-\alpha}(y_i,\theta)) = 0, \tag{9}$$

$$n(\frac{2}{\lambda} - \frac{1}{1+\lambda}) - \sum_{i=1}^{n}\left(y_i + F_L'(y_i,\lambda)\left[\frac{\alpha-1}{F_L(y_i,\lambda)} + \frac{\alpha F_L^{\alpha-1}(y_i,\lambda)}{1-F_L^{\alpha}(y_i,\lambda)}\right]\right) = 0, \tag{10}$$

$$n(\frac{2}{\theta}-\frac{1}{1+\theta})-\sum_{i=1}^{n}\left(y_i+F_L'(y_i,\theta)\left[\frac{\alpha-1}{F_L(y_i,\theta)}+\frac{\alpha F_L^{\alpha-1}(y_i,\theta)}{1-F_L^{\alpha}(y_i,\theta)}\right]\right)=0, \tag{11}$$

where $F_L(y,t)$ is distribution function of Lindley random variable with parameter t, and

$$F_L'(y,t)=\frac{\partial F_L(y,t)}{\partial t}.$$

The maximum likelihood estimators of α,θ and λ can be found by solving the system of Equations (9), (10) and (11), but clearly that is impossible to solve these equations analytically because it is very difficult to get closed forms for each parameter, these equations are impossible to solve analytically and a suitable numerical technique, such as Newton-Raphson iteration method, to obtain the estimates must be used.

2.1. Fisher information matrix

In this sub-section, we derive asymptotic confidence intervals of the parameters based on expected Fisher information matrix. According to Zheng and Gastwirth [39] the expected Fisher information in randomly censored data can be expressed in terms of hazard rate functions. The Fisher information of parameters $\xi=(\alpha,\theta,\lambda)$ contained in randomly censored sample of size n from GL distribution is given by

$$I(\xi)=n\times\begin{pmatrix} I_{1,1}(\xi) & I_{1,2}(\xi) & I_{1,3}(\xi)\\ & I_{2,2}(\xi) & I_{2,3}(\xi)\\ & & I_{3,3}(\xi)\end{pmatrix}$$

where

$$I_{1,1}(\xi)=\int_0^{\infty}\left(\frac{\partial}{\partial\alpha}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx+\int_0^{\infty}\left(\frac{\partial}{\partial\alpha}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx$$

$$I_{2,2}(\xi)=\int_0^{\infty}\left(\frac{\partial}{\partial\theta}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx+\int_0^{\infty}\left(\frac{\partial}{\partial\theta}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx$$

$$I_{3,3}(\xi)=\int_0^{\infty}\left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx+\int_0^{\infty}\left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx$$

$$\begin{aligned} I_{1,2}(\xi) &= \int_0^{\infty}\left(\frac{\partial}{\partial\alpha}\ln h_X(x)\right)\left(\frac{\partial}{\partial\theta}\ln h_X(x)\right) f_X(x)\bar{F}_T(x)dx\\ &\quad+\int_0^{\infty}\left(\frac{\partial}{\partial\alpha}\ln h_T(x)\right)\left(\frac{\partial}{\partial\theta}\ln h_T(x)\right) f_T(x)\bar{F}_X(x)dx\end{aligned}$$

$$\begin{aligned} I_{1,3}(\xi) &= \int_0^{\infty}\left(\frac{\partial}{\partial\alpha}\ln h_X(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right) f_X(x)\bar{F}_T(x)dx\\ &\quad+\int_0^{\infty}\left(\frac{\partial}{\partial\alpha}\ln h_T(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right) f_T(x)\bar{F}_X(x)dx\end{aligned}$$

$$\begin{aligned} I_{2,3}(\xi) &= \int_0^{\infty}\left(\frac{\partial}{\partial\theta}\ln h_X(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right) f_X(x)\bar{F}_T(x)dx\\ &\quad+\int_0^{\infty}\left(\frac{\partial}{\partial\theta}\ln h_T(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right) f_T(x)\bar{F}_X(x)dx\end{aligned}$$

Here, h_X and h_T are the failure rate functions of $GL(\alpha, \lambda)$ and $GL(\alpha, \theta)$, respectively. The elements of the expected Fisher information matrix $I(\xi)$ need to be computed numerically. Thus, the variance- covariance matrix of MLEs $\hat{\xi} = (\hat{\alpha}, \hat{\theta}, \hat{\lambda})$ is the inverse of Fisher information matrix and it is given by

$$I^{-1}(\xi) = \frac{1}{n}\begin{pmatrix} I_{1,1}^{-1}(\xi) & 0 & 0 \\ 0 & I_{2,2}^{-1}(\xi) & 0 \\ 0 & 0 & I_{3,3}^{-1}(\xi) \end{pmatrix} = \begin{pmatrix} Var(\hat{\alpha}) & 0 & 0 \\ 0 & Var(\hat{\theta}) & 0 \\ 0 & 0 & Var(\hat{\lambda}) \end{pmatrix}$$

The asymptotic normality of the MLEs can be used to compute the approximate confidence intervals for parameters α, θ and λ. Therefore, $(1-\eta)100\%$ confidence intervals (CIs) for parameters α, θ and λ become

$$\left(\hat{\alpha} \mp Z_{1-\eta/2}\sqrt{(\hat{\alpha})}\right), \quad \left(\hat{\theta} \mp Z_{1-\eta/2}\sqrt{(\hat{\theta})}\right), \quad \left(\hat{\lambda} \mp Z_{1-\eta/2}\sqrt{(\hat{\lambda})}\right),$$

respectively. Here, $Z_{1-\eta/2}$ is the lower $(1-\eta/2)$th percentile of the standard normal distribution. Also. the coverage probabilities for the parameters α, θ and λ are, respectively, given by

$$CP_\alpha = P\left(\left|\frac{\hat{\alpha}-\alpha}{\sqrt{(\hat{\alpha})}}\right| \leq Z_{1-\eta/2}\right), CP_\theta = P\left(\left|\frac{\hat{\theta}-\theta}{\sqrt{(\hat{\theta})}}\right| \leq Z_{1-\eta/2}\right),$$
$$CP_\lambda = P\left(\left|\frac{\hat{\lambda}-\alpha}{\sqrt{(\hat{\lambda})}}\right| \leq Z_{1-\eta/2}\right).$$

3. Bootstrap confidence intervals

Here, we propose the use of two bootstrap confidence intervals for the unknown parameters. The two bootstrap methods that are widely used in practice are (i) the percentile bootstrap (boot-p) method proposed by Efron [13], and (ii) the bootstrap-t (boot-t) method proposed by Hall [23]. We use the following steps for two bootstrap confidence intervals for α, θ and λ as suggested by Efron and Tibshirani [12].

3.1. Percentile method

1. Generate a randomly censored sample $(y, d) = (y_1, d_1), (y_2, d_2), ..., (y_n, d_n)$ of size n from the model given in (6) and compute the MLEs $\tilde{\alpha}, \tilde{\theta}$ and $\tilde{\lambda}$, of α, θ and λ, respectively.
2. Generate a bootstrap sample $(y^*, d^*) = (y_1^*, d_1^*), (y_2^*, d_2^*), ..., (y_n^*, d_n^*)$, using $\tilde{\alpha}, \tilde{\theta}$ and $\tilde{\lambda}$ as the true values of the parameters. Compute the bootstrap MLEs of $\tilde{\alpha}^*, \tilde{\theta}^*$ and $\tilde{\lambda}^*$ using the bootstrap sample.
3. Repeat step 2, B times to obtain a set of bootstrap MLEs $(\tilde{\alpha}_i^*, \tilde{\theta}_i^*, \tilde{\lambda}_i^*; i = 1, 2, ...B)$.
4. Let $(\tilde{\alpha}_{(1)}^* \leq \tilde{\alpha}_{(2)}^* \leq ... \leq \tilde{\alpha}_{(B)}^*)$, $(\tilde{\theta}_{(1)}^* \leq \tilde{\theta}_{(2)}^* \leq ... \leq \tilde{\theta}_{(B)}^*)$ and $(\tilde{\lambda}_{(1)}^* \leq \tilde{\lambda}_{(2)}^* \leq ... \leq \tilde{\lambda}_{(B)}^*)$ denote the ordered values of $\tilde{\alpha}_i^*, \tilde{\theta}_i^*$, and $\tilde{\lambda}_i^*; i = 1, 2, ...B$, respectively.

The approximate $100(1-\eta)\%$ p-boot confidence interval for α, θ and λ are given by

$$CI_{p-boot}(\alpha) = \left(\tilde{\alpha}_{(a_1)}^*, \tilde{\alpha}_{(a_2)}^*\right), CI_{p-boot}(\theta) = \left(\tilde{\theta}_{(a_1)}^*, \tilde{\theta}_{(a_2)}^*\right), CI_{p-boot}(\lambda) = \left(\tilde{\lambda}_{(a_1)}^*, \tilde{\lambda}_{(a_2)}^*\right)$$

where $a_1 = [B(\eta/2)]$ and $a_2 = [B(1-\eta/2)]$.

3.2. Bootstrap-t method

Steps 1 and 2 are the same as in p-boot method

3. Compute the t-bootstrap statistics $T^*_\alpha = \frac{\tilde{\alpha}^* - \tilde{\alpha}}{\sqrt{(\tilde{\alpha}^*)}}$, $T^*_\theta = \frac{\tilde{\theta}^* - \tilde{\theta}}{\sqrt{(\tilde{\theta}^*)}}$ and $T^*_\lambda = \frac{\tilde{\lambda}^* - \tilde{\lambda}}{\sqrt{(\tilde{\lambda}^*)}}$, for $\tilde{\alpha}^*$, $\tilde{\theta}^*$ and $\tilde{\lambda}^*$, respectively.
4. Repeat steps 2-3, B times to obtain a set of bootstrap statistics $(T^*_\alpha, T^*_\theta, T^*_\lambda; i = 1, 2, ..., B)$.
5. Let $(T^*_{\alpha_{(1)}} \leq T^*_{\alpha_{(2)}} \leq ... \leq T^*_{\alpha_{(B)}})$, $(T^*_{\theta_{(1)}} \leq T^*_{\theta_{(2)}} \leq ... \leq T^*_{\theta_{(B)}})$ and $(T^*_{\lambda_{(1)}} \leq T^*_{\lambda_{(2)}} \leq ... \leq T^*_{\lambda_{(B)}})$ denote the ordered values of $T^*_{\alpha_{(i)}}$, $T^*_{\theta_{(i)}}$ and $T^*_{\lambda_{(i)}}$, $i = 1, 2, ..., B$, respectively. The approximate $100(1-\eta)\%$ t-boot confidence intervals for α, θ and λ are given by

$$CI_{t-boot}(\alpha) = \left(\tilde{\alpha} \mp T^*_{\alpha_{([B(1-\eta/2)])}} \sqrt{(\tilde{\alpha})}\right), CI_{t-boot}(\theta) = \left(\tilde{\theta} \mp T^*_{\theta_{([B(1-\eta/2)])}} \sqrt{(\tilde{\theta})}\right)$$
$$CI_{t-boot}(\lambda) = \left(\tilde{\lambda} \mp T^*_{\lambda_{([B(1-\eta/2)])}} \sqrt{(\tilde{\lambda})}\right).$$

4. Expected time on test

In this section, we study the expected time on test (ETT) of a randomly censored life testing experiment. In real life applications, ETT is useful to have an idea about the number of items to be put on test, the expected duration and cost of the life testing experiment. Let the failure time be $X \sim GL(\alpha, \lambda)$ and the censoring time be $T \sim GL(\alpha, \theta)$. Again, let $Z = \max(Y_1, Y_2, ..., Y_n)$, then the cdf of Z is given by

$$F_Z(z) = P(Z \leq z) = P(\max(Y_1, Y_2, ..., Y_n) \leq z) = (P(Y_i \leq z))^n; \quad z > 0$$

since, $Y_i, i = 1, 2, ..., n$ are iid. Note that, for $i = 1, 2, ..., n$,

$$\begin{aligned} F_{Y_i}(z) &= P(Y_i \leq z) = P(\min(X_i, T_i) \leq z) = 1 - P(\min(X_i, T_i) \geq z) \\ &= 1 - P(X_i \geq z)P(T_i \geq z) = 1 - \bar{F}_X(z)\bar{F}_T(z), \quad z > 0 \end{aligned}$$

Therefor

$$E(Z) = \int_0^\infty 1 - P(Z > z)dz = \int_0^\infty 1 - (1 - \bar{F}_X(z)\bar{F}_T(z))^n dz$$

Now, if the failure time X follows $GL(\alpha, \lambda)$ and censoring time T follows $GL(\alpha, \theta)$, the ETT for randomly censored experiment is given by

$$E(Z) = \int_0^\infty 1 - \left[1 - \left(1 - (1 + \frac{\lambda z}{1+\lambda})e^{-\lambda z}\right)^\alpha \left(1 - (1 + \frac{\theta z}{1+\theta})e^{-\theta z}\right)^\alpha\right]^n dz \tag{12}$$

ETT obtained in Equation (12) can be computed numerically for the given values of the parameters and the sample size n. Also, the observed time on the test (OBTT) is given by $OBTT = \max(y_1, y_2, ..., y_n)$. We compute, the average absolute bias (AB) and mean squared error (MSE) for OBTT based on 1,000 randomly censored simulated samples from the model in equation (6). The values of ETT, AB and MSE for OBTT under randomly censored GL distribution for different values of the parameters and sample size n are presented in Table 1. This table shows that OBTT estimates ETT precisely for various values of the parameters α, θ and λ. The MLE of ETT can be obtained using invariance property of MLEs.

Table 1. Expected time on test (ETT) and the observed time on test (OBTT).

		$\theta = 0.5$			$\theta = 1$			$\theta = 2$		
		ETT	OBTT		ETT	OBTT		ETT	OBTT	
λ	n		AB	MSE		AB	MSE		AB	MSE
0.5	20	7.3247	7.3324	2.4997	4.6137	4.6680	1.1792	2.4875	2.5075	0.3882
	60	8.2074	8.2635	2.2085	5.2035	5.1912	1.0373	2.8376	2.8579	0.4146
	60	8.7133	8.7132	2.3806	5.5420	5.5261	1.1963	3.0395	3.0516	0.3652
	80	9.0683	9.1501	2.4091	5.7795	5.7774	1.0059	3.1816	3.1798	0.3716
1	20	4.6137	4.5819	1.1981	3.4257	3.4802	0.6278	2.1384	2.1710	0.3142
	40	5.2035	5.1600	1.0539	3.8620	3.8806	0.6719	2.4273	2.4305	0.2570
	60	5.5420	5.5073	1.1523	4.1125	4.1014	0.6028	2.5936	2.5927	0.2430
	80	5.7795	5.7455	1.0708	4.2884	4.2515	0.5769	2.7105	2.7141	0.2654
2	20	2.4875	2.4945	0.4004	2.1384	2.1496	0.2591	1.5739	1.6122	0.1613
	40	2.8376	2.8452	0.3968	2.4273	2.4655	0.2739	1.7865	1.8207	0.1445
	60	3.0395	3.0639	0.3975	2.5936	2.5864	0.2484	1.9090	1.9460	0.1462
	80	3.1816	3.2178	0.3615	2.7105	2.7335	0.2570	1.9951	2.0152	0.1431

5. Bayesian estimation

In this section, we derive the Bayes estimates of the unknown parameters of the model in equation (6) using randomly censored data. There are various ways to choose the priors. Here, we consider piecewise independent gamma priors for the parameters α, θ and λ as

$$g_1(\alpha) = \frac{b_1^{a_1} \alpha^{a_1 - 1} e^{-b_1 \alpha}}{\Gamma(a_1)}, \quad \alpha > 0, a_1, b_1 > 0,$$

$$g_2(\theta) = \frac{b_2^{a_2} \theta^{a_2 - 1} e^{-b_2 \theta}}{\Gamma(a_2)}, \quad \theta > 0, a_2, b_2 > 0,$$

$$g_3(\lambda) = \frac{b_3^{a_3} \alpha^{a_3 - 1} e^{-b_3 \lambda}}{\Gamma(a_3)}, \quad \lambda > 0, a_3, b_3 > 0,$$

respectively. Thus, the joint prior distribution of α, θ and λ can be written as

$$g(\alpha, \theta, \lambda) \propto \alpha^{a_1 - 1} e^{-b_1 \alpha} \times \theta^{a_2 - 1} e^{-b_2 \theta} \times \alpha^{a_3 - 1} e^{-b_3 \lambda}. \tag{13}$$

The joint posterior distribution of α, θ and λ is given by

$$\begin{aligned} \pi(\alpha, \theta, \lambda | \vec{y}, \vec{d}) &= \frac{L(\vec{y}, \vec{d} | \alpha, \theta, \lambda) g(\alpha, \theta, \lambda)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\vec{y}, \vec{d} | \alpha, \theta, \lambda) g(\alpha, \theta, \lambda) d\alpha d\theta d\lambda} \\ &= \alpha^{n + a_1 - 1} e^{-\alpha(b_1 + T_1)} \frac{\theta^{2*m + a_2 - 1}}{(1+\theta)^m} e^{-\theta(b_2 + T2)} \frac{\lambda^{2*(n-m) + a_3 - 1}}{(1+\lambda)^{n-m}} e^{-\theta(b_3 + T3)} \\ &\quad \times \prod_{i=1}^n \left(\frac{1 - F(y_i, \lambda, \alpha)}{F_L(y_i, \theta)} \right)^{d_i} \times \prod_{i=1}^n \left(\frac{1 - F(y_i, \theta, \alpha)}{F_L(y_i, \lambda)} \right)^{1 - d_i}, \end{aligned} \tag{14}$$

where F_L is distribution function of Lindley random variable and

$$T_1 = \sum_{i=1}^n d_i \ln F_L(y_i, \theta) + (1 - d_i) \ln F_L(y_i, \lambda),$$

$$T_2 = \sum_{i=1}^n d_i \ln F_L(y_i, \theta), \quad T_3 = \sum_{i=1}^n (1 - d_i) \ln F_L(y_i, \lambda).$$

Therefore, the Bayes estimator of any function of α, θ and λ, say, $\phi(\alpha, \theta, \lambda)$ under SELF is the posterior expectation of $\phi(\alpha, \theta, \lambda)$ and is given by

$$E[\phi(\alpha,\theta,\lambda)|\vec{y},\vec{d}] = \frac{\int_0^\infty \int_0^\infty \int_0^\infty \phi(\alpha,\theta,\lambda)L(\vec{y},\vec{d}|\alpha,\theta,\lambda)g(\alpha,\theta,\lambda)d\alpha d\theta d\lambda}{\int_0^\infty \int_0^\infty \int_0^\infty L(\vec{y},\vec{d}|\alpha,\theta,\lambda)g(\alpha,\theta,\lambda)d\alpha d\theta d\lambda}. \quad (15)$$

From the above Equation we observe that the Bayes estimator is in the form of ratio of two integrals for which closed form solution is not available. The above ratio of integrals can be solved numerically. Here, we use Lindley's approximation method and MCMC techniques like Gibbs sampling method and Metropolis-Hastings algorithm to derive Bayes estimates.

5.1. Lindleys Approximation

For the evaluation of the ratio of two integrals, Lindley [32] gave an approximation method known as Lindleys' approximation. According to this method, the approximate Bayes estimator of $\phi(\alpha, \theta, \lambda)$ under SELF is given by

$$\begin{aligned}
\widehat{\phi}_L = \; & \widehat{\phi} + \frac{1}{2}[(\phi_{11} + 2\phi_1\rho_1)\sigma_{11} + (\phi_{12} + 2\phi_1\rho_2)\sigma_{12} + (\phi_{13} + 2\phi_1\rho_3)\sigma_{13} \\
& +(\phi_{21} + 2\phi_2\rho_1)\sigma_{21} + (\phi_{22} + 2\phi_2\rho_2)\sigma_{22} + (\phi_{23} + 2\phi_2\rho_3)\sigma_{23} \\
& +(\phi_{31} + 2\phi_3\rho_1)\sigma_{31} + (\phi_{32} + 2\phi_3\rho_2)\sigma_{32} + (\phi_{33} + 2\phi_3\rho_3)\sigma_{33} \\
& +(\phi_1\sigma_{11} + \phi_2\sigma_{12} + \phi_3\sigma_{13})(l_{300}\sigma_{11} + 2l_{210}\sigma_{12} + 2l_{201}\sigma_{13} + 2l_{111}\sigma_{23} + l_{120}\sigma_{22} + l_{102}\sigma_{33}) \\
& +(\phi_1\sigma_{21} + \phi_2\sigma_{22} + \phi_3\sigma_{33})(l_{210}\sigma_{11} + 2l_{120}\sigma_{12} + 2l_{111}\sigma_{13} + 2l_{021}\sigma_{23} + l_{030}\sigma_{22} + l_{012}\sigma_{33}) \\
& +(\phi_1\sigma_{31} + \phi_2\sigma_{32} + \phi_3\sigma_{33})(l_{201}\sigma_{11} + 2l_{111}\sigma_{12} + 2l_{102}\sigma_{13} + 2l_{012}\sigma_{23} + l_{021}\sigma_{22} + l_{003}\sigma_{33})]
\end{aligned} \quad (16)$$

where,

$$\phi_1 = \frac{\partial\phi}{\partial\alpha}, \quad \phi_2 = \frac{\partial\phi}{\partial\theta}, \quad \phi_3 = \frac{\partial\phi}{\partial\lambda}, \quad \phi_{11} = \frac{\partial^2\phi}{\partial\alpha^2}, \quad \phi_{22} = \frac{\partial^2\phi}{\partial\theta^2},$$
$$\phi_{33} = \frac{\partial^2\phi}{\partial\lambda^2}, \quad \phi_{12} = \frac{\partial^2\phi}{\partial\alpha\partial\theta}, \quad \phi_{13} = \frac{\partial^2\phi}{\partial\alpha\partial\lambda}, \quad \phi_{23} = \frac{\partial^2\phi}{\partial\theta\partial\lambda}$$

$$\begin{aligned}
& l_{300} = \frac{\partial^3 l}{\partial\alpha^3}, \quad l_{030} = \frac{\partial^3 l}{\partial\theta^3}, \quad l_{003} = \frac{\partial^3 l}{\partial\lambda^3}, \\
& l_{120} = \frac{\partial^3 l}{\partial\alpha\partial\theta^2}, \quad l_{012} = \frac{\partial^3 l}{\partial\theta\partial\lambda^2}, \quad l_{102} = \frac{\partial^3 l}{\partial\alpha\partial\lambda^2}, \\
& l_{210} = \frac{\partial^3 l}{\partial\alpha^2\partial\theta}, \quad l_{021} = \frac{\partial^3 l}{\partial\theta\partial^2\lambda}, \quad l_{201} = \frac{\partial^3 l}{\partial\alpha^2\partial\lambda}, \quad l_{111} = \frac{\partial^3 l}{\partial\alpha\partial\theta\partial\lambda}.
\end{aligned}$$

$$\rho_1 = \frac{\partial \ln g_1}{\partial\alpha} = \frac{a_1 - 1}{\alpha} - b_1, \quad \rho_2 = \frac{\partial \ln g_2}{\partial\theta} = \frac{a_2 - 1}{\theta} - b_2, \quad \rho_3 = \frac{\partial \ln g_3}{\partial\alpha} = \frac{a_3 - 1}{\lambda} - b_3,$$

and $\sigma_{ij} = (i,j)th$ element of the observed variance-covariance matrix $I^{-1}(.)$ as computed in Section 3. By relation (8) we have

$$l_{300} = \frac{\partial^3 l}{\partial \alpha^3} = \frac{2n}{\alpha^3} + \sum_{i=1}^{n}(1-d_i)A_1(y_i,\theta,\alpha) + d_i A_1(y_i,\lambda,\alpha)$$
$$l_{030} = \frac{\partial^3 l}{\partial \theta^3} = m(\frac{4}{\theta^3} - \frac{2}{(1+\theta)^3}) + (\alpha-1)\sum_{i=1}^{n} d_i B_1(y_i,\theta) + (1-d_i)A_2(y_i,\lambda,\alpha)$$
$$l_{003} = \frac{\partial^3 l}{\partial \lambda^3} = (n-m)(\frac{4}{\lambda^3} - \frac{2}{(1+\lambda)^3}) + (\alpha-1)\sum_{i=1}^{n}(1-d_i)B_1(y_i,\lambda) + d_i A_2(y_i,\theta,\alpha)$$

$$l_{210} = \frac{\partial^3 l}{\partial \alpha^2 \partial \theta} = \sum_{i=1}^{n}(1-d_i)A_3(y_i,\theta,\alpha)$$
$$l_{201} = \frac{\partial^3 l}{\partial \alpha^2 \partial \lambda} = \sum_{i=1}^{n} d_i A_3(y_i,\lambda,\alpha)$$
$$l_{120} = \frac{\partial^3 l}{\partial \alpha \partial \theta^2} = \sum_{i=1}^{n} d_i B_2(y_i,\theta) + (1-d_i)A_4(y_i,\theta,\alpha))$$
$$l_{102} = \frac{\partial^3 l}{\partial \alpha \partial \lambda^2} = \sum_{i=1}^{n}(1-d_i)B_2(y_i,\lambda) + d_i A_4(y_i,\lambda,\alpha))$$

where, for $t = \theta, \lambda$

$$A_1(y,t,\alpha) = \frac{\partial \ln(1-F(y,t,\alpha))}{\partial \alpha^3}, \quad A_2(y,t,\alpha) = \frac{\partial \ln(1-F(y,t,\alpha))}{\partial t^3}$$
$$A_3(y,t,\alpha) = \frac{\partial \ln(1-F(y,t,\alpha))}{\partial \alpha^2 \partial t}, \quad A_4(y,t,\alpha) = \frac{\partial \ln(1-F(y,t,\alpha))}{\partial \alpha \partial t^2}$$
$$B_1(y,t) = \frac{\partial \ln F_L(y,t)}{\partial t^3}, \quad B_2(y,t) = \frac{\partial \ln F_L(y,t)}{\partial \partial t^2}$$

When, $\phi(\alpha,\theta,\lambda) = \alpha, \theta$ and λ, the Bayes estimators of α, θ and λ under SELF, respectively, are given by

$$\begin{aligned}\widehat{\alpha}_L &= \widehat{\alpha} + (\rho_1\sigma_{11} + \rho_2\sigma_{12} + \rho_3\sigma_{13}) \\ &+\frac{1}{2}\left[l_{300}\sigma_{11}^2 + l_{120}(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2) + l_{102}(\sigma_{33}\sigma_{22} + 2\sigma_{13}^2) + l_{030}\sigma_{22}\sigma_{21} + l_{003}\sigma_{33}\sigma_{31}\right]\end{aligned} \tag{17}$$

$$\begin{aligned}\widehat{\theta}_L &= \widehat{\theta} + (\rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{33}) \\ &+\frac{1}{2}\left[l_{300}\sigma_{11}\sigma_{12} + 3l_{120}\sigma_{12}\sigma_{22} + l_{102}(\sigma_{33}\sigma_{12} + 2\sigma_{13}\sigma_{32}) + l_{030}\sigma_{22}^2 + l_{003}\sigma_{33}\sigma_{32}\right]\end{aligned} \tag{18}$$

$$\begin{aligned}\widehat{\lambda}_L &= \widehat{\lambda} + (\rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33}) \\ &+\frac{1}{2}\left[l_{300}\sigma_{11}\sigma_{13} + l_{120}(\sigma_{22}\sigma_{13} + 2\sigma_{12}\sigma_{23}) + 3l_{102}\sigma_{13}\sigma_{33} + l_{030}\sigma_{22}\sigma_{23} + l_{003}\sigma_{33}^2\right]\end{aligned} \tag{19}$$

All the expressions on the right-hand side of equations 17, 18 and 19 are to be computed at MLEs $\widehat{\omega} = (\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda})$ of $\omega = (\alpha, \theta, \lambda)$. Using Lindley's approximation method, the Bayes estimators of the unknown parameters can be obtained easily.

5.2. *Gibbs sampling method*

In this sub-section, we propose to use the Gibbs sampling method to draw the random sample from the joint posterior distribution so that the sample based inference can be performed. The detail study about MCMC techniques can be found in Robert and Casella [35] and Gelman et al. [18]. For implementing the Gibbs sampling method, the full conditional posterior densities of α, θ and λ are, respectively given by

$$\pi_1(\theta|\alpha, \vec{y}, \vec{d}) = \frac{\theta^{2*m+a_2-1}}{(1+\theta)^m} e^{-\theta(b_2+T2)} \prod_{i=1}^{n} F(Y_i, \theta, \alpha-1)^{d_i} \prod_{i=1}^{n} (1-F(Y_i, \theta, \alpha))^{1-d_i} \tag{20}$$

$$\pi_2(\lambda|\alpha, \vec{y}, \vec{d}) = \frac{\lambda^{2*(n-m)+a_3-1}}{(1+\lambda)^{n-m}} e^{-\theta(b_3+T3)} \prod_{i=1}^{n} F(Y_i, \lambda, \alpha-1)^{1-d_i} \prod_{i=1}^{n} (1-F(Y_i, \lambda, \alpha))^{d_i} \tag{21}$$

$$\pi_3(\alpha|\theta, \lambda, \vec{y}, \vec{d}) = \alpha^{n+a_1-1} e^{-\alpha(b_1+T_1)} \prod_{i=1}^{n} (1-F(Y_i, \theta, \alpha))^{1-d_i} \prod_{i=1}^{n} (1-F(Y_i, \lambda, \alpha))^{d_i} \tag{22}$$

We use following Gibbs sampler algorithm to generate samples from the full conditional posterior densities (20), (21) and (22).

Step 1: Start with initial guess of α, θ and λ say α_0, θ_0 and λ_0.
Step 2: Set $j = 1$.
Step 3: Generate θ_j from $\pi_1(\theta|\alpha_{j-1}, \vec{y}, \vec{d})$ in (20) using MH algorithm with normal proposal density.
Step 4: Generate λ_j from $\pi_2(\lambda|\alpha_{j-1}, \vec{y}, \vec{d})$ in (21) using MH algorithm with normal proposal density.
Step 5: Generate α_j from $\pi_3(\alpha|\theta_{j-1}, \lambda_{j-1}, \vec{y}, \vec{d})$ in (22) using MH algorithm with normal proposal density.
Step 6: Set $j = j + 1$ and repeat steps $3 - 5$ for all $j = 1, 2, ..., M$ to obtain MCMC samples

$$(\alpha_1, \theta_1, \lambda_1), (\alpha_2, \theta_2, \lambda_2), ..., (\alpha_M, \theta_M, \lambda_M).$$

Now, the approximate Bayes estimator of $\phi(\alpha, \theta, \lambda)$, can be obtained as

$$\widehat{\phi}_{GS}(\alpha, \theta, \lambda) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \phi(\alpha_j, \theta_j, \lambda_j) \tag{23}$$

where, M_0 is the burn-in period i.e. a number of iterations in Markov chain before the stationary distribution is achieved. Thus, taking, $\phi(\alpha, \theta, \lambda) = \alpha, \theta$ and λ, the Bayes estimators of the parameters α, θ and λ under SELF, respectively, are given by

$$\widehat{\alpha}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \alpha_j,$$

$$\widehat{\theta}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \theta_j,$$

$$\widehat{\lambda}_{GS} = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \lambda_j.$$

Now, we construct the HPD credible intervals of α, θ, and λ using the generated importance samples. Let $\alpha_{(1)} < \alpha_{(2)} < ... < \alpha_{(M')}$ denote the ordered values of $\alpha_1, \alpha_2, ..., \alpha_{M'}$, $(M' = M - M_0)$. Then, using the algorithm proposed by Chen and Shao [6], the $100(1-\varepsilon)\%$, where $0<\varepsilon<1$, HPD credible interval for α is given by $(\alpha_{(j)}, \alpha_{(j+[(1-\varepsilon)M'])})$, where j is chosen such that

$$\alpha_{(j+[(1-\varepsilon)M'])}) - \alpha_{(j)} = \min_{1\leq i\leq \varepsilon M'} (\alpha_{(i+[(1-\varepsilon)M'])}) - \alpha_{(i)}), \quad j = 1, 2, ..., M'.$$

where $[x]$ is the largest integer less than or equal to x. Similarly, we can construct the $100(1-\varepsilon)\%$ HPD credible intervals for θ and λ.

6. Monte Carlo Simulation Study

In this section, we describe a Monte Carlo simulation study that was conducted to compare estimators developed in the previous sections. All the computations were performed using statistical software $R-4.0.3$. We consider three different sample sizes $n=20, 40$ and 60 in this simulation study. In all cases, true value of $\theta=1.0$, three different value of $\lambda=0.5, 1, 2$ and two different values of $\alpha=1, 2$ are used. For Bayesian computation non-informative as well as gamma informative priors under SELF are considered. In the case of informative priors, following values of hyper-parameters $(a_1, b_1, a_2, b_2, a_3, b_3)$ are taken so that prior means are exactly equal to the true values of the parameters:

$$(2, 2, 2, 2, 4, 2), (2, 2, 2, 2, 2, 2), (2, 2, 2, 2, 2, 4)$$

$$(4, 2, 2, 2, 4, 2), (4, 2, 2, 2, 2, 2), (4, 2, 2, 2, 2, 4)$$

For each case, the ML and Bayes estimates of the unknown parameters, survival and failure rate functions are computed. For Bayesian estimators, Lindley's approximation and Gibbs sampling methods are considered. The 95% asymptotic confidence intervals based on expected Fisher information matrix and HPD credible intervals based on Gibbs sampling method are constructed. The integrals associated with expected Fisher information matrix are solved using the integrate function of software R. We take $M=10,000$ with burn-in period $M_0=0.2M$ for Gibbs sampling method. The whole process was simulated 1,000 times and the average absolute biases (AB) with the corresponding mean squared errors (MSE) are computed for different estimators. Also, the average length (AL) and the coverage probabilities (CP) of 95% asymptotic confidence and HPD credible intervals are calculated. The results of the simulation study are presented in Tables 2, 3, 4, 5, 6, 7 and 13.

In simulation tables, the short notations LB stands for Lindley method, GS stand for Gibbs sampling method, $Prior0$ for non-informative prior and $Prior1$ for gamma informative prior. From these results the following conclusions are made:

Table 2. The Maximum Likelihood and Bayes estimates of α when $\alpha=1, \theta=1$.

		$\widehat{\alpha}_{ML}$		$\widehat{\alpha}_{LB}$				$\widehat{\alpha}_{GS}$			
				$prior0$		$prior1$		$prior0$		$prior1$	
λ	n	EV	MSE	EV	MSE	EV	MSE	EV	MSE	EV	MSE
0.5	30	1.0658	0.0528	0.9372	0.0206	0.9511	0.0043	1.0298	0.0122	0.9511	0.00276
	50	1.0391	0.0482	0.9513	0.0180	0.9854	0.0024	1.0478	0.0091	0.9855	0.00029
	70	1.0556	0.0449	0.9878	0.0084	1.0011	0.0002	1.0683	0.0096	1.0011	0.00003
1	30	1.0520	0.0453	0.9056	0.0106	0.9996	0.0089	0.9779	0.0112	0.9117	0.00780
	50	1.0305	0.0413	0.9522	0.0117	1.0705	0.0023	1.0408	0.0079	0.9671	0.00108
	70	1.0324	0.0371	0.9691	0.0053	0.9707	0.0010	1.0131	0.0047	0.9783	0.00046
2	30	1.0520	0.0389	0.8901	0.0244	0.8801	0.0125	1.0660	0.0181	1.1552	0.02449
	50	1.0374	0.0328	0.9336	0.0101	1.0554	0.0045	1.0059	0.0092	1.0955	0.00780
	70	1.0225	0.0267	0.9556	0.0051	0.9753	0.0020	0.9642	0.0067	1.0815	0.00568

Table 3. The Maximum Likelihood and Bayes estimates of θ when $\alpha = 1, \theta = 1$.

		$\widehat{\alpha}_{ML}$		$\widehat{\alpha}_{LB}$				$\widehat{\alpha}_{GS}$			
				$prior0$		$prior1$		$prior0$		$prior1$	
λ	n	EV	MSE	EV	MSE	EV	MSE	EV	MSE	EV	MSE
0.5	30	1.0400	0.0490	1.1299	0.0168	1.0697	0.0048	0.9509	0.0159	1.1030	0.0138
	50	1.0081	0.0469	1.0807	0.0065	1.0549	0.0030	1.1237	0.0238	0.9643	0.0089
	70	1.0242	0.0333	1.0714	0.0051	1.0579	0.0033	1.0179	0.0063	1.0607	0.0050
1	30	1.0398	0.0563	1.1412	0.0231	1.1513	0.0201	0.9743	0.0191	0.9750	0.0190
	50	1.0172	0.0540	1.0886	0.0126	1.1125	0.0079	0.9410	0.0140	0.9978	0.0104
	70	1.0125	0.0519	1.0760	0.0058	1.0727	0.0053	0.9675	0.0081	1.0452	0.0034
2	30	1.0490	0.0672	1.1653	0.0384	1.1722	0.0307	0.9439	0.0357	1.0240	0.0330
	50	1.0238	0.0613	1.1035	0.0209	1.1058	0.0114	0.9854	0.0196	0.9833	0.0173
	70	1.0104	0.0551	1.0808	0.0088	1.0935	0.0066	0.9530	0.0158	0.9895	0.0133

Table 4. The Maximum Likelihood and Bayes estimates of λ when $\alpha = 1, \theta = 1$.

		$\widehat{\alpha}_{ML}$		$\widehat{\alpha}_{LB}$				$\widehat{\alpha}_{GS}$			
				$prior0$		$prior1$		$prior0$		$prior1$	
λ	n	EV	MSE	EV	MSE	EV	MSE	EV	MSE	EV	MSE
0.5	30	0.5554	0.0370	0.5970	0.0135	0.6154	0.0096	0.5721	0.0120	0.5337	0.0086
	50	0.5409	0.0296	0.5608	0.0061	0.5783	0.0037	0.5443	0.0071	0.5273	0.0048
	70	0.5543	0.0262	0.5504	0.0040	0.5634	0.0025	0.5514	0.0060	0.5260	0.0039
1	30	1.0415	0.0560	1.1418	0.0202	1.1339	0.0180	1.0297	0.0203	1.0203	0.0187
	50	1.0407	0.0538	1.0991	0.0116	1.1078	0.0098	1.0009	0.0119	1.0096	0.0111
	70	1.0496	0.0467	1.0787	0.0062	1.0777	0.0060	1.0296	0.0087	1.0639	0.0059
2	30	2.0578	0.0773	2.2479	0.0615	1.8820	0.0139	1.8974	0.0704	1.9005	0.0728
	50	2.0327	0.0666	2.1422	0.0202	1.9497	0.0025	1.9108	0.0475	1.73311	0.0315
	70	2.0381	0.0471	2.1033	0.01069	1.9739	0.0006	2.1882	0.0640	1.8978	0.0355

Table 5. The Maximum Likelihood and Bayes estimates of α when $\alpha = 2, \theta = 1$.

		$\widehat{\alpha}_{ML}$		$\widehat{\alpha}_{LB}$				$\widehat{\alpha}_{GS}$			
				$prior0$		$prior1$		$prior0$		$prior1$	
λ	n	EV	MSE	EV	MSE	EV	MSE	EV	MSE	EV	MSE
0.5	30	2.0243	0.0798	1.7899	0.1322	1.6389	0.0461	1.8170	0.0747	1.8339	0.0718
	50	2.0027	0.0766	1.8734	0.0589	1.7580	0.0164	1.9954	0.0291	1.6977	0.0153
	70	1.9945	0.0732	1.9040	0.0321	1.8210	0.0093	1.9428	0.0274	1.8054	0.0113
1	30	2.0367	0.0775	1.8481	0.1179	1.6565	0.0230	2.1057	0.0544	1.9130	0.0498
	50	2.0133	0.0761	1.9182	0.0367	1.8082	0.0066	1.9929	0.0242	1.9144	0.0229
	70	2.0040	0.0748	1.9620	0.0157	1.8743	0.0014	2.0035	0.0192	1.8785	0.0102
2	30	2.0105	0.0753	1.7861	0.0220	1.8577	0.0475	2.1250	0.0617	1.7650	0.0673
	50	1.9957	0.0701	1.8610	0.0076	1.9148	0.0197	1.9719	0.0276	1.8478	0.0192
	70	1.9852	0.0697	1.8817	0.0063	1.9213	0.0141	2.1087	0.0338	1.9206	0.0054

Table 6. The Maximum Likelihood and Bayes estimates of θ when $\alpha = 2, \theta = 1$.

		$\widehat{\alpha}_{ML}$		$\widehat{\alpha}_{LB}$				$\widehat{\alpha}_{GS}$			
				$prior0$		$prior1$		$prior0$		$prior1$	
λ	n	EV	MSE	EV	MSE	EV	MSE	EV	MSE	EV	MSE
0.5	30	1.0180	0.0295	1.0387	0.0087	1.0933	0.0015	0.9351	0.0102	1.0202	0.0076
	50	0.9912	0.0289	1.0204	0.0018	1.0431	0.0004	1.0752	0.0103	1.0513	0.0068
	70	1.0072	0.0279	1.0099	0.0009	1.0301	0.0001	1.0126	0.0029	0.9637	0.0013
1	30	1.0164	0.0324	1.0812	0.0301	1.1733	0.0066	0.9162	0.0157	1.0313	0.0108
	50	0.9968	0.0319	1.0501	0.0118	1.1088	0.0025	1.0241	0.0063	1.0381	0.0053
	70	1.0161	0.0311	1.0468	0.0075	1.0865	0.0022	1.0047	0.0044	1.0401	0.0019
2	30	1.0166	0.0522	1.0928	0.0568	1.2372	0.0092	0.9360	0.0207	1.0652	0.0136
	50	1.0066	0.0447	1.0502	0.0247	1.1568	0.0027	1.0607	0.0145	1.0099	0.0077
	70	1.0209	0.0391	1.0381	0.0113	1.1061	0.0015	0.9436	0.0110	1.0340	0.0057

Table 7. The Maximum Likelihood and Bayes estimates of λ when $\alpha = 2, \theta = 1$.

		$\widehat{\alpha}_{ML}$		$\widehat{\alpha}_{LB}$				$\widehat{\alpha}_{GS}$			
				$prior0$		$prior1$		$prior0$		$prior1$	
λ	n	EV	MSE	EV	MSE	EV	MSE	EV	MSE	EV	MSE
0.5	30	0.5103	0.0201	0.5247	0.0101	0.5994	0.0018	0.4632	0.0057	0.5107	0.0045
	50	0.5132	0.0195	0.5187	0.0029	0.5538	0.0006	0.5019	0.0026	0.4475	0.0037
	70	0.5262	0.0177	0.5188	0.0013	0.5369	0.0004	0.4883	0.0022	0.5160	0.0021
1	30	1.0154	0.0316	1.0750	0.0205	1.1432	0.0056	1.0636	0.0127	1.0168	0.0102
	50	1.0195	0.0298	1.0491	0.0091	1.0952	0.0024	1.0213	0.0063	1.0031	0.0059
	70	1.0228	0.0272	1.0367	0.0038	1.0619	0.0013	1.0707	0.0093	1.0191	0.0045
2	30	2.0182	0.0583	2.1236	0.0028	2.0536	0.0152	1.9395	0.0320	1.9306	0.0221
	50	2.0148	0.0520	2.0754	0.0020	2.0456	0.0030	1.9762	0.0206	1.7680	0.0117
	70	2.0059	0.0448	2.0413	0.0004	2.0212	0.0009	2.0203	0.0137	1.8656	0.0088

- As the sample size increases, the MSE of the ML and Bayes estimators of the parameters decrease in all the cases.
- As the value of the shape parameter α increases, MSE also increase.
- Bayes estimates are better than MLEs in terms of MSEs as they include prior information. In addition Bayes estimates computed using Lindley approximation method is better than estimates using Gibbs sampling method in respect of MSE.
- The coverage probabilities based on all confidence and HPD credible intervals of the parameters attain their prescribed confidence levels almost in all cases.
- The average length of all intervals decreases as the sample size n increases. Among the confidence and HPD credible intervals, HPD credible intervals are the best in respect of average length.

7. Real data analysis

In this section, we discuss a real data set for illustration purpose. First of all, we fitted GL and some other well-known reliability distributions, viz., exponential, Lindley, Rayleigh, gamma, Weibull, Generalized Inverted

Table 8. The average length (AL) and coverage probability (CP) of 95% classical confidence/ HPD credible intervals for α, θ and λ when $\alpha = 2, \theta = 1, \lambda = 0.5$.

	n	Asymptotic Confidence Interval		Boot-p Confidence Interval		Boot-t Confidence Interval		HPD Credible Interval			
								$prior0$		$prior1$	
		AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
	30	1.1201	0.987	2.5148	0.998	2.4634	1.000	0.8372	0.985	0.8134	0.987
$\alpha = 2$	50	0.8552	0.963	1.6791	0.991	1.7765	0.993	0.6618	0.998	0.6493	0.995
	70	0.7230	0.939	1.4072	0.995	1.4731	0.991	0.5666	1.000	0.5499	0.993
	30	0.9779	0.995	0.7496	0.999	0.6993	0.989	0.3217	0.974	0.3241	0.993
$\theta = 1$	50	0.7583	0.992	0.5389	0.989	0.5244	0.981	0.2511	0.993	0.2512	0.995
	70	0.6332	0.972	0.4467	0.993	0.4476	0.972	0.2138	0.993	0.2101	0.983
	30	0.6908	0.989	0.5929	0.997	0.5403	0.999	0.2643	0.999	0.2521	0.977
$\lambda = 0.5$	50	0.5365	0.978	0.4482	0.982	0.4299	0.991	0.2033	0.999	0.1973	0.991
	70	0.4508	0.947	0.3496	0.978	0.3574	0.982	0.1697	0.996	0.1682	0.993

Exponential(GIE) and Inverse Weibull(IW) (pdfs listed below) to these data.

$$\text{Exponential}: \quad f(x,\theta) = \theta \exp(-\theta x),\ x > 0,\ \theta > 0$$

$$\text{Lindley}: \quad f(x,\theta) = \frac{\theta^2}{1+\theta}(1+x)\exp(-\theta x),\ x > 0,\ \theta > 0$$

$$\text{Gamma}: \quad f(x,\alpha,\theta) = \frac{\theta^\alpha x^{\alpha-1}\exp(-\theta x)}{\Gamma(\alpha)},\ x > 0,\ \alpha,\theta > 0$$

$$\text{Weibull}: \quad f(x,\alpha,\theta) = \alpha\theta x^{\alpha-1}\exp(-\theta x^\alpha),\ x > 0,\ \alpha,\theta > 0$$

$$\text{GIE}: \quad f(x,\alpha,\theta) = \frac{\alpha\theta}{x^2}\exp(-\theta/x)(1-\exp(-\theta/x))^{\alpha-1},\ x > 0,\ \alpha,\theta > 0$$

$$\text{IW}: \quad f(x,\alpha,\theta) = \frac{\alpha\theta}{x^{\alpha+1}}\exp(\frac{-\theta}{x^{-\alpha}}),\ x > 0,\ \alpha,\theta > 0$$

$$\text{Burr-XII}: \quad f(x,\alpha,\theta) = \frac{\theta\gamma}{\alpha^\theta}\, x^{\theta-1}\left[1+(\frac{x}{\alpha})^\theta\right]^{-\gamma-1},\ x > 0,\ \alpha,\theta,\gamma > 0.$$

The data given here arose in tests on endurance of deep groove ball bearings witch obtained from Lawless [28]. The data are the numbers of million revolution before failure for each of the 23 ball bearings in the life test and they are:

Table 9. Real data set.

17.88+	28.92	33.00	41.52+	42.12	45.60+	48.40+	51.84+
51.96+	54.12	55.56+	67.90	68.64+	68.64+	68.88	84.12+
93.12	98.64+	105.12	105.84+	127.92	127.04	173.40	

Here, plus sign (+) indicates censored observation. The maximum likelihood estimation method is used to estimate the parameters of the above distributions. These estimates, along with the data, are used to calculate estimated negative log likelihood function ($-\ln L$), the Akaike information criterion (AIC), proposed by Akaike [1], defined by $AIC = 2k - 2\ln(L)$, Bayesian information criterion (BIC) proposed by Schwarz [38], defined by $BIC = k\ln(n) - 2\ln(L)$, where k is the number of parameters in the survival model, n is the number of observations in the given data set, and L is the maximized value of the likelihood function for the estimated model and Kolmogorov-Smirnov (KS) statistic with its $p-value$. The best distribution corresponds to the lowest $-\ln L$, AIC, BIC and KS statistic values and the highest $p-value$. The KS statistic with its p-value values were obtained' using $ks.test$ function of statistical software R. These $-\ln L$, AIC, BIC and KS statistics with $p-values$ are listed in Table 10.

Table 10. Summary fit for the real data set.

Distribution	MLEs	$-\ln L$	AIC	AIC	KS Test	
					Statistic	$p-value$
Exponential	$\hat{\theta}=0.00723$ $\hat{\lambda}=0.00663$	137.3419	278.6839	280.9549	0.2635	0.0673
Lindley	$\hat{\theta}=0.01795$ $\hat{\lambda}=0.01656$	130.9637	265.9275	268.1984	0.1332	0.7604
Gamma	$\hat{\alpha}=3.39374$ $\hat{\theta}=0.03439$ $\hat{\lambda}=0.03179$	128.9603	263.9205	267.327	0.1298	0.7867
Weibull	$\hat{\alpha}=2.10430$ $\hat{\theta}=111.474$ $\hat{\lambda}=116.180$	129.5798	265.1596	268.5661	0.1507	0.6191
GIE	$\hat{\alpha}=130.115$ $\hat{\theta}=2.77602$ $\hat{\lambda}=2.5446$	129.4447	264.8894	268.2959	0.1701	0.7993
IW	$\hat{\alpha}=366.123$ $\hat{\theta}=1.41397$ $\hat{\lambda}=1.36220$	130.1906	266.3813	269.7877	0.1807	0.7952
Burr XII	$\hat{\alpha}=86.1798$ $\hat{\theta}=2.82509$ $\hat{\lambda}=0.95264$ $\hat{\gamma}=0.87325$	129.1399	266.2798	270.8218	0.1116	0.9064
GL	$\hat{\alpha}=1.88429$ $\hat{\theta}=0.02690$ $\hat{\lambda}=0.02470$	128.9194	263.8387	267.2452	0.1263	0.8124

According to -lnL, AIC and BIC, the GL distribution is the best choice among the competing distributions and according to KS test, the Burr XII distribution is slightly better than GL distribution. The main advantage of using GL distribution over the Burr XII distribution is that it involves two parameters. Hence, inferential procedures especially from computational point to view, become simple to deal with. So, it can be seen that the GL fits the data very well and also we have just plotted the empirical $S(t)$ and the fitted $S(t)$ in Fig 3. We Observed that the GL can be a good model to fitting this data.

In addition, the likelihood ratio (LR) test is applied to compare the GL distribution with its sub-model Lindley distribution. For example, a testing hypothesis of $H_0 : \alpha = 1$ versus $H_1 : \alpha \neq 1$ is equivalent to compare the GL distribution with Lindley distribution. To investigate this testing hypothesis, the LR statistic can be computed by the following equation

$$LR = 2\left[l(\widehat{\alpha}, \widehat{\theta}, \widehat{\lambda}) - l(1, \widehat{\theta^*}, \widehat{\lambda^*})\right],$$

where $\widehat{\alpha^*}, \widehat{\theta^*}$, and $\widehat{\lambda^*}$ are the MLEs of α, θ, and λ, obtained under H_0, respectively. Under H_0 and by considering the regularity conditions, the LR test statistic converges to χ^2_r in distribution, where r is equal to difference between the number of parameters estimated under H_0 and the number of parameters estimated in general, that is, under $H_0 : \alpha = 1$, we have $r = 1$.

Table 11 gives values of the LR statistic with it's corresponding p-value. From Table 11, we can see that the obtained p-value is too small. So, we can reject the null hypotheses. In other words, based on the LR criterion we conclude that fitting of the GL has better performance than Lindley distribution.

Table 11. The LR test results for the real data set.

	Hypotheses	LR	p-value
GL versus Lindley	$H_0 : \alpha = 1$	4.088738	0.04316

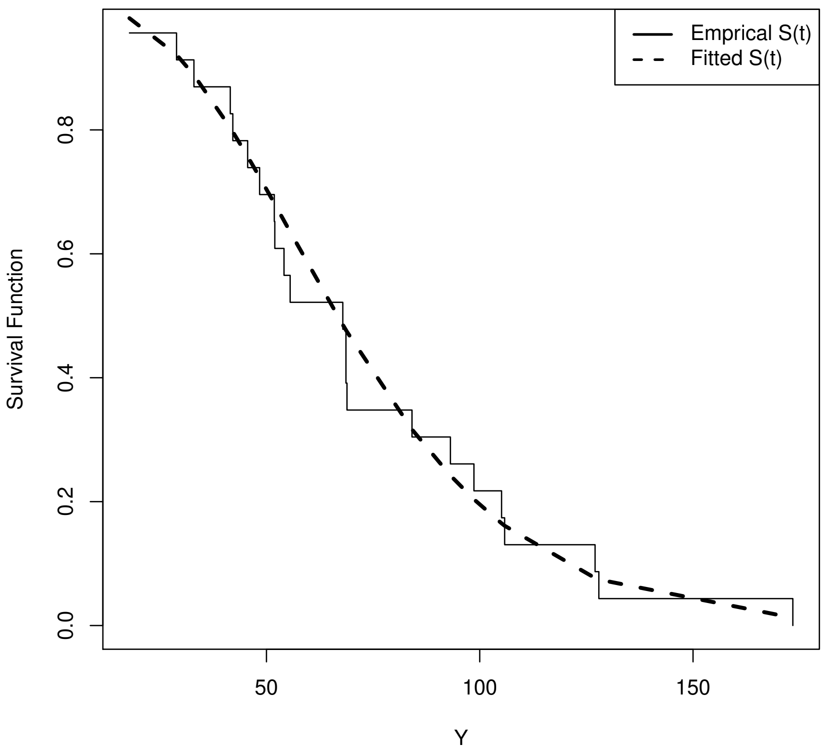

Figure 3. Empirical and fitted survival functions for real data set.

To show that the MLEs exist and are unique, the profile log-likelihood function of θ and λ was plotted (see Figure 4). It shows that the likelihood surface has curvature in both θ and λ directions, indicating that the MLEs of θ and λ exist and are unique.

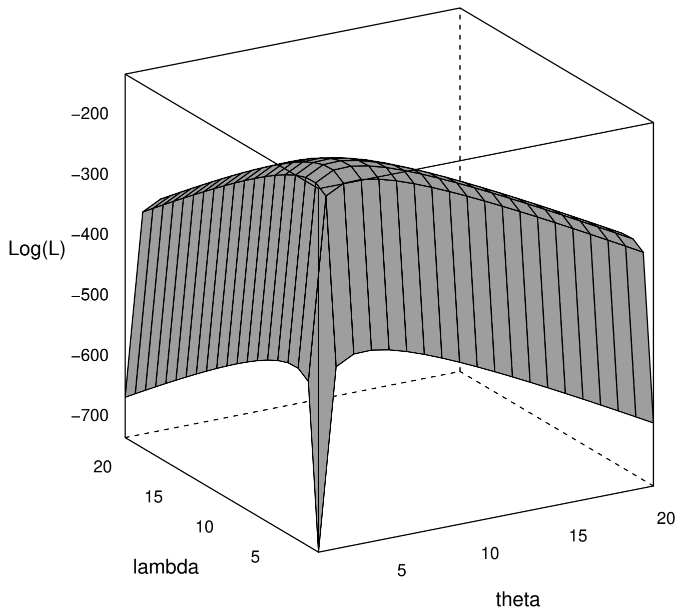

Figure 4. Log-likelihood function of the GL model for selected θ and λ.

Now, for the above data set we obtain the ML and Bayes estimates of the parameters. Since, we do not have any prior information, the Bayes estimates of the unknown parameters are computed with non-informative priors. For non-informative priors, we take hyperparameters $a_1 = b_1 = a_2 = b_2 = 0$ in equation (13). The Bayes estimates are obtained using Lindley approximation method and Gibbs sampling. For Gibbs sampling method we take $M = 10,000$ with burn-in-period $0.2M$. The ML and Bayes estimates of the parameters are given in Table 12. Also, the 95% asymptotic, boot-p, boot-t confidence intervals and HPD credible intervals for the parameters are reported in Table 13. Also, we check the convergence of the generated sequences of α, θ and λ graphically. The trace plots, autocorrelation function (ACF) plots and histograms with density plots of the 10, 000 iterations of the parameters α, θ and λ are presented in Figure 5.

Table 12. The ML and Bayes estimates of the parameters corresponding to the real data set.

Parameter	MLE	Bayes Estimate	
		LB	GS
α	1.88429	1.63671	1.84964
θ	0.02691	0.02852	0.02661
λ	0.02470	0.02622	0.02441

From Figure 5 we observe that the trace plots look like a random scatter about their mean values (represented by solid line) and fine mixing of the chains for the parameters. ACF plots show that chains have very low

Table 13. The 95% asymptotic, boot-p, boot-t confidence intervals and HPD credible intervals for the parameters corresponding the real data set.

Parameter	Asymptotic CI	Boot-p CI	Boot-t CI	HPD CI
α	$(1.3113, 2.4573)$	(1.1816, 4.4061)	(0.8280, 2.9539)	(1.4182, 2.3151)
θ	$(0.0196, 0.0343)$	(0.0179, 0.0420)	(0.0152, 0.0414)	(0.0211, 0.0323)
λ	$(0.0176, 0.0319)$	(0.0161, 0.0405)	(0.0137, 0.0388)	(0.0192, 0.0298)

autocorrelations. Histograms show that the densities of α, θ and λ are almost symmetric. Thus, from Figure 5 we can see that Gibbs sampling method is convergent.

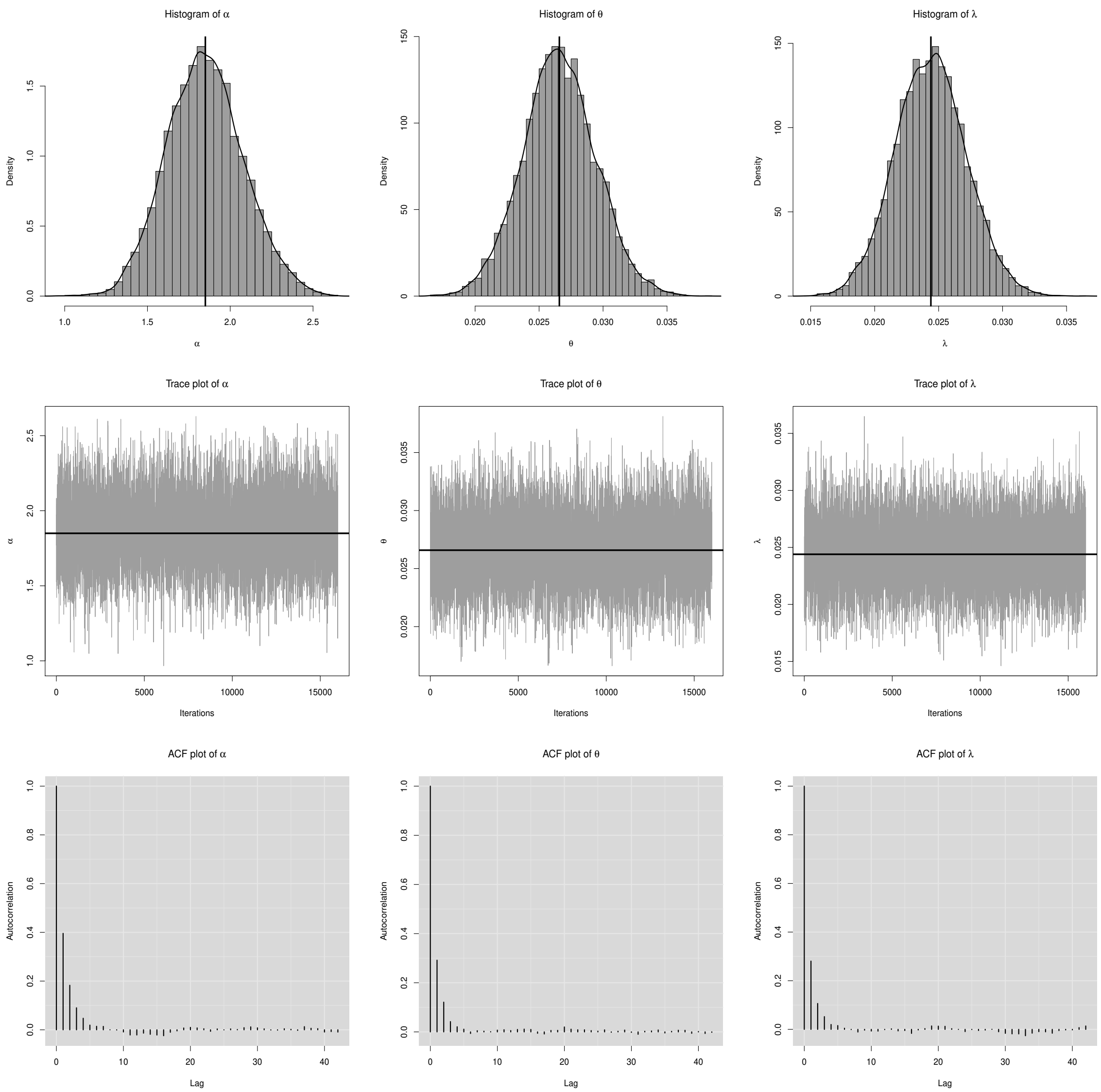

Figure 5. Histograms, trace and autocorrelation plots of the parameters α, θ and λ corresponding to the real data set.

8. Concluding remarks

The generalized Lindley (GL) distribution is an important lifetime model for representing behavior of the failure rate function. This work is helpful to statistician in analyzing randomly censored lifetime data. In this article, we considered the classical and Bayesian estimation procedures for the parameters of GL distribution under random censoring model. The MLEs of the unknown parameters were derived. Asymptotic confidence intervals for the parameters based on Fisher information and Bootstrap confidence interval are also obtained. The Bayes estimators of the parameters under SELF were approximated using Lindley's approximation and Gibbs sampling methods. ETT for randomly censored experiment were computed. The performance of these estimators was examined by extensive simulation study. The Bayes estimates based on gamma informative priors using Lindley approximation method showed minimum average absolute biases and mean squared errors than both the ML and the Bayes estimates under non-informative priors. We recommend the Bayes estimators when some prior information about parameters is available or using non-informative priors.

REFERENCES

1. Akaike H., (1974) A new look at the statistical models identification, IEEE Transactions on Automatic Control, vol. 19, pp. 716-723.
2. Ali S., Aslam M., and Kazmi S.M.A, (2013) A study of the effect of the loss function on Bayes estimate, posterior risk and hazard function for Lindley distribution, Applied Mathematical Modelling vol. 37, pp. 60686078.
3. BALAKRISHNAN N., AGGARWALA R., (2000). Progressive Censoring: Theory, Methods and Applications. Springer Science and Business Media, New York.
4. Breslow N. and Crowley J., (1974) A large sample study of the life table and product limit estimates under random censorship, The Annals of Statistics. 2, 437-453.
5. CHATURVEDI A., KUMAR N., and KUMAR K., (2018). Statistical inference for the reliability functions of a family of lifetime distributions based on progressive Type II right censoring. Statistica, 78, no. 1, pp. 81101.
6. CHEN M., H., SHAO Q. M., (1999). Monte Carlo estimation of Bayesian credible and HPD intervals. Journal of Computational and Graphical Statistics, 8, no. 1, pp. 6992.
7. Csorgo S., and Horvath L., (1981) On the Koziol-Green model for random censorship. Biometrika, 68 (1981), 391-401.
8. Danish M. Y., and Aslam M., (2013) Bayesian estimation for randomly censored generalized exponential distribution under asymmetric loss functions, Journal of Applied Statistics. 40, 1106-1119.
9. Danish M. Y., and Aslam M., (2014) Bayesian inference for the randomly censored Weibull distribution, Journal of Statistical Computation and Simulation. 84, 215-230.
10. Danish M. Y., Arshad I. A., and Aslam M., (2018) Bayesian inference for the randomly censored Burr-type XII distribution, Journal of Applied Statistics. 45, 270-283.
11. Dube M., Garg R., and Krishna H., (2016) On progressively first failure censored Lindley distribution, Computational Statistics, vol. 31, pp. 139163, 2016.
12. Efron B., and Tibshirani R. J., (1994) An Introduction to the Bootstrap, Chapman and Hall/CRC, New York, 1994.
13. Efron B., (1982) The Jacknife, the bootstrap and other re-sampling plans, CBMS-NSF Regional Conference Series in Applied Mathematics Society, SIAM 38.
14. EL-Sagheer R. M., Mohamed A. W. Mahmoud and Hasaballah M. Hasaballah. (2020) Bayesian Inference for the Randomly Censored Three-Parameter Burr XII Distribution, Applied Mathematics and Information Sciences, 14, No. 2, pp. 1-11
15. Friesl M., and Hurt J., (2007) On Bayesian estimation in an exponential distribution under random censorship, Kybernetika. 43 (2007), 45-60.
16. Garg R., Dube M., Kumar K., and Krishna H., (2016) On randomly censored generalized inverted exponential distribution, American Journal ofMathematical and Management Sciences. 35, 361-379.
17. Garg R., Dube M., and Krishna H., (2020) Estimation of Parameters and Reliability Characteristics in Lindley Distribution Using Randomly Censored Data. STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING, Vol. 8, March 2020, pp 8097.
18. GELMAN, J. B. CARLIN, H. S. STERN, D. B. DUNSON, A. VEHTARI, D. B. RUBIN (2013). Bayesian Data Analysis. Chapman and Hall/CRC, Boca Raton, FL.
19. Ghitany, M.E., Atieh, B. & Nadarajah, S. (2008). Lindley distribution and its application, *Mathematics and Computers in Simulation*, 78, 493-506.
20. Ghitany M.E., (2001) A compound Rayleigh survival model and its application to randomly censored data, Statistical Papers. 42, 437-450.
21. Ghitany M. E., and Al-Awadhi S., (2002) Maximum likelihood estimation of Burr XII distribution parameters under random censoring, Journal of Applied Statistics. 29, 955-965.
22. Gilbert J. P., (1962) Random Censorship, Ph.D. thesis, University of Chicago.
23. Hall P., (1988) Theoretical comparison of bootstrap confidence intervals, The Annals of Statistics, vol. 16, pp. 927-953, 1988.
24. Kim J. H., (1993) Chi-square goodness-of-fit tests for randomly censored data, The Annals of Statistics. 21, 1621-1639.
25. Koziol J. A., and Green S. B., (1976) A Cramer-von Mises statistic for random censored data, Biometrika. 63, 465-474.
26. Kumar K., Krishna H., and Garg R., (2015) Estimation of P(Y ¡ X) in Lindley distribution using progressively first failure censoring, International Journal of System Assurance Engineering and Management, vol. 6, pp. 330-341.

27. KUMAR K., GARG R., and KRISHNA H., (2017). Nakagami distribution as a reliability model under progressive censoring. International Journal of System Assurance Engineering and Management, 8, no. 1, pp. 109122.
28. Krishna H., and Kumar K., (2011) Reliability estimation in Lindley distribution with progressively type II right censored sample, Mathematics and Computers in Simulation, vol. 82, pp. 281-294,
29. Krishna H., Vivekanand and Kumar K.,(2015) Estimation in Maxwell distribution with randomly censored data, Journal of Statistical Computation and Simulation. 85, 3560- 3578.
30. Krishna H., and Goel N., (2017) maximum likelihood and Bayes estimation in randomly censored geometric distribution, Journal of Probability and Statistics. 3 2017, 1-12.
31. Krishna H., and Goel N., (2018) Classical and Bayesian inference in two parameter exponential distribution with randomly censored data, Computational Statistics. 33, 249-275.
32. Lindley, D.V. (1958). Fiducial distributions and Bayesian theorem. *Journal of the Royal Statistical Society B*, 20, 102-107.
33. Mazucheli J., and Achcar J. A., (2011) The Lindley distribution applied to competing risks lifetime data, Computer Methods and Programs in Biomedicine, vol. 104, pp. 188-192.
34. Nadarajah, S., Bakouch, H. S., & Tahmasbi, R. (2011). A generalized Lindley distribution. *Sankhya B*, 73, 331-359.
35. ROBERT, G. CASELLA (2004). Monte Carlo Statistical Methods. Springer Science and Business Media, New York.
36. Saleem M., and Aslam M., (2009) On Bayesian analysis of the Rayleigh survival time assuming the random censor time, Pakistan Journal of Statistics. 25, 71-82.
37. Saleem M., and Raza A., (2011) On Bayesian analysis of the exponential survival time assuming the exponential censor time, Pakistan Journal of Science. 63, 44-48.
38. Schwarz G., (1978) Estimating the dimension of a model, The Annals of Statistics, vol. 6, pp. 421-464.
39. Zheng G., and Gastwirth J. L., (2001) On the Fisher information in randomly censored data, Statistics and Probability Letters, vol. 52, pp. 421-426.