STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. **7**, December 2019, pp 731–747.
Published online in International Academic Press (www.IAPress.org)

Exponential Stability of a Transmission Problem with History and Delay

Noureddine Bahri [1], Abderrahmane Beniani [2,*]

[1] *University of Hassib Benbouali-Chlef 02000, Algeria*
[2] *Department of Mathematics, Center University of Belhadj Bouchaib -B.P. 284 RP, Ain Temouchent 46000, Algeria*

Abstract In this paper, we consider a transmission problem in the presence of history and delay terms.Under appropriate assumptions, we prove well-posedness by using the semigroup theory. Our stability estimate proves that the unique dissipation given by the history term is strong enough to stabilize exponentially the system in presence of delay by introducing a suitable Lyaponov functional.

Keywords Wave equation, Transmission problem, Past history, Delay term.

AMS 2010 subject classifications 35B37, 35L55, 74D05, 93D15, 93D20.

DOI: 10.19139/soic-2310-5070-728

1. Introduction

In this paper we study the following transmission system with a past history and a delay term

$$\begin{aligned} u_{tt}(x,t) - a u_{xx}(x,t) + \int_0^\infty g(s) u_{xx}(x,t-s)\mathrm{d}s \\ + \mu u_t(x,t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - b v_{xx}(x,t) = 0, \quad (x,t) \in (L_1,L_2) \times (0,+\infty), \end{aligned} \tag{1}$$

Under the boundary and transmission conditions

$$\begin{aligned} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i=1,2, \\ a u_x(L_i,t) - \int_0^\infty g(s) u_x(L_i,t-s)\mathrm{d}s = b v_x(L_i,t), \quad i=1,2, \end{aligned} \tag{2}$$

and the initial conditions

$$\begin{aligned} u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega, \ t \in (0,\tau), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2), \end{aligned} \tag{3}$$

where $0 < L_1 < L_2 < L_3$,$\Omega =]0, L_1[\bigcup]L_2, L_3[, a, \mu, b$ are positive constants,u_0 is given history, and $\tau > 0$ is the delay.

---

*Correspondence to: Abderrahmane Beniani (Email: a.beniani@yahoo.fr). Laboratory ACEDP, Department of Mathematics, Center University of Belhadj Bouchaib -B.P. 284 RP, Ain Temouchent 46000, Algeria.

ISSN 2310-5070 (online) ISSN 2311-004X (print)

Transmission problems arise in several applications of physics and biology. We note that problem (1)-(3) is related to the wave propagation over a body which consists of two different type of materials: the elastic part and the viscoelastic part that has the past history and time delay effect.

For wave equations with various dissipations, many results concerning stabilization of solutions have been proved. Recently, wave equations with viscoelastic damping have been investigated by many authors, see [2, 4, 3, 9, 8, 10, 16, 18] and the references therein. It is showed that the dissipation produced by the viscoelastic part can produce the decay of the solution. For example, A. Guesmia [6] studied the equation

$$u_{tt} - Au + \int_0^{\infty} g(t)Au(t-s)\mathrm{d}s + \mu u_t(t-\tau) = 0, \quad \text{in } \Omega \times (0,\infty),$$

and under the condition:

$$\exists \delta > 0, \quad g'(s) \leq -\delta g(s) \quad \forall\, s \in \mathbb{R}^+$$

the authors showed the exponential decay.

Messaoudi [12] investigated the following viscoelastic equation:

$$u_{tt} - \Delta u + \int_0^{t} g(t)\Delta u(t-s)\mathrm{d}s = 0, \quad \text{in } \Omega \times (0,\infty),$$

in a bounded domain, and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases.

In [7] the authors examined a system of wave equations with a linear boundary damping term with a delay:

$$\begin{aligned} & u_{tt}(x,t) - au_{xx}(x,t) + \int_0^{\infty} g(s)u_{xx}(x,t-s)\mathrm{d}s \\ & + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ & v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad (x,t) \in (L_1,L_2) \times (0,+\infty), \end{aligned} \tag{4}$$

and under the assumption

$$\mu_2 \leq \mu_1 \tag{5}$$

they proved that the solution is exponentially stable. On the contrary, if (5) does not hold, they found a sequence of delays for which the corresponding solution of (4) will be unstable.

In [11], authors considered the equation

$$u_{tt}(x,t) - \Delta_x u(x,t) - \mu_1 \Delta_x u_t(x,t) - \mu_2 \Delta_x u_t(x,t-\tau) = 0,$$

and under the assumption

$$|\mu_2| \leq \mu_1, \tag{6}$$

they proved the well-posedness and the exponential decay of energy.

Recently, in[19] Yadav and Jiwari considered Burgers'-Fisher equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au\frac{\partial u}{\partial x} + bu(1-u) = 0, \quad (x,t) \in (0,T) \times \Omega,$$

the authors proved existence and uniqueness of solution. Furthermore, they also presented finite element analysis and approximation.

The paper is organized as follows. The well-posedness of the problem is analyzed in Section 2 using the semigroup theory. In Section 3, we prove the exponential decay of the energy when time goes to infinity.

2. Preliminaries and assumptions

We assume that the function g satisfies the following:
A1: We assume that the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$g(0) > 0, \quad a - \int_0^\infty g(t)dt = a - g_0 = l > 0. \tag{7}$$

A2: There exists a positive constantδ,

$$g'(s) \leq -\delta g(s) \quad \forall s \in \mathbb{R}^+, \tag{8}$$

As in [14], we introduce the variable

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).$$

Then

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,\infty).$$

Following the ideal in [5], we set

$$\eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,t,s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{9}$$

Then

$$\eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t), \quad (x,t,s) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus, system (1) becomes

$$\begin{gathered} u_{tt}(x,t) - lu_{xx}(x,t) - \int_0^\infty g(s)\eta^t_{xx}(x,s)\mathrm{d}s + \mu z(x,1,t) = 0, \quad (x,t) \in \Omega \times (0,+\infty), \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad (x,t) \in (L_1,L_2) \times (0,+\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad (x,\rho,t) \in \Omega \times (0,1) \times (0,+\infty), \\ \eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t), \quad (x,s,t) \in \Omega \times (0,+\infty) \times (0,+\infty), \end{gathered} \tag{10}$$

the boundary and transmission conditions (2) become

$$\begin{gathered} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \ t \in (0,+\infty), \\ lu_x(L_i,t) + \int_0^\infty g(s)\eta^t_x(L_i,s)\mathrm{d}s = bv_x(L_i,t), \quad i = 1,2, \ t \in (0,+\infty), \end{gathered} \tag{11}$$

and the initial conditions (3) become

$$\begin{array}{lll} u(x,-t) = u_0(x,t), & u_t(x,0) = u_1(x), & x \in \Omega, \\ z(x,0,t) = u_t(x,t), & z(x,1,t) = f_0(x,t-\tau), & (x,t) \in \Omega \times (0,+\infty), \\ v(x,0) = v_0(x), & v_t(x,0) = v_1(x), & x \in (L_1,L_2), \end{array} \tag{12}$$

It is clear that

$$\begin{array}{ll} \eta^t(x,0) = 0, & \text{for all } x > 0, \\ \eta^t(0,s) = \eta^t(L_3,s) = 0, & \text{for all } s > 0, \\ \eta^0(x,s) = \eta_0(s), & \text{for all } s > 0. \end{array} \tag{13}$$

Let $V := (u, v, \varphi, \psi, z, \eta^t)^T$, then V satisfies the problem

$$\begin{gathered} V_t = (\mathscr{A} + \mathscr{B})V(t), \quad t > 0, \\ V(0) = V_0, \end{gathered} \tag{14}$$

where $V_0 := (u_0(\cdot,0), v_0, u_1, v_1, f_0(\cdot,-\tau), \eta_0)^T$ The operator $\mathscr{A}$ and $\mathscr{B}$ are linear and defined by

$$\mathscr{A}\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \\ w \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ lu_{xx} + \int_0^{+\infty} g(s)w_{xx}(s)\mathrm{d}s - \mu\varphi - \mu z(.,1) \\ bv_{xx} \\ -\frac{1}{\tau}z_\rho \\ -w_s + \varphi \end{pmatrix}$$

and

$\mathscr{B}(u, v, \varphi, \psi, z, \eta^t)^T = \mu(0, 0, \varphi, 0, 0, 0)^T$

where

$$X_* = \Big\{(u,v) \in H^1(\Omega) \times H^1(L_1, L_2) : u(0,t) = u(L_3,t) = 0, u(L_i,t) = v(L_i,t),$$
$$lu_x(L_i,t) + \int_0^\infty g(s)\eta_x^t(L_i,s)\mathrm{d}s = bv_x(L_i,t), i = 1,2\Big\}$$

and $L_g^2(\mathbb{R}_+, H^1(\Omega))$ denotes the Hilbert space of H^1-valued functions on $\mathbb{R}_+$, endowed with the inner product

$$(\phi, \vartheta)_{L_g^2(\mathbb{R}_+, H^1(\Omega))} = \int_\Omega \int_0^{+\infty} g(s)\phi_x(s)\vartheta_x(s)\mathrm{d}s\mathrm{d}x.$$

Set

$$V = (u, v, \varphi, \psi, z, w)^T, \quad \bar{V} = (\bar{u}, \bar{v}, \bar{\varphi}, \bar{\psi}, \bar{z}, \bar{w})^T.$$

We define the inner product in the energy space $\mathscr{H}$,

$$\langle V, \bar{V}\rangle_{\mathscr{H}} = \int_\Omega \varphi\bar{\varphi}\mathrm{d}x + \int_{L_1}^{L_2} \psi\bar{\psi}\mathrm{d}x + \int_\Omega lu_x\bar{u}_x\mathrm{d}x + \int_{L_1}^{L_2} bv_x\bar{v}_x\mathrm{d}x$$
$$+ \int_\Omega \int_0^{+\infty} g(s)w_x(s)\bar{w}_x(s)\mathrm{d}s\mathrm{d}x + \tau\mu \int_\Omega \int_0^1 z\bar{z}\mathrm{d}\rho\mathrm{d}x.$$

The domain of $\mathscr{A}$ is

$$D(\mathscr{A}) = \Big\{(u, v, \varphi, \psi, z, w)^T \in \mathscr{H} : (u,v) \in \big\{\big(H^2(\Omega) \times H^2(L_1, L_2)\big) \cap X_*\big\},$$
$$\varphi \in H^1(\Omega), \psi \in H^1(L_1, L_2), w \in L_g^2\left(\mathbb{R}_+, H^2(\Omega) \cap H^1(\Omega)\right), w_s \in \left(\mathbb{R}_+, H^1(\Omega)\right),$$
$$z_\rho \in L^2((0,1), L^2(\Omega)), w(x,0) = 0, z(x,0) = \varphi(x)\Big\}.$$

and $D(\mathscr{B}) = \mathscr{H}$ The well-posedness of problem (10)-(11) is ensured by the following theorem.

Theorem 1
Assume that (A1),(A2) hold. Let $V_0 \in \mathscr{H}$, then there exists a unique weak solution $V \in C(\mathbb{R}_+, \mathscr{H})$ of problem (14). Moreover, if $V_0 \in D(\mathscr{A})$, then

$$V \in C(\mathbb{R}_+, D(\mathscr{A})) \cap C^1(\mathbb{R}_+, \mathscr{H}).$$

Proof
We use the semigroup approach. So, first, we prove that the operator $\mathscr{A}$ is dissipative. In fact, for $(u, v, \varphi, \psi, z, w)^T \in D(\mathscr{A})$, where $\varphi(L_i) = \psi(L_i), i = 1,2$, we have

$$
\begin{aligned}
\langle \mathscr{A}V, V\rangle_{\mathscr{H}} = &\int_\Omega l u_{xx}\varphi \mathrm{d}x + \int_\Omega \Big(\int_0^{+\infty} g(s) w_{xx}(s)\mathrm{d}s - \mu\varphi - \mu z(.,1)\Big)\varphi \mathrm{d}x \\
&+ \int_\Omega l u_x \varphi_x \mathrm{d}x + \int_{L_1}^{L_2} b v_x \psi_x \mathrm{d}x + \int_{L_1}^{L_2} b v_{xx}\psi \mathrm{d}x \\
&+ \int_\Omega \int_0^{+\infty} g(s) w_x(s)(-w_s+\varphi)_x \mathrm{d}s\mathrm{d}x \\
&- \mu \int_\Omega \int_0^1 z z_\rho(x,\rho)\mathrm{d}\rho\mathrm{d}x.
\end{aligned} \tag{15}
$$

For the last term of the right side of (15), we obtain

$$
\mu \int_\Omega \int_0^1 z z_\rho(x,\rho)\mathrm{d}\rho\mathrm{d}x = \mu \int_\Omega \int_0^1 \frac{1}{2}\frac{\partial}{\partial \rho} z^2(x,\rho)\mathrm{d}\rho\mathrm{d}x = \frac{\mu}{2}\int_\Omega (z^2(x,1) - z^2(x,0))\mathrm{d}x.
$$

Noticing that $z(x,0,t) = \varphi(x,t)$, $w(x,0) = 0$ and $\varphi(L_i) = \psi(L_i), i = 1,2$, we obtain

$$
\begin{aligned}
\langle \mathscr{A}V, V\rangle_{\mathscr{H}} = &\Big[l u_x\varphi + \int_0^{+\infty} g(s) w_x(s)\mathrm{d}s\varphi\Big]_{\partial\Omega} + [b v_x\psi]_{L_1}^{L2} \\
&+ \int_\Omega (-\mu\varphi - \mu z(.,1))\varphi \mathrm{d}x - \frac{1}{2}\int_\Omega \Big[g(s)|w_x(x,s)|^2 \mathrm{d}s\Big]_0^{+\infty} \\
&+ \frac{1}{2}\int_\Omega \int_0^{+\infty} g'(s)|w_x(x,s)|^2\mathrm{d}s\mathrm{d}x - \frac{\mu}{2}\int_\Omega (z^2(x,1) - \varphi^2(x)\mathrm{d}x,
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
&[l u_x\varphi + \int_0^{+\infty} g(s) w_x(s)\mathrm{d}s\varphi]_{\partial\Omega} \\
&= \Big(l u_x(L_1,t) + \int_0^{+\infty} g(s) w_x(L_1,s)\mathrm{d}s\Big)\varphi(L_1,t) \\
&\quad - \Big(l u_x(L_2,t) + \int_0^{+\infty} g(s) w_x(L_2,s)\mathrm{d}s\Big)\varphi(L_2,t) \\
&= -[b v_x\psi]_{L_1}^{L2}
\end{aligned}
$$

Using Young's inequality, we have

$$
\langle \mathscr{A}V, V\rangle_{\mathscr{H}} = \frac{1}{2}\int_\Omega \int_0^{+\infty} g'(s)|w_x(x,s)|^2\mathrm{d}s\mathrm{d}x.
$$

Consequently, taking (A2) into account, we conclude that

$$
\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \le 0;
$$

that is, $\mathscr{A}$ is dissipative.

Next, we prove that $-\mathscr{A}$ is maximal. Actually, let $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathscr{H}$, we prove that there exists $V = (u, v, \varphi, \psi, z, w)^T \in D(\mathscr{A})$ satisfying

$$
(\lambda I - \mathscr{A})V = F, \tag{16}
$$

which is equivalent to

$$\begin{aligned} \lambda u-\varphi&=f_1,\\ \lambda v-\psi&=f_2,\\ \lambda\varphi-lu_{xx}-\int_0^\infty g(s)w_{xx}(s)\mathrm{d}s+\mu\varphi+\mu z(.,t)&=f_3,\\ \lambda\psi-bv_{xx}&=f_4,\\ \lambda z+\frac{1}{\tau}z_\rho&=f_5,\\ \lambda w+w_s-\varphi&=f_6. \end{aligned}\tag{17}$$

Assume that with the suitable regularity we have found u and v, then

$$\begin{aligned} \varphi&=\lambda u-f_1,\\ \psi&=\lambda v-f_2. \end{aligned}\tag{18}$$

So we have $\varphi\in H^1(\Omega)$ and $\psi\in H^1(L_1,L_2)$. Moreover, we can find z with

$$z(x,0)=\varphi(x),\quad \text{for } x\in\Omega.$$

Using the equation in (17), we obtain

$$z(x,\rho)=\varphi(x)e^{-\lambda\rho\tau}+\tau e^{-\lambda\rho\tau}\int_0^\rho f_5(x,\sigma)e^{\lambda\sigma\tau}\mathrm{d}\sigma.$$

From (18), we obtain

$$z(x,\rho)=\lambda ue^{-\lambda\rho\tau}-f_1e^{-\lambda\rho\tau}+\tau e^{-\lambda\rho\tau}\int_0^\rho f_5(x,\sigma)e^{\lambda\sigma\tau}\mathrm{d}\sigma.\tag{19}$$

It is easy to see that the last equation in (17) with $w(x,0)=0$ has a unique solution

$$\begin{aligned} w(x,s)&=\Big(\int_0^s e^{\lambda y}(f_6(x,y)+\varphi(x))\mathrm{d}y\Big)e^{-\lambda s}\\ &=\Big(\int_0^s e^{\lambda y}(f_6(x,y)+\lambda u(x)-f_1(x))\mathrm{d}y\Big)e^{-\lambda s}. \end{aligned}\tag{20}$$

By using (17), (18) and (20), the functions u and v satisfy

$$\begin{aligned} \left(\lambda^2+\mu\lambda+\mu\lambda e^{-\lambda\tau}\right)u-\tilde{l}u_{xx}&=\tilde{f},\\ \lambda^2 v-bv_{xx}&=f_4+\lambda f_2, \end{aligned}\tag{21}$$

where

$$\tilde{l}=l+\lambda\int_0^\infty g(s)e^{-\lambda s}\Big(\int_0^s e^{\lambda y}\mathrm{d}y\Big)\mathrm{d}s$$

and

$$\begin{aligned} \tilde{f}=&\int_0^\infty g(s)e^{-\lambda s}\Big(\int_0^s e^{\lambda y}(f_6(x,y)-f_1(x,y))_{xx}\mathrm{d}y\Big)\mathrm{d}s\\ &-\mu\tau e^{-\lambda\tau}\int_0^1 f_5(x,\sigma)e^{\lambda\sigma\tau}\mathrm{d}\sigma+\left(\lambda+\mu+\mu e^{-\lambda\tau}\right)f_1+f_3. \end{aligned}$$

We just need to prove that (21) has a solution $(u,v)\in X_*$ and replace in (18), (19) and (20) to get $V=(u,v,\varphi,\psi,z,w)^T\in D(\mathscr{A})$ satisfying (16). Consequently, problem (21) is equivalent to the problem

$$\Phi((u,v),(\omega_1,\omega_2))=l(\omega_1,\omega_2),\tag{22}$$

where the bilinear form $\Phi : (X_*, X_*) \to \mathbb{R}$ and the linear form $l : X_* \to \mathbb{R}$ are defined by

$$\Phi((u,v),(\omega_1,\omega_2)) = \int_\Omega \left[\left(\lambda^2 + \mu\lambda + \mu\lambda e^{-\lambda\tau}\right) u\omega_1 + \tilde{l}u_x(\omega)_x\right]\mathrm{d}x - [\tilde{l}u_x\omega_1]_{\partial\Omega} + \int_{L_1}^{L_2} \left(\lambda^2 v\omega_2 + bv_x(\omega_2)_x\right)\mathrm{d}x - [bv_x\omega_2]_{L_1}^{L_2}$$

and

$$l(\omega_1,\omega_2) = \int_\Omega \tilde{f}\omega_1\mathrm{d}x + \int_{L_1}^{L_2}(f_4 + \lambda f_2)\omega_2\mathrm{d}x.$$

Using the properties of the space X_*, it is easy to see that Φ is continuous and coercive, and l is continuous. Applying the Lax-Milgram theorem, we infer that for all $(\omega_1,\omega_2) \in X_*$, problem (22) has a unique solution $(u,v) \in X_*$. It follows from (21) that $(u,v) \in \left\{\left(H^2(\Omega) \times H^2(L_1,L_2)\right) \cap X_*\right\}$. Thence, the operator $\lambda I - \mathscr{A}$ is surjective for any $\lambda > 0$. That mean $\mathscr{A}$ is maximal monotone operator.Then,using Lummer-Phillips theorem [15],we dedece that $\mathscr{A}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$.
On the other hand, it is clear that the linear operator $\mathscr{B}$ is Lipschitz continuous.Finally,also $\mathscr{A} + \mathscr{B}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$. Consequently (14) is well-posed in the sense of Theorem 1(see [15]).

□

3. Exponential stability

In this section, we consider a decay result of problem (1)-(3). In fact using the energy method to produce a suitable Lyapunov functional

Theorem 2
Let (u,v) be the solution of (1)-(3). Assume that (A1),(A2) hold, and that

$$a > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2}l, \quad b > \frac{8(L_2 - L_1)}{L_1 + L_3 - L_2}l \tag{23}$$

then there exist two constants $\gamma_1, \gamma_2 > 0$ such that,

$$E(t) \le \gamma_2 e^{-\gamma_1 t}, \forall t \in \mathbb{R}_+ \tag{24}$$

For the proof of Theorem 2, we need some lemmas.
For a solution of (1)-(3), we define the energy

$$E(t) = \frac{1}{2}\int_\Omega [u_t^2(x,t) + lu_x^2(x,t)]\mathrm{d}x + \frac{1}{2}\int_{L_1}^{L_2}[v_t^2(x,t) + bv_x^2(x,t)]\mathrm{d}x + \frac{1}{2}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x + \frac{\tau\mu}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x, \tag{25}$$

Lemma 1
Let (u,v,η,z) be the solution of (10)-(11). Then we have the inequality

$$\frac{d}{dt}E(t) \le \mu\int_\Omega u_t^2(x,t)\mathrm{d}x + \frac{1}{2}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x. \tag{26}$$

Proof
We have

$$\begin{aligned}\frac{d}{dt}E(t) & \int_\Omega \Big(u_t u_{tt} + l u_x u_{xt} + \int_0^\infty g(s)\eta_x^t(x,s)\eta_{xt}^t \mathrm{d}s\Big)\mathrm{d}x \\ & + \int_{L_1}^{L_2} (v_t v_{tt} + b v_x v_{xt})\mathrm{d}x + \tau|\mu| \int_\Omega \int_0^1 z_t(x,\rho,t) z(x,\rho,t)\mathrm{d}\rho\mathrm{d}x \\ & = \Big[\Big(l u_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)u_t\Big]_{\partial\Omega} - [b v_x v_t]_{L_1}^{L_2} \\ & \quad - \int_\Omega \int_0^\infty g(s)\eta_x^t(x,s)\eta_{xs}^t(x,s)\mathrm{d}s\mathrm{d}x \\ & \quad - \mu \int_\Omega u_t z(x,1,t)dx + \frac{\mu}{2}\int_\Omega u_t^2(x,t)\mathrm{d}x - \frac{\mu}{2}\int_\Omega z^2(x,1,t)\mathrm{d}x \\ & = \quad \frac{1}{2}\int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x - \mu\int_\Omega u_t z(x,1,t)dx + \frac{\mu}{2}\int_\Omega u_t^2(x,t)\mathrm{d}x \\ & - \frac{\mu}{2}\int_\Omega z^2(x,1,t)\mathrm{d}x \end{aligned} \tag{27}$$

where we have used that

$$\begin{aligned} & \Big[\Big(l u_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)u_t\Big]_{\partial\Omega} \\ & = \Big(l u_x(L_1,t) + \int_0^\infty g(s)\eta_x^t(L_1,s)\mathrm{d}s\Big)u_t(L_1,t) \\ & \quad - \Big(l u_x(L_2,t) + \int_0^\infty g(s)\eta_x^t(L_2,s)\mathrm{d}s\Big)u_t(L_2,t) \\ & = -[b v_x v_t]_{L_1}^{L_2}, \end{aligned}$$

and

$$\Big[\frac{1}{2}\int_\Omega g(s)|\eta_x^t(x,s)|^2\mathrm{d}x\Big]_0^\infty = 0,$$

and

$$\frac{\tau\mu}{2}\frac{d}{dt}\int_\Omega \int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x = -\frac{\mu}{2\tau}\int_\Omega (z^2(x,1) - z^2(x,0))\mathrm{d}x. \tag{28}$$

Young's inequality gives us

$$\frac{d}{dt}E(t) \le \mu \int_\Omega u_t^2(x,t)\mathrm{d}x + \frac{1}{2}\int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.$$

□

Now, we define the functional

$$\mathscr{D}(t) = \int_\Omega u u_t \mathrm{d}x + \int_{L_1}^{L_2} v v_t \mathrm{d}x,$$

then we have the following lemma.

Lemma 2
The functional $\mathscr{D}(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}\mathscr{D}(t) \le & \int_\Omega u_t^2\mathrm{d}x + \int_{L_1}^{L_2} v_t^2\mathrm{d}x + (L^2\varepsilon + \varepsilon - l)\int_\Omega u_x^2\mathrm{d}x - \int_{L_1}^{L_2} b v_x^2\mathrm{d}x \\ & + \frac{g_0}{4\varepsilon}\int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x + \frac{\mu^2}{4\varepsilon}\int_\Omega z^2(x,1,t)\mathrm{d}x. \end{aligned} \tag{29}$$

Proof
Taking the derivative of $\mathscr{D}(t)$ with respect to t and using (10), we have

$$\begin{aligned}\frac{d}{dt}\mathscr{D}(t) &= \int_\Omega u_t^2 \mathrm{d}x - l\int_\Omega u_x^2 \mathrm{d}x - \mu\int_\Omega z(x,1,t)u\mathrm{d}x + [bv_x v]_{L_1}^{L_2} + \int_{L_1}^{L_2} v_t^2 \mathrm{d}x \\ &\quad + \Big[\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)u\Big]_{\partial\Omega} \\ &\quad - \int_\Omega u_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\mathrm{d}x - \int_{L_1}^{L_2} bv_x^2\mathrm{d}x \\ &= \int_\Omega u_t^2 \mathrm{d}x - l\int_\Omega u_x^2 \mathrm{d}x - \mu\int_\Omega z(x,1,t)u\mathrm{d}x + \int_{L_1}^{L_2} v_t^2 \mathrm{d}x \\ &\quad - \int_{L_1}^{L_2} bv_x^2\mathrm{d}x - \int_\Omega u_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\mathrm{d}x,\end{aligned} \tag{30}$$

where we used that

$$\begin{aligned}\Big[\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)u\Big]_{\partial\Omega} &= \Big(lu_x(L_1,t) + \int_0^\infty g(s)\eta_x^t(L_1,s)\mathrm{d}s\Big)u(L_1,t) \\ &\quad - \Big(lu_x(L_2,t) + \int_0^\infty g(s)\eta_x^t(L_2,s)\mathrm{d}s\Big)u(L_2,t) \\ &= -[bv_x v_t]_{L_1}^{L_2}.\end{aligned}$$

By the boundary conditions (2), we have

$$\begin{aligned}u^2(x,t) &= \Big(\int_0^x u_x(x,t)\mathrm{d}x\Big)^2 \le L_1\int_0^{L_1} u_x^2(x,t)\mathrm{d}x, \quad x\in[0,L_1], \\ u^2(x,t) &\le (L_3-L_2)\int_{L_2}^{L_3} u_x^2(x,t)\mathrm{d}x, \quad x\in[L_2,L_3],\end{aligned}$$

which implies

$$\int_\Omega u^2(x,t)\mathrm{d}x \le L^2\int_\Omega u_x^2\mathrm{d}x, \quad x\in\Omega, \tag{31}$$

where $L=\max\{L_1, L_3-L_2\}$. By making use of Young's inequality and (31), for any $\varepsilon>0$, we obtain

$$\mu\int_\Omega z(x,1,t)u\mathrm{d}x \le \frac{\mu^2}{4\varepsilon}\int_\Omega z^2(x,1,t)\mathrm{d}x + L^2\varepsilon\int_\Omega u_x^2\mathrm{d}x. \tag{32}$$

Young's inequality, Hölder's inequality and (A2) imply that

$$\int_\Omega u_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\mathrm{d}x \le \varepsilon\int_\Omega u_x^2(x,t)\mathrm{d}x + \frac{g_0}{4\varepsilon}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x. \tag{33}$$

Inserting the estimates (32) and (33) into (30), then (29) is fulfilled. □

Next, enlightened by [13], we introduce the functional

$$q(x) = \begin{cases} x - \frac{L_1}{2}, & x\in[0,L_1], \\ \frac{L_1}{2} - \frac{L_1+L_3-L_2}{2(L_2-L_1)}(x-L_1), & x\in(L_1,L_2), \\ x - \frac{L_2+L_3}{2}, & x\in[L_2,L_3]. \end{cases}$$

It is easy to see that $q(x)$ is bounded: $|q(x)| \leq M$, where $M = \max\{\frac{L_1}{2}, \frac{L_3-L_2}{2}\}$.

We define the functionals

$$\mathscr{F}_1(t) = -\int_\Omega q(x)u_t\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x, \quad \mathscr{F}_2(t) = -\int_{L_1}^{L_2} q(x)v_x v_t \mathrm{d}x,$$

then we have the following results.

Lemma 3

The functionals $\mathscr{F}_1(t)$ and $\mathscr{F}_2(t)$ satisfy

$$\begin{aligned}
&\frac{d}{dt}\mathscr{F}_1(t) \\
&\leq \Big(\frac{l+g_0}{2} + \varepsilon_1 M^2\Big)\int_\Omega u_t^2 \mathrm{d}x + \big(l^2 + l^2\varepsilon_1\big)\int_\Omega u_x^2 \mathrm{d}x \\
&\quad + \frac{M^2\mu^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)\mathrm{d}x + (g_0 + g_0\varepsilon_1)\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x \\
&\quad - \frac{g(0)}{4\varepsilon_1}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x - \Big[\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega} \\
&\quad - \Big[\frac{q(x)}{2}\Big(lu_x(x,t) + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\Big]_{\partial\Omega}
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
\frac{d}{dt}\mathscr{F}_2(t) \leq & -\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\Big(\int_{L_1}^{L_2} v_t^2 \mathrm{d}x + \int_{L_1}^{L_2} b v_x^2 \mathrm{d}x\Big) + \frac{L_1}{4}v_t^2(L_1) \\
& + \frac{L_3 - L_2}{4}v_t^2(L_2) + \frac{b}{4}\big((L_3 - L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)\big).
\end{aligned} \tag{35}$$

Proof

Taking the derivative of $\mathscr{F}_1(t)$ with respect to t and using (10), we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathscr{F}_1(t) \\
&= -\int_\Omega q(x)u_{tt}\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x \\
&\quad - \int_\Omega q(x)u_t\Big(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)\mathrm{d}s\Big)\mathrm{d}x \\
&= -\int_\Omega q(x)\Big(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s\Big)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x \\
&\quad + \mu\int_\Omega q(x)z(x,1,t)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x \\
&\quad - \int_\Omega q(x)u_t\Big(lu_{xt} + \int_0^\infty g(s)\eta_{xt}^t(x,s)\mathrm{d}s\Big)\mathrm{d}x.
\end{aligned} \tag{36}$$

We pay attention to

$$\begin{aligned}
&-\int_\Omega q(x)\Big(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s\Big)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)\mathrm{d}x \\
&= \frac{1}{2}\int_\Omega q'(x)\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\mathrm{d}x \\
&\quad - \Big[\frac{q(x)}{2}\Big(lu_x + \int_0^\infty g(s)\eta_x^t(x,s)\mathrm{d}s\Big)^2\Big]_{\partial\Omega}.
\end{aligned} \tag{37}$$

The last term in (36) can be treated as follows

$$\begin{aligned}
&-\int_\Omega q(x)u_t\Big(lu_{xt}+\int_0^\infty g(s)\eta^t_{xt}(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&=-l\int_\Omega q(x)u_tu_{xt}\mathrm{d}x-\int_\Omega q(x)u_t\int_0^\infty g(s)\eta^t_{xt}(x,s)\mathrm{d}s\mathrm{d}x\\
&=\Big[-\frac{l}{2}q(x)u_t^2\Big]_{\partial\Omega}+\frac{l}{2}\int_\Omega q'(x)u_t^2\mathrm{d}x\\
&\quad-\int_\Omega q(x)u_t\int_0^\infty g(s)\left(u_t-\eta^t_s\right)_x\mathrm{d}s\mathrm{d}x\\
&=\Big[-\frac{l}{2}q(x)u_t^2\Big]_{\partial\Omega}+\frac{l}{2}\int_\Omega q'(x)u_t^2\mathrm{d}x-g_0\int_\Omega q(x)u_tu_{tx}\mathrm{d}x\\
&\quad+\int_\Omega q(x)u_t\int_0^\infty g(s)\eta^t_{sx}(x,s)\mathrm{d}s\mathrm{d}x\\
&=\Big[-\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega}+\frac{l+g_0}{2}\int_\Omega q'(x)u_t^2\mathrm{d}x\\
&\quad-\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta^t_x\mathrm{d}s\mathrm{d}x,
\end{aligned}\tag{38}$$

where we used that

$$-\Big[\int_\Omega q(x)u_tg(s)\eta^t_x(x,s)\mathrm{d}x\Big]_0^\infty=0.$$

Inserting (37) and (38) in (36), we arrive at

$$\begin{aligned}
&\frac{d}{dt}\mathscr{F}_1(t)\\
&=-\Big[\frac{q(x)}{2}\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\mathrm{d}s\Big)^2\Big]_{\partial\Omega}-\Big[\frac{l+g_0}{2}q(x)u_t^2\Big]_{\partial\Omega}\\
&\quad+\frac{1}{2}\int_\Omega q'(x)\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\mathrm{d}s\Big)^2\mathrm{d}x\\
&\quad+\mu\int_\Omega q(x)z(x,1,t)\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\mathrm{d}s\Big)\mathrm{d}x\\
&\quad+\frac{l+g_0}{2}\int_\Omega q'(x)u_t^2\mathrm{d}x-\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta^t_x\mathrm{d}s\mathrm{d}x.
\end{aligned}\tag{39}$$

Using Minkowski and Young's inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\mathrm{d}s\Big)^2\mathrm{d}x\\
&\le l^2\int_\Omega u_x^2\mathrm{d}x+g_0\int_\Omega\int_0^\infty g(s)|\eta^t_x(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned}\tag{40}$$

Young's inequality gives us that for any $\varepsilon_1>0$,

$$\begin{aligned}
&\Big|\mu\int_\Omega q(x)z(x,1,t)\Big(lu_x+\int_0^\infty g(s)\eta^t_x(x,s)\mathrm{d}s\Big)\mathrm{d}x\Big|\\
&\le\frac{M^2\mu^2}{4\varepsilon_1}\int_\Omega z^2(x,1,t)\mathrm{d}x+l^2\varepsilon_1\int_\Omega u_x^2(x,t)\mathrm{d}x\\
&\quad+g_0\varepsilon_1\int_\Omega\int_0^\infty g(s)|\eta^t_x(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned}\tag{41}$$

It is clear that

$$\begin{aligned}&\left|\int_\Omega q(x)u_t\int_0^\infty g'(s)\eta_x^t \mathrm{d}s\mathrm{d}x\right|\\ &\leq \varepsilon_1 M^2\int_\Omega u_t^2\mathrm{d}x-\frac{g(0)}{4\varepsilon_1}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.\end{aligned}\tag{42}$$

Inserting (40)-(42) into (39), we obtain (34).

By the same method, taking the derivative of $\mathscr{F}_1(t)$ with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_2(t)&=-\int_{L_1}^{L_2} q(x)v_{xt}v_t\mathrm{d}x-\int_{L_1}^{L_2} q(x)v_xv_{tt}\mathrm{d}x\\ &=\Big[-\frac{1}{2}q(x)v_t^2\Big]_{L_1}^{L_2}+\frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2\mathrm{d}x+\frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2\mathrm{d}x\\ &\quad+\Big[-\frac{b}{2}q(x)v_x^2\Big]_{L_1}^{L_2}\\ &\leq -\frac{L_1+L_3-L_2}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2\mathrm{d}x+\int_{L_1}^{L_2} bv_x^2\mathrm{d}x\Big)+\frac{L_1}{4}v_t^2(L_1)\\ &\quad+\frac{L_3-L_2}{4}v_t^2(L_2)+\frac{b}{4}\left((L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)\right).\end{aligned}$$

Thus, the proof of Lemma 3 is complete. □

We define the functional

$$\mathscr{F}_3(t)=\tau\int_\Omega\int_0^1 e^{-\tau\rho}z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x,$$

then we have the following estimate.

Lemma 4

The functionals $\mathscr{F}_3(t)$ satisfies

$$\frac{d}{dt}\mathscr{F}_3(t)\leq -c_2\Big(\int_\Omega z^2(x,1,t)\mathrm{d}x+\tau\int_\Omega\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\Big)+\int_\Omega u_t^2(x,t)\mathrm{d}x.$$

Proof

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_3(t)&=2\tau\int_0^1\int_\Omega e^{-\tau\rho}z_t(x,\rho,t)z(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\\ &=-2\int_0^1\int_\Omega e^{-\tau\rho}z_\rho(x,\rho,t)z(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\\ &=-\int_0^1\int_\Omega e^{-\tau\rho}\frac{\partial}{\partial\rho}\Big(z^2(x,\rho,t)\Big)\mathrm{d}\rho\mathrm{d}x\\ &=-\tau\int_0^1\int_\Omega e^{-\tau\rho}z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x+\int_\Omega u_t^2(x,t)\mathrm{d}x-e^{-\tau}\int_\Omega z^2(x,1,t)\mathrm{d}x\\ &\leq -e^{-\tau}\Big(\tau\int_0^1\int_\Omega z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x+\int_\Omega z^2(x,1,t)\mathrm{d}x\Big)+\int_\Omega u_t^2(x,t)\mathrm{d}x.\end{aligned}$$

□

We define the functional

$$\mathscr{F}_4(t) = -\int_\Omega u_t \int_0^\infty g(s)(u(t) - u(t-s))\mathrm{d}s\mathrm{d}x,$$

then we have the following estimate.

Lemma 5
The functional $\mathscr{F}_4(t)$ satisfies

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_4(t) \leq& -(g_0 - \delta_2)\int_\Omega u_t^2\mathrm{d}x + \delta_2 l^2 \int_\Omega u_x^2 \mathrm{d}x + \delta_2\mu\int_\Omega z^2(x,1,t)\mathrm{d}x \\ &+ \Big(g_0 + \frac{g_0}{4\delta_2} + \frac{\mu g_0 L^2}{2\delta_2}\Big)\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x \\ &- \frac{g(0)L^2}{\delta_2}\int_\Omega\int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{43}$$

Proof

Taking the derivative of $\mathscr{F}_4(t)$ with respect to t and using (10), we have

$$\begin{aligned} &\frac{d}{dt}\mathscr{F}_4(t) \\ &= -\int_\Omega \Big(lu_{xx} + \int_0^\infty g(s)\eta_{xx}^t(x,s)\mathrm{d}s - \mu z(x,1,t)\Big) \\ &\quad \times \int_0^\infty g(s)(u(t)-u(t-s))\mathrm{d}s\mathrm{d}x - \int_\Omega u_t\int_0^\infty g(s)(u_t(t) - u_t(t-s))\mathrm{d}s\mathrm{d}x \\ &= \int_\Omega lu_x \int_0^\infty g(s)(u_x(t) - u_x(t-s))\mathrm{d}s\mathrm{d}x - g_0\int_\Omega u_t^2\mathrm{d}x \\ &\quad + \int_\Omega u_t\int_0^\infty g(s)\eta_s^t(s)\mathrm{d}s\mathrm{d}x + \int_\Omega\Big(\int_0^\infty g(s)(u_x(t)-u_x(t-s))\mathrm{d}s\Big)^2\mathrm{d}x \\ &\quad + \int_\Omega \mu z(x,1,t)\int_0^\infty g(s)(u(t)-u(t-s))\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{44}$$

Using Young's inequality and (31), we obtain for any $\delta_2 > 0$,

$$\begin{aligned} &\int_\Omega lu_x\int_0^\infty g(s)(u_x(t)-u_x(t-s))\mathrm{d}s\mathrm{d}x \\ &\leq \delta_2 l^2\int_\Omega u_x^2\mathrm{d}x + \frac{g_0}{4\delta_2}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x, \end{aligned} \tag{45}$$

$$\begin{aligned} &\int_\Omega \mu z(x,1,t)\int_0^\infty g(s)(u(t)-u(t-s))\mathrm{d}s\mathrm{d}x \\ &\leq \delta_2\mu\int_\Omega z^2(x,1,t)\mathrm{d}x + \frac{\mu g_0 L^2}{4\delta_2}\int_\Omega\int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x. \end{aligned} \tag{46}$$

We notice that

$$
\begin{aligned}
&\int_\Omega \Big(\int_0^\infty g(s)(u_x(t)-u_x(t-s))\mathrm{d}s \Big)^2 \mathrm{d}x \\
&= \int_\Omega \Big(\int_0^\infty \sqrt{g(s)}\sqrt{g(s)}(u_x(t)-u_x(t-s))\mathrm{d}s \Big)^2 \mathrm{d}x \\
&\le \int_\Omega \int_0^\infty g(s)\mathrm{d}s \Big(\int_0^\infty g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s \Big)\mathrm{d}x \\
&\le g_0 \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x
\end{aligned}
\tag{47}
$$

and

$$
\begin{aligned}
\int_\Omega u_t \int_0^\infty g(s)\eta_s^t(s)\mathrm{d}s\mathrm{d}x &= -\int_\Omega u_t \int_0^\infty g'(s)\eta^t(s)\mathrm{d}s\mathrm{d}x \\
&\le \delta_2 \int_\Omega u_t^2 \mathrm{d}x - \frac{g(0)L^2}{4\delta_2}\int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned}
\tag{48}
$$

Inserting the estimates (45)-(48) into (44), we obtain (43). The proof is complete.

□

Proof
We define the Lyapunov functional

$$
\mathscr{L}(t) = N_1 E(t) + N_2 \mathscr{D}(t) + \mathscr{F}_1(t) + N_4 \mathscr{F}_2(t) + N_5 \mathscr{F}_3(t) + N_6 \mathscr{F}_4(t),
\tag{49}
$$

where N_1, N_2, N_4, N_5 and N_6 are positive constants that will be fixed later.
Taking the derivative of (49) with respect to t and taking advantage of the above lemmas, we have

$$
\begin{aligned}
\frac{d}{dt}\mathscr{L}(t) \le &-\Big\{ N_6(g_0-\delta_2) - N_2 - \Big(\frac{l+g_0}{2} + \varepsilon_1 M^2\Big) \\
&- N_5 - N_1\mu \Big\} \int_\Omega u_t^2 \mathrm{d}x \\
&- \Big\{ N_5 c_2 - \frac{N_2\mu^2}{4\varepsilon} - \frac{M^2\mu^2}{4\varepsilon_1} - N_6\delta_2\mu \Big\} \int_\Omega z^2(x,1,t)\mathrm{d}x \\
&- \{ N_2(l - L^2\varepsilon - \varepsilon) - (l^2 + l^2\varepsilon_1) - N_6\delta_2 l^2 \} \int_\Omega u_x^2 \mathrm{d}x \\
&- \{ \frac{b(L_1+L_3-L_2)}{4(L_2-L_1)} N_4 + N_2 b \} \int_{L_1}^{L_2} v_x^2 \mathrm{d}x \\
&- \Big\{ \frac{L_1+L_3-L_2}{4(L_2-L_1)} N_4 - N_2 \Big\} \int_{L_1}^{L_2} v_t^2 \mathrm{d}x \\
&- (b - N_4)\frac{b}{4}\big((L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t) \big) \\
&- (a - N_4)\Big[\frac{L_1}{4} v_t^2(L_1,t) + \frac{L_3-L_2}{4} v_t^2(L_2,t) \Big] \\
&+ c(N_2,N_6) \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x \\
&+ \Big(\frac{N_1}{2} - \frac{g(0)}{4\varepsilon_1} - \frac{N_6 g(0)L^2}{4\delta_2} \Big) \int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2\mathrm{d}s\mathrm{d}x.
\end{aligned}
\tag{50}
$$

At this moment, we wish all coefficients except the last two in (50) will be negative. We want to choose N_2 and N_4 to ensure that

$$a - N_4 \geq 0, \quad b - N_4 \geq 0,$$
$$\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} N_4 - N_2 > 0.$$

. For this purpose, since $\frac{8l(L_2-L_1)}{L_1+L_3-L_2} < \min\{a, b\}$ we first choose N_4 satisfying

$$\frac{8l(L_2 - L_1)}{L_1 + L_3 - L_2} < N_4 \leq \min\{a, b\}.$$

Once N_4 is fixed, we pick N_2 satisfying

$$2l < N_2 < \frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} N_4.$$

Then we take ε,ε_1 and ε_1 small enough, and $\delta_2 < \frac{1}{2N_6}$ we have

$$N_2(l - L^2\varepsilon - \varepsilon) - 2l^2\varepsilon_1 > \frac{3}{2}l^2.$$

Once ε and ε_1 are fixed, we take N_5 satisfying

$$N_5 > \max\left\{\frac{2N_2\mu^2}{\varepsilon c_2}, \frac{2M^2\mu^2}{\varepsilon_1 c_2}\right\}$$

and $\delta_2 < \frac{N_5 c_2}{8N_6\mu}$ such that

$$N_5 c_2 - \frac{N_2\mu^2}{4\varepsilon} - \frac{M^2\mu^2}{4\varepsilon_1} > \frac{3}{8}N_5 c_2.$$

Further, we take $\delta_2 < \frac{g_0}{2}$ we choose N_6 satisfying

$$N_6 > \frac{2N_2}{g_0} + \frac{l + g_0}{g_0} + \frac{2\varepsilon_1 M^2}{g_0} + \frac{2N_5}{g_0} + \frac{2N_1\mu}{g_0}.$$

Then we have

$$N_6 > \max\left\{\frac{2N_2}{g_0}, \frac{l + g_0}{g_0} + \frac{2\varepsilon_1 M^2}{g_0}, \frac{2N_5}{g_0}, \frac{2N_1\mu}{g_0}\right\}.$$

Then, we pick δ_2 satisfying

$$\delta_2 < \min\left\{\frac{g_0}{2}, \frac{N_5 c_2}{8N_6\mu}, \frac{1}{2N_6}\right\},$$

$$\left\{N_5 c_2 - \frac{N_2\mu^2}{4\varepsilon} - \frac{M^2\mu^2}{4\varepsilon_1} - N_6\delta_2\mu\right\} \geq 0.$$

Once

$$\left\{N_2(l - L^2\varepsilon - \varepsilon) - \left(l^2 + l^2\varepsilon_1\right) - N_6\delta_2 l^2\right\} \geq 0.$$

Finally, choosing N_1 large enough such that the first and the last coefficients in (50) is positive.

From the above, we deduce that there exist two positive constants α_1 and α_2 such that (50) becomes

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t) &\leq -\alpha_1 E(t) + \alpha_2 \int_\Omega \int_0^\infty g(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x \\ &\leq -\alpha_1 E(t) - \frac{\alpha_2}{\delta} \int_\Omega \int_0^\infty g'(s)|\eta_x^t(x,s)|^2 \mathrm{d}s\mathrm{d}x \\ &\leq -\alpha_1 E(t) - \alpha_3 E'(t).\end{aligned} \tag{51}$$

That is

$$\big(\mathscr{L}(t)+\alpha_3 E(t)\big)' \le -\alpha_1 E(t) \tag{52}$$

where $\alpha_3>0$. Denote $\mathcal{E}(t)=\mathscr{L}(t)+\alpha_3 E(t)$, then it is easy to see that

$$\mathcal{E}(t)\sim E(t),$$

i.e., there exist two positive constants β_1,β_2:

$$\beta_1\, E(t)\le \mathcal{E}(t)\le \beta_2 E(t), \quad \forall t\ge 0. \tag{53}$$

Combining (52) and (53), we deduce that there exists $\gamma_1>0$ for which the estimate

$$\frac{d\mathcal{E}(t)}{dt}\le -\gamma_1\mathcal{E}(t), \quad \forall t\ge 0, \tag{54}$$

since

$$\mathcal{E}(t)(t)\le \mathcal{E}(0)e^{-\gamma_1 t}, \quad \forall t\ge 0. \tag{55}$$

Consequently, using (55)and (53), we find

$$E(t)\le \frac{1}{\beta_1}\mathcal{E}(t)\le \frac{1}{\beta_1}\mathcal{E}(0)e^{-\gamma_1 t}, \quad \forall t\ge 0. \tag{56}$$

Thus, the proof of Theorem 2 is complete. □

4. Conclusion

In this paper we study the following transmission system with a past history and a delay term. Under assumptions on initial data and boundary conditions, past history and a delay term, we focused our study on the existence and asymptotic behavior of solutions where we obtained exponential decay of solutions for transmission problems.

REFERENCES

1. A. Benseghir, *Existence and exponential decay of solutions for transmission problems with delay.* Electronic Journal of Differential Equations, 212 (2014) , 1–11.
2. S. Berrimi, S. A. Messaoudi; *Existence and decay of solutions of a viscoelastic equation with a nonlinear source*, Nonlinear Anal., 64 (2006), no. 10, 2314–2331.
3. M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano; *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, Electron. J. Differential Equations,(2002), No. 44, 14 pp.
4. M. M. Cavalcanti et al., *Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping*, Differential Integral Equations, **14** (2001), no. 1, 85–116.
5. C. M. Dafermos, *Asymptotic stability in viscoelasticity.* Archive for rational mechanics and analysis, 37(4) (1970), 297-308.
6. A. Guesmia, *Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay.* IMA Journal of Mathematical Control and Information, 30(4) (2013), 507-526.
7. G.Li,D. Wang, , B. Zhu, *Well-posedness and decay of solutions for a transmission problem with history and delay.* Electronic Journal of Differential Equations. Vol. 2016 (2016), No. 23, pp. 1–21.
8. W. J. Liu, K. W. Chen, *Existence and general decay for nondissipative distributed systems with boundary frictional and memory dampings and acoustic boundary conditions*, Z. Angew. Math. Phys., **66** (2015), no. 4, 1595–1614.
9. W. J. Liu, *Arbitrary rate of decay for a viscoelastic equation with acoustic boundary conditions*, Appl. Math. Lett., **38** (2014), 155–161.
10. W. J. Liu, Y. Sun, *General decay of solutions for a weak viscoelastic equation with acoustic boundary conditions*, Z. Angew. Math. Phys., **65** (2014), no. 1, 125–134.
11. S. A.Messaoudi, A.Fareh and N. Doudi, *Well posedness and exponential stability in a wave equation with a strong damping and a strong delay*, Journal of Mathematical Physics, vol. 57, no 11, p. 111501(2016).
12. S. A. Messaoudi, *General decay of solutions of a viscoelastic equation*, J. Math. Anal. Appl., 341 (2008), 1457-1467.
13. A. Marzocchi, J. E. Muñoz Rivera, M. G. Naso, *Asymptotic behaviour and exponential stability for a transmission problem in thermoelasticity,* Math. Methods Appl. Sci., (2002), no. 11, 955–980.
14. S. Nicaise, C. Pignotti, *Stability and instability results of the wave equation with a delay*, 45 (2006), no. 5, 1561–1585.

15. A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* Springer New York 198 (1983).
16. S. B. Gazi Karakoc *A Quartic Subdomain Finite Element Method for the Modified KdV Equation,* Stat., Optim. Inf. Comput., Vol. 6, December 2018, pp 609-618.
17. S. B. Gazi Karakoc and Halil Zeybek, *A cubic B-spline Galerkin approach for the numerical simulation of the GEW equation,* Stat., Optim. Inf. Comput., Vol. 4, March 2016, pp 30-41.
18. F. Tahamtani, A. Peyravi, *Asymptotic behavior and blow-up of solutions for a nonlinear viscoelastic wave equation with boundary dissipation*, Taiwanese J. Math., **17** (2013), no. 6, 1921–1943.
19. O. P. Yadav and R. Jiwari, *Finite element analysis and approximation of Burgers-Fisher equation*, Numerical Methods for Partial Differential Equations, 33 (5), 1652-1677 (2017).