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# The Marshall-Olkin Odd Exponential Half Logistic-G Family of Distributions: Properties and Applications

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**Abstract** We develop a new family of distributions, referred to as the Marshall-Olkin odd exponential half logistic-G, which is a linear combination of the exponential-G family of distributions. The family of distributions can handle heavy tailed data and has non-monotonic hazard rate functions. We also conducted a simulation study to assess the performance of the proposed model. Real data examples are provided to demonstrate the usefulness of the proposed model in comparison with several other existing models.

**Keywords** Marshall-Olkin-G, Half-Logistic-G, Maximum Likelihood Estimation

**AMS 2010 subject classifications** 60E05, 62E15, 62F30.

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## 1. Introduction

Marshall and Olkin [20], introduced a new flexible distribution that can be used to generalize other known distributions. The Marshall-Olkin distribution applies to data that has monotonic or non-monotonic hazard rates. This made the distribution receive much attention from statisticians in areas of biology, medicine, engineering, hydrology, and economics. The distribution performs better in data fitting compared to widely used distributions like the exponential, Weibull and Rayleigh distributions, because the later distribution exhibits both monotonic and non-monotonic hazard rate functions. The Marshall-Olkin-G (MO-G) distribution has its cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{MO-G}(x;\delta,\underline{\xi}) = 1 - \frac{\delta\bar{G}(x;\underline{\xi})}{1-\bar{\delta}\bar{G}(x;\underline{\xi})}, \quad (1)$$

and

$$f_{MO-G}(x;\delta,\underline{\xi}) = \frac{\delta g(x;\underline{\xi})}{\left[1-\bar{\delta}\bar{G}(x;\underline{\xi})\right]^2}, \quad (2)$$

respectively, where $\delta$ is the tilt parameter and $G(x;\underline{\xi})$ is the baseline cdf.

Some generalizations of the Marshall-Olkin distribution include: beta Marshall-Olkin-G (BMO-G) by Alizadeh et al. [4], Kumaraswamy Marshall-Olkin-G (KwMO-G) by Alizadeh et al. [5], Marshall-Olkin Log-logistic Extended Weibull (MOLLEW) family of distributions by Lepetu et al. [19], Marshall-Olkin extended Weibull

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family of distributions by Santos et al. [29], Marshall-Olkin Kumaraswamy-G distribution by Chakraborty et al. [9], Marshall-Olkin extended generalized Gompertz distribution by Lazhar et al. [18] and Marshall-Olkin extended Burr Type III distribution by Kumar et al. [16]. Other new generalizations available in the literature are bivariate gamma distribution by Rafiei et al. [27], generalized modification of the Kumaraswamy distribution by Alshkaki [3] and Ristic-Balakrishnan odd log-logistic family of distributions by Esmaeili et al. [10].

Furthermore, Cordeiro et al. [13], developed the type 1 half-logistic family of distributions with the cdf and pdf given by

$$\begin{aligned} F_{TI-HL-G}(x;\lambda,\underline{\xi}) &= \int_0^{-ln(1-G(x;\underline{\xi}))} \frac{2\lambda\exp\{-\lambda x\}}{(1+\exp\{-\lambda x\})^2}dx \\ &= \frac{1-[1-G(x;\underline{\xi})]^\lambda}{1+[1-G(x;\underline{\xi})]^\lambda}, \end{aligned} \tag{3}$$

where $G(x;\underline{\xi})$ is the cdf of the baseline distribution and $\lambda > 0$ is the shape parameter. We obtain a special case, namely, half-logistic-G (HL-G) model, with cdf

$$F_{_{HL-G}}(x;\underline{\xi}) = \frac{G(x;\underline{\xi})}{1+\overline{G}(x;\underline{\xi})}, \tag{4}$$

by setting $\lambda = 1$ in equation (3). The corresponding pdf of the HL-G model is given by

$$f_{_{HL-G}}(x;\underline{\xi}) = \frac{2g(x;\underline{\xi})}{(1+\overline{G}(x;\underline{\xi}))^2}. \tag{5}$$

Afify et al. [1] generalized the HL-G distribution to develop a new family of distributions referred to as the Exponentiated Odd Exponential Half Logistic-G (EOEHL-G) family of distributions. Their generalization exhibit interesting shapes for both the density and hazard rate function, demonstrating its usefulness in lifetime data analysis. Other generalizations of the HL-G distribution include the exponentiated half-logistic generated family by Cordeiro et al. [14], Kumaraswamy type 1 half logistic family of distributions with applications by El-sayed [15], the type I generalized half logistic distribution based on upper record values by Kumar et al. [17] and generalized half-logistic Poisson distributions by Muhammad et al. [21]. The Exponentiated Odd Exponential Half Logistic-G (EOEHL-G) distribution has cdf and pdf given by

$$F_{_{EOEHL-G}}(x;\alpha,\lambda,\underline{\xi}) = \left\{\frac{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}\right\}^\alpha \tag{6}$$

and

$$f_{_{EOEHL-G}}(x;\alpha,\lambda,\underline{\xi}) = \frac{2\alpha\lambda g(x;\underline{\xi})\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big(1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big)^{\alpha-1}}{\bar{G}^2(x;\underline{\xi})\big(1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big)^{\alpha+1}}, \tag{7}$$

where $\mathbf{O}_\lambda(x;\underline{\xi}) = \frac{\lambda G(x;\underline{\xi})}{\bar{G}(x;\underline{\xi})}$, for $\alpha, \lambda > 0$, $g(x;\underline{\xi}) = \frac{dG(x;\underline{\xi})}{dx}$ and $G(x;\underline{\xi})$ is the baseline distribution. When $\alpha = 1$, we have the Odd Exponential Half Logistic-G (OEHL-G) distribution.

We were motivated by the desirable properties of the MO-G and the OEHL-G distributions to develop a new family of distribution which is a combination of these two distributions. We hope the new distribution will receive much attention from statisticians since the distribution can be applied to

- heavy-tailed data and;
- data that have non-monotonic hazard rate functions.

The new distribution is an infinite linear combination of the exponentiated distribution and this property allows for the derivation of other statistical properties from this distribution.

In this note, we develop the new family of distributions, namely, Marshall-Olkin-odd exponential half logistic-G (MO-OEHL-G) family of distributions. We present in Section 2, the new generalized family of distributions and density expansion. Section 3, show some sub-families of the MO-OEHL-G family of distributions. In Section 4, we present some structural properties which include the distribution of order statistics, Rényi entropy, moments, generating functions, and quantile function. Estimation via maximum likelihood estimation and the Fisher's information matrix are presented in Section 5. We conduct Monte Carlo simulation study and the results are shown in Section 6. Applications of the proposed model to real data are given in Section 7, followed by concluding remarks.

## 2. Marshall-Olkin-Odd Exponential Half Logistic-G Family of Distributions

We develop the MO-OEHL-G distribution using the generalization proposed by Marshall and Olkin [20], and taking the baseline distribution to be the OEHL-G distribution. Therefore, the cdf and pdf of the MO-OEHL-G family of distributions is given by

$$F_{MO-OEHL-G}(x;\delta,\lambda,\underline{\xi}) = 1 - \frac{\delta\left\{1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}\right\}}{1-\bar{\delta}\left\{1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}\right\}} \tag{8}$$

and

$$f_{MO-OEHL-G}(x;\delta,\lambda,\underline{\xi}) = \frac{2\delta\lambda g(x;\underline{\xi})\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]\left\{1+\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]\right\}^{-2}}{\bar{G}^2(x;\underline{\xi})\left\{1-\bar{\delta}\left(1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}\right)\right\}^2}, \tag{9}$$

respectively, for $\delta > 0$, $\bar{\delta} = 1-\delta$, $\lambda > 0$ and $\underline{\xi}$ is a vector of parameters from the baseline distribution function G(.).

### 2.1. Expansion of Density Function

We use the general results for the Marshall and Olkin's family of distributions by Barreto-Souza et al. [7] to derive the series expansion of the pdf of the MO-OEHL-G family of distributions. Considering

$$f(x;\delta,\lambda,\underline{\xi}) = \frac{\delta f_{OEHL-G}(x;\lambda,\underline{\xi})}{(1-\bar{\delta}\bar{F}_{OEHL-G}(x;\lambda,\xi))^2}, \tag{10}$$

we can write equation (10) as

$$f(x;\delta,\lambda,\underline{\xi}) = \frac{f_{OEHL-G}(x;\lambda,\underline{\xi})}{\delta[1-\frac{\delta-1}{\delta}F_{OEHL-G}(x;\lambda,\underline{\xi})]^2}, \tag{11}$$

where $f_{OEHL-G}(x;\lambda,\underline{\xi})$ and $F_{OEHL-G}(x;\lambda,\underline{\xi})$ are as given in equations (7) and (6), respectively. Also we apply the series expansion

$$(1-z)^{-k} = \sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j, \tag{12}$$

which is valid for $|z| < 1$ and $k > 0$. If $\delta \in (0,1)$, we can obtain

$$f(x;\delta,\lambda,\underline{\xi}) = f_{OEHL-G}(x;\lambda,\underline{\xi}) \sum_{j=0}^{\infty}\sum_{k=0}^{j} w_{j,k} F_{OEHL-G}(x;\lambda,\underline{\xi})^{j-k}, \tag{13}$$

where $w_{j,k} = w_{j,k}(\delta) = \delta(j+1)(1-\delta)^j(-1)^{j-k}\binom{j}{k}$. For $\delta > 1$, we have

$$f(x;\delta,\lambda,\underline{\xi}) = f_{OEHL-G}(x;\lambda,\underline{\xi}) \sum_{j=0}^{\infty} v_j F^{j}_{OEHL-G}(x;\lambda,\underline{\xi}), \tag{14}$$

where $v_j = v_j(\delta) = \frac{(j+1)(1-1/\delta)}{\delta}$. For $\delta \in (0,1)$, equation (10) becomes

$$f(x;\delta,\lambda,\underline{\xi}) = \sum_{j=0}^{\infty}\sum_{k=0}^{j} w_{j,k} \frac{2\lambda g(x;\underline{\xi}) \exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{(\bar{G}(x;\underline{\xi}))^2\big(1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big)^2} \left\{\frac{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}\right\}^{j-k}.$$

Applying the following series expansions

$$\big\{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big\}^{-(j-k+2)} = \sum_{m=0}^{\infty}\binom{-(j-k+2)}{m}\exp\big[-m\mathbf{O}_\lambda(x;\underline{\xi})\big],$$

$$\Big\{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\Big\}^{(j-k)} = \sum_{n=0}^{\infty}(-1)^n\binom{j-k}{n}\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big],$$

$$\exp\big[-(m+n+1)\mathbf{O}_\lambda(x;\underline{\xi})\big] = \sum_{p=0}^{\infty}\frac{(-1)^p\lambda^p[m+n+1]^p}{p!}\big[\mathbf{O}(x;\underline{\xi})\big]^p,$$

and

$$[\bar{G}(x;\underline{\xi})]^{-(p+2)} = [1-G(x;\underline{\xi})]^{-(p+2)} = \sum_{q=0}^{\infty}(-1)^q\binom{-(p+2)}{q}G^q(x;\underline{\xi}),$$

we can write

$$\begin{aligned} f(x;\delta,\lambda,\underline{\xi}) &= \sum_{j,m,n,p,q=0}^{\infty}\sum_{k=0}^{j}\frac{2(-1)^{n+p}\lambda^{p+1}[m+n+1]^p}{p!(p+q+1)}w_{j,k} \\ &\times \binom{-(j-k+2)}{m}\binom{j-k}{n}\binom{-(p+2)}{q} \\ &\times (p+q+1)g(x;\underline{\xi})[G(x;\underline{\xi})]^{p+q} \\ &= \sum_{p,q=0}^{\infty} w^*_{p,q} g_{p+q+1}(x;\underline{\xi}). \end{aligned} \tag{15}$$

It follows that for $\delta \in (0,1)$, the MO-OEHL-G family of distributions can be expressed as an infinite linear combination of the exponentiated-G (Exp-G) distribution with power parameter $(p+q+1)$ and linear component

$$\begin{aligned} w^*_{p,q} &= \sum_{j,m,n=0}^{\infty}\sum_{k=0}^{j}\frac{2(-1)^{n+p}\lambda^{p+1}[m+n+1]^p}{p!(p+q+1)}w_{j,k}\binom{-(j-k+2)}{m} \\ &\times \binom{j-k}{n}\binom{-(p+2)}{q}. \end{aligned} \tag{16}$$

Furthermore, for $\delta > 1$ equation (10) can be written as

$$f(x;\delta,\lambda,\underline{\xi}) = \frac{2\lambda g(x;\underline{\xi})\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{(\bar{G}(x;\underline{\xi}))^2\big(1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big)^2}\sum_{j=0}^{\infty}v_j\left[\frac{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}\right]^j.$$

Applying the following series expansions

$$\Big\{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\Big\}^{-(j+2)} = \sum_{m=0}^{\infty}\binom{-(j+2)}{m}\exp\big[-m\mathbf{O}_\lambda(x;\underline{\xi})\big],$$

$$\Big\{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\Big\}^{j} = \sum_{n=0}^{\infty}(-1)^n\binom{j}{n}\exp\big[-n\mathbf{O}_\lambda(x;\underline{\xi})\big],$$

$$\exp\big[-(m+n+1)\mathbf{O}_\lambda(x;\underline{\xi})\big] = \sum_{p=0}^{\infty}\frac{(-1)^p\lambda^p[m+n+1]^p}{p!}\big[\mathbf{O}(x;\underline{\xi})\big]^p,$$

and

$$[\bar{G}(x;\underline{\xi})]^{-(p+2)} = [1-G(x;\underline{\xi})]^{-(p+2)} = \sum_{q=0}^{\infty}(-1)^q\binom{-(p+2)}{q}G^q(x;\underline{\xi}),$$

we can write

$$\begin{aligned} f(x;\delta,\lambda,\underline{\xi}) &= \sum_{j,m,n,p,q=0}^{\infty}\frac{2(-1)^{n+p}\lambda^{p+1}[m+n+1]^p}{p!(p+q+1)}v_j\binom{-(j+2)}{m}\binom{j}{n} \\ &\times \binom{-(p+2)}{q}(p+q+1)g(x;\underline{\xi})[G(x;\underline{\xi})]^{p+q} \\ &= \sum_{p,q=0}^{\infty}v^*_{p,q}g_{p+q+1}(x;\underline{\xi}). \end{aligned} \tag{17}$$

Therefore, for $\delta > 1$, the MO-OEHL-G family of distributions can be expressed as an infinite linear combination of the Exp-G distribution with power parameter $(p+q+1)$ and linear component

$$v^*_{p,q} = \sum_{j,m,n=0}^{\infty}\frac{2(-1)^{n+p}\lambda^{p+1}[m+n+1]^p}{p!(p+q+1)}v_j\binom{-(j+2)}{m}\binom{j}{n}\binom{-(p+2)}{q}. \tag{18}$$

## 3. Some Special Cases

In this Section, we present some special cases of the MO-OEHL-G family of distributions. We considered cases when the baseline distributions are uniform, log-logistic and Weibull distributions.

### *3.1. Marshall-Olkin Odd Exponential Half Logistic-Uniform Distribution*

Consider the uniform distribution as the baseline distribution. The uniform distribution has pdf and cdf given by $g(x) = 1/\theta$ and $G(x,\theta) = x/\theta$, respectively, for $0 < x < \theta$. Therefore, the MO-OEHL-U distribution has cdf and pdf given by

$$F(x;\delta,\lambda,\theta) = 1 - \frac{\delta\left\{1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\theta)\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\theta)\right]}\right\}}{1-\bar{\delta}\left\{1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\theta)\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\theta)\right]}\right\}}$$

and

$$f(x;\delta,\lambda,\theta) = \frac{2\delta\lambda\exp\left[-\mathbf{O}_\lambda(x;\theta)\right]\left\{1+\exp\left[-\mathbf{O}_\lambda(x;\theta)\right]\right\}^{-2}}{\theta(1-x/\theta)\left\{1-\bar{\delta}\left(1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\theta)\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\theta)\right]}\right)\right\}^2},$$

respectively, where $\left[\mathbf{O}_\lambda(x;\theta)\right] = \left(\lambda\frac{x}{\theta-x}\right)$, for $\lambda$, $\delta$ and $\theta > 0$.

Figure 1 shows plots of the pdfs and hrfs of the MO-OEHL-U distribution for selected parameters values.

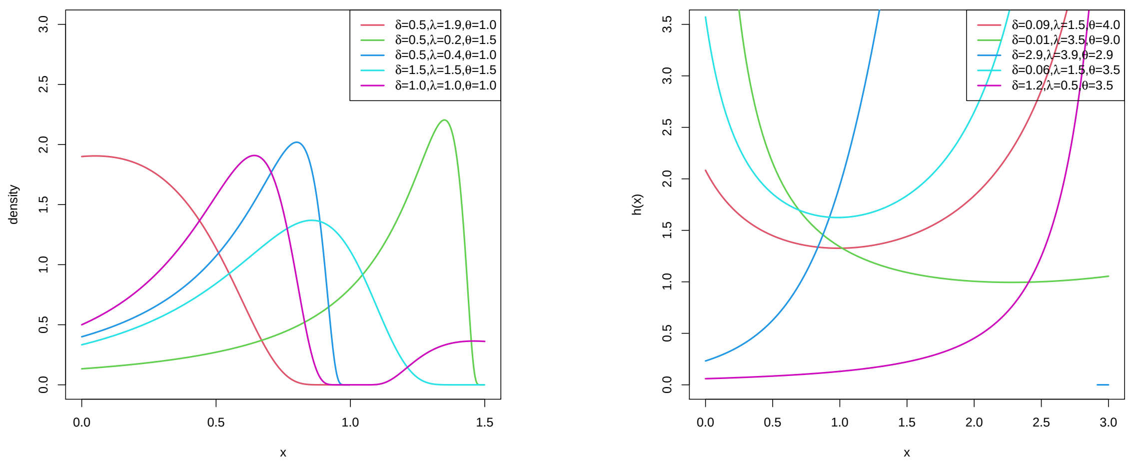



Figure 1. Plots of the pdf and hrf for the MO-OEHL-U distribution

It shows that the new distribution can handle data that is left or right-skewed. The MO-OEHL-U model can fit data sets that have increasing, decreasing and bathtub hazard rate shapes.

### *3.2. Marshall-Olkin Odd Exponential Half Logistic-Log-Logistic Distribution*

Consider the log-logistic distribution as the baseline distribution. The log-logistic distribution has pdf and cdf given by $g(x) = cx^{c-1}(1+x^c)^{-2}$ and $G(x) = 1-(1+x^c)^{-1}$, for $c > 0$, respectively. Therefore, the MO-OEHL-LL distribution has cdf and pdf given by

$$F(x;\delta,\lambda,c) = 1-\frac{\delta\left[1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;c)\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;c)\right]}\right]}{1-\bar{\delta}\left\{1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;c)\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;c)\right]}\right\}}$$

and

$$f(x;\delta,\lambda,c) = \frac{2\delta\lambda cx^{c-1}(1+x^c)^{-2}\exp\left[-\mathbf{O}_\lambda(x;c)\right]}{(1+x^c)^{-2}\left\{1-\bar{\delta}\left(1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;c)\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;c)\right]}\right)\right\}^2}\left\{1+\exp\left[-\mathbf{O}_\lambda(x;c)\right]\right\}^{-2},$$

respectively, where $\left[\mathbf{O}_\lambda(x;c)\right] = \left(\lambda\frac{[1-(1+x^c)^{-1}]}{(1+x^c)^{-1}}\right)$, for $\delta, \lambda, c > 0$.

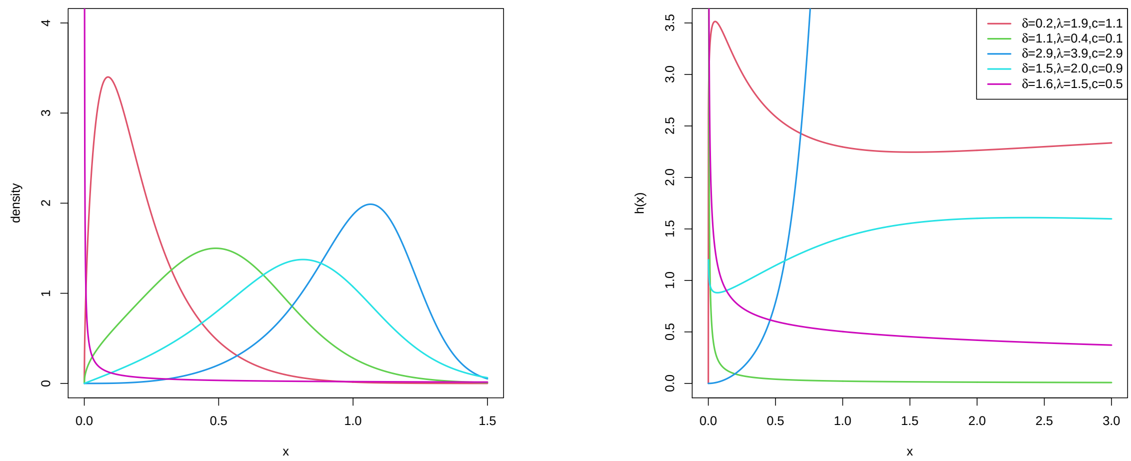



Figure 2. Plots of the pdf and hrf for the MO-OEHL-LL distribution

The pdf of the MO-OEHL-LL distribution takes various shapes including reverse-J, left and right-skewed. The hazard rate function also exhibits various shapes that include reverse-J, bathtub, upside bathtub followed by bathtub.

### *3.3. Marshall-Olkin Odd Exponential Half Logistic-Weibull Distribution*

Consider the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x;\gamma,\omega) = \gamma\omega x^{\omega-1}e^{-\gamma x^{\omega}}$ and $G(x;\gamma,\omega) = 1 - e^{-\gamma x^{\omega}}$, respectively, for $\gamma,\omega > 0$. The cdf and pdf of the MO-OEHL-W distribution are given by

$$F(x;\delta,\lambda,\gamma,\omega) = 1 - \frac{\delta\left[1 - \frac{1-\exp\left[-\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right]}{1+\exp\left[-\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right]}\right]}{1-\bar{\delta}\left\{1 - \frac{1-\exp\left[-\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right]}{1+\exp\left[-\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right]}\right\}}$$

and

$$f(x;\delta,\lambda,\gamma,\omega) = \frac{2\delta\lambda\gamma\omega x^{\omega-1}e^{-\gamma x^{\omega}}\exp\left[-\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right]}{(e^{-\gamma x^{\omega}})^{2}\left\{1-\bar{\delta}\left(1 - \frac{1-\exp\left[-\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right]}{1+\exp\left[-\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right]}\right)\right\}^{2}}\left\{1+\exp\left[-\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right]\right\}^{-2},$$

respectively, where $\left[\mathbf{O}_{\lambda}(x;\lambda,\gamma)\right] = \left(\lambda\frac{[1-e^{-\gamma x^{\omega}}]}{e^{-\gamma x^{\omega}}}\right)$, for $\delta,\lambda,\gamma,\omega > 0$.

Plots of the MO-OEHL-W pdf shows that the distribution can take various shapes that include reverse-J, almost symmetric, left or right-skewed. The hazard rate function exhibits both monotonic and non-monotonic hazards rate shapes.

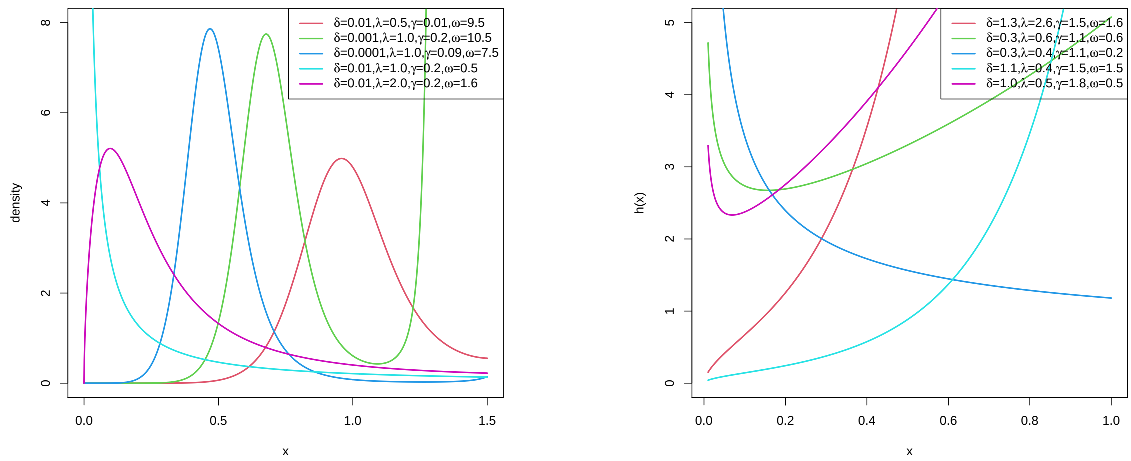



Figure 3. Plots of the pdf and hrf for the MO-OEHL-W distribution

## 4. Statistical Properties

### *4.1. Distribution of Order Statistics*

Suppose that $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d) random variables distributed according to (9). The pdf of the $i^{th}$ order statistic $X_{i:n}$, is given by

$$f_{i:n}(x;\delta,\lambda,\underline{\xi}) = \delta n! f_{OEHL-G}(x;\underline{\xi}) \sum_{l=0}^{n-i} \frac{(-1)^l}{(i-1)!(n-i)!} \frac{F_{OEHL-G}^{l+i-1}(x;\lambda,\underline{\xi})}{[1-\bar{\delta}F_{OEHL-G}(x;\lambda,\underline{\xi})]^{l+i-1}}. \tag{19}$$

If $\delta \in (0,1)$, we have

$$f_{i:n}(x;\delta,\lambda,\underline{\xi}) = f_{OEHL-G}(x;\lambda,\underline{\xi}) \sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j} U_{j,l,k} F_{OEHL-G}^{j+l-k+i-1}(x;\lambda,\underline{\xi}), \tag{20}$$

where

$$U_{j,l,k} = U_{j,l,k}(\delta) = \frac{\delta n!(-1)^l(1-\delta)^j(-1)^{j-k}}{(i-1)!(n-i)!}\binom{j}{k}\binom{l+i+j}{j}. \tag{21}$$

For $\delta > 1$, we write $1-\bar{\delta}\bar{F}_{OEHL-G}(x;\lambda,\underline{\xi}) = \delta\left\{1-(\delta-1)F_{OEHL-G}(x;\lambda,\underline{\xi})/\delta\right\}$, such that

$$f_{i:n}(x;\delta,\lambda,\underline{\xi}) = f_{OEHL-G}(x;\lambda,\underline{\xi}) \sum_{j=0}^{\infty}\sum_{l=0}^{n-i} C_{j,l} F_{OEHL-G}^{j+l+i-1}(x;\lambda,\underline{\xi}), \tag{22}$$

where

$$C_{j,l} = C_{j,l}(\delta) = \frac{(-1)^l(\delta-1)^j n!}{\delta^{l+j+i}(i-1)!(n-i)!}\binom{l+i+j}{j}. \tag{23}$$

For $\delta \in (0,1)$, using equation (20) and substituting the pdf and cdf of the OEHL-G distribution, we get

$$\begin{aligned} f_{i:n}(x;\delta,\lambda,\underline{\xi}) &= \frac{2\lambda g(x;\underline{\xi})\big(\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{\bar{G}^2(x;\underline{\xi})\big(1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big)^2} \sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j} U_{j,l,k} \\ &\times \left\{\frac{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}\right\}^{j+l-k+i-1}. \end{aligned}$$

By applying the following series expansions

$$\Big\{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\Big\}^{-(j+l-k+i+1)} = \sum_{m=0}^{\infty}\binom{-(j+l-k+i+1)}{m}\exp\big[-m\mathbf{O}_\lambda(x;\underline{\xi})\big],$$

$$\begin{aligned}\Big\{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\Big\}^{j+l-k+i-1} &= \sum_{n=0}^{\infty}(-1)^n\binom{(j+l-k+i-1)}{n}\\ &\times \exp\big[-n\mathbf{O}_\lambda(x;\underline{\xi})\big],\end{aligned}$$

$$\exp\big[-(m+n+1)\mathbf{O}_\lambda(x;\underline{\xi})\big]=\sum_{p=0}^{\infty}\frac{(-1)^p\lambda^p[m+n+1]^p}{p!}\big[\mathbf{O}(x;\underline{\xi})\big]^p$$

and

$$[\bar{G}(x;\underline{\xi})]^{-(p+2)}=[1-G(x;\underline{\xi})]^{-(p+2)}=\sum_{q=0}^{\infty}(-1)^q\binom{-(p+2)}{q}G^q(x;\underline{\xi}),$$

we can write

$$\begin{aligned}f_{i:n}(x;\delta,\lambda,\underline{\xi}) &= \sum_{j,m,n,p,q=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}\frac{2(-1)^{n+p}\lambda^{p+1}[m+n+1]^p}{p!(p+q+1)}U_{j,l,k}\\ &\times \binom{-(j+l-k+i+1)}{m}\binom{(j+l-k+i-1)}{n}\binom{m}{p}\\ &\times (p+q+1)g(x;\underline{\xi})[G(x;\underline{\xi})]^{p+q}\\ &= \sum_{p,q=0}^{\infty}U^*_{p,q}g_{p+q+1}(x;\underline{\xi}), \qquad (24)\end{aligned}$$

where $g_{p+q+1}(x;\underline{\xi})=(p+q+1)g(x;\underline{\xi})[G(x;\underline{\xi})]^{p+q}$ is an Exp-G distribution with power parameter $(p+q+1)$ and

$$\begin{aligned}U^*_{p,q} &= \sum_{j,m,n=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}\frac{2(-1)^{n+p}\lambda^{p+1}[m+n+1]^p}{p!(p+q+1)}U_{j,l,k}\\ &\times \binom{-(j+l-k+i+1)}{m}\binom{(j+l-k+i-1)}{n}\binom{m}{p}. \qquad (25)\end{aligned}$$

Furthermore, for $\delta>1$, we get

$$f_{i:n}(x;\delta,\lambda,\underline{\xi}) = \frac{2\lambda g(x;\underline{\xi})\big\{\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big\}}{\bar{G}^2(x;\underline{\xi})\big(1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\big)^2}\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}C_{j,l}\left\{\frac{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}\right\}^{j+l+i-1}.$$

By applying the following series expansions

$$\Big\{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\Big\}^{-(j+l+i+1)} = \sum_{m=0}^{\infty}\binom{-(j+l+i+1)}{m}\exp\big[-m\mathbf{O}_\lambda(x;\underline{\xi})\big],$$

$$\begin{aligned}\Big\{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\Big\}^{j+l+i-1} &= \sum_{n=0}^{\infty}(-1)^n\binom{(j+l+i-1)}{n}\\ &\times \exp\big[-n\mathbf{O}_\lambda(x;\underline{\xi})\big],\end{aligned}$$

$$\exp\left[-(m+n+1)\mathbf{O}_\lambda(x;\underline{\xi})\right] = \sum_{p=0}^{\infty} \frac{(-1)^p\lambda^p[m+n+1]^p}{p!}\left[\mathbf{O}(x;\underline{\xi})\right]^p$$

and

$$[\bar{G}(x;\underline{\xi})]^{-(p+2)} = [1-G(x;\underline{\xi})]^{-(p+2)} = \sum_{q=0}^{\infty}(-1)^q\binom{-(p+2)}{q}G^q(x;\underline{\xi}),$$

we can write

$$\begin{aligned} f_{i:n}(x;\delta,\lambda,\underline{\xi}) &= \sum_{j,m,n,p,q=0}^{\infty}\sum_{l=0}^{n-i}\frac{2(-1)^{n+p}\lambda^{p+1}[m+n+1]^p}{p!(p+q+1)}C_{j,l} \\ &\times \binom{-(j+l+i+1)}{m}\binom{(j+l+i-1)}{n}\binom{m}{p} \\ &\times (p+q+1)g(x;\underline{\xi})[G(x;\underline{\xi})]^{p+q} \\ &= \sum_{p,q=0}^{\infty} C^*_{p,q}\, g_{p+q+1}(x;\underline{\xi}), \end{aligned} \tag{26}$$

where $g_{p+q}(x;\underline{\xi}) = (p+q+1)g(x;\underline{\xi})[G(x;\underline{\xi})]^{p+q}$ is an Exp-G distribution with power parameter $(p+q+1)$ and

$$\begin{aligned} C^*_{p,q} &= \sum_{j,m,n=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}\frac{2(-1)^{n+p}\lambda^{p+1}[m+n+1]^p}{p!(p+q+1)}C_{j,l} \\ &\times \binom{-(j+l+i+1)}{m}\binom{(j+l+i-1)}{n}\binom{m}{p}. \end{aligned} \tag{27}$$

### *4.2. Entropy*

An Entropy is a measure of variation of uncertainty for a random variable X with pdf g(x). There are two famous measures of entropy, namely Shannon entropy [30] and Rényi entropy [28]. Rényi entropy is defined by

$$I_R(\nu) = (1-\nu)^{-1}\log\left[\int_0^\infty g^\nu(x)dx\right],$$

where $\nu > 0$ and $\nu \neq 1$. Using expansion (12), for $\delta \in (0,1)$

$$f^\nu(x;\delta,\lambda,\underline{\xi}) = \frac{\delta^\nu f^\nu_{OEHL-G}(x;\lambda,\underline{\xi})}{\Gamma(2\nu)}\sum_{j=0}^{\infty}(1-\delta)^j\Gamma(2\nu+j)\frac{[1-F_{OEHL-G}(x;\lambda,\underline{\xi})]^j}{j!}$$

and for $\delta > 1$

$$f^\nu(x;\delta,\lambda,\underline{\xi}) = \frac{f^\nu_{OEHL-G}(x;\lambda;\underline{\xi})}{\delta^\nu\Gamma(2\nu)}\sum_{j=0}^{\infty}(\delta-1)^j\Gamma(2\nu+j)\frac{F^j_{OEHL-G}(x;\lambda,\underline{\xi})}{j!}.$$

Thus, Rényi entropy for $\delta \in (0,1)$ and $\delta > 1$ are given by

$$I_R(\nu) = (1-\nu)^{-1}\log\left(\sum_{j=0}^{\infty}e_j\int_0^\infty f^\nu_{OEHL-G}(x;\lambda,\underline{\xi})(1-F_{OEHL-G}(x;\lambda,\underline{\xi}))^j dx\right) \tag{28}$$

and

$$I_R(\nu) = (1-\nu)^{-1}\log\left(\sum_{j=0}^{\infty}h_j\int_0^\infty f^\nu_{OEHL-G}(x;\underline{\xi})F^j_{OEHL-G}(x;\lambda,\underline{\xi})dx\right), \tag{29}$$

where

$$e_j = e_j(\delta) = \frac{\delta^{\nu}(1-\delta)^j\Gamma(2\nu+j)}{\Gamma(2\nu)j!}$$

and

$$h_j = h_j(\delta) = \frac{(\delta-1)^j\Gamma(2\nu+j)}{\delta^{\nu+j}\Gamma(2\nu)j!}.$$

Now, for $\delta \in (0,1)$ and using equation (28), we have

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\log\Bigg[\sum_{j=0}^{\infty} e_j \int_0^{\infty} \frac{2^{\nu}g^{\nu}(x;\xi)\exp\big[-\nu\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}{[\bar{G}(x;\xi)]^2\big(1+\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]\big)^{2\nu}} \\ &\times \left\{1-\frac{1-\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}\right\}^j dx\Bigg]. \end{aligned}$$

By considering the following expansions

$$\left\{1-\frac{1-\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}\right\}^j = \sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(j+1)}{\Gamma(j+1-m)m!}\left\{\frac{1-\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}\right\}^m,$$

$$\Big(1+\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]\Big)^{-(2\nu+m)} = \sum_{n=0}^{\infty}\binom{-(2\nu+m)}{n}\exp\big[-n\mathbf{O}_{\lambda}(x;\underline{\xi})\big],$$

$$\Big(1-\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]\Big)^{m} = \sum_{w=0}^{\infty}(-1)^w\binom{m}{w}\exp\big[-w\mathbf{O}_{\lambda}(x;\underline{\xi})\big],$$

$$\exp\big[-(n+w+\nu)\mathbf{O}_{\lambda}(x;\underline{\xi})\big] = \sum_{p=0}^{\infty}\frac{(-1)^p\lambda^p[n+w+\nu]^p}{p!}\big[\mathbf{O}(x;\underline{\xi})\big]^p$$

and

$$[\bar{G}(x;\underline{\xi})]^{-(p+2\nu)} = [1-G(x;\underline{\xi})]^{-(p+2\nu)} = \sum_{q=0}^{\infty}\binom{-(p+2\nu)}{q}G^q(x;\underline{\xi}),$$

we can write

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\log\Bigg[\sum_{j,m,n,p,q,w=0}^{\infty} e_j\frac{2^{\nu}(-1)^{m+w+p}\lambda^{p+\nu}[m+n+\nu]^p\Gamma(j+1)}{\Gamma(j+1-m)m!p!} \\ &\times \binom{-(m+2\nu)}{n}\binom{m}{p}\binom{-(p+2\nu)}{q}\frac{1}{\left(\frac{p+q}{\nu}+1\right)^{\nu}} \\ &\times \int_0^{\infty}\left((\frac{p+q}{\nu}+1)g(x;\underline{\xi})[G(x;\underline{\xi})]^{\frac{p+q}{\nu}}\right)^{\nu}dx\Bigg] \\ &= (1-\nu)^{-1}\log\Bigg[\sum_{p,q=0}^{\infty}e^{*}_{p,q}\exp(1-\nu)I_{REG}\Bigg], \end{aligned} \quad (30)$$

where

$$\begin{aligned} e^{*}_{p,q} &= \sum_{j,m,n,w=0}^{\infty} e_j\frac{2^{\nu}(-1)^{m+w+p}\lambda^{p+\nu}[m+n+\nu]^p\Gamma(j+1)}{\Gamma(j+1-m)m!p!} \\ &\times \binom{-(m+2\nu)}{n}\binom{m}{p}\binom{-(p+2\nu)}{q}\frac{1}{\left(\frac{p+q}{\nu}+1\right)^{\nu}} \end{aligned} \quad (31)$$

and $I_{REG}=\int_0^\infty\left(\left(\frac{p+q}{\nu}+1\right)g(x;\underline{\xi})[G(x;\underline{\xi})]^{\frac{p+q}{\nu}}\right)^{\nu}dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\left(\frac{p+q}{\nu}+1\right)$.

Furthermore, for $\delta>0$

$$
\begin{aligned}
I_R(\nu) &= (1-\nu)^{-1}\log\Bigg[\sum_{j=0}^{\infty}h_j\int_0^\infty\frac{2^\nu g^\nu(x;\underline{\xi})\exp\big[-\nu\mathbf{O}_\lambda(x;\underline{\xi})\big]}{[\bar{G}(x;\underline{\xi})]^2\big(1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]^{2\nu}}\\
&\times\left\{\frac{1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]}\right\}^j dx\Bigg].
\end{aligned}
$$

By considering the following expansions

$$
\left(1+\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\right)^{-(2\nu+j)}=\sum_{m=0}^{\infty}\binom{-(2\nu+j)}{m}\exp\big[-m\mathbf{O}_\lambda(x;\underline{\xi})\big],
$$

$$
\left(1-\exp\big[-\mathbf{O}_\lambda(x;\underline{\xi})\big]\right)^{j}=\sum_{n=0}^{\infty}(-1)^n\binom{j}{n}\exp\big[-n\mathbf{O}_\lambda(x;\underline{\xi})\big],
$$

$$
\exp\big[-(m+n+\nu)\mathbf{O}_\lambda(x;\underline{\xi})\big]=\sum_{p=0}^{\infty}\frac{(-1)^p\lambda^p[m+n+\nu]^p}{p!}\big[\mathbf{O}(x;\underline{\xi})\big]^p
$$

and

$$
[\bar{G}(x;\underline{\xi})]^{-(p+2\nu)}=[1-G(x;\underline{\xi})]^{-(p+2\nu)}=\sum_{q=0}^{\infty}\binom{-(p+2\nu)}{q}G^q(x;\underline{\xi}),
$$

we can write

$$
\begin{aligned}
I_R(\nu) &= (1-\nu)^{-1}\log\Bigg[\sum_{j,m,n,p,q=0}^{\infty}h_j\frac{2^\nu(-1)^{m+w+p}\lambda^{p+\nu}[m+n+\nu]^p}{p!}\\
&\times\binom{-(j+2\nu)}{m}\binom{j}{n}\binom{-(p+2\nu)}{q}\frac{1}{\left(\frac{p+q}{\nu}+1\right)^\nu}\\
&\times\int_0^\infty\left(\left(\frac{p+q}{\nu}+1\right)g(x;\underline{\xi})[G(x;\underline{\xi})]^{\frac{p+q}{\nu}}\right)^{\nu}dx\Bigg]\\
&= (1-\nu)^{-1}\log\Bigg[\sum_{p,q=0}^{\infty}h^*_{p,q}\exp(1-\nu)I_{REG}\Bigg],
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
h^*_{p,q} &= \sum_{j,m,n=0}^{\infty}h_j\frac{2^\nu(-1)^{m+w+p}\lambda^{p+\nu}[m+n+\nu]^p}{p!}\\
&\times\binom{-(j+2\nu)}{m}\binom{j}{n}\binom{-(p+2\nu)}{q}\frac{1}{\left(\frac{p+q}{\nu}+1\right)^\nu}
\end{aligned}
\tag{33}
$$

and $I_{REG}=\int_0^\infty\left(\left(\frac{p+q}{\nu}+1\right)g(x;\underline{\xi})[G(x;\underline{\xi})]^{\frac{p+q}{\nu}}\right)^{\nu}dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\left(\frac{p+q}{\nu}+1\right)$.

### 4.3. Moments and Generating Functions

Let $X \sim MO-OEHL-G(\delta, \lambda, \underline{\xi})$, then the $r^{th}$ moment can be obtained from equations (15) and (17). For $\delta \in (0,1)$,

$$E(X^r) = \sum_{p,q=0}^{\infty} w^*_{p,q} E(W^r_{p+q+1}),$$

where $w^*_{p,q}$ is as defined in equation (16) and $E(W^r_{p+q+1})$ denotes the $r^{th}$ moment of $W_{p+q+1}$ which follows an Exp-G distribution with power parameter $(p+q+1)$. For $\delta > 1$

$$E(X^r) = \sum_{p,q=0}^{\infty} v^*_{p,q} E(W^r_{p+q+1}), \tag{34}$$

where $v^*_{p,q}$ is as defined in equation (18) and $E(W^r_{p+q+1})$ denotes the $r^{th}$ moment of $W_{p+q+1}$ which follows an Exp-G distribution with power parameter $(p+q+1)$. The incomplete moments can be obtained as follows:
For $\delta \in (0,1)$

$$I_X(t) = \int_0^t x^s f(x; \delta, \lambda, \underline{\xi}))dx = \sum_{p,q=0}^{\infty} w^*_{p,q} I_{p+q+1}(t),$$

where $I_{p+q+1}(t) = \int_0^t x^r g_{p+q+1}(x;\xi)dx$ and $w^*_{p,q}$ is as defined in equation (16). Also, For $\delta > 1$

$$I_X(t) = \int_0^t x^s f(x; \delta, \lambda, \underline{\xi}))dx = \sum_{p,q=0}^{\infty} v^*_{p,q} I_{p+q+1}(t),$$

where $I_{p+q+1}(t) = \int_0^t x^r g_{p+q+1}(x;\underline{\xi}))dx$ and $v^*_{p,q}$ is as defined in equation (18). The moment generating function (mgf) of $X$ is given by:
For $\delta \in (0,1)$

$$M_X(t) = \sum_{p,q=0}^{\infty} w^*_{p,q} E(e^{tW_{p+q+1}}),$$

where $E(e^{tW_{p+q+1}})$ is the mgf of the Exp-G distribution with power parameter $(p+q+1)$ and $w^*_{p,q}$ is as defined in equation (16). For $\delta > 1$

$$M_X(t) = \sum_{p,q=0}^{\infty} v^*_{p,q} E(e^{tW_{p+q+1}}),$$

where $E(e^{tW_{p+q+1}})$ is the mgf of the Exp-G distribution with power parameter $(p+q+1)$ and $v^*_{p,q}$ is as defined in equation (18). Furthermore, we can obtain the characteristic function and is given by $\phi(t) = E(e^{itX})$, where $i = \sqrt{-1}$, for $\delta \in (0,1)$

$$\phi(t) = \sum_{qp,=0}^{\infty} w^*_{p,q} \phi_{p+q+1}(t),$$

where $\phi_{p+q+1}(t)$ is the characteristic function of Exp-G distribution with power parameter $(p+q+1)$ and $w^*_{p,q}$ is as defined in equation (16). For $\delta > 1$

$$\phi(t) = \sum_{p,q=0}^{\infty} v^*_{p,q} \phi_{p+q+1}(t),$$

where $\phi_{p+q+1}(t)$ is the characteristic function of Exp-G distribution with power parameter $(p+q+1)$ and $v^*_{p,q}$ is as defined in equation (18).

We present in Figures 4 and 5, 3D plots of skewness and kurtosis of the MO-OEHL-LL distribution. We observe that

- When we fix the parameters $\lambda$ and $c$, the skewness and kurtosis of the MO-OEHL-LL decreases as $\delta$ increases.
- When we fix the parameters $\lambda$ and $\delta$, the skewness and kurtosis of the MO-OEHL-LL increases as $c$ increases.
- When we fix the parameters $c$ and $\lambda$, the skewness and kurtosis of the MO-OEHL-LL increases as $\delta$ increases.
- When we fix the parameters $c$ and $\delta$, the skewness and kurtosis of the MO-OEHL-LL increases as $\lambda$ increases.

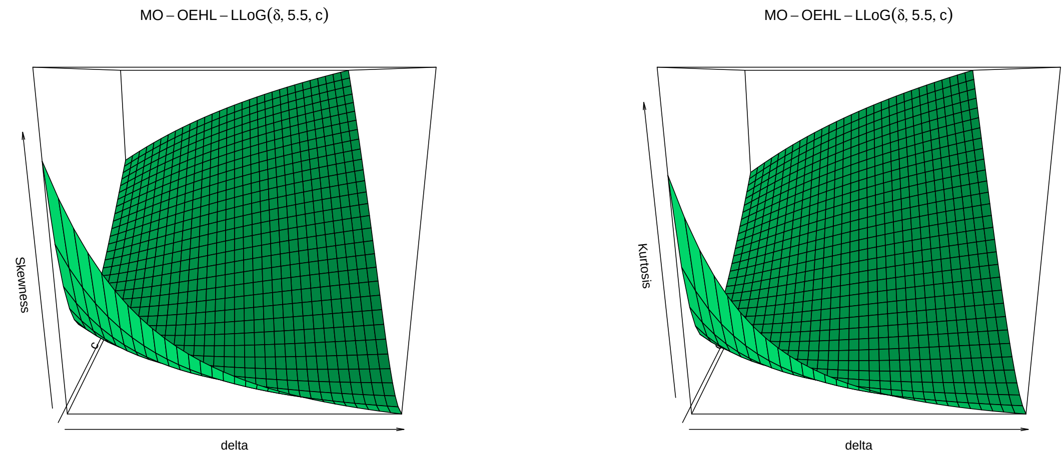



Figure 4. 3 D Plots of skewness and kurtosis for the MO-OEHL-LL distribution $(\delta, 1, c)$

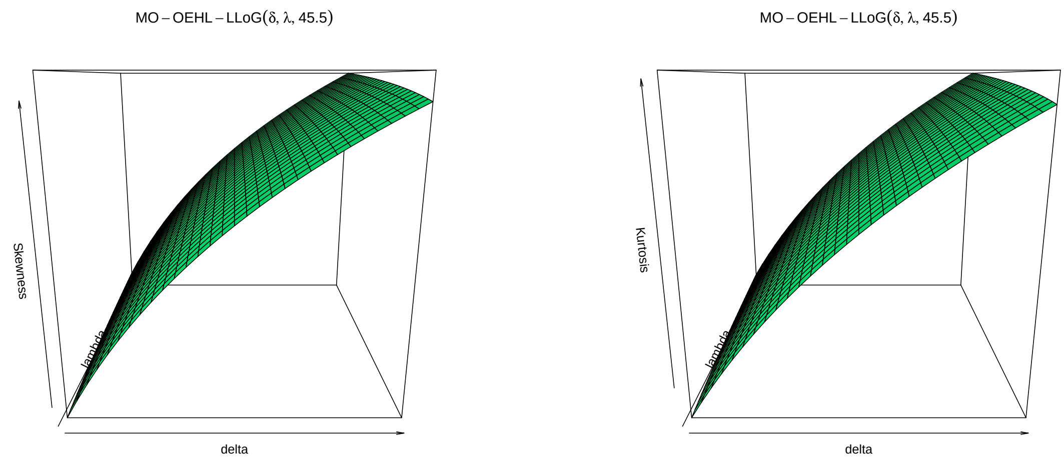



Figure 5. 3 D Plots of skewness and kurtosis for the MO-OEHL-LL distribution $(\delta, \lambda, 1)$

### 4.4. Quantile Function

We invert the MO-OEHL-G cdf to obtain the quantile function. Note that

$$F(x;\delta,\lambda,\xi)=1-\frac{\delta\Big\{1-\frac{1-\exp\left[-\mathbf{O}_{\lambda}(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_{\lambda}(x;\underline{\xi})\right]}\Big\}}{1-\bar{\delta}\Big\{1-\frac{1-\exp\left[-\mathbf{O}_{\lambda}(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_{\lambda}(x;\underline{\xi})\right]}\Big\}}=u$$

for $0\leq u\leq 1$, that is,

$$z=\frac{\Big\{1-\frac{1-\exp\left[-\mathbf{O}_{\lambda}(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_{\lambda}(x;\underline{\xi})\right]}\Big\}}{1-\bar{\delta}\Big\{1-\frac{1-\exp\left[-\mathbf{O}_{\lambda}(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_{\lambda}(x;\underline{\xi})\right]}\Big\}},$$

where $z=\left(\frac{1-u}{\delta}\right)$ Further simplifying the expression we get

$$\frac{z}{1+z\bar{\delta}}=\left[1-\frac{1-\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}\right]$$

which results in

$$\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]=\frac{\left[\frac{z}{1+z\bar{\delta}}\right]}{\left[2-\frac{z}{1+z\bar{\delta}}\right]}$$

so that

$$G(x;\xi)=\left\{\frac{\ln\left(\frac{\left[\frac{z}{1+z\bar{\delta}}\right]}{\left[2-\frac{z}{1+z\bar{\delta}}\right]}\right)}{\ln\left[\frac{\left(\frac{z}{1+z\bar{\delta}}\right)}{\left(2-\frac{z}{1+z\bar{\delta}}\right)}\right]-\lambda}\right\}.$$

Therefore, the quantiles of the MO-OEHL-G family of distributions may be determined by solving the non-linear equation

$$x(u)=G^{-1}\left\{\frac{\ln\left(\frac{\left[\frac{z}{1+z\bar{\delta}}\right]}{\left[2-\frac{z}{1+z\bar{\delta}}\right]}\right)}{\ln\left(\frac{\left[\frac{z}{1+z\bar{\delta}}\right]}{\left[2-\frac{z}{1+z\bar{\delta}}\right]}\right)-\lambda}\right\}, \tag{35}$$

via iterative methods in R or Matlab software.

## 5. Maximum Likelihood Estimation

If $X_i\sim MO-OEHL-G(\delta,\lambda;\underline{\xi})$ with the parameter vector $\Delta=(\delta,\lambda;\underline{\xi})^T$ . The total log-likelihood $\ell=\ell(\Delta)$ from a random sample of size $n$ is given by

$$\begin{aligned}\ell &= n\log(2\delta)+n\log(\lambda)+\sum_{i=1}^{n}\log[g(x_i;\underline{\xi})]-\lambda\sum_{i=1}^{n}\big[\mathbf{O}_(x;\underline{\xi})\big]\\ &- 2\sum_{i=1}^{n}\log\left[1-\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]\right]-2\sum_{i=1}^{n}\log[\bar{G}(x_i;\underline{\xi})]\\ &- 2\sum_{i=1}^{n}\log\left[1-\bar{\delta}\left(1-\frac{1-\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}{1+\exp\big[-\mathbf{O}_{\lambda}(x;\underline{\xi})\big]}\right)\right].\end{aligned}$$

The score vector $U = \left(\frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\underline{\xi}_k}\right)$ has elements given by:

$$\frac{\partial\ell}{\partial\delta} = \frac{n}{\delta} + 2\sum_{i=1}^{n} \frac{2\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{\left(\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]\right) - 2\delta\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]},$$

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} &= \frac{n}{\lambda} - \sum_{i=1}^{n}\left[\mathbf{O}(x;\underline{\xi})\right] - 2\sum_{i=1}^{n}\frac{G(x_i;\underline{\xi})\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{\bar{G}(x_i;\underline{\xi})\left(1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]\right)} \\ &- 2\sum_{i=1}^{n}\frac{2\bar{\delta}G(x_i;\underline{\xi})\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{\bar{G}(x_i;\underline{\xi})\left(1-\bar{\delta}\left[1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}\right]\right)\left(1+\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]\right)^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\underline{\xi}_k} &= \sum_{i=1}^{n}\frac{1}{g(x_i;\underline{\xi})}\frac{\partial g(x_i;\underline{\xi})}{\partial\underline{\xi}_k} - \lambda\sum_{i=1}^{n}\frac{G(x_i;\underline{\xi})\frac{\partial\bar{G}(x_i;\underline{\xi})}{\partial\underline{\xi}_k} - \bar{G}(x_i;\underline{\xi})\frac{\partial G(x_i;\underline{\xi})}{\partial\underline{\xi}_k}}{\bar{G}^2(x_i;\underline{\xi})} \\ &- 2\sum_{i=1}^{n}\frac{1}{\left(1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]\right)}\frac{\partial\left(1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]\right)}{\partial\underline{\xi}_k} \\ &- 2\sum_{i=1}^{n}\frac{1}{\left[1-\bar{\delta}\left(1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}\right)\right]}\frac{\partial\left[1-\bar{\delta}\left(1-\frac{1-\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}{1+\exp\left[-\mathbf{O}_\lambda(x;\underline{\xi})\right]}\right)\right]}{\partial\underline{\xi}_k} \\ &- 2\sum_{i=1}^{n}\frac{1}{\bar{G}(x_i;\underline{\xi})}\frac{\partial\bar{G}(x_i;\underline{\xi})}{\partial\underline{\xi}_k},\end{aligned}$$

respectively. These partial derivatives are not in closed form and can be solved using R, MATLAB and SAS software by use of iterative methods.

We use the information matrix

$$J(\Delta) = \begin{pmatrix} J_{\delta\delta}(\Delta) & J_{\delta\lambda}(\Delta & J_{\delta\underline{\xi}}(\Delta) \\ J_{\lambda\delta}(\Delta) & J_{\lambda\lambda}(\Delta) & J_{\lambda\underline{\xi}}(\Delta) \\ J_{\underline{\xi}\delta}(\Delta) & J_{\underline{\xi}\lambda}(\Delta) & J_{\underline{\xi}\underline{\xi}}(\Delta) \end{pmatrix}, \tag{36}$$

where $J_{i,j} = \frac{-\partial^2\ell(\Delta)}{\partial i\partial j}$, for $i,j = \delta,\lambda,\underline{\xi}$, to obtain confidence intervals for model parameters $(\delta,\lambda,\underline{\xi})$ and testing the hypotheses concerning these parameters. Under the usual regularity conditions $\hat{\Delta}$ is asymptotically normal distributed, that is $\hat{\Delta} \sim N(\underline{0}, I^{-1}(\Delta))$ as $n \longrightarrow \infty$, where $I(\Delta)$ is the expected information matrix. The asymptotic behaviour remains valid if $I(\Delta)$ is replaced by $J(\hat{\Delta})$, the information matrix evaluated at $\hat{\Delta}$.

## 6. Simulation Study

We conduct simulation study to evaluate consistency of the maximum likelihood estimates. We simulated for N=1000 times with sample size $n$= 100, 200,400,800 and 1000. Simulation results are shown in Table 1 and we conclude that our model produces consistent results when estimating parameters.

Table 1. Monte Carlo Simulation Results for MO-OEHL-LLoG Distribution: Mean, RMSE and Average Bias

| | | $\delta = 0.01, \lambda = 0.01, c = 0.9$ | | | $\delta = 0.05, \lambda = 0.05, c = 1.0$ | | |
|---|---|---|---|---|---|---|---|
| Parameter | n | Mean | RMSE | Bias | Mean | RMSE | Bias |
| $\delta$ | 100 | 0.035321 | 0.049296 | 0.025321 | 0.166226 | 1.660734 | 0.116226 |
| | 200 | 0.024151 | 0.026966 | 0.014151 | 0.074865 | 0.058053 | 0.024865 |
| | 400 | 0.016067 | 0.011162 | 0.006067 | 0.061143 | 0.030488 | 0.011143 |
| | 800 | 0.013791 | 0.006669 | 0.003791 | 0.057245 | 0.019982 | 0.007245 |
| | 1000 | 0.013070 | 0.005398 | 0.003070 | 0.054585 | 0.016587 | 0.004585 |
| $\lambda$ | 100 | 0.034037 | 0.046530 | 0.024037 | 0.093277 | 0.172627 | 0.043277 |
| | 200 | 0.023641 | 0.026086 | 0.013641 | 0.071868 | 0.051718 | 0.021868 |
| | 400 | 0.015877 | 0.011088 | 0.005877 | 0.060005 | 0.028708 | 0.010005 |
| | 800 | 0.013706 | 0.006530 | 0.003706 | 0.056825 | 0.019015 | 0.006825 |
| | 1000 | 0.012973 | 0.005312 | 0.002973 | 0.054293 | 0.016067 | 0.004293 |
| c | 100 | 0.874298 | 0.074671 | -0.025702 | 0.980042 | 0.115285 | -0.019958 |
| | 200 | 0.883726 | 0.055320 | -0.016274 | 0.984486 | 0.082672 | -0.015514 |
| | 400 | 0.892954 | 0.032723 | -0.007046 | 0.992936 | 0.058627 | -0.007064 |
| | 800 | 0.895159 | 0.017814 | -0.004841 | 0.992920 | 0.040520 | -0.007080 |
| | 1000 | 0.896523 | 0.015429 | -0.003477 | 0.997091 | 0.036534 | -0.002909 |
| | | $\delta = 0.9, \lambda = 0.01, c = 0.9$ | | | $\delta = 0.01, \lambda = 1.0, c = 1.5$ | | |
| $\delta$ | 100 | 2.294744 | 5.326255 | 1.394744 | 0.017725 | 0.031076 | 0.007725 |
| | 200 | 1.662573 | 4.255223 | 0.762573 | 0.014122 | 0.013170 | 0.004122 |
| | 400 | 1.180349 | 1.469468 | 0.280349 | 0.012078 | 0.006625 | 0.002078 |
| | 800 | 1.097065 | 1.947694 | 0.197065 | 0.011644 | 0.003934 | 0.001644 |
| | 1000 | 0.962967 | 0.151690 | 0.062967 | 0.011479 | 0.003742 | 0.001479 |
| $\lambda$ | 100 | 0.035561 | 0.088457 | 0.025561 | 1.355495 | 1.129026 | 0.355495 |
| | 200 | 0.024582 | 0.065339 | 0.014582 | 1.235544 | 0.758367 | 0.235544 |
| | 400 | 0.015767 | 0.032293 | 0.005767 | 1.135394 | 0.423490 | 0.135394 |
| | 800 | 0.013073 | 0.032368 | 0.003073 | 1.120320 | 0.300849 | 0.120320 |
| | 1000 | 0.011061 | 0.003011 | 0.001061 | 1.116915 | 0.305543 | 0.116915 |
| c | 100 | 0.833577 | 0.137037 | -0.066423 | 1.488913 | 0.127810 | -0.011087 |
| | 200 | 0.851294 | 0.104692 | -0.048706 | 1.488094 | 0.091135 | -0.011906 |
| | 400 | 0.873073 | 0.068210 | -0.026927 | 1.491398 | 0.061355 | -0.008602 |
| | 800 | 0.886983 | 0.042890 | -0.013017 | 1.491914 | 0.039685 | -0.008086 |
| | 1000 | 0.887436 | 0.028464 | -0.012564 | 1.495486 | 0.035293 | -0.004514 |

## 7. Applications

Three real data examples are presented to demonstrate the usefulness of the MO-OEHL-LLoG distribution compared to other competing known non-nested models. We assessed model performance using goodness-of-fit statistics, that includes -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer von Mises ($W^*$) and Andersen-Darling ($A^*$) as described by Chen and Balakrishnan [12], Kolmogorov-Sminorv (K-S) and its p-value. The model with the smaller values of these statistics is the better model.

We used the nlm function in R software to estimate the model parameters and the package AdequacyModel in R for goodness-of-fit test. Model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the three data sets are shown in Tables 2, 3 and 4. We also provide plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. [11]) to show how well our model fits the observed data sets. Also, provided in this section are estimated cdf, Kaplan Meier (KM) survival curves, hazard rate function (HRF) and TTT-transform plots.

The MO-OEHL-LL distribution was compared to other competing three parameter non-nested models, namely, the exponentiated-Fréchet (EFr) distribution by Nadarajah and Kotz [23], other two non-nested models studied by Barreto-Souza et al. [6], namely, Marshall-Olkin extended Fréchet (MOEFr) and Marshall-Olkin extended generalized exponential (MOEGE) distributions, Marshall-Olkin extended inverse Weibull (IWMO) by Pakungwati et al. [25], exponentiated Weibull by Pal et al. [26] and alpha power Weibull (APW) by Nassar et al. [24] distributions. The pdfs of the non-nested models are given by:

$$f_{IWMO}(x;\alpha,\theta,\lambda)=\frac{\alpha\lambda\theta^{-\lambda}x^{-\lambda-1}e^{-(\theta x)^{-\lambda}}}{[\alpha-(\alpha-1)e^{-(\theta x)^{-\lambda}}]^2},$$

for $\alpha,\theta,\lambda>0$,

$$f_{EF}(x;\alpha,\lambda,\delta)=\alpha\lambda\delta^{\lambda}\left[1-e^{-(\delta/x)^{\lambda}}\right]^{\alpha-1}x^{-(1+\lambda)}e^{-(\lambda+1)(\delta/x)^{\lambda}},$$

for $\alpha,\lambda,\delta>0$,

$$f_{MOEF}(x;\alpha,\lambda,\delta)=\frac{\alpha\lambda\delta^{\lambda}x^{-(\lambda+1)}e^{-(\delta/x)^{\lambda}}}{[1-\bar{\alpha}(1-e^{-(\delta/x)^{\lambda}})]^2},$$

for $\alpha,\lambda,\delta>0$,

$$f_{MOEGE}(x;\alpha,\gamma,\lambda)=\frac{\alpha\gamma\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\gamma-1}}{(1-\bar{\alpha}[1-e^{-\lambda x}]^{\gamma})^2},$$

for $\alpha,\gamma,\lambda>0$,

$$f_{EW}(x;\alpha,\beta,\delta)=\alpha\beta\delta x^{\beta-1}e^{-\alpha x^{\beta}}(1-e^{-\alpha x^{\beta}})^{\delta},$$

for $\alpha,\beta,\delta>0$, and

$$f_{APW}(x;\alpha,\beta,\theta)=\frac{\log(\alpha)}{(\alpha-1)}\beta\theta x^{\beta-1}e^{-\theta x^{\beta}}\alpha^{1-e^{-\theta x^{\beta}}},$$

for $\alpha,\beta,\theta>0$.

### 7.1. Strengths of 1.5 cm Glass Fibres Data

The first data set is on strengths of 1.5 cm glass fibres. The data set was also analyzed by Bourguignon et al. [8] and Smith and Naylor [31]. The data are 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

Table 2. Parameter estimates and goodness-of-fit statistics for various models fitted for 1.5 cm glass fibres data set

| | Estimates | | | Statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Model | $\delta$ | $\lambda$ | $c$ | $-2\log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$ | $A^*$ | K-S | p-value |
| MO-OEHL-LL | 8.3217 | 0.6948 | 3.2018 | 24.1 | 30.1 | 30.5 | 36.5 | 0.1057 | 0.5911 | 0.0999 | 0.5548 |
| | (10.3885) | (0.5651) | (0.9478) | | | | | | | | |
| | $\alpha$ | $\gamma$ | $\lambda$ | | | | | | | | |
| MOEGE | 1.3296 $\times10^{-3}$ | 10.7200 | 5.8366 | 31.9 | 37.9 | 38.4 | 44.4 | 0.4410 | 2.4309 | 0.9995 | $<2.200\times10^{-16}$ |
| | (0.3306) | (0.1849) | (0.2071) | | | | | | | | |
| | $\alpha$ | $\delta$ | $\lambda$ | | | | | | | | |
| EFr | 0.04621 | 0.4993 | 20.1145 | 189.1 | 195.1 | 195.5 | 201.6 | 1.1986 | 6.3098 | 0.4279 | 1.9210 $\times10^{-10}$ |
| | (0.0153) | (0.0147) | (6.1482) | | | | | | | | |
| MOEFr | 54074 | 0.3858 | 7.9253 | 45.6 | 51.6 | 51.9 | 58.0 | 19.2509 | 122.7666 | 0.9997 | $<2.200\times10^{-16}$ |
| | (3.8277 $\times10^{-8}$) | (6.0532 $\times10^{-2}$) | (0.8731) | | | | | | | | |
| | $\alpha$ | $\beta$ | $\delta$ | - | | | | | | | |
| EW | 1.8741 | 1.3803 | 15.3624 | 64.9 | 70.9 | 71.3 | 77.3 | 0.7088 | 3.8788 | 0.2487 | 0.0008 |
| | (0.5608) | (0.3216) | (9.0935) | | | | | | | | |
| | $\alpha$ | $\lambda$ | $\theta$ | | | | | | | | |
| IWMO | 52636 | 7.9256 | 2.5828 | 45.6 | 51.6 | 51.9 | 58.0 | 0.4974 | 2.7509 | 0.1536 | 0.1020 |
| | (9.7035) | (0.1041) | (54.9189) | | | | | | | | |
| | $\alpha$ | $\beta$ | $\theta$ | | | | | | | | |
| APW | 10.8558 | 4.4836 | 0.1948 | 26.9 | 32.9 | 33.4 | 39.4 | 0.1686 | 0.9272 | 0.1225 | 0.3009 |
| | (12.7241) | (0.7632) | (0.1083) | | | | | | | | |

The estimated variance-covariance matrix for MO-OEHL-LL model on 1.5 cm glass fibres data set is given by

$$\begin{bmatrix}107.9211 & 5.6729 & -9.0627\\ 5.6729 & 0.3193 & -0.5279\\ -9.0627 & -0.5279 & 0.8983\end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta\in[8.3217\pm20.3615]$, $\lambda\in[0.6948\pm1.1076]$ and $c\in[3.2018\pm1.8577]$. Based on the results shown in Table 2, we conclude that the MO-OEHL-LL model perform better that the selected models on strength of 1.5 cm glass fibre data set since it has the smallest values for the goodness-of-fit statistics. Plots of estimated pdf, cdf, PP, KM survival curves, HRF and TTT-transform are shown in Figures 6, 7 and 8 for 1.5 cm glass fibres.

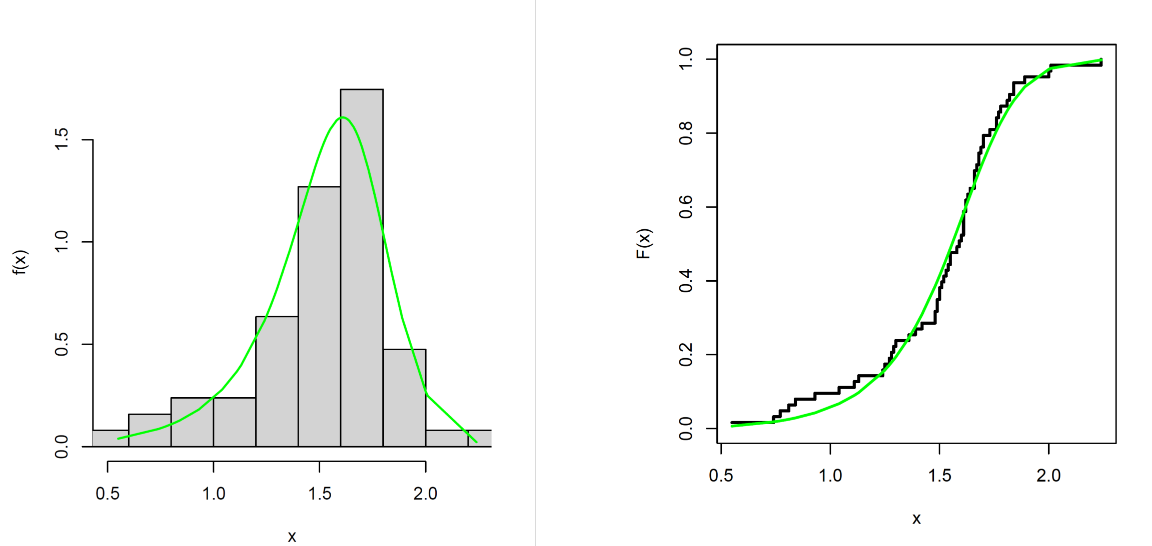



Figure 6. Fitted pdf and cdf for glass fibres data

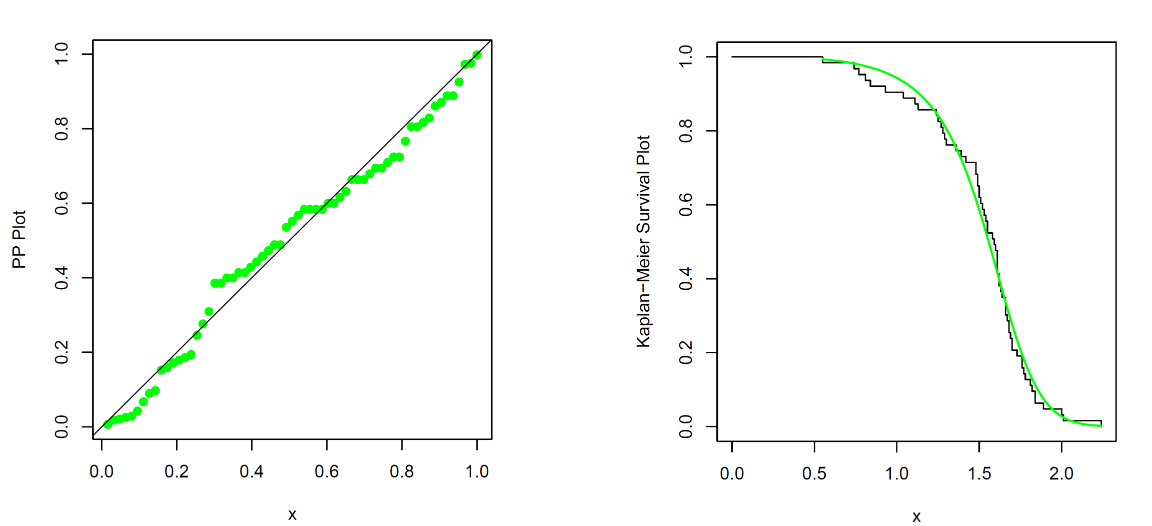
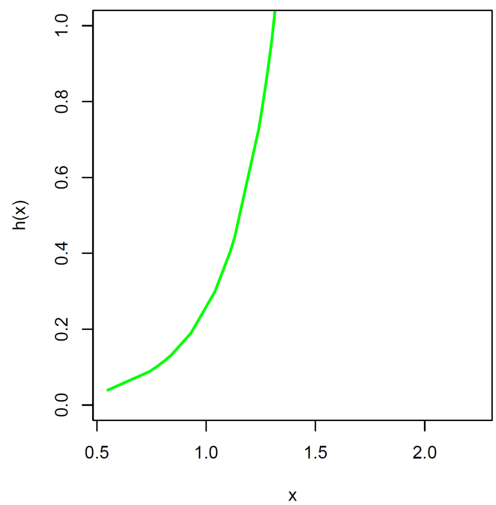



Figure 7. Probability plots and Kaplan Meier curves for glass fibres data

### *7.2. Silicon Nitride Data*

The second data set is on fracture toughness of silicon nitride measured in MPa $m^{1/2}$. The data set was also analysed by Nadarajah and Kotz [22] and by Ali et al. [2]. The data are 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30,



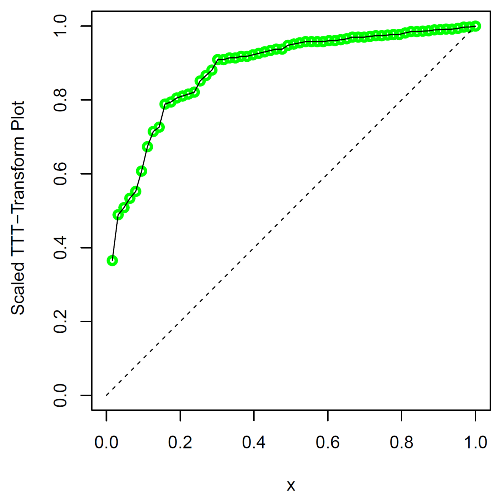




Figure 8. HRF and TTT plots for glass fibres data

5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.
The estimated variance-covariance matrix for MO-OEHL-LL model on silicon nitride data set is given by

$$\begin{bmatrix} 21.4461 & 0.1710 & -6.2915 \\ 0.1710 & 0.0014 & -0.0517 \\ -6.2915 & -0.0517 & 1.9154 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [2.8087 \pm 9.0767]$, $\lambda \in [0.0134 \pm 0.0733]$ and $c \in [3.3255 \pm 2.7126]$. Results in Table 3 shows that the MO-OEHL-LL distribution performs better than the selected models on silicon nitride data, based on the smallest values of the goodness-of-fit statistics. Plots of estimated pdf, cdf, probability plots (PP), Kaplan Meier (KM) survival curves, hazard rate function (HRF) and TTT-transform are shown in Figures 9, 10 and 11 for silicon nitride data set.

Table 3. Parameter estimates and goodness-of-fit statistics for various models fitted for silicon nitride data set

| | Estimates | | | Statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Model | $\delta$ | $\lambda$ | $c$ | $-2\log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$ | $A^*$ | K-S | p-value |
| MO-OEHL-LL | 2.8088 | 0.0134 | 3.3255 | 335.6 | 341.6 | 341.8 | 349.9 | 0.0506 | 0.3047 | 0.0523 | 0.9010 |
| | (4.6309) | (0.0374) | (1.3839) | | | | | | | | |
| | $\alpha$ | $\gamma$ | $\lambda$ | | | | | | | | |
| MOEGE | $1.0718\times10^{-2}$ | 20.7600 | 1.7321 | 340.3 | 346.3 | 346.5 | 354.6 | 0.2326 | 1.4637 | 0.9926 | $< 2.2000\times10^{-16}$ |
| | $(7.0715\times10^{-3})$ | $(3.5685\times10^{-5})$ | (0.1434) | | | | | | | | |
| | $\alpha$ | $\delta$ | $\lambda$ | | | | | | | | |
| EFr | 0.0555 | 1.5719 | 18.4091 | 586.9 | 592.9 | 593.2 | 601.3 | 1.3837 | 7.6851 | 0.3861 | $7.772\times10^{-16}$ |
| | (0.0244) | (0.0359) | (7.7522) | | | | | | | | |
| MOEFr | 2407.7 | 1.4344 | 7.0579 | 356.6 | 362.6 | 362.9 | 370.9 | 38.2495 | 235.4782 | 0.9989 | $< 2.2000\times10^{-16}$ |
| | $(7.7867\times10^{-6})$ | (0.1296) | (0.5495) | | | | | | | | |
| | $\alpha$ | $\beta$ | $\delta$ | - | | | | | | | |
| EW | 0.7015 | 1.1441 | 23.9388 | 381.5 | 387.5 | 387.7 | 395.8 | 0.6594 | 3.8924 | 0.1676 | 0.0025 |
| | (0.2973) | (0.2012) | (12.5289) | - | | | | | | | |
| | $\alpha$ | $\lambda$ | $\theta$ | | | | | | | | |
| IWMO | 2407.7 | 7.0579 | 0.6972 | 356.6 | 362.6 | 362.9 | 370.9 | 0.3596 | 2.2543 | 0.0804 | 0.4241 |
| | $(8.9422\times10^{-7})$ | (0.5495) | (0.0630) | | | | | | | | |
| | $\alpha$ | $\beta$ | $\theta$ | | | | | | | | |
| APW | 7.4246 | 3.8975 | 0.0039 | 335.8 | 341.8 | 342.0 | 350.2 | 0.0571 | 0.3591 | 0.0560 | 0.8492 |
| | (0.0201) | (0.2965) | (0.0019) | | | | | | | | |

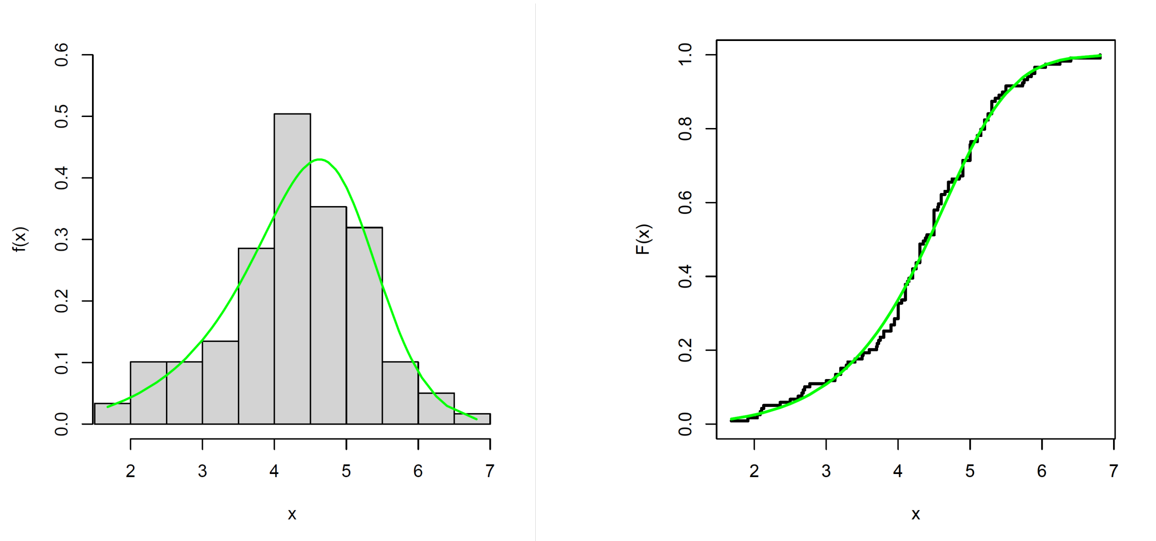



Figure 9. Fitted pdf and cdf for silicon nitride data

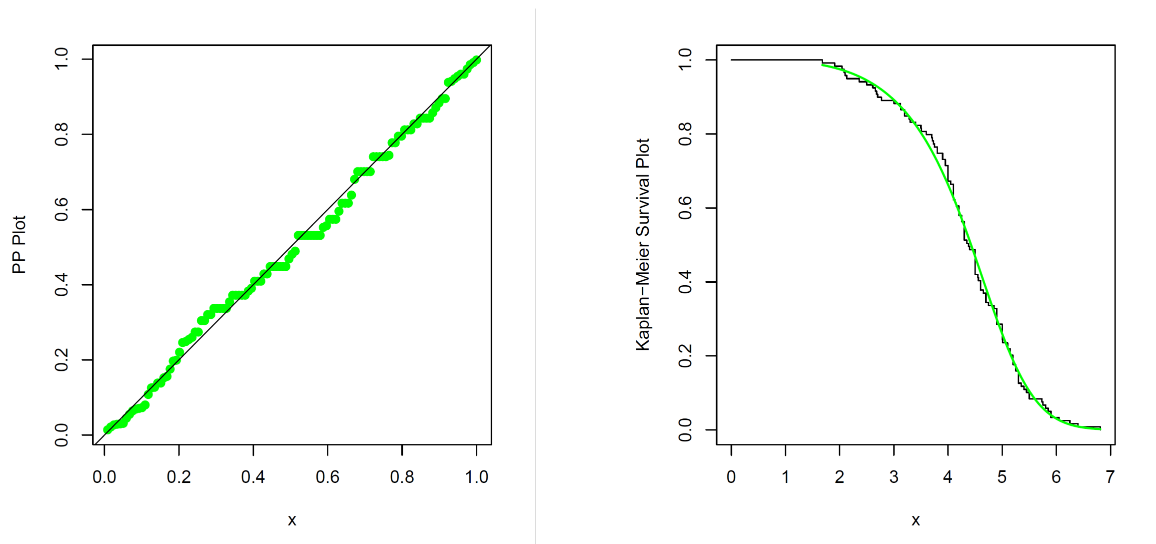



Figure 10. Probability plots and Kaplan Meier curves for silicon nitride data

### *7.3. Turbocharger Failure Times Data*

The third data set represents failure times $(10^3 h)$ of turbocharger of one type of engine as report by Xu et al [32]. The data are 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.
The estimated variance-covariance matrix for MO-OEHL-LL model on turbocharger failure times data set is given by

$$\begin{bmatrix} 8.0245 & 0.0529 & -2.1839 \\ 0.0529 & 0.0004 & -0.0177 \\ -2.1839 & -0.0177 & 0.7617 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [2.4294 \pm 5.5522]$, $\lambda \in [0.0096 \pm 0.0399]$ and $c \in [2.7877 \pm 1.7106]$. Furthermore, results in Table 4 confirms that the MO-OEHL-LL model indeed perform better than

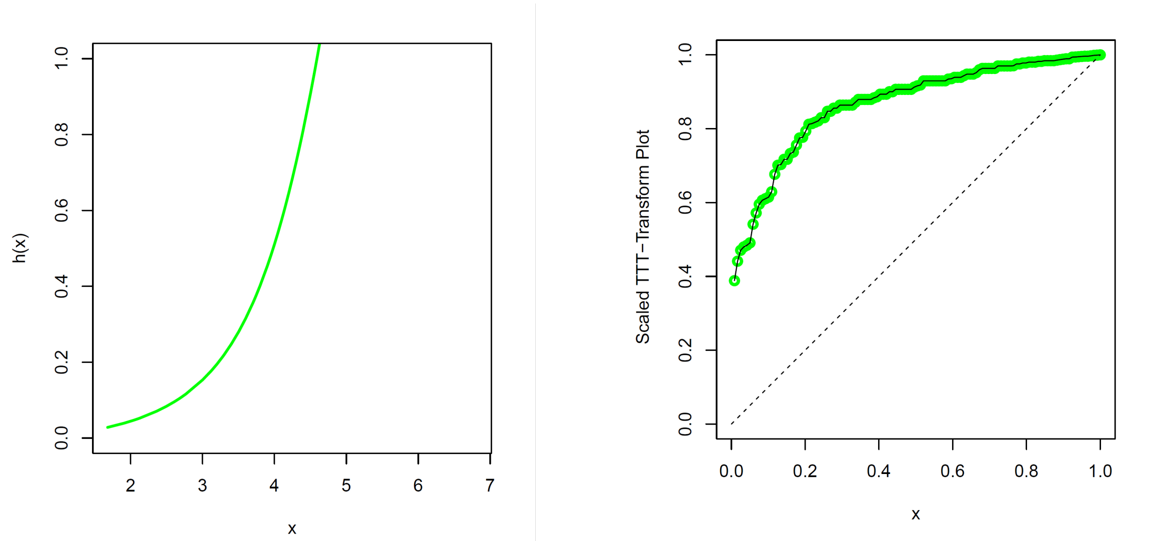



Figure 11. HRF and TTT plots for silicon nitride data

Table 4. Parameters estimates and goodness-of-fit statistics for various models fitted for turbocharger failure times data set

| | Estimates | | | Statistics | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Model | $\delta$ | $\lambda$ | $c$ | $-2\log L$ | $AIC$ | $AICC$ | $BIC$ | $W^*$ | $A^*$ | K-S | p-value |
| MO-OEHL-LL | 2.4294 | 0.0096 | 2.7877 | 162.6 | 168.6 | 169.3 | 173.7 | 0.0496 | 0.3766 | 0.0918 | 0.8889 |
| | (2.8327) | (0.0203) | (0.8728) | | | | | | | | |
| | $\alpha$ | $\gamma$ | $\lambda$ | | | | | | | | |
| MOEGE | 0.116 | 3.1108 | 0.8755 | 167.1 | 173.1 | 173.8 | 178.2 | 0.1183 | 0.8382 | 0.9932 | $< 2.2000 \times 10^{-16}$ |
| | (0.0280) | (0.6155) | (0.1246) | | | | | | | | |
| | $\alpha$ | $\delta$ | $\lambda$ | | | | | | | | |
| EF | 0.0398 | 1.4314 | 18.2212 | 253.7 | 259.7 | 260.3 | 264.7 | 0.6684 | 3.7808 | 0.4145 | $2.1430 \times 10^{-6}$ |
| | (0.0065) | (0.0414) | (0.6291) | | | | | | | | |
| MOEF | 4836.2 | 1.0782 | 4.8391 | 177.4 | 183.4 | 184.1 | 188.5 | 12.8752 | 78.7089 | 0.9987 | $< 2.2000 \times 10^{-16}$ |
| | $(1.1593 \times 10^{-5})$ | (0.2678) | (0.6512) | | | | | | | | |
| | $\alpha$ | $\beta$ | $\delta$ | - | | | | | | | |
| EW | 0.7549 | 0.8389 | 17.9156 | 187.4 | 193.4 | 194.1 | 198.5 | 0.3195 | 1.9997 | 0.1672 | 0.2134 |
| | (0.6721) | (0.0400) | (22.6988) | - | | | | | | | |
| | $\alpha$ | $\lambda$ | $\theta$ | | | | | | | | |
| IWMO | 4836.3 | 4.8391 | 0.9274 | 177.4 | 183.4 | 18401 | 188.5 | 0.2153 | 1.4129 | 0.1438 | 0.3796 |
| | $(8.4139 \times 10^{-6})$ | (0.6512) | (0.2303) | | | | | | | | |
| | $\alpha$ | $\beta$ | $\theta$ | | | | | | | | |
| APW | 4.3051 | 3.3068 | 0.0024 | 163.8 | 169.8 | 170.5 | 174.9 | 0.0621 | 0.4691 | 0.0986 | 0.8320 |
| | (0.0444) | (0.4576) | (0.0022) | | | | | | | | |

the selected models. Plots of estimated pdf, cdf, PP, KM survival curves, HRF and TTT-transform are shown in Figures 12, 13 and 14 for silicon nitride data set.

## 8. Concluding Remarks

A new model was developed, which is referred to as the Marshall-Olkin-Odd Exponential Half Logistic-G family of distributions. The new distribution is an infinite linear combination of the Exp-G distribution. The new proposed distribution can be applied to heavily skewed data and also data that have non-monotonic hazard rate shapes. From real data examples presented in Tables 2, 3 and 4, we conclude that the MO-OEHL-G distribution performs better than several competing non-nested models.

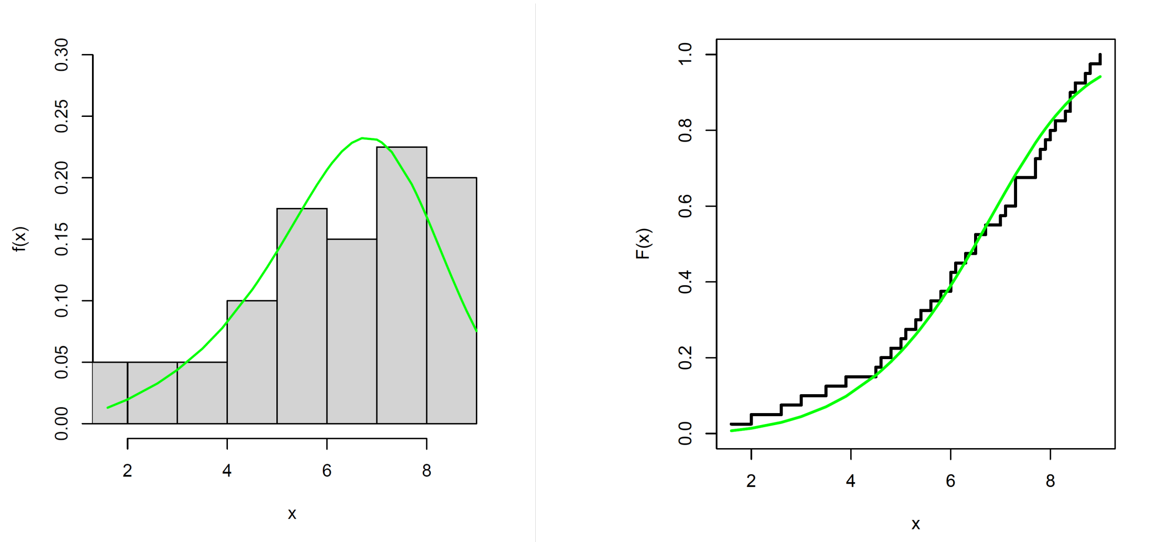



Figure 12. Fitted pdf and cdf for turbocharger failure times data

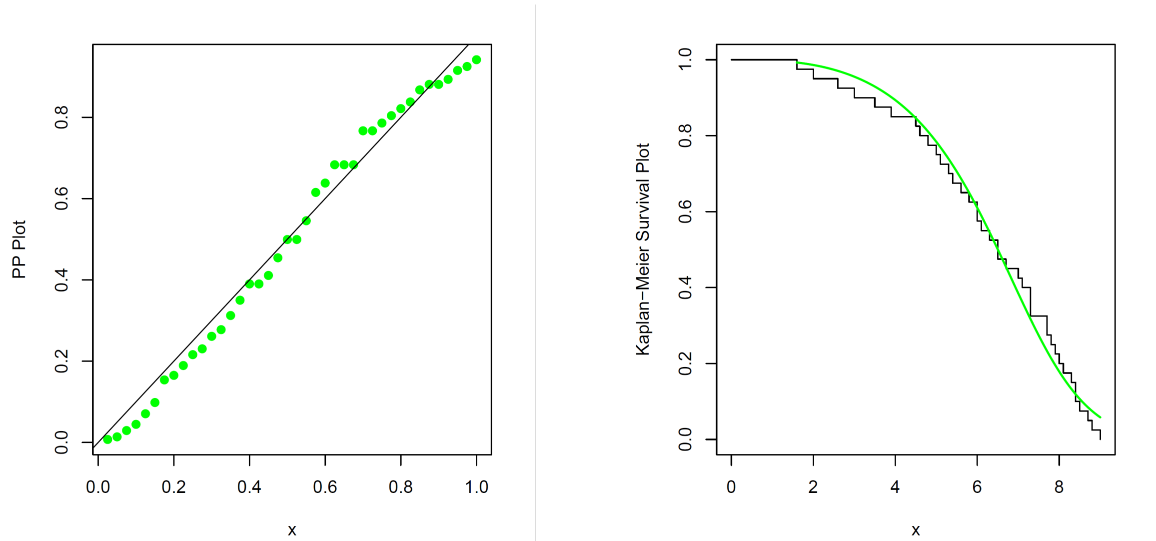



Figure 13. Probability plots and Kaplan Meier curves for glass fibres data

## Appendix

Simulation algorithm
https://drive.google.com/file/d/1tXDS-AqiAzbj7QagdXKbRFOMUu6HIGG2/view?usp=sharing

## Acknowledgement

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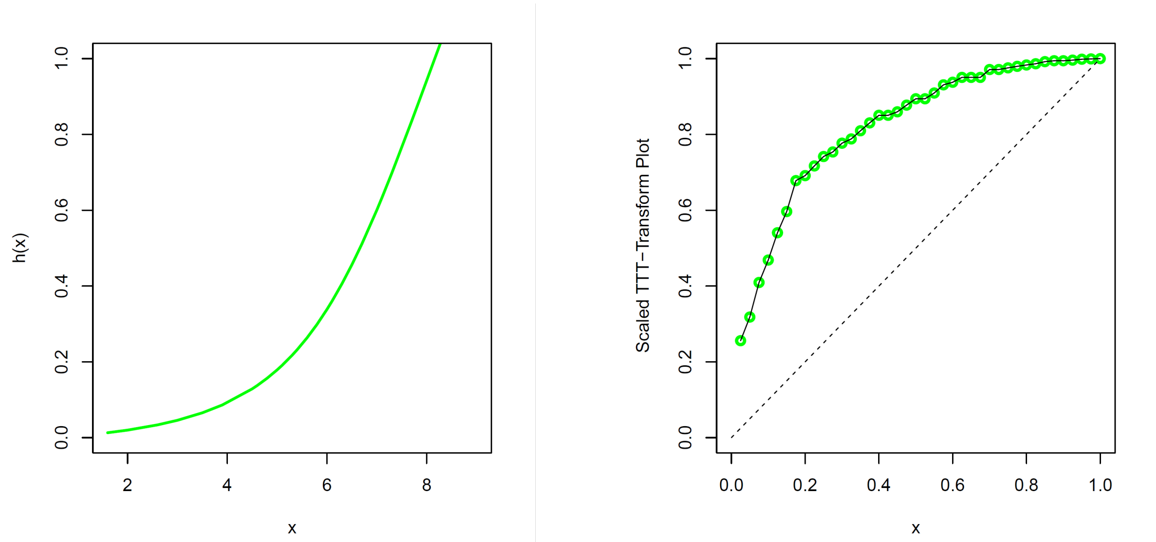


Figure 14. HRF and TTT plots for turbocharger failure times data


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