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Buys-Ballot estimators of the parameters of the cubic polynomial trend model and their statistical properties

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Abstract Time series, especially those with the cubic trend component, are encountered in many data analysis situations. The decomposition of such series into various components requires a method that can adequately estimate the cubic trend as well as other components of the series. In this study, the chain base, fixed base and classical methods of decomposition of time series with the cubic trend component are discussed with emphasis on the additive model. Chain base and fixed base estimators of the additive model parameters are derived. Basic properties of these two classes of estimators are equally determined. The derived chain base variables have the autocorrelation structure of an invertible third-order moving average model. The chain base estimators are found to be pairwise-negatively correlated estimators. Though the classical method and chain base method are both used for time series decomposition, the chain base method is recommended when a case of multicollinearity has been established.

Keywords Buys-Ballot methods, cubic trend, autocorrelation structure, pairwise-negatively correlated estimators, invertible third-order moving average model

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1. Introduction

One of the tasks frequently performed by time series analysts is the decomposition of a given time series into its various components. The classical decomposition method is the first known method of decomposing time series. Its application is often predicated on the additive and multiplicative models. The objectives of the classical decomposition method have been mentioned in numerous studies. It helps us to investigate the presence of trend, seasonal and cyclical effects in a time series. Estimates of the four components of time series which include trend, seasonal, cyclical and irregular components are found with the help of this method . Classical decomposition models are also used for short term forecasting.

Inspite of its uses, the classical decomposition method has some limitations. Notable among the demerits is the tedious nature of the method since the components are estimated one after the other. Another disadvantage of this method is its frequent poor forecasting performance [3, 6]. The least squares estimation procedure, which is used to estimate the trend component of a series in the classical decomposition approach, may not be reliable in the presence of multicollinearity. The high level of multicollinearity between two or more powers of the time variable(t) in a polynomial trend model often results in wrong inferences and model selection based on the least squares estimates of the concerned parameters [4, 24]. As a consequence, the Buys-Ballot estimation procedure proposed in [12], which is capable of yielding estimates that are robust to multicollinearity, is considered when the multicollinearity problem exists [21]. The Buys-Ballot approach is primarily used for the decomposition

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of a relatively short series such that the trend and cyclical components are jointly estimated. The additive and multiplicative Buys-Ballot decomposition models are stated in Equations (1) and (2) respectively:

$$X_t = M_t + S_t + e_t \tag{1}$$

$$X_t = M_t S_t e_t \tag{2}$$

where X_t is the observed value of the time series at time t, M_t is the trend- cycle component at time t, S_t is the seasonal component at t and e_t is the irregular component or the error term at time t. In (1), $e_t \sim N(0, \sigma_1^2)$ while in (2), $e_t \sim N(1, \sigma_2^2)$.

Apart from the work of [12] in which the chain base and fixed base estimation techniques were used in accordance with the linear trend-cycle component, the additive and multiplicative models, several studies have been subsequently undertaken within the context of the Buys-Ballot method of analysing time series data. In this regard, [17] developed the Buys-Ballot procedure of analysing time series data with the quadratic trend-cycle component. Their specific contributions include the derivation of the chain base and fixed base estimators of the parameters of each of the additive and multiplicative models. The Buys-Ballot estimates for exponential and s-shaped curves were derived in [13]. Certain properties of the chain base and fixed estimators have been discussed with respect to the linear trend-cycle. The unbiasedness and consistency of both estimators were established in [15]. According to [16], if the trend-cycle component is linear, only the derived chain base variables are stationary with the autocorrelation structure of the moving average process of order one. These authors equally found the best linear unbiased estimate of the slope parameter using the variables associated with the chain base estimation. Works carried out in other areas of research interest in the Buys-Ballot Method include the development of the procedures through which one can determine when each of the additive and multiplicative models should be considered [14], test for the presence of seasonal variations [22] and the influence of the mis-specification of error distribution on the prediction accuracy of the fitted Buy-Ballot model [21].

Though the trends in many time series can be represented by the linear , quadratic trend or exponential trend models, there are cases where the cubic time trend model is inevitably applicable [3, 1]. In particular, the cubic trend model has found applications in Agronomy [19], Computational Statistics [11], Epidermology [20], Fishery [8], Meteorology [9] and Psychology [25, 18].

Motivated by the wide applicability of the cubic trend model and robustness of the Buys-Ballot method to multicollinearity, in this study, we discuss the Buys-Ballot method of decomposing a time series with a cubic trend-cycle component. The chain base and fixed base estimators of the parameters of the cubic trend-cyle component and seasonal component of the additive model are derived. We also pay attention to the properties of the derived estimators of the parameters of the cubic trend-cycle component. The subsequent parts of this work are arranged in the following manner. Section 2 deals with the overview of the least squares estimation of the parameters of the cubic trend model. The theoretical results on the Buys-Ballot method are presented in Section 3. Section 4 deals with the properties of the estimators derived in Section 3. In Section 5, we apply the results in Section 3 to real life time series data. Also considered in this section, is the comparison of the prediction performances of the fitted additive Buys-ballot model and the model based on the least squares approach. The conclusion of this work is given in Section 5.

2. Overview of the Cubic Trend Estimation by Least Squares Method

Let $\{X_t\}$ and $\{T_t\}$ denote a given time series and the corresponding cubic trend model such that

$$T_t = \sum_{r=0}^{3} a_r t^r, r = 0, 1, 2, 3. \tag{3}$$

Suppose the least squares estimator of T_t is

$$\hat{T}_t = \sum_{r=0}^{3} \hat{a}_r t^r, r = 0, 1, 2, 3. \tag{4}$$

Then the least squares estimators $\hat{a}_0$, $\hat{a}_1$, $\hat{a}_2$ and $\hat{a}_3$ of a_0, a_1, a_2 and a_3 respectively, minimise the sum of squared deviations (S) of X_t from $\hat{T}_t$. For $S = \sum_{t=1}^{n}(X_t - \hat{T}_t)^2$, we evaluate $\frac{\partial S}{\partial \hat{a}_r} = 0$ to obtain the normal equations:

$$\hat{a}_0 n + \hat{a}_1 \sum_{t=1}^{n} t + \hat{a}_2 \sum_{t=1}^{n} t^2 + \hat{a}_3 \sum_{t=1}^{n} t^3 = \sum_{t=1}^{n} X_t, \tag{5}$$

$$\hat{a}_0 \sum_{t=1}^{n} t + \hat{a}_1 \sum_{t=1}^{n} t^2 + \hat{a}_2 \sum_{t=1}^{n} t^3 + \hat{a}_3 \sum_{t=1}^{n} t^4 = \sum_{t=1}^{n} tX_t, \tag{6}$$

$$\hat{a}_0 \sum_{t=1}^{n} t^2 + \hat{a}_1 \sum_{t=1}^{n} t^3 + \hat{a}_2 \sum_{t=1}^{n} t^4 + \hat{a}_3 \sum_{t=1}^{n} t^5 = \sum_{t=1}^{n} t^2 X_t, \tag{7}$$

$$\hat{a}_0 \sum_{t=1}^{n} t^3 + \hat{a}_1 \sum_{t=1}^{n} t^4 + \hat{a}_2 \sum_{t=1}^{n} t^5 + \hat{a}_3 \sum_{t=1}^{n} t^6 = \sum_{t=1}^{n} t^3 X_t. \tag{8}$$

In matrix form, the system of linear equations in (5), (6), (7) and (8) becomes

$$\mathbf{B} = \mathbf{A}^{-1} D, \tag{9}$$

where

$$\mathbf{B}=\begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3. \end{pmatrix}, \mathbf{A}=\begin{pmatrix} n & \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2 & \sum_{t=1}^{n} t^3 \\ \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2 & \sum_{t=1}^{n} t^3 & \sum_{t=1}^{n} t^4 \\ \sum_{t=1}^{n} t^2 & \sum_{t=1}^{n} t^3 & \sum_{t=1}^{n} t^4 & \sum_{t=1}^{n} t^5 \\ \sum_{t=1}^{n} t^3 & \sum_{t=1}^{n} t^4 & \sum_{t=1}^{n} t^5 & \sum_{t=1}^{n} t^6 \end{pmatrix} \text{ and } \mathbf{D}=\begin{pmatrix} \sum_{t=1}^{n} X_t \\ \sum_{t=1}^{n} tX_t \\ \sum_{t=1}^{n} t^2 X_t \\ \sum_{t=1}^{n} t^3 X_t \end{pmatrix}.$$

The estimated variance-covarinace matrix for the estimators is

$$V(\mathbf{B}) = \hat{\sigma}^2 \mathbf{A}^{-1} \tag{10}$$

Here, $\hat{\sigma}^2 = \frac{S}{n-4}$. The decision to include any of the parameter estimates in the cubic trend model is made in line with the test of significance of the parameter. While a t-test may be appropriate for testing the significance of a single parameter, the significance of the overall cubic trend model can be investigated through the analysis of variance technique [7]. It is noteworthy that the regression outputs from many statistical packages contain the requisite summary of the test results and conclusions may be drawn on the basis of the computed p-values.

Once the cubic trend model parameters have been estimated, the *classical decomposition method* (CDM) may be employed in estimating the seasonal indices. Before the estimation of seasonal indices by this approach, the original series has to be detrended. Assuming the additive model, the process of detrending the series deals with the subtraction of the trend component from the series. If the detrended series is arranged in accordance with the seasons(months or quarters as the case may be), the seasonal averages can be obtained. These averages are used to find the seasonal indices.

3. Methods

3.1. Preliminary Results Based on the Buys-Ballot Table

An important aspect of the time series analysis using the Buys-Ballot approach is the arrangement of the observed seasonal time series data in a Buys-Ballot table as shown in Table 1. All the derivations made in this section are based on the systematic component of the additive Buys-Ballot decomposition model.

Table 1.

The Buys-Ballot Tabular Arrangement of Time Series Data.

	Season)								
Period	1	2	$\cdots$	j	$\cdots$	s	T_i	$\bar{X}_i$	$\hat{\sigma}_i$
1	X_1	X_2	$\cdots$	X_j	$\cdots$	X_s	T_i	$\bar{X}_{1}.$	$\hat{\sigma}_{1}.$
2	X_{s+1}	X_{s+2}	$\cdots$	X_{s+j}	$\cdots$	X_{2s}	T_2	$\bar{X}_{2}.$	$\hat{\sigma}_{2}.$
3	X_{2s+1}	X_{2s+2}	$\cdots$	X_{2s+j}	$\cdots$	X_{3s}	T_3	$\bar{X}_{3}.$	$\hat{\sigma}_{3}.$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
i	$X_{(i-1)s+1}$	$X_{(i-1)s+2}$	$\cdots$	$X_{(i-1)s+j}$	$\cdots$	$X_{(i-1)s+s}$	$T_{i}.$	$\bar{X}_{i}.$	$\hat{\sigma}_{i}.$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
m	$X_{(m-1)s+1}$	$X_{(m-1)s+2}$	$\cdots$	$X_{(m-1)s+j}$	$\cdots$	$X_{(m-1)s+s}$	$T_{m}.$	$\bar{X}_{m}.$	$\hat{\sigma}_{m}.$
T_j	T_1	T_2	$\cdots$	T_j	$\cdots$	T_s	$T_{..}$		
$\bar{X}_{.j}$	$\bar{X}_{.1}$	$\bar{X}_{.2}$	$\cdots$	$\bar{X}_{.j}$	$\cdots$	$\bar{X}_{.s}$		$\bar{X}_{..}$	
$\hat{\sigma}_{.j}$	$\hat{\sigma}_{.1}$	$\hat{\sigma}_{.2}$	$\cdots$	$\hat{\sigma}_{.j}$	$\cdots$	$\hat{\sigma}_{.s}$			$\hat{\sigma}_{..}$

Source: Iwueze and Nwogu (2004)

If we make the substitution $M_t = a + bt + ct^2 + dt^3$ into (1) and consider only the systematic part of (1), we have

$$X_t = a + bt + ct^2 + dt^3 + S_t \tag{11}$$

From Table 1, the ith row total is given as

$$\begin{aligned}
T_{i.} &= \sum_{j=1}^{s} X_{(i-1)s+j} \\
&= \sum_{j=1}^{s} [a + b((i-1)s + j) + c((i-1)s + j)^2 + d((i-1)s + j)^3 + S_j] \\
&= \Big[as + b(i-1)s^2 + \frac{bs(s+1)}{2} + c(i-1)^2 s^3 + cs^2(s+1)(i-1) + \frac{cs(s+1)(2s+1)}{6} \\
&+ d(i-1)^3 s^4 + \frac{3d(i-1)^2 s^3(s+1)}{2} + \frac{3d(i-1)^2 s^2(s+1)(2s+1)}{6} + d\Big(\frac{s(s+1)}{2}\Big)^2\Big] \\
&= \Big[as + \frac{bs((2i-1)s+1)}{2} + cs^2(i-1)(is+1) + \frac{cs(s+1)(2s+1)}{6} \\
&+ \frac{ds^2(i-1)(is(2is-s+3)+1)}{2} + d\Big(\frac{s(s+1)}{2}\Big)^2\Big]
\end{aligned} \tag{12}$$

In deriving (12), we made use of the assumption $\sum_{j=1}^{s} S_j = 0$. Now, the ith row average is

$$\begin{aligned}
\bar{X}_{i.} &= \frac{T_{i.}}{s} \\
&= \Big[a + \frac{b((2i-1)s+1)}{2} + cs(i-1)(is+1) + \frac{c(s+1)(2s+1)}{6} \\
&+ \frac{ds(i-1)(is(2is-s+3)+1)}{2} + ds\Big(\frac{(s+1)}{2}\Big)^2\Big]
\end{aligned} \tag{13}$$

Next, we derive an expression for each of the jth column total and mean. With $\sum_{i=1}^{m} S_j = mS_j$, the jth column total becomes

$$
\begin{aligned}
T_{.j} &= \sum_{i=1}^{m} X_{(i-1)s+j} \\
&= am + b\sum_{i=1}^{m}\Big((i-1)s+j\Big) + c\sum_{i=1}^{m}\Big((i-1)s+j\Big)^2 + d\sum_{i=1}^{m}\Big((i-1)s+j\Big)^3 + mS_j \\
&= \Big[am + \frac{m(m-1)bs}{2} + mbj + \frac{m(m-1)(2m-1)cs^2}{6} + m(m-1)csj \\
&+ mcj^2 + ds^3\Big(\frac{m(m-1)}{2}\Big)^2 + 3ds^2j\Big(\frac{m(m-1)(2m-1)}{6}\Big) + 3dsj^2\Big(\frac{m(m-1)}{2}\Big) + dmj^3 + mS_j] \qquad (14)
\end{aligned}
$$

Dividing both sides of (14) by m, the jth column mean is obtained to be

$$
\begin{aligned}
\bar{X}_{.j} \quad &= \Big[a + \tfrac{(ms-s+2j)b}{2} + cj(j+(m-1)s) + \tfrac{(m-1)(2m-1)cs^2}{6} + \\
&+dj\Big(j^2 + \tfrac{(m-1)(2m-1)s^2}{2} + \tfrac{3sj(m-1)}{2}\Big) + \tfrac{dm(m-1)^2s^3}{4} + S_j\Big] \qquad (15)
\end{aligned}
$$

Furthermore, the grand total of the observations is given as

$$
\begin{aligned}
T_{..} &= \sum_{i=1}^{m} T_{.i} = \sum_{j=1}^{s} T_{.j} \\
&\Rightarrow T_{..} = \sum_{j=1}^{s} T_{.j} \\
&= \sum_{j=1}^{s}\Big[am + \frac{m(m-1)bs}{2} + mbj + \frac{m(m-1)(2m-1)cs^2}{6} + m(m-1)csj \\
&+ mcj^2 + ds^3\Big(\frac{m(m-1)}{2}\Big)^2 + 3ds^2j\Big(\frac{m(m-1)(2m-1)}{6}\Big) + 3dsj^2\Big(\frac{m(m-1)}{2}\Big) + dmj^3 + mS_j\Big] \\
&= an + \frac{bn(n+1)}{2} + \frac{cn(n+1)(2n+1)}{6} + d\Big(\frac{n(n+1)}{2}\Big)^2 \qquad (16)
\end{aligned}
$$

For $ms = n$, we have

$$
T_{..} \quad = an + \tfrac{bn(n+1)}{2} + \tfrac{cn(n+1)(2n+1)}{6} + d\Big(\tfrac{n(n+1)}{2}\Big)^2 \qquad (17)
$$

The grand mean is obtained by dividing (17) by n. Thus,

$$
\bar{X}_{..} = a + \tfrac{b(n+1)}{2} + \tfrac{c(n+1)(2n+1)}{6} + dn\Big(\tfrac{(n+1)}{2}\Big)^2 \qquad (18)
$$

3.2. The Proposed Buys-Ballot Estimators of the Cubic Trend-Cycle and Seasonal Components of the Additive Buys-Ballot Decomposition Model

Two classes of estimators, namely chain base and fixed base estimators of the requisite parameters are derived following the procedure of [17]. For the purpose of deriving the estimators, we find the first, second and third differences of the ith row mean series. From (13), we obtain

$$
\begin{aligned}
\bar{X}_{(i+1).} \quad &= \Big[a + \tfrac{b(2is+s+1)}{2} + cs(i^2s + is + i) + \tfrac{c(s+1)(2s+1)}{6} \\
&+ \tfrac{ds(2i^3s^2+3i^2s^2+2is^2+3i^2s+3is+i)}{2} + ds\Big(\tfrac{(s+1)}{2}\Big)^2\Big] \qquad (19)
\end{aligned}
$$

$$\bar{X}_{(i+2).} = \Big[a + \tfrac{b(2is+3s+1)}{2} + cs(i^2s + 3is + i + 2s + 1) + \tfrac{c(s+1)(2s+1)}{6} + \tfrac{ds(2i^3s^2+9i^2s^2+14is^2+3i^2s+9is+i+6s+7s^2+1)}{2} + ds\Big(\tfrac{(s+1)}{2}\Big)^2\Big] \tag{20}$$

$$\bar{X}_{(i+3).} = \Big[a + \tfrac{b(2is+5s+1)}{2} + cs(i^2s + 5is + i + 6s + 2) + \tfrac{c(s+1)(2s+1)}{6} + \tfrac{ds(2i^3s^2+15i^2s^2+38is^2+3i^2s+15is+i+18s+32s^2+2)}{2} + ds\Big(\tfrac{(s+1)}{2}\Big)^2\Big] \tag{21}$$

Given the forward difference operator $\triangle$, then the first forward difference of $\bar{X}_i.$ has the following representation

$$\begin{aligned} Y_{i.} &= \triangle\bar{X}_{i.} = \bar{X}_{(i+1).} - \bar{X}_{i.} \\ &= \Big[\frac{b(2is+1-2is+s+1)}{2} + cs(i^2s + is + i - i^2s - i + is + 1) \\ &+ \frac{ds(2i^3s^2 + 3i^2s^2 + 2is^2 + 3i^2s + 3is + i - 2i^3s^2 + 3i^2s^2 - 2is^2 - 3i^2s - i + 3is + s^2 + 1)}{2}\Big] \\ &= bs + cs(2is+1) + \frac{ds}{2}(6i^2s^2 + 6is + s^2 + 1) \end{aligned} \tag{22}$$

Hence,

$$\begin{aligned} \sum_{i=1}^{m-1} Y_{i.} &= \sum_{i=1}^{m-1}(\bar{X}_{(i+1).} - \bar{X}_{i.}) \\ &= \sum_{i=1}^{m-1}\Big[bs + cs(2is+1) + \frac{ds}{2}(6i^2s^2 + 6is + s^2 + 1)\Big] \\ &= \Big[bs(m-1) + 2cs^2\sum_{i=1}^{m-1} i + cs(m-1) + 3ds^3\sum_{i=1}^{m-1} i^2 + 3ds^2\sum_{i=1}^{m-1} i + \frac{ds^3}{2}(m-1) + \frac{ds}{2}(m-1)\Big] \\ &= \Big[b(n-s) + cs^2m(m-1) + cs(m-1) + \frac{ds^3}{2}m(m-1)(2m-1) \\ &+ \frac{3ds^2}{2}m(m-1) + \frac{ds^3}{2}(m-1) + \frac{ds}{2}(m-1)\Big] \\ &= (n-s)b + c(n+1)(n-s) + \frac{d(n-s)}{2}(n(2n-s+3) + s^2 + 1) \end{aligned} \tag{23}$$

From the foregoing, the following relationship between the estimators of b, c and d can be easily deduced

$$\hat{b} = \frac{\bar{X}_m - \bar{X}_1}{n-s} - \hat{c}(n+1) - \frac{\hat{d}(n(2n-s+3) + s^2 + 1)}{2} \tag{24}$$

Equation (24) simply reveals that unless the estimators of c and d are known, it is practically impossible to estimate b within the context of the Buys-Ballot methods used in this research work. Consequently, one may wish to know if the estimator of c is a function of that of d. To obtain the estimator of c, we evaluate the second difference of $\bar{X}_{i.}$. For this purpose, we have

$$\begin{aligned} Z_{i.} &= \triangle^2\bar{X}_{i.} \\ &= \bar{X}_{(i+2).} - 2\bar{X}_{(i+1).} + \bar{X}_{i.} \end{aligned} \tag{25}$$

If we substitute (13), (19) and (20) into (25) and simplify the resulting expression completely, the result in (26) is obtained

$$Z_{i.} = cs^2 + 3ds^2(2is + s + 1) \tag{26}$$

It follows from (26) that

$$\hat{c} = \frac{\sum_{i=1}^{m-2} Z_{i.}}{2s(n-2s)} - \frac{3\hat{d}(n+1)}{2} \tag{27}$$

The estimator of c in (27) certainly depends on that of d. To find the estimator of d, we first evaluate the third difference of $\bar{X}_{i.}$. Let

$$\begin{aligned} Q_{i.} &= \triangle^3 \bar{X}_{i.} \\ &= \bar{X}_{(i+3).} - 3\bar{X}_{(i+2).} + 3\bar{X}_{(i+1).} - \bar{X}_{i.} \end{aligned} \tag{28}$$

Simplifying (28) after $\bar{X}_{i.}$, $\bar{X}_{(i+1).}$, $\bar{X}_{(i+2).}$ and $\bar{X}_{(i+3).}$ have been replaced by (13), (19), (20) and (21) respectively, yields

$$Q_{i.} = 6ds^3 \tag{29}$$

Definition 1: Let $\hat{d}_i^{CBE} = \frac{Q_{i.}}{6s^3}, i = 1, 2, \cdots, m-3$. The chain base estimator of d in (29), denoted by $\hat{d}^{CBE}$ is the simple average of d_i. That is

$$\hat{d}^{CBE} = \frac{\sum_{i=1}^{m-3} Q_{i.}}{6(m-3)s^3} \tag{30}$$

Let $W_{i.} = Z_{(i+1).} - Z_{i.}$. Then,

$$\begin{aligned} W_{i.} &= 2cs^2 + 3ds^2(2(i+1)s + s + 1) - (2cs^2 + 3ds^2(2s + s + 1)) \\ \therefore W_{i.} &= 6dis^3 \end{aligned} \tag{31}$$

Definition 2: A simple average of the variable $\hat{d}_{i.}^{FBE} = \frac{W_{i.}}{6is^3}, i = 1, 2, \cdots, m-3$ is called the fixed base estimator of d. Hence, the fixed base estimator of d is

$$\begin{aligned} \hat{d}^{FBE} &= \frac{\sum_{i=1}^{m-3} \hat{d}_{i.}^{\;FBE}}{(m-3)} \\ &= \left(\frac{1}{6s^3(m-3)}\right) \sum_{i=1}^{m-3} \left(\frac{W_{i.}}{i}\right) \end{aligned} \tag{32}$$

The fixed base estimator of d may also be considered to be a weighted average of a Buys-Ballot derived variable $\hat{d}_{i.}^{*} = \frac{W_{i.}}{6s^3}$, such that the weights are $i^{-1}, i = 1, 2, 3, \cdots, m-3$. In line with (18), an estimator of a is generally given as

$$\hat{a} = \bar{X}_{..} - \frac{\hat{b}(n+1)}{2} - \frac{\hat{c}(n+1)(2n+1)}{6} - \hat{d}n\left(\frac{n+1}{2}\right)^2 \tag{33}$$

By substituting $n = ms$ into (15) and using (33), the estimator of S_j is found to be

$$\begin{aligned} \hat{S}_j &= \Big[\bar{X}_{.j} - \bar{X}_{..} + \tfrac{\hat{b}(1+s-2j)}{2} + \tfrac{\hat{c}(n^2+3n+3ns-s^2-6nj+6js-6j^2+1)}{6} \\ &\quad + \hat{d}\Big(\tfrac{2n^2+n-4j^3-6j^2n+6j^2s-4jn^2+6jns-2js^2+2n^2s-ns^2}{4}\Big)\Big] \end{aligned} \tag{34}$$

Owing to the fact that the derived estimators of a, b, c and S_j depend on that of d, those estimators are said to be chain base estimators if there are all functions of the chain base estimator of d. On the other hand, there are called fixed base estimators if there are determined using the fixed base estimator of d. Observe that in the absence of trend, $\hat{a} = \hat{b} = \hat{c} = \hat{d} = 0$ and

$$\hat{S}_j = \bar{X}_j - \bar{X}_{..} \tag{35}$$

4. Properties of the Estimators of Parameters of the Cubic Trend-Cycle Component

When two or more estimators of a particular parameter exist, as it is the case in this study, efforts are made to determine the best among the competing estimators. To determine a better class of estimators between the chain base and fixed base families of estimators, we derive and compare properties of the estimators belonging to the two classes. For an estimator to be unbiased for a particular parameter, its expectation must be equal to the given parameter. Taking expectation of both sides of (30) leads to

$$\begin{aligned} E\Big(\hat{d}_i^{CBE}\Big) &= E\Big(\frac{\bar{X}_{(i+3).} - 3\bar{X}_{(i+2).} + 3\bar{X}_{(i+1).} - \bar{X}_{i.}}{6s^3}\Big) \\ &= E\Big(\frac{6ds^3 + \bar{e}_{(i+3).} - 3\bar{e}_{(i+2).} + 3\bar{e}_{(i+1).} - \bar{e}_{i.}}{6s^3}\Big) \end{aligned}$$

We can deduce from the assumption of e_t associated with (1) that $e_{i.} \sim N(0, \frac{\sigma_1^2}{s})$. Thus,

$$E\Big(\hat{d}_i^{CBE}\Big) = d \tag{36}$$

Also,

$$\begin{aligned} E\Big(\hat{d}^{CBE}\Big) &= E\Big(\frac{\sum_{i=1}^{m-3} \hat{d}_{i.}^{CBE}}{(m-3)}\Big) \\ &= E\Big(\frac{\sum_{i=1}^{m-3} Q_{i.}}{6(m-3)s^3}\Big) \\ &= E\Big(d + \frac{\sum_{i=1}^{m-3}(\bar{e}_{(i+3).} - 3\bar{e}_{(i+2).} + 3\bar{e}_{(i+1).} - \bar{e}_{i.})}{6(m-3)s^3}\Big) \\ &= E\Big(d + \frac{-\bar{e}_{1.} + 2\bar{e}_{2.} - \bar{e}_{3.} + \bar{e}_{(m-2).} - 2\bar{e}_{(m-1).} + \bar{e}_{m.}}{6(m-3)s^3}\Big) \\ &= d \end{aligned} \tag{37}$$

It is now evident that both $\hat{d}_i^{CBE}$ and $\hat{d}^{CBE}$ are unbiased for d.

The variances of these estimators are obtained as follows:

$$\begin{aligned} var\Big(\hat{d}_i^{CBE}\Big) &= E\Big(\hat{d}_i^{CBE} - d\Big)^2 \\ &= E\Big(d + \frac{\bar{e}_{(i+3).} - 3\bar{e}_{(i+2).} + 3\bar{e}_{(i+1).} - \bar{e}_{i.}}{6(m-3)s^3} - d\Big)^2 \end{aligned}$$

Acknowledging the fact that the errors are uncorrelated, we get

$$Var\Big(\hat{d}_i^{CBE}\Big) = \frac{5\sigma_1^2}{9s^7} \tag{38}$$

$$\begin{aligned} Var\Big(\hat{d}^{CBE}\Big) &= E\Big(\hat{d}^{CBE} - d\Big)^2 \\ &= E\Big(d + \frac{\sum_{i=1}^{m-3}(\bar{e}_{(i+3).} - 3\bar{e}_{(i+2).} + 3\bar{e}_{(i+1).} - \bar{e}_{i.})}{6(m-3)s^3} - d\Big)^2 \\ &= E\Big(\frac{-\bar{e}_{1.} + 2\bar{e}_{2.} - \bar{e}_{3.} + \bar{e}_{(m-2).} - 2\bar{e}_{(m-1).} + \bar{e}_{m.}}{6(m-3)s^3}\Big)^2 \\ &= \frac{\sigma_1^2}{3(m-3)^2 s^7} \end{aligned} \tag{39}$$

Having knowledge of the covariance between $\hat{d}_i^{CBE}$ and $\hat{d}_h^{CBE}$, where $i \neq h$ helps us to determine if the chain base derived variables are independent or not. It will also help us to derive the autocorrelation function for the derived variables. With the autocorrelation function, one can figure out whether the variables are generated by a stationary series or a nonstationary series. Notationally,

$$\begin{aligned} Cov\Big(\hat{d}_i^{CBE}, \hat{d}_h^{CBE}\Big) &= \frac{1}{36s^6} E\Big[\bar{e}_{(i+3).}\bar{e}_{(h+3).} - 3\bar{e}_{(i+3).}\bar{e}_{(h+2).} + 3\bar{e}_{(i+3).}\bar{e}_{(h+1).} - \bar{e}_{(i+3).}\bar{e}_{h.} \\ &- 3\bar{e}_{(i+2).}\bar{e}_{(h+3).} + 9\bar{e}_{(i+2).}\bar{e}_{(h+2).} - 9\bar{e}_{(i+2).}\bar{e}_{(h+1).} + 3\bar{e}_{(i+2).}\bar{e}_{h.} + 3\bar{e}_{(i+1).}\bar{e}_{(h+3).} - 9\bar{e}_{(i+1).}\bar{e}_{(h+2).} \\ &+ 9\bar{e}_{(i+1).}\bar{e}_{(h+1).} - 3\bar{e}_{(i+1).}\bar{e}_{h.} - \bar{e}_{i}\bar{e}_{(h+3).} + 3\bar{e}_{i}\bar{e}_{(h+2).} - 3\bar{e}_{i.}\bar{e}_{(h+1).} + \bar{e}_{i.}\bar{e}_{h.}\Big] \end{aligned} \tag{40}$$

Let $k = h - i$. Then the autocovariance function for $\hat{d}_i^{CBE}$ is

$$R^{CBE}(k) = \begin{cases} \frac{5}{9s^7}\sigma_1^2, & \text{if } k = 0; \\ -\frac{5}{12s^7}\sigma_1^2, & \text{if } k = \pm 1; \\ \frac{1}{6s^7}\sigma_1^2, & \text{if } k = \pm 2; \\ -\frac{1}{36s^7}\sigma_1^2, & \text{if } k = \pm 3; \\ 0, & k \neq 0, \pm 1, \pm 2, \pm 3 \end{cases}$$

The following autocorrelation function for $\hat{d}_i^{CBE}$, is derived using $R^{CBE}(k)$

$$\rho^{CBE}(k) = \begin{cases} 1, & \text{if } k = 0; \\ -\frac{3}{4}, & \text{if } k = \pm 1; \\ \frac{3}{10}, & \text{if } k = \pm 2; \\ -\frac{1}{20}, & \text{if } k = \pm 3; \\ 0, & k \neq 0, \pm 1, \pm 2, \pm 3 \end{cases}$$

It is now certain that the CBE derived variables $\hat{d}_i^{CBE}, i = 1, 2, 3, ., ., ., m-3$ have the autocorrelation structure of a third-order moving average process, indicating that there are generated by a stationary process. We can also deduce from the work of [**?**] that the moving average process is invertible.

Next, we consider the properties of $\hat{c}^{CBE}$.

$$\begin{aligned} E\Big(\hat{c}^{CBE}\Big) &= E\Big(\frac{\sum_{i=1}^{m-2} Z_{i.}}{2s(n-2s)} - \frac{3}{2}(n+1)\hat{d}^{CBE}\Big) \\ &= E\Big(\frac{2cs^2(m-2) + 3ds^2\sum_{i=1}^{m-2}(2is+s+1) + \bar{e}_{(i+2).} - 2\bar{e}_{(i+1).} + \bar{e}_{i.})}{2s(n-2s)} - \frac{3}{2}(n+1)\hat{d}^{CBE}\Big) \\ &= E\Big(c + \frac{\sum_{i=1}^{m-2}(\bar{e}_{(i+2).} - 2\bar{e}_{(i+1).} + \bar{e}_{i.})}{2(m-2)s^2} - \frac{3}{2}(n+1)(\hat{d}^{CBE} - d)\Big) \\ &= E\Big(c + \frac{\bar{e}_{1.} - \bar{e}_{2.} - \bar{e}_{(m-1).} + \bar{e}_{m.}}{2(m-2)s^2} - \frac{3}{2}(n+1)(\hat{d}^{CBE} - d)\Big) \\ &= c \end{aligned} \tag{41}$$

$$
\begin{aligned}
\therefore Var(\hat{c}^{CBE}) &= E(\hat{c}^{CBE}-c)^2\\
&= E\Big(c+\frac{\bar{e}_{1.}-\bar{e}_{2.}-\bar{e}_{(m-1).}+\bar{e}_{m.}}{2(m-2)s^2}-\frac{3}{2}(n+1)(\hat{d}^{CBE}-d)-c\Big)^2\\
&= \Big[\frac{E\Big([\bar{e}_{1.}-\bar{e}_{2.}-\bar{e}_{(m-1).}+\bar{e}_{m.}]\Big)^2}{4(m-2)^2s^4}-\frac{3(n+1)E([\bar{e}_{1.}-\bar{e}_{2.}-\bar{e}_{(m-1).}+\bar{e}_{m.}][\hat{d}^{CBE}-d])}{2(m-2)s^2}\\
&+\frac{9(n+1)^2var(\hat{d}^{CBE})}{4}\Big]\\
&= \Big[\frac{\sigma_1^2}{(m-2)^2s^5}-\frac{3(n+1)E([\bar{e}_{1.}-\bar{e}_{2.}-\bar{e}_{(m-1).}+\bar{e}_{m.}][-\bar{e}_{1.}+2\bar{e}_{2.}-\bar{e}_{3.}+\bar{e}_{(m-2).}-2\bar{e}_{(m-1).}+\bar{e}_{m.}])}{12(m-2)(m-3)s^5}\\
&+\frac{3(n+1)^2\sigma_1^2}{4(m-3)^2s^7}\Big]\\
&= \frac{\sigma_1^2(4(m-3)^2s^2+3(n+1)^2(m-2)^2)}{4(m-2)^2(m-3)^2s^7} \qquad (42)
\end{aligned}
$$

In order to determine the nature of the relationship between $\hat{c}^{CBE}$ and $\hat{d}^{CBE}$, we need to find the covariance ($Cov(\hat{c}^{CBE},\hat{d}^{CBE})$) of $\hat{c}^{CBE}$ and $\hat{d}^{CBE}$. This covariance will be needed for the derivation of $Var(\hat{b}^{CBE})$.

$$
\begin{aligned}
Cov(\hat{c}^{CBE},\hat{d}^{CBE}) &= E\Big((\hat{c}^{CBE}-c)(\hat{d}^{CBE}-d)\Big)\\
&= E\Big((\frac{\bar{e}_{1.}-\bar{e}_{2.}-\bar{e}_{(m-1).}+\bar{e}_{m.}}{2(m-2)s^2}-\frac{3}{2}(n+1)(\hat{d}^{CBE}-d))(\hat{d}^{CBE}-d)\Big)\\
&= -\frac{3}{2}(n+1)var(\hat{d}^{CBE})\\
&= -\frac{\sigma_1^2}{2(m-3)^2s^7}(n+1) \qquad (43)
\end{aligned}
$$

Basic properties of $\hat{b}^{CBE}$ are given below:

$$
\begin{aligned}
E\Big(\hat{b}^{CBE}\Big) &= E\Big(\frac{\sum_{i=1}^{m-1}Y_{i.}}{(n-s)}-(n+1)\hat{c}^{CBE}-\frac{(n(2n-s)+3n+s^2+1)}{2}\hat{d}^{CBE}\Big)\\
&= E\Big(\frac{\sum_{i=1}^{m-1}(X_{(i+1.)}-X_{i.})}{(n-s)}-(n+1)\hat{c}^{CBE}-\frac{(n(2n-s)+3n+s^2+1)}{2}\hat{d}^{CBE}\Big)\\
&= E\Big(\frac{(n-s)b+c(n+1)(n-s)+\sum_{i=1}^{m-1}(\bar{e}_{(i+1).}-\bar{e}_{i.})}{n-s}+\frac{d(n(2n-s)+3n+s^2+1)(n-s)}{2(n-s)}\\
&-\hat{c}(n+1)-\frac{(n(2n-s)+3n+s^2+1)}{2}\hat{d}^{CBE}\Big)\\
&= E\Big(b+\frac{\sum_{i=1}^{m-1}(\bar{e}_{(i+1).}-\bar{e}_{i.})}{n-s}-(n+1)(\hat{c}^{CBE}-c)-\frac{(n(2n-s)+3n+s^2+1)}{2}(\hat{d}^{CBE}-d)\Big)\\
&= E\Big(b+\frac{(\bar{e}_{m.}-\bar{e}_{1.})}{n-s}-(n+1)(\hat{c}^{CBE}-c)-\frac{(n(2n-s)+3n+s^2+1)}{2}(\hat{d}^{CBE}-d)\Big)\\
&= b \qquad (44)
\end{aligned}
$$

$$\begin{aligned}
Var\Big(\hat{b}^{CBE}\Big) &= E\Big(\hat{b}^{CBE} - b\Big)^2 \\
&= E\Big(b + \frac{(\bar{e}_{m.} - \bar{e}_{1.})}{n-s} - (n+1)(\hat{c}^{CBE} - c) - \frac{(n(2n-s) + 3n + s^2 + 1)}{2}(\hat{d}^{CBE} - d) - b\Big)^2 \\
&= E\Big(\frac{(\bar{e}_{m.} - \bar{e}_{1.})}{n-s} - (n+1)\Big(\frac{\bar{e}_{1.} - \bar{e}_{2.} - \bar{e}_{(m-1).} + \bar{e}_{m.}}{2(m-2)s^2}\Big) \\
&- \frac{(n(2n-s) + 3n + s^2 + 1)}{2}\Big(\frac{-\bar{e}_{1.} + 2\bar{e}_{2.} - \bar{e}_{3.} + \bar{e}_{(m-2).} - 2\bar{e}_{(m-1).} + \bar{e}_{m.}}{6(m-3)s^3}\Big)\Big)^2 \\
&= E\Big(\frac{(\bar{e}_{m.} - \bar{e}_{1.})}{n-s}\Big)^2 + E\Big((n+1)\Big(\frac{\bar{e}_{1.} - \bar{e}_{2.} - \bar{e}_{(m-1).} + \bar{e}_{m.}}{2(m-2)s^2}\Big)\Big)^2 \\
&+ E\Big(\frac{n(2n-s) + 3n + s^2 + 1}{2}\Big(\frac{-\bar{e}_{1.} + 2\bar{e}_{2.} - \bar{e}_{3.} + \bar{e}_{(m-2).} - 2\bar{e}_{(m-1).} + \bar{e}_{m.}}{6(m-3)s^3}\Big)\Big)^2 \\
&- 2(n+1)E\Big(\frac{(\bar{e}_{m.} - \bar{e}_{1.})}{n-s}\Big(\frac{\bar{e}_{1.} - \bar{e}_{2.} - \bar{e}_{(m-1).} + \bar{e}_{m.}}{2(m-2)s^2}\Big)\Big) \\
&- 2E\Big(\Big(\frac{(\bar{e}_{m.} - \bar{e}_{1.})}{n-s}\Big)\Big(\frac{(n(2n-s) + 3n + s^2 + 1)}{2}\Big)\Big(\frac{-\bar{e}_{1.} + 2\bar{e}_{2.} - \bar{e}_{3.} + \bar{e}_{(m-2).} - 2\bar{e}_{(m-1).} + \bar{e}_{m.}}{6(m-3)s^3}\Big)\Big) \\
&+ 2(n+1)E\Big(\Big(\frac{\bar{e}_{1.} - \bar{e}_{2.} - \bar{e}_{(m-1).} + \bar{e}_{m.}}{2(m-2)s^2}\Big)\Big(\frac{n(2n-s) + 3n + s^2 + 1}{2}\Big) \\
&\Big(\frac{-\bar{e}_{1.} + 2\bar{e}_{2.} - \bar{e}_{3.} + \bar{e}_{(m-2).} - 2\bar{e}_{(m-1).} + \bar{e}_{m.}}{6(m-3)s^3}\Big)\Big) \\
&= \frac{\sigma_1^2}{6(m-1)^2(m-2)^2(m-3)^2s^7}\Big[12(m-2)^2(m-3)^2s^4 + 6(n+1)^2 \\
&(m-1)^2(m-3)^2s^2 + (n(2n-s) + 3n + s^2 + 1)^2(m-1)^2(m-2)^2 \\
&- 2(n(2n-s) + 3n + s^2 + 1)(m-1)(m-2)^2(m-3)s^2\Big]
\end{aligned} \tag{45}$$

$$\begin{aligned}
&Cov(\hat{b}^{CBE}, \hat{c}^{CBE}) = E(\hat{b}^{CBE} - b)(\hat{c}^{CBE} - c)) \\
&= E\Big(\Big(\frac{(\bar{e}_{m.} - \bar{e}_{1.})}{n-s} - (n+1)(\hat{c}^{CBE} - c) - \frac{(n(2n-s) + 3n + s^2 + 1)}{2}(\hat{d}^{CBE} - d)\Big) \\
&\Big(\frac{\bar{e}_{1.} - \bar{e}_{2.} - \bar{e}_{(m-1).} + \bar{e}_{m.}}{2(m-2)s^2} - \frac{3}{2}(n+1)(\hat{d}^{CBE} - d)\Big)\Big) \\
&= -\frac{\sigma_1^2(n+1)}{4(m-1)(m-2)^2(m-3)^2s^7}R
\end{aligned} \tag{46}$$

where

$$R = \Big[2(m-2)^2(m-3)s^2 + (m-1)(4(m-3)^2s^2 + 3(n+1)^2(m-2)^2) - (m-1)(m-2)^2(n(2n-s) + 3n + s^2 + 1)\Big]$$

$$\begin{aligned}
Cov(\hat{b}^{CBE}, \hat{d}^{CBE}) &= E(\hat{b}^{CBE} - b)(\hat{d}^{CBE} - d)) \\
&= E\Big(\Big(\frac{(\bar{e}_{m.} - \bar{e}_{1.})}{n-s} - (n+1)(\hat{c}^{CBE} - c) - \frac{(n(2n-s) + 3n + s^2 + 1)}{2}\Big)(\hat{d}^{CBE} - d)\Big)(\hat{d}^{CBE} - d)) \\
&= \sigma_1^2\Big[\frac{2(m-3)s^2 - 3(n+1)^2(m-1) - (n(2n-s) + 3n + s^2 + 1)(m-1)}{6(m-1)(m-3)^2s^7}\Big]
\end{aligned} \tag{47}$$

The properties of $\hat{a}^{CBE}$ are derived as follows

$$
\begin{aligned}
E\left(\hat{a}^{CBE}\right) &= E\left(\bar{X}_{..} - \frac{(n+1)}{2}\hat{b}^{CBE} - \frac{(n+1)(2n+1)}{6}\hat{c}^{CBE} - n\left(\frac{n+1}{2}\right)^2\hat{d}^{CBE}\right) \\
&= E\left(a - \frac{(n+1)}{2}(\hat{b}^{CBE} - b) - \frac{(n+1)(2n+1)}{6}(\hat{c}^{CBE} - c) - n\left(\frac{n+1}{2}\right)^2(\hat{d}^{CBE} - d)\right) \\
&= a
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
Var\left(\hat{a}^{CBE}\right) &= E\left(\hat{a}^{CBE} - a\right)^2 \\
&= E\left(-\frac{(n+1)}{2}(\hat{b}^{CBE} - b) - \frac{(n+1)(2n+1)}{6}(\hat{c}^{CBE} - c) - n\left(\frac{n+1}{2}\right)^2(\hat{d}^{CBE} - d)\right)^2 \\
&= \left(\frac{n+1}{2}\right)^2 Var\left(\hat{b}^{CBE}\right) + \left(\frac{(n+1)(2n+1)}{6}\right)^2 Var\left(\hat{c}^{CBE}\right) + n^2\left(\frac{n+1}{2}\right)^4 Var\left(\hat{d}^{CBE}\right) \\
&+ \left(\frac{(n+1)^2(2n+1)}{6}\right) Cov\left(\hat{b}^{CBE}, \hat{c}^{CBE}\right) + \left(\frac{n(n+1)^3}{4}\right) Cov\left(\hat{b}^{CBE}, \hat{d}^{CBE}\right) \\
&+ \left(\frac{n(n+1)^3(2n+1)}{12}\right) Cov\left(\hat{c}^{CBE}, \hat{d}^{CBE}\right)
\end{aligned} \tag{49}
$$

where $Var\left(\hat{b}^{CBE}\right), Var\left(\hat{c}^{CBE}\right), Var\left(\hat{d}^{CBE}\right), Cov\left(\hat{b}^{CBE}, \hat{c}^{CBE}\right), Cov\left(\hat{b}^{CBE}, \hat{d}^{CBE}\right)$ and $Cov\left(\hat{c}^{CBE}, \hat{d}^{CBE}\right)$ are given in (45), (42), (39), (46), (47) and (43) respectively. For the covariance of $\hat{a}^{CBE}$ with the other chain base estimators, we have

$$
\begin{aligned}
Cov\left(\hat{a}^{CBE}, \hat{b}^{CBE}\right) &= E\left((\hat{a}^{CBE} - a)(\hat{b}^{CBE} - b)\right) \\
&= E\bigg(\bigg(-\frac{(n+1)}{2}(\hat{b}^{CBE} - b) - \frac{(n+1)(2n+1)}{6}(\hat{c}^{CBE} - c) \\
&- n\left(\frac{n+1}{2}\right)^2(\hat{d}^{CBE} - d)\bigg)\left(\hat{b}^{CBE} - b\right)\bigg) \\
&= -\frac{(n+1)}{12}\bigg(6Var\left(\hat{b}^{CBE}\right) + (4n+2)Cov\left(\hat{b}^{CBE}, \hat{c}^{CBE}\right) \\
&+ (3n^2+3n)Cov\left(\hat{b}^{CBE}, \hat{d}^{CBE}\right)\bigg)
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
Cov\left(\hat{a}^{CBE}, \hat{c}^{CBE}\right) &= E\left((\hat{a}^{CBE} - a)(\hat{c}^{CBE} - c)\right) \\
&= E\bigg(\bigg(-\frac{(n+1)}{2}(\hat{b}^{CBE} - b) - \frac{(n+1)(2n+1)}{6}(\hat{c}^{CBE} - c) \\
&- n\left(\frac{n+1}{2}\right)^2(\hat{d}^{CBE} - d)\bigg)\left(\hat{c}^{CBE} - c\right)\bigg) \\
&= -\frac{(n+1)}{12}\bigg(6Cov\left(\hat{b}^{CBE}, \hat{c}^{CBE}\right) + (4n+2)Var\left(\hat{c}^{CBE}\right) \\
&+ (3n^2+3n)Cov\left(\hat{c}^{CBE}, \hat{d}^{CBE}\right)\bigg)
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
Cov\left(\hat{a}^{CBE}, \hat{d}^{CBE}\right) &= E\left((\hat{a}^{CBE} - a)(\hat{d}^{CBE} - c)\right) \\
&= E\bigg(\bigg(-\frac{(n+1)}{2}(\hat{b}^{CBE} - b) - \frac{(n+1)(2n+1)}{6}(\hat{c}^{CBE} - c) \\
&- n\left(\frac{n+1}{2}\right)^2(\hat{d}^{CBE} - d)\bigg)\left(\hat{d}^{CBE} - d\right)\bigg) \\
&= -\frac{(n+1)}{12}\bigg(6Cov\left(\hat{b}^{CBE}, \hat{d}^{CBE}\right) + (4n+2)Cov\left(\hat{c}^{CBE}, \hat{d}^{CBE}\right) \\
&+ (3n^2+3n)Var\left(\hat{d}^{CBE}\right)\bigg)
\end{aligned} \tag{52}
$$

The chain base estimator ($\hat{S}_j^{CBE}$) of the seasonal component formulated in Equation (35) is easily seen to be unbiased for (S_j^{CBE}).

Let $\mathbf{V}_c$ be the variance- covariance matrix pertaining to the derived chain base estimators. Then

$$\mathbf{V}_c = \begin{pmatrix} Var(\hat{a}^{CBE}) & Cov(\hat{a}^{CBE}, \hat{b}^{CBE}) & Cov(\hat{a}^{CBE}, \hat{c}^{CBE}) & Cov(\hat{a}^{CBE}, \hat{d}^{CBE}) \\ Cov(\hat{a}^{CBE}, \hat{b}^{CBE}) & Var(\hat{b}^{CBE}) & Cov(\hat{b}^{CBE}, \hat{c}^{CBE}) & Cov(\hat{b}^{CBE}, \hat{d}^{CBE}) \\ Cov(\hat{a}^{CBE}, \hat{c}^{CBE}) & Cov(\hat{b}^{CBE}, \hat{c}^{CBE}) & Var(\hat{c}^{CBE}) & Cov(\hat{c}^{CBE}, \hat{d}^{CBE}) \\ Cov(\hat{a}^{CBE}, \hat{d}^{CBE}) & Cov(\hat{b}^{CBE}, \hat{d}^{CBE}) & Cov(\hat{c}^{CBE}, \hat{d}^{CBE}) & Var(\hat{d}^{CBE}) \end{pmatrix} \tag{53}$$

Efficiency comparison based on determinants of variance-covariance matrices has been discussed by [10]. Given two point estimation methods, then one of the methods is said to be more efficient than the other if its corresponding variance- covariance matrix has a smaller determinant. We may observe that $\mathbf{V}_c$ depends on σ_1^2. Since σ_1^2 is often not known, it is estimated by the mean squared error(MSE). If we replace σ_1^2 by MSE, we obtain the estimated variance-covariance matrix.

So far, we have focused on the properties of the chain base estimators based on (1). In what follows, properties of fixed base estimators are discussed. Considering (32) and the associated error terms, we have

$$\begin{aligned} E\Big(\hat{d}_i^{FBE}\Big) &= E\Big(\frac{6dis^3 + \bar{e}_{(i+2).} - 2\bar{e}_{(i+1).} + \bar{e}_{i.} - \bar{e}_{3.} + 2\bar{e}_{2.} - \bar{e}_{1.}}{6is^3}\Big) \\ &= d \end{aligned} \tag{54}$$

$$\begin{aligned} Var\Big(\hat{d}_i^{FBE}\Big) &= E\Big(\hat{d}_i^{CBE} - d\Big)^2 \\ &= E\Big(d + \frac{\bar{e}_{(i+2).} - 2\bar{e}_{(i+1).} + \bar{e}_{i.} - \bar{e}_{3.} + 2\bar{e}_{2.} - \bar{e}_{1.}}{6is^3} - d\Big)^2 \\ &= \frac{\sigma_1^2}{3i^2 s^7} \end{aligned} \tag{55}$$

$$\begin{aligned} E\Big(\hat{d}^{FBE}\Big) &= E\Big(\frac{\sum_{i=1}^{m-3} \hat{d}_{i.}^{FBE}}{(m-3)}\Big) = E\Big(\frac{\sum_{i=1}^{m-3} W_{i.}}{6i(m-3)s^3}\Big) \\ &= E\Big(d + \frac{\sum_{i=1}^{m-3}(\bar{e}_{(i+2).} - 2\bar{e}_{(i+1).} + \bar{e}_{i.} - \bar{e}_{3.} + 2\bar{e}_{2.} - \bar{e}_{1.})}{6i(m-3)s^3}\Big) \\ &= E\Big(d + \frac{-(m-4)\bar{e}_{1.} + (2m-7)\bar{e}_{2.} - (m-3)\bar{e}_{3.} + \bar{e}_{(m-1).} - \bar{e}_{(m-2).} + \bar{e}_{m.}}{6i(m-3)s^3}\Big) \\ &= d \end{aligned} \tag{56}$$

$$\begin{aligned} \therefore Var\Big(\hat{d}^{FBE}\Big) &= E\Big(\hat{d}^{FBE} - d\Big)^2 \\ &= E\Big(\frac{-(m-4)\bar{e}_{1.} + (2m-7)\bar{e}_{2.} - (m-3)\bar{e}_{3.} + \bar{e}_{(m-1).} - \bar{e}_{(m-2).} + \bar{e}_{m.}}{6i(m-3)s^3}\Big)^2 \\ &= \frac{\sigma_1^2((m-4)^2 + (2m-7)^2 + (m-3)^2 + 3)}{36i^2(m-3)^2 s^7} \end{aligned} \tag{57}$$

Both $\hat{d}_i^{FBE}$ and $\hat{d}^{FBE}$ are unbiased for d. Generally, the fixed base estimators are unbiased for their respective parameters. Variances of $\hat{d}_i^{FBE}$ and $\hat{d}^{FBE}$ are functions of i, indicating nonstationarity of the fixed base derived variables and $\hat{d}^{FBE}$. For this reason, an empirical example based on the fixed base method will not be considered.

After estimating parameters of the cubic trend-cycle model, it will be expedient to test hypothesis about the individual parameters. Notably, each of the concerned chain base estimators of the cubic trend- cycle model is a linear function of the observed time series. Suppose we wish to test the null hypothesis $H_0 : g = g_0$ against $H_1 : g \neq g_0$, where g is any of the parameters a, b, c and d. For a Guassian white noise process and unknown σ_1^2, the t-test statistic

$$t = \frac{\hat{g} - g_0}{est.std(\hat{g})} \tag{58}$$

may be used, where $est.std(\hat{g})$ is the estimated standard deviation of the given parameter. At α level of significance, we reject H_0 if $|t| \geq t_{\frac{\alpha}{2}, n-4}$.

5. Empirical Results

A numerical example is given to illustrate the *chain base method* (CBM). We also compare the method with the *classical decomposition method* (CDM) through the decomposition of a real time series data set. The prediction accuracy measures employed in this work are the mean squared error (MSE), mean absolute error(MAE) and mean absolute percentage error (MAPE).

5.1. Real Life Example

As a practical application of CDM and CBM, we consider the time series decomposition of the monthly concentrations of atmospheric co_2 .The monthly atmospheric concentrations of co_2 data set for the period January, 1959 to December, 1997 is available in R package.

Following [5]and [2], it can be deduced that co_2 data exhibit trend and seasonal variation. In particular, the monthly atmospheric concentrations of co_2 data set for the period January, 1959 to December have been shown to have cubic trend and seasonal variations [26].

The estimated least squares cubic trend model for the time series is $\hat{T}_t = 316.265 + 0.0290513t + 0.000292787t^2 - 2.90208 \times 10^{-7}t^3$. Having fitted a cubic trend model, we proceed to investigate the significance of the overall regression using the results in Table 2.

Table 2.

Analysis of Variance Results for
the Cubic Trend Model for the Time Series
Data on Atmospheric Carbon (IV) Oxide.

Source	DF	SS	MS	F	P
Regression	3	102536	34178.6	7674.04	0.000
Error	464	2067	4.5		
Total	467	104602			

With a p-value of 0.000, it is obvious that the the fitted cubic trend model is statistically significant at 5 per cent significance level. This shows that the time series has a cubic trend component. Consequently, it is imperative to determine the coefficients that should be included in the fitted cubic trend model for the time series data. This calls for tests for significance of coefficients of the model parameters. Results based on these tests, are summarised in terms of p-values in Table 3.

Table 3.

Estimates of the Parameters
of the Cubic Trend Model
and their Corresponding P-values.

Term	Coefficient	P
Constant	316.265	0.000
t	0.0290513	0.000
t^2	0.00029278	0.000
t^3	-2.90208×10^{-7}	0.000

From all indications, it is necessary to consider a cubic trend model containing all the four coefficients. Trend parameter estimates as well as seasonal indices are given in Table 4 for CDM and CBM.

Table 4.

Decomposition Methods, Trend Parameter Estimates, Seasonal Indices and Associated MSE, MAE and MAPE for the Monthly Atmospheric Carbon (IV) Oxide Series.

Method	Cubic Trend Parameter Estimates	Month	Estimates of $SI's$	MSE	MAE	MAPE
CDM	316.265	Jan	-0.0601	0.251115	0.388781	0.114726
	0.0290513	Feb	0.6108			
	0.000292787	Mar	1.3602			
	-2.90208×10^{-7}	Apr	2.4979			
		May	2.9833			
		Jun	2.3270			
		Jul	0.8126			
		Aug	-1.2486			
		Sep	-3.0667			
		Oct	-3.2435			
		Nov	-2.0480			
		Dec	-0.9249			
CBM	315.237	Jan	-0.04783	277.228	11.1467	3.16444
	0.0451198	Feb	0.61908			
	0.000228465	Mar	1.37398			
	-2.13569×10^{-7}	Apr	2.49476			
		May	2.98169			
		Jun	2.33897			
		Jul	0.81557			
		Aug	-1.24826			
		Sep	-3.05251			
		Oct	-3.24903			
		Nov	-2.06545			
		Dec	-0.96097			

It is not surprising that minimum values of MSE, MAE and MAPE are associated with the classical decomposition method (CDM) since the chain base method (CBM) is basically meant for short series. The issue is then how short should the series be for CBM to be a good rival of CDM, when assumptions of the latter hold. To address this issue, we examine the following Figures. From Figure 1, it is evident that the series has a cubic trend

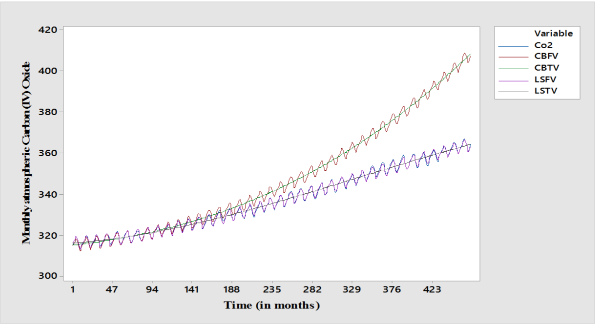

Figure 1. Plot of monthly co_2 concentrations (ppm), cubic trend values and fitted values

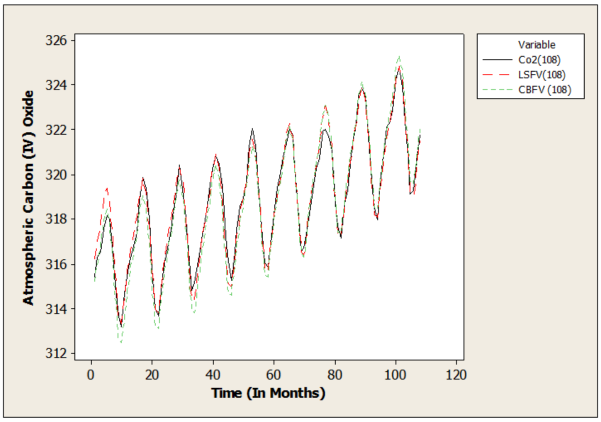

Figure 2. Time Plot of Monthly Co_2 Concentrations(ppm) and Fitted Values based on the first 108 Observations

and seasonal components. The Figure 2 shows that fits based on CDM are quite close to the actual observations for all time points. On the other hand, fits based on CBM are close to the actual observations from January of 1959 to December, 1967. Within this period, fits based on the two methods are also close, as indicated in Figure 2. The closeness of the two sets of fitted values, may be attributed to closeness of the corresponding least squares and chain base estimates in Table 4. Beyond the period, the two cubic trend curves (LSTV and CBTV) as well as their corresponding fits (LSFV and CBFV) in Figure 2, become more and more divergent as time increases. Lack of parallelism of least squares cubic trend curve and chain base cubic trend curve, can be deduced from Figure 3.

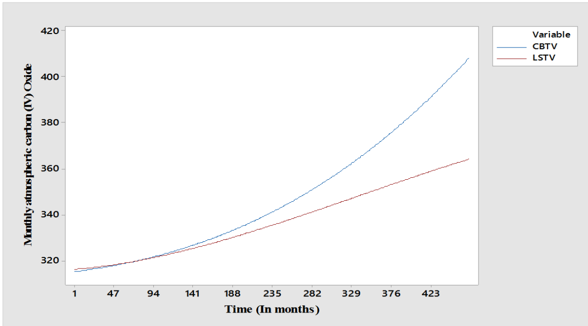

Figure 3. Trend Estimates based on CDM and CBM

6. Conclusion

Presented in this paper, are the Buys-Ballot and classical methods of decomposing a time series with a cubic trend component. Two sets of estimators, namely chain base and fixed base estimators have been derived and their properties investigated. While the two sets of estimators are generally unbiased, there is still a remarkable difference between their properties. The chain base estimators are functions of stationary variables with the third-order moving average model autocorrelation structure whereas the fixed base estimators depend on nonstationary variables. This is in agreement with the findings in [15, 16]. One might think of differencing the nonstationary variables to obtain the modified fixed base estimators based on stationary variables. We may note that the first difference of the derived fixed base variables yields the derived chain base variables which are stationary. As a result, the modified fixed base estimators will not be different from the already derived estimators of chain base type.

Being a linear combination of the derived chain base variables, each of the chain base estimators, is normally distributed when the underlying time series is normally distributed. It follows that hypothesis testing about the significance of a cubic model parameter can be carried out using a t statistic and an appropriate chain base estimate. There is no doubt that the chain base estimators are not mutually independent. Undoubtedly, the estimators are pairwise-negatively correlated.

We have graphically illustrated that for the first 108 obervations on the time series, CBM competes favourably with CDM. However, if there is a case of multicollinearity, CBM may be considered.

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