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Cauchy Formula for Affine Stochastic Differential Equation with Skorohod Integral

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Abstract The Cauchy representation formula enables to obtain a solution to a nonhomogeneous equation with the help of the linear homogeneous part solution and nonhomogeneities. In case of known asymptotics of the linear homogeneous part solution, we can establish some properties of behavior of a solution to nonhomogeneous equation. For diffusion equations the Cauchy formula was ascertained and successfully applied for different cases. In this paper, the Cauchy representation formula for a solution to a multidimentional affine stochastic differential equation with the Skorohod integral is established. Conditions for inclusion of the solution into generalized Wiener functional spaces are given.

Keywords Affine stochastic differential equation, Cauchy representation formula, Skorohod integral, Wick product, S-transform.

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1. Introduction

The Cauchy representation formula is an efficient tool in research of an affine stochastic differential equation (ASDE). In the diffusion case it was established in [11]. It gives possibility to connect behavior of the stochastic semigroup generated by the homogenous equation with behavior of nonhomogeneities. The case of the stable stochastic semigroup and different types of nonhomogeneities is considered in [5]-[8]. In the case of bounded nonhomogeneities the solution of ASDE is stochastically bounded as has been shown in [5, 6]. If nonhomogeneities are periodic then the solution of ASDE is periodic too (see [8]). If nonhomogeneities vanish quite quickly when $t \to \infty$ then the solution of ASDE vanishes as well (see [7]).

In the paper [1] R. Buckdahn and D. Nualart obtained explicit form of the solution to anticipating linear SDEs with the Skorohod integral and proved, specifically, inclusion of this solution into spaces of generalized Wiener functionals. These results open perspectives in constructing solutions of other kinds of SDEs.

This paper deals with multidimensional anticipative ASDEs with the Skorohod integral. In a way analogous to that used in [1] the Cauchy formula is proved, that is, the solution to the ASDE is represented explicitly with the help of the Cauchy matrix of the corresponding linear equation and additive summands of the initial equation. To describe this solution, some proper stochastic spaces are introduced.

The one-dimensional case is considered in [9] by means of the Girsanov transform.

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2. Preliminaries

Let $L(V)$ be a set of linear operators in a vector space V; I - the identical operator; $\mathbb{T} := [0,1]$; $|f|_p$ - the usual norm of $f \in L^p(\mathbb{T}), p \geq 1$.

We denote by $w_t, t \in \mathbb{T}$, a one-dimensional standard Brownian motion defined on the canonical probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is, $\Omega = C_0(\mathbb{T})$ is a set of continuous functions, $t \in \mathbb{T}$, such that $x(0) = 0$ and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Here $\mathcal{F} = \overline{\mathcal{B}}^{\mathbb{P}}(\Omega)$ is the Borel $\sigma-$algebra $\mathcal{B}(\Omega)$ completed with respect to $\mathbb{P}$. In this context $w_t(\omega) \equiv \omega_t$ is the Brownian motion path. Put $\|F\| = (\mathbb{E}|F|^2)^{1/2}$.

Let us denote by *Dom* δ the domain of definition of the definite *Skorohod integral* (see [4, 12]). Let $\mathbf{1}_{[0,t]}u \in$ *Dom* δ for each $t \in \mathbb{T}$. Then the Skorohod integral process $\int_0^t u(s)\, dw_s = \int_0^1 \mathbf{1}_{[0,t]}(s)u(s)\, dw_s$ is defined correctly.

By $L_s^2(\mathbb{T}^n)$ we mean a subspace of $L^2(\mathbb{T}^n)$ that consists of symmetric functions. Let $f_n \in L^2(\mathbb{T}^n)$ and $g_m \in L^2(\mathbb{T}^m)$. Then, $f_n \widetilde{\otimes} g_m$ is the symmetrized tensor product of $f_n \otimes g_m$.

Denote by $I_n(f_n) = \int_0^1 \cdots \int_0^1 f_n(s_1, \ldots, s_n)\, dw_{s_1} \ldots dw_{s_n}$, $f_n \in L^2(\mathbb{T}^n)$, the multiple stochastic integral. If $F \in L^2(\Omega)$, then the Wiener chaos expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$, $f_n \in L_s^2(\mathbb{T}^n)$, holds true, where $I_0(f_0) = \mathbb{E}F$. For such F we have $\|F\|^2 = \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} n!\, |f_n|_2^2$. Let D_t, $t \in \mathbb{T}$, denote the stochastic derivative.

Put $\varepsilon(h) = \exp\left\{i\int_0^1 h_s\, dw_s + \frac{1}{2}\int_0^1 h_s^2\, ds\right\} = \sum_{n=0}^{\infty} \frac{i^n}{n!} I_n(h^{\otimes n})$, $h \in L^2(\mathbb{T})$, $i = \sqrt{-1}$. So, $\varepsilon(h)$ is a complex version of stochastic exponent.

3. Functional spaces, S-transform, Wick product

Set $V = \mathbb{R}^d$ or $L(\mathbb{R}^d)$. Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of kernels, $f_n \in L_s^2(\mathbb{T}^n; V)$. Consider now the formal expansion $F = \sum_{n=0}^{\infty} I_n(f_n)$. For $0 < \lambda < \infty$ we put $\|F\|_\lambda^2 = \sum_{n=0}^{\infty} n!\, \lambda^{2n}|f_n|_2^2$. Let

$$H_\lambda = H_\lambda(V) = \{F : \|F\|_\lambda < \infty\}.$$

Thus, H_λ is a space of such F that the seminorm $\|F\|_\lambda$ is finite. Let $F_\lambda = \sum_{n=0}^{\infty} I_n(\lambda^n f_n)$. Then $F^t = F_1^t$. Since $\|F_\lambda\|^2 = \sum_{n=0}^{\infty} n!\, \lambda^{2n}|f_n|_2^2 = \|F\|_\lambda^2$, we have $F \in H_\lambda$ if and only if $F_\lambda \in L^2(\Omega)$. We have $H_{\lambda_2} \subset H_{\lambda_1}$ for $\lambda_1 < \lambda_2$. For $1 < \lambda < \infty$ the space $H_\lambda \subseteq L^2(\Omega)$. That is, H_λ consists of convergent Wiener chaos expansions. If $0 < \lambda < 1$ the space H_λ is considered as generalized Wiener functionals because it contains divergent Wiener chaos expansions. Put $H_\infty = \bigcap_{\lambda \geq 1} H_\lambda$. The set H_∞ is called the space of analytic functionals (see [1]). Put $H_{0+} = \bigcup_{\lambda > 0} H_\lambda$.

Let $F = \sum_{n=0}^{\infty} I_n(f_n) \in H_{0+}$. We determine $S-$*transform* of a generalized Wiener functional F as

$$S_h(F) = \mathbb{E}\big(F\varepsilon(h)\big) = \sum_{n=0}^{\infty} i^n \big(f_n, h^{\otimes n}\big)_{L^2(\mathbb{T}^n)},\ h \in L^2(\mathbb{T}).$$

Since $|S_h(F)| \leq \|\varepsilon(h/\lambda)\|\|F\|_\lambda$, the $S-$transform is a correct operation. Taking into account that $F \in H_\lambda$ is equal to $F_\lambda \in L^2(\Omega)$, the $S-$transform characterizes the generalized functional F as an element H_{0+}, namely if $S_h(F) = 0$ for all $h \in L^2(\mathbb{T})$, then $F = 0$ as an element H_{0+}.

For $F^t \in H_{0+}$, $t \in \mathbb{T}$, with a sequence of kernels $\{f_n^t\}_{n=0}^{\infty}$, $f_n^t \in L_s^2(\mathbb{T}^n; V)$, the Skorohod integral is defined as $\int_0^1 F^s\, dw_s = \sum_{n=0}^{\infty} I_{n+1}(\widetilde{f}_n)$ and it is shown in [1] that $\int_0^1 F^s\, dw_s \in H_{0+}$.

The *Wick product* of two multiple stochastic integrals $I_n(f_n)$ and $I_m(g_m)$, $f_n \in L_s^2(\mathbb{T}^n; V)$, $g_m \in L_s^2(\mathbb{T}^m; V)$, is denoted by $\diamond$ and determined by means of equality $I_n(f_n) \diamond I_m(g_m) = I_{n+m}(f_n \widetilde{\otimes} g_m)$. With the help of linear property the Wick product can be carried over to the case of finite sums of multiple integrals. The Wick product is a closed operation in the spaces H_{0+} and H_∞. For the Wick product and the $S-$transform the following properties are valid:

(i) if $S_h(F) = S_h(G)$ for all $h \in L^2(\mathbb{T})$, then $F = G$;
(ii) $S_h(F \diamond G) = S_h(F)S_h(G)$;
(iii) $S_h\big(\int_0^1 \psi_s\, dw_s\big) = i\int_0^1 S_h(\psi_s)\, h_s\, ds$;

(iv) $\|\varphi \diamond \psi\|_\lambda \leqq \|\varphi\|_{2\lambda} \|\psi\|_{2\lambda}$ for $\lambda > 0$ and $\varphi, \psi \in H_{2\lambda}$;
(v) $F \diamond \int_0^1 \psi_s \, dw_s = \int_0^1 F \diamond \psi_s \, dw_s$.

Consider a sequence of kernels $\{f_n^t\}_{n=0}^{\infty}$, $t \in \mathbb{T}$. Let $F_\lambda^t = \sum_{n=0}^{\infty} I_n(\lambda^n f_n^t)$ be the formal Wiener chaos expansion for each t and $\lambda > 0$. Denote

$$H_{1,\lambda} = \left\{ F : \|F\|_{1,\lambda} = \int_0^1 \|F^s\|_\lambda \, ds < \infty \right\};$$

$$\widetilde{H}_{1,\lambda} = \left\{ F : \langle F \rangle_\lambda^2 = \sum_{n=0}^{\infty} n! \, \lambda^{2n} \left(\int_0^1 \left| f_n^s \right|_2 ds \right)^2 < \infty \right\};$$

$$H_{2,\lambda} = \left\{ F : \|F\|_{2,\lambda}^2 = \int_0^1 \|F^s\|_\lambda^2 \, ds < \infty \right\};$$

$H_{1,\infty} = \bigcap_{\lambda \geq 1} H_{1,\lambda}$; $H_{1,0+} = \bigcup_{\lambda > 0} H_{1,\lambda}$; $\widetilde{H}_{1,\infty} = \bigcap_{\lambda \geq 1} \widetilde{H}_{1,\lambda}$; $\widetilde{H}_{1,0+} = \bigcup_{\lambda > 0} \widetilde{H}_{1,\lambda}$; $H_{2,\infty} = \bigcap_{\lambda \geq 1} H_{1,\lambda}$; $H_{2,0+} = \bigcup_{\lambda > 0} H_{2,\lambda}$.

Lemma 3.1
Let $f_n^\bullet \in L^1\big(\mathbb{T}; L_s^2(\mathbb{T}^n; V)\big)$ and $F \in H_{1,\lambda} \bigcap \widetilde{H}_{1,\lambda}$. Then

$$\int_0^1 F_\lambda^s \, ds = \sum_{n=0}^{\infty} I_n\left(\lambda^n \int_0^1 f_n^s \, ds \right); \tag{1}$$

$$\left\| \int_0^1 F^s \, ds \right\|_\lambda^2 = \sum_{n=0}^{\infty} n! \, \lambda^{2n} \left| \int_0^1 f_n^s \, ds \right|_2^2 \leq \langle F \rangle_\lambda^2. \tag{2}$$

Proof
If $F \in H_{1,\lambda}$, then $\int_0^1 F_\lambda^s \, ds \in L^2(\Omega)$ because of inequality

$$\left\| \int_0^1 F_\lambda^s \, ds \right\| \leq \int_0^1 \left\| F_\lambda^s \right\| ds = \int_0^1 \left\| F^s \right\|_\lambda ds < \infty.$$

So, we can obtain the Wiener chaos expansion for $\int_0^1 F_\lambda^s \, ds$. Now, we must show that

$$\int_0^1 F_\lambda^s \, ds = \sum_{n=0}^{\infty} I_n\left(\lambda^n \int_0^1 f_n^s \, ds \right).$$

It is proved in [10] that under condition $f_n^\bullet \in L^1\big(\mathbb{T}; L_s^2(\mathbb{T}^n)\big)$ the Fubini theorem for multiple stochastic integrals holds true, namely $\int_0^1 I_n\big(f_n^s\big) \, ds = I_n\big(\int_0^1 f_n^s \, ds \big)$ $(\mathbb{P} = 1)$. Next, since $F \in H_{1,\lambda}$, we have $\left\| F^s \right\|_\lambda < \infty$ for almost all $s \in \mathbb{T}$ and as a result

$$\lim_{N \to \infty} \left\| \sum_{n=N+1}^{\infty} I_n(\lambda^n f_n^s) \right\| = 0 \text{ for almost all } s \in \mathbb{T}.$$

Then, if $N \to \infty$

$$\left\| \int_0^1 \sum_{n=0}^{\infty} I_n(\lambda^n f_n^s) \, ds - \sum_{n=0}^{N} I_n\left(\lambda^n \int_0^1 f_n^s \, ds \right) \right\| =$$

$$\left\| \int_0^1 \sum_{n=N+1}^{\infty} I_n(\lambda^n f_n^s) \, ds \right\| \leq \int_0^1 \left\| \sum_{n=N+1}^{\infty} I_n(\lambda^n f_n^s) \right\| ds \to 0$$

according to the Lebesgue dominated convergent theorem. Hence, the equality (1) is fulfilled and $\int_0^1 F^s \, ds = \sum_{n=0}^{\infty} I_n\big(\int_0^1 f_n^s \, ds \big)$. The validity of right-hand side of (2) is obvious because of inequality $\left| \int_0^1 f_n^s \, ds \right|_2 \leq \int_0^1 \left| f_n^s \right|_2 ds$ and assumption that $F \in \widetilde{H}_{1,\lambda}$. □

Lemma 3.2
If $F \in H_{2,\,\lambda\sqrt{2}}$, then $\left\| \int_0^1 F^s\, dw_s \right\|_\lambda^2 \le \lambda^2 \|F\|^2_{2,\,\lambda\sqrt{2}}$.

Proof
Lemma 3.2 is actually proved in [1]. □

Remark 3.1
Spaces H_λ and $H_{2,\,\lambda}$ are made good use in [1].

Remark 3.2
A future application of the $S-$transform is admissible operation because of Lemma 3.1 and Lemma 3.2.

Definition 3.1
Let $A \in L\big((L^2(\Omega))^n\big)$. An operator $B^{\diamond(-1)} \in L\big((L^2(\Omega))^n\big)$ such that

$$A \diamond B^{\diamond(-1)} = B^{\diamond(-1)} \diamond A = I$$

will be called the Wick inverse of A.

Suppose that $A \in L^1\big(\mathbb{T}; L(\mathbb{R}^d)\big)$ and $B \in L^2\big(\mathbb{T}; L(\mathbb{R}^d)\big)$. Consider the stochastic semigroup U_s^t , $0 \le s \le t \le 1$, defined by the linear stochastic differential equation

$$U_s^t = I + \int_s^t A_v U_s^v\, dv + \int_s^t B_v U_s^v\, dw_v. \tag{3}$$

Lemma 3.3
There exists the Wick inverse of U_s^t which fulfills equation

$$(U_s^t)^{\diamond(-1)} = I + \int_s^t (U_s^v)^{\diamond(-1)}(-A_v)\, dv + \int_s^t (U_s^v)^{\diamond(-1)}(-B_v)\, dw_v \tag{4}$$

and has the Wiener chaos expansion

$$(U_s^t)^{\diamond(-1)} = \sum_{n=1}^{\infty} I_n(v_n^{s,t}),\ |v_n^{s,t}|_2^2 \le K_1 \frac{K_2^n}{(n!)^2}, \tag{5}$$

with $K_1 = \sqrt{d}, K_2 = e^{2|A|_1}|B|_2^2$. In this case we have

$$S_h\big((U_s^t)^{\diamond(-1)}\big) = \big(S_h(U_s^t)\big)^{-1}. \tag{6}$$

Proof
Suppose that U_s^t has a Wiener chaos exprassion of the form $U_s^t = \sum_{n=0}^{\infty} I_n(u_n^{s,t})$. The kernels sequence $\{v_n^{t,s}\}_{n=0}^{\infty}$ of the Wiener chaos expansion for $(U_s^t)^{\diamond(-1)}$ is defined by the following system (see [3])

$$v_0^{t,s} u_0^{t,s} = I, \sum_{k=0}^{n} v_k^{t,s} \widetilde{\otimes}\, u_{n-k}^{t,s} = 0, n = 1, 2, \dots . \tag{7}$$

Considering that U_s^t satisfies (3), $u_0^{t,s} = \mathbb{E} U_s^t$ and $\mathbb{E} \int_0^1 F^s\, dw_s = 0$, $F \in H_{2,\,0+}$, one concludes that $u_0^{t,s}$ is a solution to equation

$$u_0^{t,s} = I + \int_s^t A_v u_0^{v,s}\, dv.$$

The system (7) has a unique solution if and only if $u_0^{t,s}$ is nondegenerate. It is true because of the Liouville theorem (see [2]). We have $v_n^{t,s} = -\sum_{k=0}^{n-1} v_k^{t,s} \widetilde{\otimes}\, u_{n-k}^{t,s} (u_0^{t,s})^{-1}$ for $n = 1, 2, \dots$ immediately from (7). So, there exists the unique $(U_s^t)^{\diamond(-1)}$. Now, we find an equation for $(U_s^t)^{\diamond(-1)}$. The relation (3) implies

$$S_h(U_s^t) = I + \int_s^t A_v S_h(U_s^v)\, dv + i \int_s^t B_v S_h(U_s^v) h_v\, dv. \tag{8}$$

Let G_s^t be determined by the following equation

$$G_s^t = I + \int_s^t G_s^v(-A_v)\, dv + \int_s^t G_s^v(-B_v)\, dw_v.$$

Then

$$S_h(G_s^t) = I + \int_s^t S_h(G_s^v)(-A_v)\, dv + i \int_s^t S_h(G_s^v)(-B_v) h_v\, dv.$$

Therefore for $S_h(G_s^t)S_h(U_s^t)$ we get

$$S_h(G_s^t)S_h(U_s^t) = I + \int_s^t \Big[S_h(G_s^v)\big(-A_v + i(-B_v)h_v\big)S_h(U_s^v) + S_h(G_s^v)\big(A_v + iB_v h_v\big)S_h(U_s^v)\Big]\, dv = I.$$

For this reason $S_h(G_s^t \diamond U_s^t) = S_h(G_s^t)S_h(U_s^t) = I = S_h(I)$. Thus, $G_s^t \diamond U_s^t = I$, that is, $G_s^t = (U_s^t)^{\diamond(-1)}$. Hence, equalities (4) and (6) are valid.

To establish (5) it suffices to remark that relation (4) implies the following equation for $\big((U_s^t)^{\diamond(-1)}\big)^T$

$$\big((U_s^t)^{\diamond(-1)}\big)^T = I + \int_s^t (-A_v)^T \big((U_s^v)^{\diamond(-1)}\big)^T dv + \int_s^t (-B_v)^T \big((U_s^v)^{\diamond(-1)}\big)^T dw_v.$$

The Wiener chaos expansion of solution U_s^t of the equation (3) and kernels estimation are obtained in [1], namely

$$U_s^t = \sum_{n=1}^{\infty} I_n\big(u_n^{t,s}\big),\ |u_n^{t,s}|_2^2 \le K_1 \frac{K_2^n}{(n!)^2}, \tag{9}$$

where $K_1 = \sqrt{d}$, $K_2 = e^{2|A|_1}|B|_2^2$. Consequently,

$$\big((U_s^t)^{\diamond(-1)}\big)^T = \sum_{n=0}^{\infty} I_n\big((v_n^{s,t})^T\big),\ |v_n^{s,t}|_2^2 = |(v_n^{s,t})^T|_2^2 \le K_1 \frac{M^n}{(n!)^2},$$

$K_1 = \sqrt{d}, M = e^{2|-A^T|_1}|-B^T|_2^2 = K_2.$ □

4. Main result

Let $x_0 : \Omega \to \mathbb{R}^d$ and $\varphi, \psi : \mathbb{T} \times \Omega \to \mathbb{R}^d$. Consider the following stochastic differential equation

$$x_t = x_0 + \int_0^t (A_s x_s + \varphi_s)\, ds + \int_0^t (B_s x_s + \psi_s)\, dw_s\,. \tag{10}$$

Definition 4.1
A process $x_t\,,\ t \in \mathbb{T}$ is called the solution of the equation (10), if $\mathbf{1}_{[0,t]}(\bullet)(B_\bullet x_\bullet + \psi_\bullet) \in Dom\,\delta$ for each $t \in \mathbb{T}$ and the equality (10) holds true with probability 1 for every $t \in \mathbb{T}$.

Remark 4.1
As it is shown in [1], in the case of $\varphi = \psi = 0$ the solution of the Cauchy problem with a random initial condition $x_0(\omega) \in H_{0+}$ is of the following form $x_t = U_0^t \diamond x_0$.

Theorem 4.1
Suppose that $x_0 \in H_{0+}(\mathbb{R}^d)$, $\varphi \in H_{1,0+}(\mathbb{R}^d) \bigcap \widetilde{H}_{1,0+}(\mathbb{R}^d)$ and $\psi \in H_{2,0+}(\mathbb{R}^d)$. Then there exists the unique solution x_t of the system (10). Uniqueness of the solution means that given kernels sequences of x_0, φ and ψ under condition we come to a unique kernels sequence of the solution x_t. This solution is of the following form

$$x_t = U_0^t \diamond \left(x_0 + \int_0^t (U_0^s)^{\diamond(-1)} \diamond \varphi_s \, ds + \int_0^t (U_0^s)^{\diamond(-1)} \diamond \psi_s \, dw_s \right) = \\ U_0^t \diamond x_0 + \int_0^t U_s^t \diamond \varphi_s \, ds + \int_0^t U_s^t \diamond \psi_s \, dw_s. \tag{11}$$

In this case $x_t \in H_{0+}(\mathbb{R}^d)$. In addition, if $x_0 \in H_\infty(\mathbb{R}^d)$, $\varphi \in H_{1,\infty}(\mathbb{R}^d) \bigcap \widetilde{H}_{1,\infty}(\mathbb{R}^d)$ and $\psi \in H_{2,\infty}(\mathbb{R}^d)$, then $x_t \in H_\infty(\mathbb{R}^d)$.

Proof
We shall first prove equality of right-hand side of (11) and the term in the middle of (11). To this end we show that

$$U_s^t U_0^s = U_s^t \diamond U_0^s \,, \; 0 \le s \le t \le 1. \tag{12}$$

Fix an arbitrary $h \in L^2(\mathbb{T})$. The following equalities hold true

$$S_h\big(U_s^t U_0^s\big) = \mathbb{E}\big(U_s^t U_0^s \, \varepsilon(h)\big) = \mathbb{E}\big(U_s^t \varepsilon(\mathbf{1}_{[s,1]} h) \, U_0^s \varepsilon(\mathbf{1}_{[0,s]} h)\big) = \\ \mathbb{E}\big(U_s^t \varepsilon(\mathbf{1}_{[s,1]} h)\big) \, \mathbb{E}\varepsilon(\mathbf{1}_{[0,s]} h) \, \mathbb{E}\varepsilon(\mathbf{1}_{[s,1]} h) \, \mathbb{E}\big(U_0^s \varepsilon(\mathbf{1}_{[0,s]} h)\big) = \\ \mathbb{E}\big(U_s^t \varepsilon(\mathbf{1}_{[0,1]} h)\big) \, \mathbb{E}\big(U_0^s \varepsilon(\mathbf{1}_{[0,1]} h)\big) = S_h\big(U_s^t\big) S_h\big(U_0^s\big).$$

This implies (12).

Since $U_s^t \in H_\infty(L(\mathbb{R}^d))$ (see [1]), we have equality $U_0^t \diamond \int_0^1 F^s \, dw_s = \int_0^1 U_0^t \diamond F^s \, dw_s$, $F \in H_{2,0+}$, (see, for example, [3]). Linearity of the Wick product implies $U_0^t \diamond \int_0^1 F^s \, ds = \int_0^1 U_0^t \diamond F^s \, ds$ for $F \in H_{1,0+}(\mathbb{R}^d) \bigcap \widetilde{H}_{1,0+}(\mathbb{R}^d)$. Taking into account (12), we finally obtain

$$U_0^t \diamond \left(x_0 + \int_0^t (U_0^s)^{\diamond(-1)} \diamond \varphi_s \, ds + \int_0^t (U_0^s)^{\diamond(-1)} \diamond \psi_s \, dw_s \right) = \\ U_0^t \diamond x_0 + U_0^t \diamond \int_0^t (U_0^s)^{\diamond(-1)} \diamond \varphi_s \, ds + U_0^t \diamond \int_0^t (U_0^s)^{\diamond(-1)} \diamond \psi_s \, dw_s = \\ U_0^t \diamond x_0 + \int_0^t U_0^t \diamond (U_0^s)^{\diamond(-1)} \diamond \varphi_s \, ds + \int_0^t U_0^t \diamond (U_0^s)^{\diamond(-1)} \diamond \psi_s \, dw_s = \\ U_0^t \diamond x_0 + \int_0^t U_s^t U_0^s \diamond (U_0^s)^{\diamond(-1)} \diamond \varphi_s \, ds + \int_0^t U_s^t U_0^s \diamond (U_0^s)^{\diamond(-1)} \diamond \psi_s \, dw_s = \\ U_0^t \diamond x_0 + \int_0^t U_s^t \diamond U_0^s \diamond (U_0^s)^{\diamond(-1)} \diamond \varphi_s \, ds + \int_0^t U_s^t \diamond U_0^s \diamond (U_0^s)^{\diamond(-1)} \diamond \psi_s \, dw_s = \\ U_0^t \diamond x_0 + \int_0^t U_s^t \diamond \varphi_s \, ds + \int_0^t U_s^t \diamond \psi_s \, dw_s.$$

Now, we must show that all summands in (11) are elements of $H_{0+}(\mathbb{R}^d)$. As it is proved in [1], $U_0^t \diamond x_0 \in H_{0+}(\mathbb{R}^d)$. To verify that $\int_0^t U_s^t \diamond \varphi_s \, ds \in H_{0+}(\mathbb{R}^d)$, it is should be noted that U_s^t is an analytical functional for each $0 \le s \le t \le 1$. Indeed, from (9) we have $\|U_s^t\|_\lambda^2 = \sum_{n=0}^\infty \lambda^{2n} n! \, |u_n^{t,s}|_2^2 \le K_1 e^{K_2 \lambda^2} = L_\lambda < \infty$. Thus, $U_s^t \in H_\lambda$ for each $0 < \lambda < \infty$, that is, $U_s^t \in H_\infty$. Next, it may be proved that $U_\bullet^t \diamond \varphi_\bullet \in H_{1,\lambda}$ for $\varphi \in H_{1,2\lambda}$. For this purpose we verify that $\int_0^t \|U_s^t \diamond \varphi_s\|_\lambda\, ds < \infty$. In virtue of the Wick product property $\|\varphi \diamond \psi\|_\lambda \leqq \|\varphi\|_{2\lambda}\,\|\psi\|_{2\lambda}$, $\lambda > 0$, $\varphi, \psi \in H_{2\lambda}$, and the estimation of $\|U_\bullet^t\|_{2\lambda}$, we have

$$\int_0^t \|U_s^t \diamond \varphi_s\|_\lambda\, ds \le \int_0^t \|U_s^t\|_{2\lambda}\|\varphi_s\|_{2\lambda}\, ds \le \sqrt{L_{2\lambda}} \int_0^t \|\varphi_s\|_{2\lambda}\, ds < \infty.$$

In order to check that $\langle U_\bullet^t \diamond \varphi_\bullet\rangle_\lambda < \infty$, we should use the estimation of $u_n^{t,s}$ (9). Finally, we get

$$\begin{aligned}
&\langle U_\bullet^t \diamond \varphi_\bullet\rangle_\lambda^2 = \sum_{n=0}^\infty n!\,\lambda^{2n}\Big(\int_0^t \big|\sum_{k=0}^n u_{n-k}^{t,s}\widetilde{\otimes}\varphi_k^s\big|_2\, ds\Big)^2 \le \\
&\sum_{n=0}^\infty n!\,\lambda^{2n}(n+1)\sum_{k=0}^n\Big(\int_0^t \big|u_{n-k}^{t,s}\big|_2\big|\varphi_k^s\big|_2\, ds\Big)^2 \le \\
&K_1\sum_{n=0}^\infty\sum_{k=0}^n (n-k)!\frac{(2\lambda^2K_2)^{(n-k)}}{((n-k)!)^2}\; k!\,(2\lambda^2)^k\Big(\int_0^t \big|\varphi_k^s\big|_2\, ds\Big)^2 \le \\
&K_1\sum_{n=0}^\infty\sum_{k=0}^n \frac{(2\lambda)^{2(n-k)}K_2^{(n-k)}}{(n-k)!}\; k!\,(2\lambda)^{2k}\Big(\int_0^t \big|\varphi_k^s\big|_2\, ds\Big)^2 \le \\
&K_1\sum_{n=0}^\infty\Big((2\lambda)^{2n}\frac{K_2^n}{n!}\Big)\sum_{k=0}^\infty\Big(k!\,(2\lambda)^{2k}\Big(\int_0^t \big|\varphi_k^s\big|_2\, ds\Big)^2\Big) = L_{2\lambda}\langle\varphi\rangle_{2\lambda}^2 < \infty.
\end{aligned}$$

In view of (2), we have $\big\|\int_0^t U_s^t \diamond \varphi_s\, ds\big\|_\lambda \le \langle U_\bullet^t \diamond \varphi_\bullet\rangle_\lambda < \infty$ for $\varphi \in H_{1,2\lambda} \bigcap \widetilde{H}_{1,2\lambda}$ and, on account of Lemma 3.1, $\int_0^t U_s^t \diamond \varphi_s\, ds \in H_{0+}(\mathbb{R}^d)$.

Now, we shall prove that $\int_0^t U_s^t \diamond \psi_s\, dw_s \in H_{0+}(\mathbb{R}^d)$. For this purpose, by Lemma 3.2, it suffices to show that $\mathbf{1}_{[0,t]}(\bullet)U_\bullet^t \diamond \psi_\bullet \in H_{2,\lambda\sqrt{2}}$ if $\psi \in H_{2,2^{3/2}\lambda}$. We have

$$\begin{aligned}
&\big\|\mathbf{1}_{[0,t]}(\bullet)U_\bullet^t \diamond \psi_\bullet\big\|_{2,\lambda\sqrt{2}}^2 = \int_0^t \|U_s^t \diamond \psi_s\|_{\lambda\sqrt{2}}^2\, ds \le \\
&\int_0^t \|U_s^t\|_{2^{3/2}\lambda}^2\|\psi_s\|_{2^{3/2}\lambda}^2\, ds \le L_{2^{3/2}\lambda}\int_0^1 \|\psi_s\|_{2^{3/2}\lambda}^2\, ds < \infty.
\end{aligned}$$

To complete the proof it still remains to show that the expression (11) is a solution to the equation (10). We shall first apply S-transform to z_t determined by the right-hand side of (11), namely to

$$z_t = U_0^t \diamond x_0 + U_0^t \diamond \int_0^t (U_0^s)^{\diamond(-1)} \diamond \varphi_s\, ds + U_0^t \diamond \int_0^t (U_0^s)^{\diamond(-1)} \diamond \psi_s\, dw_s. \tag{13}$$

Since $S_h(F \diamond G) = S_h(F)S_h(G)$, $F, G \in H_{0+}$, (see [3]), $S_h\big(\int_0^1 F^s\, dw_s\big) = i\int_0^1 S_h(F^s)\, h_s\, ds$ for $F \in H_{2,0+}$ (see [3]) and by (6), we have

$$\begin{aligned}
S_h(z_t) =& S_h(U_0^t)S_h(x_0) + \\
& S_h(U_0^t)\int_0^t \big(S_h(U_0^s)\big)^{-1}\big(S_h(\varphi_s) + iS_h(\psi_s)h_s\big)\, ds.
\end{aligned} \tag{14}$$

Considering (8), it is easy to calculate that the right-hand side of (14) satisfies equation

$$\begin{aligned}
S_h(z_t) =& S_h(x_0) + \\
&\int_0^t \Big\{\big(A_s + iB_sh_s\big)S_h(z_s) + \big(S_h(\varphi_s) + iS_h(\psi_s)h_s\big)\Big\}\, ds.
\end{aligned} \tag{15}$$

Next, let us apply S-transform to each term of equation (10). Taking into account the S-transform properties given above, we come to

$$S_h(x_t) = S_h(x_0) + \int_0^t \Big\{ A_s S_h(x_s) + S_h(\varphi_s) + \big(B_s i S_h(x_s) h_s + i S_h(\psi_s) h_s \big) \Big\} \, ds. \tag{16}$$

Inasmuch as equations (15) and (16) are equal up to the notation and order, we obtain $S_h(z_t) = S_h(x_t)$ for all $h \in L^2(\mathbb{T})$. It implies that $z_t = x_t$ as elements of $H_{0+}(\mathbb{R}^d)$, $t \in \mathbb{T}$. Thus, the expression (11) determines a solution to the equation (10).

The assertion of the case $x_0 \in H_\infty(\mathbb{R}^d)$, $\varphi \in H_{1,\infty}(\mathbb{R}^d) \bigcap \widetilde{H}_{1,\infty}(\mathbb{R}^d)$ and $\psi \in H_{2,\infty}(\mathbb{R}^d)$ follows immediately from the above reasoning.

Uniqueness of the solution is a direct consequence of linearity of the equation (10) and corresponding result for nonhomogeneous case established in [1]. □

Corollary 4.1
Let x_0 be a non-random initial condition. Denote $\mathcal{F}_s^t = \sigma\big\{w_v - w_u : 0 \le s \le u \le v \le t \le 1\big\}$. Suppose that φ_t and ψ_t are $\mathcal{F}_0^t$ measurable ($\varphi_t, \psi_t \sim \mathcal{F}_0^t$). Under these circumstances, as has been stated in [11], the solution of the system (10) can be written in the form

$$x_t = U_0^t x_0 + \int_0^t U_s^t \big(\varphi_s - B_s \psi_s\big) \, ds + U_0^t \int_0^t (U_0^s)^{(-1)} \psi_s \, dw_s, \tag{17}$$

where the stochastic integral is interpreted in Itô sense.

Proof
We shall first prove that (17) is a direct consequence of (11). It should be recalled that in this case the Skorohod integral coincides with the Ito integral. By property of the Skorohod integral

$$U_0^t \int_0^t (U_0^s)^{(-1)} \psi_s \, dw_s = \int_0^t U_0^t (U_0^s)^{(-1)} \psi_s \, dw_s + \int_0^t \big(D_s U_0^t\big)(U_0^s)^{(-1)} \psi_s \, ds,$$

where D_t, $t \in \mathbb{T}$, denotes the stochastic derivative (see [3, 4]). By simple computation,

$$D_s U_0^t = \begin{cases} 0, & 0 \leqq t < s, \\ U_s^t B_s U_0^s, & 0 \leqq s \leqq t. \end{cases}$$

After making the substitution, we get

$$\int_0^t U_s^t \psi_s \, dw_s = U_0^t \int_0^t (U_0^s)^{(-1)} \psi_s \, dw_s - \int_0^t U_s^t B_s \psi_s \, ds.$$

Since $\varphi_s, \psi_s \sim \mathcal{F}_0^s$ and $H_s^t \sim \mathcal{F}_s^t$, equalities $U_s^t \diamond \varphi_s = U_s^t \varphi_s$ and $U_s^t \diamond \psi_s = U_s^t \psi_s$ can be proved in the same manner as in the case of (12). Finally, (11) can be transformed like that

$$\begin{aligned} x_t = U_0^t \diamond x_0 + &\int_0^t U_s^t \diamond \varphi_s \, ds + \int_0^t U_s^t \diamond \psi_s \, dw_s = \\ U_0^t x_0 + &\int_0^t U_s^t \varphi_s \, ds + \int_0^t U_s^t \psi_s \, dw_s = \\ U_0^t x_0 + &\int_0^t U_s^t \big(\varphi_s - B_s \psi_s\big) \, ds + U_0^t \int_0^t (U_0^s)^{(-1)} \psi_s \, dw_s. \end{aligned}$$

□

Remark 4.2
In the one-dimensional case $(d = 1)$ the Cauchy formula can be written, as it is shown in [9], by means of the family of transformation $Q^t,\ R^t : \Omega \to \Omega$, $t \in \mathbb{T}$, defined as

$$Q^t(\omega)_s = \omega_s + \int_0^{t\wedge s} B_u\,du; \quad R^t(\omega)_s = \omega_s - \int_0^{t\wedge s} B_u\,du,\ s,t \in \mathbb{T},\ \omega \in \Omega.$$

The solution of (10) takes the form

$$x_t = U_0^t x_0(R^t) + \int_0^t U_s^t \varphi_s(R^t Q^s)\,ds + \int_0^t U_s^t \psi_s(R^t Q^s)\,dw_s.$$

5. Conclusion

In this paper the solution to the affine stochastic differential equation with the Skorohod integral in multidimensional case represents by means of the stochastic semigroup generated by the corresponding linear homogenous equation and additive nonhomogeneities of the initial equation.

Properties of the solution are delineated in terms of the Wiener chaos expansion. Some stochastic spaces are introduced and it is ascertained that if nonhomogeneities are elements of these spaces then the solution is a generalized Wiener functional.

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