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Some Confidence Regions for Traffic Intensity Vector

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Abstract Using the Consistent and Asymptotically Normal(CAN) estimator and its covariance matrix $(A,)$ $100(1-\alpha)\%$ confidence region for traffic intensity vector $\underline{\rho}$ with no assumption of arrival and service time distribution is constructed in this paper. Also Standard Bootstrap (SB), Bayesian Bootstrap(BB) and percentile bootstrap (PB) are applied to develop the confidence regions for traffic intensity vector $\underline{\rho}$ with confidence level $100(1-\alpha)\%$. Simulation study is undertaken to evaluate the performances of the confidence regions in terms of their coverage area percentage, average area and relative coverage area. Calibration technique is used to improve the coverage area percentages of confidence regions.

Keywords Traffic intensity vector, Coverage percentage, Relative coverage, Relative average length, Calibration.

AMS 2010 subject classifications: 60K20,60K25,68M20,90B22.

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1. Introduction

Consider an open queueing network model as shown in Figure 1 which consists of two nodes with respective service rates μ_1 and μ_2. The external arrival rate to node-1 is λ.

Figure 1. Two stage open queueing network

Traffic intensity vector $\underline{\rho}$ is defined as follows:

$$\underline{\rho} = (\rho_1\ ,\ \rho_2)' = \left(\frac{\lambda}{\mu_1},\quad \frac{\mu_1}{\mu_2}\right)' \tag{1}$$

and $1/\lambda$ represent mean inter-arrival time and $1/\mu_1$, $1/\mu_2$ denotes mean service times at node-1 and node-2 respectively. Traffic intensity vector $\underline{\rho}$ can be interpreted as expected number of arrivals per mean service.

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Jackson [14] presented queueing networks with arrival process that can depend on the state of the system and closed queueing networks with exponential servers. Disney [4] introduces basic properties of queueing networks. Open queueing networks are useful in studying the behavior of computer communication networks (Kleinrock [18]). Thiruvaiyaru, Basawa and Bhat [23] considered the problem of Maximum likelihood estimation for Jackson networks with Poisson arrival and exponential service time at each node. Bootstrap technique are discussed in Efron and Tibshirani [5]. Besides the standard bootstrap technique, Rubin [22] presented the Bayesian bootstrap technique of resampling. Ke and Chu [16] proposed a nonparametric approach of intensity for a queueing system with distribution free inter-arrival and service times. Gedam and Pathare [7] proposed CAN estimator and different bootstrap approaches to develop the confidence intervals of intensities. Gedam and Pathare [8] constructed an calibrated CAN, Exact-t, Variance-stabilized Bootstrap-t, and different bootstrap confidence intervals for intensity parameters of open queueing network model with feedback. Gedam and Pathare [9] used calibration technique to construct confidence intervals for intensity parameters. Numerical simulation study is conducted to demonstrate performances of the calibrated confidence intervals. Pathare and Gedam [21] proposed a consistent and asymptotically normal estimator for intensity parameters for a queueing network. Using this estimator and its estimated variance, asymptotic confidence interval for intensities is constructed. Bootstrap approaches are applied to develop the confidence intervals for intensity parameters. Gedam and Pathare [10] used data based recurrence relation to compute a sequence of response time. The sample means from those response times, denoted by $\hat{r}_1$ and $\hat{r}_2$ are used to estimate true mean response time r_1 and r_2. Confidence intervals for mean response times r_1 and r_2 are constructed. Gedam and Pathare [11] constructed various confidence intervals for mean response times of an open queueing network model with feedback using the calibration approach.

The organization of the paper is as follows: The calibration technique is given in section 2. In section 3, we discuss statistical inference of traffic intensity vector and construct different confidence regions for traffic intensity vector. Section 4 is devoted to evaluate the performance of four confidence regions in terms of simulation analysis. The performances of the confidence regions are assessed in terms of their coverage area percentage,average area and relative coverage area. Calibration technique is used to improve the coverage percentage area of confidence regions. Finally some concluding remarks are given in section 5.

2. Calibration Technique

The actual coverage of confidence region is rarely equal to the desired level. Hence to improve the coverage accuracy of confidence region we use calibration technique. First use bootstrap to estimate the true coverage of confidence region and the region is then adjusted by comparing with the target nominal level. The general theory of calibration is reviewed in Efron and Tibshirani [6], following ideas of Loh [19], Beran [2], Hall [13], Hall and Martin [12]. The bootstrap calibration technique was introduced by Loh [20]. To illustrate, first find $\hat{\gamma}$ for the confidence region for $\underset{\sim}{\rho}$ with γ. Then set

$$\begin{aligned} \gamma_1 &= \frac{\gamma^2}{\hat{\gamma}}, && if \quad \hat{\gamma} \geq \gamma \\ &= \gamma + \frac{(1-\gamma)(\gamma-\hat{\gamma})}{(1-\hat{\gamma})}, && if \quad \hat{\gamma} < \gamma \end{aligned} \tag{2}$$

That is we get the point (γ_1, γ) by linearly interpolating between

$$\begin{array}{lllll} (i) & (0,0) & and & (\gamma,\hat{\gamma}) & if \quad \hat{\gamma} \geq \gamma \\ (ii) & (\gamma,\hat{\gamma}) & and & (1,1) & if \quad \hat{\gamma} < \gamma \end{array}$$

Suppose we want a 95% confidence region for the $\underset{\sim}{\rho}$. Suppose by using $\alpha = 0.025$ such that $\gamma = 1 - 2\alpha = 0.95$, we find coverage area $\hat{\gamma} = 0.87$, that is $\hat{\gamma} = 0.87$ and $\gamma = 0.95$. Now we want to increase coverage area $\hat{\gamma} = 0.87$

to 0.95. Here $\hat{\gamma} < \gamma$ hence we get the point (γ_1, γ) by linearly interpolating between $(\gamma, \hat{\gamma})$ and $(1, 1)$. That is

$$\begin{aligned}\gamma_1 &= \gamma + \frac{(1-\gamma)(\gamma-\hat{\gamma})}{(1-\hat{\gamma})} \\ &= 0.95 + \frac{(1-0.95)(0.95-0.87)}{(1-0.87)} \\ &= 0.98\end{aligned}$$

Hence we will set $\gamma_1 = 0.98$. Therefore the calibrated confidence region for $\underline{\rho}$ is with $\gamma = \gamma_1$.

3. Statistical Inference of Traffic Intensity Vector

Let (X, Y) be nonnegative random variables representing inter-arrival time and service time of node-1 and (Y, Z) be nonnegative random variables representing inter-arrival time and service time to node-2. The traffic intensity vector $\underline{\rho}$ is defined as follows:

$$\underline{\rho} = (\rho_1\ ,\ \rho_2)' = \left(\frac{\mu_Y}{\mu_X}\quad , \quad \frac{\mu_Z}{\mu_Y}\right)' \tag{3}$$

where μ_X and μ_Y denote the mean inter-arrival time and mean service time of node-1. Also μ_Y and μ_Z denote the mean inter-arrival time and mean service time of node-2. Equation (3) is equivalent to equation (1).

3.1. Estimation of traffic intensity vector

Assume that $(X_1, X_2, \cdots, X_n)$ and $(Y_1, Y_2, \cdots, Y_n)$ are random samples drawn from X and Y respectively. Let $(Y_1, Y_2, \cdots, Y_n)$ and $(Z_1, Z_2, \cdots, Z_n)$ be random samples drawn from Y and Z. Let $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ be the sample means of X ,Y and Z respectively.

According to the Strong Law of Large Numbers, $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ are strongly consistent estimator of μ_X, μ_Y, and μ_Z respectively. Thus strongly consistent estimator of $\underline{\rho}$ is given by

$$\underline{\hat{\rho}} = (\hat{\rho}_1\ ,\ \hat{\rho}_2)' = \left(\frac{\overline{Y}}{\overline{X}}\ ,\ \frac{\overline{Z}}{\overline{Y}}\right) \tag{4}$$

As true distributions of X, Y and Z are not known in practice the asymptotic distribution of $\underline{\rho}$ can be developed as follows.

Suppose $T_m \xrightarrow{D} N_m(\theta, \Sigma)$. Let $g : R^m \longrightarrow R^k$ be such that $\underline{g}(u_1, u_2, \cdots u_m) = (g_1(u_1,$ $u_2, \cdots u_m), g_2(u_1, u_2, \cdots u_m), \cdots, g_n(u_1, u_2, \cdots u_m))$. Assume $g_1, g_2, \cdots g_k$ are totally differentiable functions then $g(T_m) \xrightarrow{D} N_k(g(\theta),\ M\Sigma M')$ (Kale [15]) where

$$M = \begin{bmatrix} \frac{\partial g_1}{\partial\theta_1} & \frac{\partial g_1}{\partial\theta_2} & \frac{\partial g_1}{\partial\theta_3} & \cdots & \frac{\partial g_1}{\partial\theta_m} \\ \frac{\partial g_2}{\partial\theta_1} & \frac{\partial g_2}{\partial\theta_2} & \frac{\partial g_2}{\partial\theta_3} & \cdots & \frac{\partial g_2}{\partial\theta_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_k}{\partial\theta_1} & \frac{\partial g_k}{\partial\theta_2} & \frac{\partial g_k}{\partial\theta_3} & \cdots & \frac{\partial g_k}{\partial\theta_m} \end{bmatrix}$$

If $\underline{X_1}, \underline{X_2}, \cdots, \underline{X_k}$ are independent and identically distributed random vectors with mean $\underline{\mu} \in R^k$ and covariance matrix Σ where Σ is positive definite and has finite elements, then $\sqrt{n}(\underline{X_n} - \underline{\mu}) \xrightarrow{D} N_k(\underline{0}, \Sigma)$ where $\xrightarrow{D}$ denotes convergence in distribution (Kale [15]).

By Theorem 3.2 we have,

$$\sqrt{n}\begin{bmatrix} \overline{X}-\mu_X \\ \overline{Y}-\mu_Y \\ \overline{Z}-\mu_Z \end{bmatrix} \xrightarrow{D} N_3\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \sum \right]$$

where,

$$\sum = \begin{bmatrix} \sigma_x^2 & 0 & 0 \\ 0 & \sigma_y^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}.$$

σ_x^2 , σ_y^2 and σ_z^2 are variances of X,Y and Z respectively.
Now consider $g : R^3 \longrightarrow R^2$ such that

$$\begin{aligned} \underline{g}(U_1,U_2,U_3) &= (g_1(U_1,U_2,U_3), \quad g_2(U_1,U_2,U_3)) \\ &= \left(\frac{U_2}{U_1}, \quad \frac{U_3}{U_2}\right) \\ &= (\hat{\rho}_1, \hat{\rho}_2) \end{aligned}$$

By Theorem 3.1 we have,

$$\sqrt{n}\begin{pmatrix} \hat{\rho}_1-\rho_1 \\ \hat{\rho}_2-\rho_2 \end{pmatrix} \xrightarrow{D} N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad M\Sigma M' \right]$$

$$\text{where } M\Sigma M' = \begin{bmatrix} \frac{U_2^2}{U_1^4}\sigma_X^2 + \frac{1}{U_1^2}\sigma_Y^2 & -\frac{U_3}{U_1U_2^2}\sigma_Y^2 \\ -\frac{U_3}{U_1U_2^2}\sigma_Y^2 & \frac{U_3^2}{U_2^4}\sigma_Y^2 + \frac{1}{U_2^2}\sigma_Z^2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} -\frac{U_2}{U_1^2} & \frac{1}{U_1} & 0 \\ 0 & -\frac{U_3}{U_2^2} & \frac{1}{U_2} \end{bmatrix}.$$

M' is transpose of M. Let $A = M\Sigma M'$. Again by Theorem 3.2 we have,

$$\sqrt{n}\left[\begin{pmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \end{pmatrix} - \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}\right] \xrightarrow{D} N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A\right]$$

That is

$$\sqrt{n}\left(\underline{\hat{\rho}} - \underline{\rho}\right) \xrightarrow{D} N_2\,(\,\underline{0}\ ,\ \ A)\,.$$

where

$$A = \begin{bmatrix} \frac{\mu_Y^2}{\mu_X^4}\sigma_X^2 + \frac{1}{\mu_X^2}\sigma_Y^2 & -\frac{\mu_Z}{\mu_X\mu_Y^2}\sigma_Y^2 \\ -\frac{\mu_Z}{\mu_X\mu_Y^2}\sigma_Y^2 & \frac{\mu_Z^2}{\mu_Y^4}\sigma_Y^2 + \frac{1}{\mu_Y^2}\sigma_Z^2 \end{bmatrix}$$

It follows that $n\left(\underline{\hat{\rho}} - \underline{\rho}\right)' A^{-1}\left(\underline{\hat{\rho}} - \underline{\rho}\right)$ has a χ^2 - distribution with two degrees of freedom (Anderson [1]).

Let $T_n = n\left(\underline{\hat{\rho}} - \underline{\rho}\right)' A^{-1}\left(\underline{\hat{\rho}} - \underline{\rho}\right) \xrightarrow{D} \chi_2^2$

If A is unknown then using the sample estimates of $\mu_X, \mu_Y, \mu_Z, \sigma_X^2, \sigma_Y^2$ and σ_Z^2 we get estimator $\hat{A}$ of A as follows:

$$\hat{A} = \begin{bmatrix} \frac{\overline{Y}^2}{\overline{X}^4}S_X^2 + \frac{1}{\overline{X}^2}S_Y^2 & -\frac{\overline{Z}}{\overline{X}\overline{Y}^2}S_Y^2 \\ -\frac{\overline{Z}}{\overline{X}\overline{Y}^2}S_Y^2 & \frac{\overline{Z}^2}{\overline{Y}^4}S_Y^2 + \frac{1}{\overline{Y}^2}S_Z^2 \end{bmatrix}$$

where

$$S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2, \quad S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\overline{Y})^2 \qquad S_Z^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Z_i-\overline{Z})^2$$

If covariance matrix A is unknown then $n\left(\underline{\hat{\rho}}-\underline{\rho}\right)'\hat{A}^{-1}\left(\underline{\hat{\rho}}-\underline{\rho}\right)\xrightarrow{D}\chi_2^2$.
Proof:
We know that Let $T_n = n\left(\underline{\hat{\rho}}-\underline{\rho}\right)'A^{-1}\left(\underline{\hat{\rho}}-\underline{\rho}\right)\xrightarrow{D}\chi_2^2$.
Let $S_n = n\left(\underline{\hat{\rho}}-\underline{\rho}\right)'\hat{A}^{-1}\left(\underline{\hat{\rho}}-\underline{\rho}\right)$.
Now we show that $T_n - S_n \xrightarrow{P} 0$. Consider

$$T_n - S_n = n\left(\underline{\hat{\rho}}-\underline{\rho}\right)'(A^{-1}-\hat{A}^{-1})\left(\underline{\hat{\rho}}-\underline{\rho}\right)$$

We know that $\overline{X}\xrightarrow{P}\mu_X$, $\overline{Y}\xrightarrow{P}\mu_Y$, $\overline{Z}\xrightarrow{P}\mu_Z$, $S_X^2\xrightarrow{P}\sigma_X^2$, $S_Y^2\xrightarrow{P}\sigma_Y^2$ and $S_Z^2\xrightarrow{P}\sigma_Z^2$. Then we have, $\frac{\overline{Y}^2}{\overline{X}^4}S_X^2+\frac{1}{\overline{X}^2}S_Y^2\xrightarrow{P}\frac{\mu_Y^2}{\mu_X^4}\sigma_X^2+\frac{1}{\mu_X^2}\sigma_Y^2$; $\frac{\overline{Z}}{\overline{XY}^2}S_Y^2\xrightarrow{P}\frac{\mu_Z}{\mu_X\mu_Y^2}\sigma_Y^2$; $\frac{\overline{Z}^2}{\overline{Y}^4}S_Y^2+\frac{1}{\overline{Y}^2}S_Z^2\xrightarrow{P}\frac{\mu_Z^2}{\mu_Y^4}\sigma_Y^2+\frac{1}{\mu_Y^2}\sigma_Z^2$. Thus $\hat{A}$ converges component wise in probability to A and hence $\hat{A}^{-1}\xrightarrow{P}A^{-1}$.
Therefore, $T_n - S_n = n\left(\underline{\hat{\rho}}-\underline{\rho}\right)'(A^{-1}-\hat{A}^{-1})\left(\underline{\hat{\rho}}-\underline{\rho}\right)\xrightarrow{P}0$
But we know that $T_n\xrightarrow{D}\chi_2^2$ Therefore $n\left(\underline{\hat{\rho}}-\underline{\rho}\right)'\hat{A}^{-1}\left(\underline{\hat{\rho}}-\underline{\rho}\right)\xrightarrow{D}\chi_2^2$.

3.2. Different Confidence Regions for Traffic Intensity Vector

In this section we construct different confidence regions for traffic intensity vector.

3.2.1. **Consistent and Asymptotically Normal Confidence Region:** If A is unknown, then replace it by $\hat{A}$ (Anderson [1]). By using Theorem 3.3, $100(1-\alpha)\%$ CAN confidence region (CR) for $\underline{\rho}$ is given by,

$$CR = \left\{\underline{\rho}\mid n\left(\underline{\hat{\rho}}-\underline{\rho}\right)'\hat{A}^{-1}\left(\underline{\hat{\rho}}-\underline{\rho}\right)\le\chi_{2,\alpha}^2\right\} \tag{5}$$

3.2.2. **Standard Bootstrap Confidence Region:** Using standard bootstrap procedure, a simple random sample $x^* = (x_1^*, x_2^*, \cdots, x_n^*)'$ can be taken from the empirical distribution function of $x = (x_1, x_2, \cdots, x_n)'$. Similarly we can draw a bootstrap samples $y^* = (y_1^*, y_2^*, \cdots, y_n^*)'$ and $z^* = (z_1^*, z_2^*, \cdots, z_n^*)'$ from $y = (y_1, y_2, \cdots, y_n)'$ and $z = (z_1, z_2, \cdots, z_n)'$ respectively. Then estimator of traffic intensity vector is denoted by $\underline{\hat{\rho}}^* = (\hat{\rho}_1^*\ ,\ \hat{\rho}_2^*)' = \left(\frac{\overline{y}^*}{\overline{x}^*}\ ,\ \frac{\overline{z}^*}{\overline{y}^*}\right)'$ and can be calculated from bootstrap samples where $\overline{x}^*$, $\overline{y}^*$ and $\overline{z}^*$ be the sample means of $x^* = (x_1^*, x_2^*, \cdots, x_n^*)'$, $y^* = (y_1^*, y_2^*, \cdots, y_n^*)'$ and $z^* = (z_1^*, z_2^*, \cdots, z_n^*)'$. Let $\underline{\hat{\rho}}^*$ be called a bootstrap estimator of $\underline{\rho}$. The above resampling process can be repeated N times.The N bootstrap estimates

$$\underline{\hat{\rho}_1^*} = \begin{pmatrix}\hat{\rho}_{11}^*\\ \hat{\rho}_{21}^*\end{pmatrix}, \underline{\hat{\rho}_2^*} = \begin{pmatrix}\hat{\rho}_{12}^*\\ \hat{\rho}_{22}^*\end{pmatrix}, \cdots, \underline{\hat{\rho}_N^*} = \begin{pmatrix}\hat{\rho}_{1N}^*\\ \hat{\rho}_{2N}^*\end{pmatrix}$$

can be computed from the bootstrap resamples. Averaging the N bootstrap estimates we get $\underline{\hat{\rho}_N}$ called bootstrap estimate of $\underline{\rho}$. That is, $\underline{\hat{\rho}_N} = (\hat{\rho}_{N1}\ ,\ \hat{\rho}_{N2})' = \left(\frac{1}{N}\sum_{i=1}^{N}\hat{\rho}_{1i}^*\ ,\ \frac{1}{N}\sum_{i=1}^{N}\hat{\rho}_{2i}^*\right)'$ and the covariance matrix of $\underline{\hat{\rho}}$ using standard bootstrap can be estimated by

$$\tilde{A}^* = \begin{bmatrix}\frac{1}{N-1}\sum_{j=1}^{n}(\hat{\rho}_{1j}^*-\hat{\rho}_{N1})^2 & \frac{1}{N-1}\sum_{j=1}^{n}(\hat{\rho}_{1j}^*-\hat{\rho}_{N1})(\hat{\rho}_{2j}^*-\hat{\rho}_{N2})\\ \frac{1}{N-1}\sum_{j=1}^{n}(\hat{\rho}_{1j}^*-\hat{\rho}_{N1})(\hat{\rho}_{2j}^*-\hat{\rho}_{N_2}) & \frac{1}{N-1}\sum_{j=1}^{n}(\hat{\rho}_{2j}^*-\hat{\rho}_{N2})^2\end{bmatrix}$$

Using the estimator of A as $\tilde{A}^{*}$, a $100(1-\alpha)\%$ SB confidence region for $\underline{\rho}$ is given by

$$CR = \left\{ \underline{\rho} \mid n \left(\underline{\hat{\rho}} - \underline{\rho} \right)' \tilde{A}^{*-1} \left(\underline{\hat{\rho}} - \underline{\rho} \right) \leq \chi^2_{2,\alpha} \right\} \tag{6}$$

3.2.3. **Bayesian Bootstrap Confidence Region:** Using Bayesian bootstrap procedure we calculate $\overline{x}^{**} = \sum_{i=1}^{n} u_i x_i$ for μ_x (the mean of X,) where $u' = (u_1, u_2, \cdots, u_n)$ is the vector of probabilities attached to the inter-arrival data $x_1, x_2, \cdots, x_n$. Similarly we calculate $\overline{y}^{**} = \sum_{i=1}^{n} v_i y_i$ for μ_y (the mean of Y) and $\overline{z}^{**} = \sum_{i=1}^{n} w_i z_i$ for μ_z (the mean of Z). where $v' = (v_1, v_2, \cdots, v_n)$ and $w' = (w_1, w_2, \cdots, w_n)$ are vector of probabilities attached to the data values $y_1, y_2, \cdots, y_n$ and $z_1, z_2, \cdots, z_n$ respectively. Then an estimate of traffic intensity vector is denoted by $\underline{\hat{\rho}}^{**} = (\hat{\rho}_1^{**}\ ,\ \hat{\rho}_2^{**})' = \left(\frac{\overline{y}^{**}}{\overline{x}^{**}}\ ,\ \frac{\overline{z}^{**}}{\overline{y}^{**}} \right)'$ and can be calculated from BB replications. Let $\underline{\hat{\rho}}^{**}$ be called a Bayesian bootstrap estimator of $\underline{\rho}$. The above BB process can be repeated N times. The N BB estimates $\underline{\hat{\rho}_1^{**}} = \begin{pmatrix} \hat{\rho}_{11}^{**} \\ \hat{\rho}_{21}^{**} \end{pmatrix}, \underline{\hat{\rho}_2^{**}} = \begin{pmatrix} \hat{\rho}_{12}^{**} \\ \hat{\rho}_{22}^{**} \end{pmatrix}, \cdots, \underline{\hat{\rho}_N^{**}} = \begin{pmatrix} \hat{\rho}_{1N}^{**} \\ \hat{\rho}_{2N}^{**} \end{pmatrix}$ can be computed from the BB replications. Averaging the N BB estimates we get $\underline{\hat{\rho}_{BB}}$ called BB estimate of $\underline{\rho}$. That is, $\underline{\hat{\rho}_{BB}} = (\hat{\rho}_{BB_1}\ ,\ \hat{\rho}_{BB_2})' = \left(\frac{1}{N} \sum_{i=1}^{N} \hat{\rho}_{1i}^{**}\ ,\ \frac{1}{N} \sum_{i=1}^{N} \hat{\rho}_{2i}^{**} \right)'$. And the covariance matrix of $\underline{\hat{\rho}}$ using Bayesian bootstrap can be estimated by

$$\tilde{A}^{**} = \begin{bmatrix} \frac{1}{N-1} \sum_{j=1}^{n} (\hat{\rho}_{1j}^{**} - \hat{\rho}_{BB_1})^2 & \frac{1}{N-1} \sum_{j=1}^{n} (\hat{\rho}_{1j}^{**} - \hat{\rho}_{BB_1})(\hat{\rho}_{2j}^{**} - \hat{\rho}_{BB_2}) \\ \frac{1}{N-1} \sum_{j=1}^{n} (\hat{\rho}_{1j}^{**} - \hat{\rho}_{BB_1})(\hat{\rho}_{2j}^{**} - \hat{\rho}_{BB_2}) & \frac{1}{N-1} \sum_{j=1}^{n} (\hat{\rho}_{2j}^{**} - \hat{\rho}_{BB_2})^2 \end{bmatrix}$$

Using the estimator of A as $\tilde{A}^{**}$ a $100(1-\alpha)\%$ BB confidence region for traffic intensity vector $\underline{\rho}$ is given by

$$CR = \left\{ \underline{\rho} \mid n \left(\underline{\hat{\rho}} - \underline{\rho} \right)' \tilde{A}^{**-1} \left(\underline{\hat{\rho}} - \underline{\rho} \right) \leq \chi^2_{2,\alpha} \right\} \tag{7}$$

3.2.4. **Percentile Bootstrap Confidence Region:** Now consider $\underline{\hat{\rho}_1}^{*} = \begin{pmatrix} \hat{\rho}_{11}^{*} \\ \hat{\rho}_{21}^{*} \end{pmatrix}, \underline{\hat{\rho}_2}^{*} = \begin{pmatrix} \hat{\rho}_{12}^{*} \\ \hat{\rho}_{22}^{*} \end{pmatrix}, \cdots, \underline{\hat{\rho}_N}^{*} = \begin{pmatrix} \hat{\rho}_{1N}^{*} \\ \hat{\rho}_{2N}^{*} \end{pmatrix}$ the bootstrap distribution of $\underline{\hat{\rho}}$. To arrange $\underline{\hat{\rho}_1}^{*},\ \underline{\hat{\rho}_2}^{*},\ \underline{\hat{\rho}_3}^{*}, \cdots,\ \underline{\hat{\rho}_N}^{*}$ we use Euclidian distance, where Euclidian distance is given by, $d_j = \sqrt{(\hat{\rho}_{1j}^{*} - \hat{\rho}_{N1})^2 + (\hat{\rho}_{2j}^{*} - \hat{\rho}_{N2})^2}\ ,\quad j = 1, 2 \cdots, N$ where $\hat{\rho}_{N1} = \frac{1}{N} \sum_{i=1}^{N} \hat{\rho}_{1i}^{*}$, and $\hat{\rho}_{N2} = \frac{1}{N} \sum_{i=1}^{N} \hat{\rho}_{2i}^{*}$. Hence $\underline{\hat{\rho}_1}^{*}(1),\ \underline{\hat{\rho}_2}^{*}(2),\ \underline{\hat{\rho}_3}^{*}(3), \cdots,\ \underline{\hat{\rho_N}}^{*}(N)$ is the ordered arrangement of $\underline{\hat{\rho}_1}^{*},\ \underline{\hat{\rho}_2}^{*},\ \underline{\hat{\rho}_3}^{*}, \cdots,\ \underline{\hat{\rho_N}}^{*}$. Then utilizing the $100(1-\alpha)^{th}$ percentage point of the bootstrap distribution, $100(1-\alpha)\%$ PB confidence region for $\underline{\rho}$ is given by

$$CR = \left\{ \underline{\rho_j} \mid d_j \leq ([N(1-\alpha)]) \right\} \tag{8}$$

where $[x]$ denotes the greatest integer less than or equal to x.

4. Simulation Study

A simulation study was performed to examine the relevance of confidence regions constructed in equations (5) to (8). The performances of the confidence regions are assessed in terms of their coverage area percentage,average area and relative coverage area.Relative coverage area is defined as the ratio of coverage area percentage to average area of confidence region. We have simulated $M/E_4/1$ to $E_4/H_4^{Pe}/1$, $M/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$, $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$ and $E_4/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$ queueing network models, where M : exponential distribution, E_4 : 4-stage Erlang distribution, H_4^{Pe} : 4-stage hyperexponential distribution and H_4^{Po} : 4-stage hypoexponential distribution. The values of $(\rho_1\ ,\ \rho_2)$ are set to $(0.2\ ,\ 0.8)$. Random samples of arrival times and service times are drawn. Next $N = 1000$ bootstrap resamples are drawn from the original samples, as well as $N = 1000$ BB replications are simulated for the original samples. The above simulation process is replicated $N = 1000$ times. We compute coverage area percentage, average area and relative coverage area of the confidence regions. Calibration technique is used to improve the coverage percentage area of confidence regions obtained in equations (5) to (8). Based on the different estimation approaches coverage area percentage,average area and relative coverage area of $\underline{\rho}$ without calibration and with calibration are shown in Tables 1 to 5. More on simulation technique for confidence intervals or hypothesis testing, we refer our readers to Kibria and Banik [17] and Banik and Kibria [3] among others. Figure 2 shows that as sample size increases from 5 to 100 coverage area percentage are approaches to 95 %.

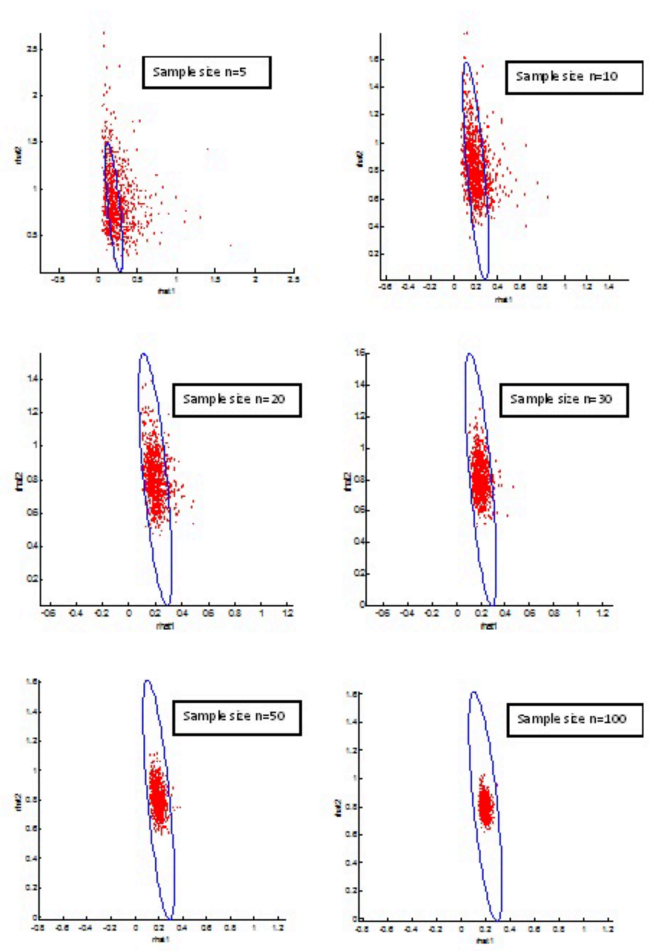

Figure 2. Confidence Regions for Traffic Intensity Vector of a $M/E_4/1$ to $E_4/H_4^{Pe}/1$ Queueing Network Model without feedback

5. Conclusions

Estimation approaches CAN, SB, BB and PB are used to construct various confidence regions for traffic intensity vector. From Tables 1 to 4 we observed that average area are decreasing as n increases from 5 to 100 but both coverage area percentage and relative coverage area are increasing as n increases from 5 to 100. Coverage area percentage are approaches to 95 % when n increases to 100. Also we observed that, with calibration technique relative coverage area is comparatively more than without calibration technique. It is observed that, the estimation approach **Bayesian Bootstrap** has the greatest relative coverage area without as well as with calibration for all queueing network models. From Table 5 it is observed that, due to calibration technique maximum increase in coverage area percentage of confidence regions is 14.7% for $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$ queueing network model. These approaches are successfully and efficiently applied to practical queueing network models. Also calibration technique can be used to improve the coverage area percentage of confidence regions.

Table 1. Simulation results for confidence regions of $M/E_4/1$ to $E_4/H_4^{Pe}/1$

Coverage Area Percentages for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Estimation	Before Calibration						After Calibration					
Approches	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$
Chi	0.826	0.896	0.909	0.928	0.935	0.928	0.896	0.926	0.953	0.941	0.932	0.950
SB	0.844	0.922	0.917	0.939	0.947	0.930	0.908	0.931	0.958	0.940	0.931	0.952
BB	0.796	0.884	0.901	0.926	0.935	0.924	0.878	0.929	0.955	0.942	0.932	0.954
PB	0.852	0.878	0.881	0.893	0.892	0.894	0.973	0.980	0.970	0.972	0.968	0.978
Average Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Chi	1.926	1.937	1.898	1.932	1.926	1.919	1.902	1.871	1.904	1.905	1.919	1.912
SB	0.827	0.240	0.104	0.068	0.040	0.020	0.868	0.231	0.105	0.067	0.040	0.019
BB	0.380	0.188	0.092	0.063	0.038	0.019	0.374	0.182	0.093	0.062	0.038	0.019
PB	0.484	0.333	0.231	0.190	0.145	0.103	0.480	0.329	0.230	0.189	0.146	0.103
Relative Coverage areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Chi	0.429	0.463	0.479	0.480	0.486	0.484	0.471	0.495	0.500	0.494	0.486	0.497
SB	1.021	3.838	8.820	13.753	23.788	47.781	1.046	4.030	9.163	13.960	23.462	48.971
BB	**2.093**	**4.708**	**9.768**	**14.736**	**24.778**	**48.696**	**2.346**	**5.115**	**10.309**	**15.214**	**24.745**	**50.532**
PB	1.760	2.637	3.812	4.705	6.152	8.703	2.028	2.976	4.214	5.138	6.623	9.463

Table 2. Simulation results for confidence regions of $M/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$

Coverage Area Percentages for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Estimation	Before Calibration						After Calibration					
Approches	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$
Chi	0.833	0.897	0.922	0.927	0.937	0.930	0.898	0.916	0.945	0.953	0.950	0.957
SB	0.864	0.924	0.931	0.937	0.945	0.932	0.908	0.926	0.953	0.954	0.949	0.959
BB	0.813	0.895	0.915	0.920	0.930	0.926	0.884	0.913	0.944	0.952	0.955	0.960
PB	0.852	0.878	0.885	0.891	0.891	0.882	0.979	0.974	0.981	0.974	0.975	0.979
Average Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Chi	1.937	1.888	1.908	1.917	1.898	1.932	1.980	1.880	1.930	1.894	1.931	1.925
SB	0.899	0.241	0.106	0.068	0.039	0.020	1.391	0.242	0.108	0.067	0.040	0.020
BB	0.397	0.186	0.093	0.063	0.037	0.019	0.405	0.185	0.094	0.062	0.038	0.019
PB	0.497	0.334	0.234	0.190	0.147	0.104	0.512	0.331	0.235	0.189	0.148	0.104
Relative Coverage Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Chi	0.430	0.475	0.483	0.484	0.494	0.481	0.454	0.487	0.490	0.503	0.492	0.497
SB	0.961	3.827	8.816	13.699	23.998	47.442	0.653	3.835	8.858	14.175	23.677	49.025
BB	**2.049**	**4.815**	**9.806**	**14.715**	**24.907**	**48.424**	**2.183**	**4.931**	**9.998**	**15.405**	**25.173**	**50.467**
PB	1.714	2.632	3.784	4.680	6.048	8.468	1.912	2.943	4.180	5.162	6.594	9.396

Table 3. Simulation results for confidence regions of $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$

Coverage Percentage Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Estimation	Before Calibration						After Calibration					
Approches	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$
Chi	0.770	0.865	0.889	0.910	0.930	0.948	0.851	0.898	0.921	0.936	0.949	0.926
SB	0.771	0.862	0.891	0.901	0.931	0.947	0.854	0.895	0.919	0.939	0.945	0.928
BB	0.713	0.842	0.876	0.899	0.924	0.950	0.817	0.879	0.916	0.934	0.946	0.919
PB	0.769	0.851	0.872	0.869	0.872	0.889	0.916	0.929	0.936	0.966	0.960	0.959
Average Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Chi	1.601	1.750	1.841	1.849	1.891	1.903	1.633	1.761	1.813	1.870	1.887	1.893
SB	0.333	0.177	0.092	0.062	0.038	0.019	0.338	0.178	0.091	0.063	0.038	0.019
BB	0.254	0.156	0.086	0.059	0.037	0.019	0.259	0.156	0.085	0.060	0.037	0.019
PB	0.588	0.449	0.326	0.270	0.212	0.151	0.606	0.441	0.322	0.272	0.211	0.150
Relative Coverage Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Chi	0.481	0.494	0.483	0.492	0.492	0.498	0.521	0.510	0.508	0.501	0.503	0.489
SB	2.313	4.880	9.648	14.596	24.579	49.761	2.530	5.036	10.100	15.006	25.068	49.080
BB	**2.809**	**5.403**	**10.141**	**15.197**	**25.077**	**50.669**	**3.158**	**5.620**	**10.730**	**15.594**	**25.735**	**49.272**
PB	1.309	1.896	2.673	3.218	4.121	5.879	1.512	2.106	2.907	3.553	4.556	6.385

Table 4. Simulation results for confidence regions of $E_4/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$

Coverage Percentage Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Estimation	Before Calibration						After Calibration					
Approches	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$
Chi	0.804	0.882	0.907	0.912	0.950	0.946	0.872	0.916	0.942	0.942	0.950	0.937
SB	0.812	0.880	0.907	0.911	0.951	0.945	0.876	0.921	0.945	0.946	0.942	0.936
BB	0.750	0.856	0.897	0.905	0.947	0.943	0.848	0.911	0.940	0.943	0.948	0.934
PB	0.816	0.860	0.872	0.887	0.904	0.913	0.950	0.966	0.968	0.963	0.975	0.960
Average Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Chi	1.1157	1.1738	1.2219	1.2062	1.2597	1.2620	1.1353	1.1965	1.2378	1.2419	1.2471	1.2468
SB	0.2447	0.1216	0.0619	0.0405	0.0253	0.0126	0.2454	0.1247	0.0628	0.0416	0.0251	0.0125
BB	0.1826	0.1055	0.0577	0.0386	0.0246	0.0125	0.1843	0.1081	0.0585	0.0397	0.0243	0.0123
PB	0.4571	0.3305	0.2419	0.1950	0.1547	0.1102	0.4601	0.3352	0.2422	0.1994	0.1537	0.1094
Relative Coverage Areas for $\rho_1 = 0.2$ and $\rho_2 = 0.8$												
Chi	0.721	0.751	0.742	0.756	0.754	0.750	0.768	0.766	0.761	0.759	0.762	0.752
SB	3.319	7.235	14.657	22.501	37.622	74.805	3.569	7.385	15.055	22.734	37.580	74.981
BB	**4.108**	**8.111**	**15.536**	**23.434**	**38.539**	**75.680**	**4.602**	**8.427**	**16.074**	**23.737**	**39.014**	**75.911**
PB	1.785	2.602	3.605	4.548	5.842	8.288	2.065	2.882	3.996	4.830	6.343	8.776

Table 5. Maximum percentage(%) increase in coverage percentage area due to calibration technique

Increase in Coverage Percentage area for $\rho_1 = 0.2$ and $\rho_2 = 0.8$						
Queueing Network Model	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$	$n=100$
$M/E_4/1$ to $E_4/H_4^{Pe}/1$	12.10	10.20	8.90	7.90	7.60	8.40
$M/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$	12.70	9.60	9.60	8.30	8.40	9.70
$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$	14.70	7.80	6.40	9.70	8.80	7.00
$E_4/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$	13.40	10.60	9.60	7.60	7.10	4.70

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