STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING
Stat., Optim. Inf. Comput., Vol. 10, June 2022, pp 858–872.
Published online in International Academic Press (www.IAPress.org)

Bootstrap Confidence Intervals for Common Signal-to-noise Ratio of Two-parameter Exponential Distributions

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Abstract Signal-to-noise ratio (SNR) is a reciprocal of coefficient of variation. The SNR is a measure of mean relative to the variability. Confidence procedures for common SNR of two-parameter exponential distributions were developed using generalized confidence interval (GCI) approach, large sample (LS) approach, adjusted method of variance estimates recovery (Adjusted MOVER) approach, and bootstrap approaches based on standard bootstrap (SB) and parametric bootstrap (PB). The performances of all approaches are measured by coverage probability and average length. Simulation studies show that all approaches have the coverage probabilities below the nominal confidence level of 0.95 when the common SNR is negative value. However, the coverage probabilities of all approaches are greater than the nominal confidence level of 0.95 when the common SNR is positive value. Moreover, the LS and AM approaches are the conservative confidence intervals. In addition, the GCI and PB approaches provide the confidence intervals with coverage probabilities close to the nominal confidence level of 0.95 when the sample sizes are large and the common SNR is positive value. The GCI and PB approaches are recommended to estimate the confidence intervals for the common SNR of two-parameter exponential distributions. Finally, all proposed approaches are employed in the data of the survival days of lung cancer patients for a demonstration.

Keywords Signal-to-Noise Ratio, Simulation, Two-Parameter Exponential Distribution, Coverage Probability, Average Length

AMS 2010 subject classifications 62F25

DOI: 10.19139/soic-2310-5070-931

1. Introduction

Two-parameter exponential distribution is used as model in lifetime data analysis and reliability analysis. The two-parameter exponential distribution is often used to illustrate concepts such as parameter estimation and hypothesis testing in probability, mathematical statistics, and reliability. Various other motivations and applications of the two-parameter exponential distribution can be found in Lawless [1], Baten and Kamil [2], Petropoulos [3], Jiang and Wong [4], Thangjai and Niwitpong [5], Saothayanun and Thangjai [6], Thangjai et al. [7], Thangjai and Niwitpong [8], and Chesneau et al. [9].

Coefficient of variation (CV) is a measure of variability relative to the mean. The CV is free from the unit of measurement. Therefore, the CV is used rather than standard deviation for measuring of relative variability. The CV has been used in many fields such as science, medicine, life insurance and climatology. Signal-to-noise ratio (SNR) is a reciprocal of the CV. It is defined as the ratio of the mean

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ISSN 2310-5070 (online) ISSN 2311-004X (print)

to the standard deviation. The SNR is important in medicine, quality control, finance, economics, and other fields. The SNR is a measure of the signal strength compared to background noise in analog and digital communications, whereas it discusses the magnitude of the mean of a process relative to variation in quality control. There are numerous publications for estimating SNRs based on two-parameter exponential data, for example, Sharma and Krishna [10], George and Kibria [11], Albatineh et al. [12], Albatineh et al. [13], Saothayanun and Thangjai [6], Niwitpong [14], Thangjai and Niwitpong [15], Thangjai and Niwitpong [16], and Thangjai and Niwitpong [17].

The generalized confidence interval (GCI) approach uses generalized pivotal quantity (GPQ) to construct the confidence interval. As an advantage, the GPQ of the parameters is based on the maximum likelihood estimates. Since the scale and location parameters in the two-parameter exponential distribution are easy to find the maximum likelihood estimators. Furthermore, the GCI approach can be used to estimate the confidence interval for complex parameters. However, the numerical simulation of the GCI approach is based on the maximum likelihood estimate only. According to Weerahandi [18], the GPQ is based on the following two properties. Property A: the random quantity has a probability distribution that is free of unknown parameters. Property B: the observed value of the random quantity does not depend on nuisance parameters. The GCI approach is successfully constructed the confidence interval for common parameter such as the research paper of Tian [19], Tian and Wu [20], Ye et al. [21], and Ng [22]. The large sample (LS) approach uses the concept of the central limit theorem (CLT) which is the most fundamental theory in statistics. The statistics of sample obtained based on a random sampling with replacement are normal distribution with the parameter when the sample size is sufficiently large. The LS approach has the advantage of being easy to construct the confidence interval using the exact formula. For disadvantage, the LS approach should use the large sample size to estimate the confidence interval. The method of variance estimates recovery (MOVER) approach is used to estimate the confidence interval for two parameters case; see Zou and Donner [23] and Zou et al. [24]. The adjusted MOVER approach is motivated based on the concepts of the large sample and MOVER approaches. This approach is used in many studies for constructing the confidence interval of common parameter, for example, see Thangjai et al. [25] and Thangjai and Niwitpong [5]. The advantage of the adjusted MOVER approach is easy to use the exact formula for computing the confidence interval, whereas the disadvantage is that the adjusted MOVER approach is based on the initial confidence interval of a single parameter. The bootstrap approach approximates the sampling distribution of the statistics by resampling with replacement. The bootstrap samples are repeatedly drawn from population. The bootstrap samples are used to represent the population. As an advantage, the bootstrap approach is simple and reasonably accurate confidence interval. For disadvantage, the distribution of estimates around the true values needs to know because the sampling distribution follows the distribution of the data since estimates are function of the data. Thangjai and Niwitpong [8] has used the parametric bootstrap approach to construct the simultaneous confidence intervals for differences of CVs of two-parameter exponential distributions. Chachi [26] provided the bootstrap approach to statistical testing of hypotheses about variance of a fuzzy random variable. Therefore, the confidence interval for the common parameter based on the bootstrap approach is of interest.

Saothayanun and Thangjai [6] proposed the GCI approach, the LS approach, the MOVER for interval estimation of the single SNR of two-parameter exponential distribution. In this study, new approaches are proposed for estimating confidence interval for common SNR of k two-parameter exponential distributions. These approaches are GCI approach, LS approach, adjusted MOVER approach, and bootstrap approaches using standard bootstrap (SB) and parametric bootstrap (PB).

The rest of this study is organized as follows. Section 2 proposes the computational procedures to construct the confidence intervals for common SNR. Section 3 conducts the simulation studies. Section 4 illustrates the proposed approaches with real example. Finally, Section 5 concludes the paper with some discussion and final remarks.

2. Confidence intervals for common SNR

Consider k independent two-parameter exponential distributions with a common SNR δ. Let $X_i = (X_{i1}, X_{i2}, ..., X_{in_i})$ be a random variable from the i-th two-parameter exponential population. Let λ_i and β_i be scale parameter and location parameter of i-th sample, respectively. The mean and variance of X_i are $E(X_i) = \lambda_i + \beta_i$ and $Var(X_i) = \lambda_i^2$, respectively.

For $i = 1, 2, ..., k$, the common SNR defined as

$$\delta = \frac{\lambda_i + \beta_i}{\lambda_i}. \tag{1}$$

From the i-th sample, the estimator of δ is

$$\begin{aligned} \hat{\delta} &= \frac{\hat{\lambda}_i + \hat{\beta}_i}{\hat{\lambda}_i} \\ &= \frac{\bar{X}_i}{\bar{X}_i - X_{(1)i}}, \end{aligned} \tag{2}$$

where $\hat{\beta}_i = X_{(1)i} = \min(X_{i1}, X_{i2}, ..., X_{in_i})$ and $\hat{\lambda}_i = \bar{X}_i - X_{(1)i}$.

The variance of $\hat{\delta}_i$ has the following form

$$Var(\hat{\delta}_i) = \frac{2n_i^2\lambda_i^2 - n_i\lambda_i^2 + 2n_i^2\lambda_i\beta_i + n_i^2\beta_i^2}{(n_i - 1)^3\lambda_i^2}. \tag{3}$$

2.1. Generalized confidence interval

The pivots for estimating β_i and λ_i based on the i-th sample are

$$W_{1i} = \frac{2n_i(\hat{\beta}_i - \beta_i)}{\lambda_i} \sim \chi_2^2 \text{ and } W_{2i} = \frac{2n_i\hat{\lambda}_i}{\lambda_i} \sim \chi_{2n_i-2}^2, \tag{4}$$

where χ_2^2 and $\chi_{2n_i-2}^2$ denote chi-squared distribution with 2 and $2n_i - 2$ degrees of freedom, respectively.

Thus, the generalized pivotal quantities for estimating β_i and λ_i defined as

$$R_{\beta_i} = \hat{\beta}_i - \frac{W_{1i}R_{\lambda_i}}{2n_i} \text{ and } R_{\lambda_i} = \frac{2n_i\hat{\lambda}_i}{W_{2i}}. \tag{5}$$

The generalized pivotal quantity for estimating δ_i based on the i-th sample is

$$R_{\delta_i} = 1 + \frac{1}{2n_i}\left(\frac{\hat{\beta}_i W_{2i}}{\hat{\lambda}_i} - W_{1i}\right). \tag{6}$$

According to Ye et al. [21], the generalized pivotal quantity for the common SNR is a weighted average of the generalized pivot R_{δ_i} based on k individual samples as

$$R_\delta = \sum_{i=1}^{k} \frac{R_{\delta_i}}{R_{Var(\hat{\delta}_i)}} \Bigg/ \sum_{i=1}^{k} \frac{1}{R_{Var(\hat{\delta}_i)}}, \tag{7}$$

where

$$R_{Var(\hat{\delta}_i)} = \frac{2n_i^2R_{\lambda_i}^2 - n_iR_{\lambda_i}^2 + 2n_i^2R_{\lambda_i}R_{\beta_i} + n_i^2R_{\beta_i}^2}{(n_i - 1)^3R_{\lambda_i}^2}.$$

Therefore, the $100(1-\alpha)\%$ generalized confidence interval for common SNR is obtained by

$$\begin{aligned} CI_{\delta.GCI} &= [L_{\delta.GCI}, U_{\delta.GCI}] \\ &= [R_\delta(\alpha/2), R_\delta(1-\alpha/2)], \end{aligned} \tag{8}$$

where $R_\delta(\alpha/2)$ and $R_\delta(1-\alpha/2)$ denote the $100(\alpha/2)$-th and the $100(1-\alpha/2)$-th percentiles of R_δ, respectively.

2.2. Large sample confidence interval

Let $x_i = (x_{i1}, x_{i2}, ..., x_{in_i})$ be observed value of $X_i = (X_{i1}, X_{i2}, ..., X_{in_i})$. Also, let $\bar{x}_i$ and $x_{(1)i}$ be observed values of $\bar{X}_i$ and $X_{(1)i}$, respectively. The estimates of $\hat{\delta}_i$ and $Var(\hat{\delta}_i)$ are

$$\hat{\delta}_i = \frac{\bar{x}_i}{\bar{x}_i - x_{(1)i}} \tag{9}$$

and

$$\hat{Var}(\hat{\delta}_i) = \frac{2n_i^2(\bar{x}_i - x_{(1)i})^2 - n_i(\bar{x}_i - x_{(1)i})^2 + 2n_i^2(\bar{x}_i - x_{(1)i})x_{(1)i} + n_i^2 x_{(1)i}^2}{(n_i-1)^3(\bar{x}_i - x_{(1)i})^2}. \tag{10}$$

According to Saothayanun and Thangjai [6], the expectation of $\hat{\delta}_i$ is

$$E(\hat{\delta}_i) = \frac{n_i^2(\lambda_i + \beta_i)}{(n_i-1)^2\lambda_i}. \tag{11}$$

It can be seen that the $\hat{\delta}_i$ is biased estimator of δ_i. Thus, an unbiased estimator of δ_i can be written as

$$\tilde{\delta}_i = \frac{(n_i-1)^2}{n_i^2}\hat{\delta}_i. \tag{12}$$

The variance of the unbiased estimator is

$$Var(\tilde{\delta}_i) = \frac{(n_i-1)^4}{n_i^4}Var(\hat{\delta}_i). \tag{13}$$

Following Graybill and Deal [27], the large sample estimate of the SNR is a pooled estimated unbiased estimator of the SNR based on k individual samples as

$$\tilde{\delta} = \sum_{i=1}^{k} \frac{\tilde{\delta}_i}{\hat{Var}(\tilde{\delta}_i)} \Bigg/ \sum_{i=1}^{k} \frac{1}{\hat{Var}(\tilde{\delta}_i)}, \tag{14}$$

where

$$\tilde{\delta}_i = \frac{(n_i-1)^2}{n_i^2}\left(\frac{\bar{x}_i}{\bar{x}_i - x_{(1)i}}\right),$$

$$\hat{Var}(\tilde{\delta}_i) = \frac{(n_i-1)^4}{n_i^4}\hat{Var}(\hat{\delta}_i),$$

and

$$\hat{Var}(\hat{\delta}_i) = \frac{2n_i^2(\bar{x}_i - x_{(1)i})^2 - n_i(\bar{x}_i - x_{(1)i})^2 + 2n_i^2(\bar{x}_i - x_{(1)i})x_{(1)i} + n_i^2 x_{(1)i}^2}{(n_i-1)^3(\bar{x}_i - x_{(1)i})^2}.$$

Let $z_{1-\alpha/2}$ be the $100(1-\alpha/2)$-th percentile of the standard normal distribution. Therefore, the $100(1-\alpha)\%$ large sample confidence interval for common SNR is obtained by

$$CI_{\delta.LS} = [L_{\delta.LS}, U_{\delta.LS}], \tag{15}$$

where

$$L_{\delta.LS} = \tilde{\delta} - z_{1-\alpha/2}\sqrt{1\Bigg/\sum_{i=1}^{k}\frac{1}{\hat{Var}(\tilde{\delta}_i)}}$$

and

$$U_{\delta.LS} = \tilde{\delta} + z_{1-\alpha/2}\sqrt{1\Bigg/\sum_{i=1}^{k}\frac{1}{\hat{Var}(\tilde{\delta}_i)}}$$

2.3. Adjusted MOVER confidence interval

For $i = 1, 2, ..., k$, the lower limit and the upper limit for δ_i are

$$l_i = \frac{(\hat{\lambda}_i + \hat{\beta}_i)\hat{\lambda}_i - \sqrt{((\hat{\lambda}_i + \hat{\beta}_i)\hat{\lambda}_i)^2 - l_{1i}u_{2i}(2(\hat{\lambda}_i + \hat{\beta}_i) - l_{1i})(2\hat{\lambda}_i - u_{2i})}}{u_{2i}(2\hat{\lambda}_i - u_{2i})} \quad (16)$$

and

$$u_i = \frac{(\hat{\lambda}_i + \hat{\beta}_i)\hat{\lambda}_i + \sqrt{((\hat{\lambda}_i + \hat{\beta}_i)\hat{\lambda}_i)^2 - u_{1i}l_{2i}(2(\hat{\lambda}_i + \hat{\beta}_i) - u_{1i})(2\hat{\lambda}_i - l_{2i})}}{l_{2i}(2\hat{\lambda}_i - l_{2i})}, \quad (17)$$

where

$$l_{1i} = \hat{\lambda}_i + \hat{\beta}_i - \sqrt{\left(\hat{\lambda}_i - \frac{n_i\hat{\lambda}_i}{z_{\alpha/2}\sqrt{n_i-1} + (n_i-1)}\right)^2 + \left(\frac{\hat{\lambda}_i}{n_i}\ln(\alpha/2)\right)^2},$$

$$u_{1i} = \hat{\lambda}_i + \hat{\beta}_i + \sqrt{\left(\frac{n_i\hat{\lambda}_i}{-z_{\alpha/2}\sqrt{n_i-1} + (n_i-1)} - \hat{\lambda}_i\right)^2 + \left(\frac{\hat{\lambda}_i}{n_i}\ln(1-\alpha/2)\right)^2},$$

$$l_{2i} = \frac{n_i\hat{\lambda}_i}{\sqrt{n_i-1}(z_{\alpha/2} + \sqrt{n_i-1})},$$

and

$$u_{2i} = \frac{n_i\hat{\lambda}_i}{\sqrt{n_i-1}(-z_{\alpha/2} + \sqrt{n_i-1})}.$$

Let $z_{\alpha/2}$ and $z_{1-\alpha/2}$ be the $(\alpha/2)$-th and the $(1-\alpha/2)$-th quantiles of the standard normal distribution, respectively, $\tilde{\delta}$, l_i, and u_i are defined in Equation (14), Equation (16), and Equation (17), respectively. Therefore, the $100(1-\alpha)\%$ adjusted MOVER confidence interval for common SNR is obtained by

$$CI_{\delta.AM} = [L_{\delta.AM}, U_{\delta.AM}], \quad (18)$$

where

$$L_{\delta.AM} = \tilde{\delta} - z_{1-\alpha/2}\sqrt{1\Bigg/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(\tilde{\delta}_i - l_i)^2}}$$

and

$$U_{\delta.AM} = \tilde{\delta} + z_{1-\alpha/2}\sqrt{1\Bigg/\sum_{i=1}^{k}\frac{z_{\alpha/2}^2}{(u_i - \tilde{\delta}_i)^2}}.$$

Table 1. Coverage probabilities of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions: 3 sample cases.

(n_1, n_2, n_3)	$(\lambda_1, \lambda_2, \lambda_3)$	θ	GCI	LS	AM	SB	PB
(10,10,10)	(1,1,1)	-10	0.9006	0.8076	0.9160	0.9054	0.9376
		-2	0.8948	0.7336	NA	0.8844	0.9254
		2	0.9570	0.9996	0.9998	0.9544	0.9648
		10	0.9140	0.8920	0.8734	0.9162	0.9460
(15,15,15)	(1,1,1)	-10	0.9138	0.8376	0.8988	0.9160	0.9384
		-2	0.9182	0.7644	0.9644	0.9084	0.9400
		2	0.9552	1.0000	0.9992	0.9548	0.9554
		10	0.9356	0.9282	0.8846	0.9294	0.9452
(30,30,30)	(1,1,1)	-10	0.9252	0.8724	0.8982	0.9304	0.9408
		-2	0.9270	0.7986	0.9206	0.9216	0.9376
		2	0.9532	1.0000	0.9992	0.9520	0.9498
		10	0.9430	0.9472	0.9078	0.9392	0.9486
(10,15,30)	(1,1,1)	-10	0.9200	0.8416	0.8994	0.9190	0.9364
		-2	0.9190	0.7798	NA	0.9116	0.9350
		2	0.9492	1.0000	0.9968	0.9484	0.9476
		10	0.9274	0.9254	0.8486	0.9272	0.9408
(30,30,50)	(1,1,1)	-10	0.9310	0.8856	0.9046	0.9312	0.9434
		-2	0.9296	0.8196	0.9000	0.9278	0.9430
		2	0.9594	1.0000	0.9968	0.9596	0.9552
		10	0.9468	0.9544	0.8682	0.9434	0.9522
(50,50,50)	(1,1,1)	-10	0.9338	0.8908	0.9058	0.9306	0.9432
		-2	0.9340	0.8160	0.8842	0.9332	0.9454
		2	0.9528	0.9998	0.9992	0.9510	0.9518
		10	0.9426	0.9578	0.9252	0.9418	0.9470
(15,30,50)	(1,1,1)	-10	0.9330	0.8768	0.9038	0.9316	0.9410
		-2	0.9228	0.7920	NA	0.9176	0.9366
		2	0.9552	1.0000	0.9948	0.9560	0.9584
		10	0.9376	0.9442	0.8302	0.9342	0.9464
(50,50,100)	(1,1,1)	-10	0.9384	0.8966	0.9096	0.9388	0.9476
		-2	0.9430	0.8290	0.8842	0.9396	0.9482
		2	0.9546	1.0000	0.9990	0.9516	0.9540
		10	0.9426	0.9582	0.9136	0.9444	0.9470
(100,100,100)	(1,1,1)	-10	0.9470	0.9126	0.9230	0.9456	0.9500
		-2	0.9386	0.8398	0.8692	0.9380	0.9422
		2	0.9542	1.0000	0.9996	0.9550	0.9536
		10	0.9502	0.9654	0.9400	0.9506	0.9518
(30,50,100)	(1,1,1)	-10	0.9408	0.9000	0.9132	0.9402	0.9460
		-2	0.9372	0.8344	0.8838	0.9382	0.9482
		2	0.9488	1.0000	0.9982	0.9482	0.9498
		10	0.9458	0.9584	0.8970	0.9454	0.9510
(100,100,200)	(1,1,1)	-10	0.9410	0.9082	0.9154	0.9402	0.9434
		-2	0.9436	0.8414	0.8620	0.9434	0.9472
		2	0.9494	0.9998	0.9996	0.9510	0.9502
		10	0.9428	0.9616	0.9456	0.9448	0.9466
(200,200,200)	(1,1,1)	-10	0.9414	0.9126	0.9138	0.9428	0.9478
		-2	0.9502	0.8486	0.8616	0.9478	0.9518
		2	0.9502	1.0000	0.9994	0.9478	0.9470
		10	0.9534	0.9704	0.9592	0.9528	0.9532

2.4. Bootstrap confidence interval

Let $X_i^* = (X_{i1}^*, X_{i2}^*, , X_{in_i}^*)$ be bootstrap sample with replacement from $X_i = (X_{i1}, X_{i2}, , X_{in_i})$ with sample size n_i and let $\bar{X}_i^*$ and $X_{(1)i}^*$ be mean and minimum of X_i^*, respectively. Let $x_i^* = (x_{i1}^*, x_{i2}^*, , x_{in_i}^*)$ be the observed values of $X_i^* = (X_{i1}^*, X_{i2}^*, , X_{in_i}^*)$ and let $\bar{x}_i^*$ and $x_{(1)i}^*$ be the observed values of $\bar{X}_i^*$ and $X_{(1)i}^*$, respectively. The estimates of $\hat{\delta}_i^*$ and $Var(\hat{\delta}_i^*)$ are

$$\hat{\delta}_i^* = \frac{\bar{x}_i^*}{\bar{x}_i^* - x_{(1)i}^*} \tag{19}$$

Table 2. Average lengths of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions: 3 sample cases.

(n_1, n_2, n_3)	$(\lambda_1, \lambda_2, \lambda_3)$	θ	GCI	LS	AM	SB	PB
(10,10,10)	(1,1,1)	-10	9.5513	7.5766	8.0025	9.1360	9.0505
		-2	2.3445	1.6401	NaN	2.3397	2.3035
		2	0.8672	1.6822	3.1759	0.9549	0.9411
		10	7.5719	7.6590	11.3934	7.5651	7.4776
(15,15,15)	(1,1,1)	-10	7.6439	6.0808	6.2240	7.2217	7.2040
		-2	1.9389	1.3241	1.6992	1.8795	1.8649
		2	0.6627	1.3512	6.7038	0.7092	0.7045
		10	5.9920	6.1209	31.1239	5.8211	5.7943
(30,30,30)	(1,1,1)	-10	5.1362	4.2186	4.2520	4.9187	4.9222
		-2	1.3625	0.9285	1.0367	1.3117	1.3096
		2	0.4386	0.9388	2.4485	0.4556	0.4547
		10	4.0429	4.2416	11.2792	3.9547	3.9514
(10,15,30)	(1,1,1)	-10	6.9989	5.4464	5.5520	6.4621	6.4539
		-2	1.7393	1.1994	NaN	1.6948	1.6825
		2	0.5873	1.2150	3.0289	0.6176	0.6146
		10	5.3802	5.4760	13.4070	5.1908	5.1728
(30,30,50)	(1,1,1)	-10	4.5968	3.8058	3.8286	4.4123	4.4170
		-2	1.2286	0.8415	0.9176	1.1849	1.1840
		2	0.3930	0.8470	1.6494	0.4059	0.4051
		10	3.6173	3.8204	7.4647	3.5434	3.5412
(50,50,50)	(1,1,1)	-10	3.8384	3.2467	3.2600	3.7260	3.7283
		-2	1.0381	0.7179	0.7646	1.0079	1.0078
		2	0.3318	0.7226	1.1654	0.3398	0.3396
		10	3.0438	3.2570	5.2528	2.9910	2.9910
(15,30,50)	(1,1,1)	-10	5.0880	4.1024	4.1338	4.7956	4.8002
		-2	1.3170	0.9049	NaN	1.2721	1.2687
		2	0.4262	0.9134	1.7717	0.4413	0.4405
		10	3.9324	4.1156	7.8894	3.8234	3.8191
(50,50,100)	(1,1,1)	-10	3.2762	2.8014	2.8092	3.1887	3.1903
		-2	0.8880	0.6214	0.6500	0.8650	0.8654
		2	0.2847	0.6242	0.8585	0.2900	0.2900
		10	2.6033	2.8093	3.8257	2.5653	2.5657
(100,100,100)	(1,1,1)	-10	2.6130	2.2824	2.2866	2.5698	2.5709
		-2	0.7125	0.5067	0.5223	0.6989	0.6989
		2	0.2306	0.5087	0.6322	0.2335	0.2334
		10	2.0982	2.2872	2.8312	2.0765	2.0768
(30,50,100)	(1,1,1)	-10	3.4927	2.9567	2.9657	3.3767	3.3804
		-2	0.9386	0.6541	0.6870	0.9127	0.9126
		2	0.3004	0.6579	0.9180	0.3063	0.3062
		10	2.7593	2.9620	4.0585	2.7119	2.7117
(100,100,200)	(1,1,1)	-10	2.2412	1.9751	1.9776	2.2119	2.2136
		-2	0.6118	0.4385	0.4482	0.6024	0.6027
		2	0.1987	0.4399	0.5122	0.2004	0.2004
		10	1.8047	1.9780	2.2861	1.7901	1.7907
(200,200,200)	(1,1,1)	-10	1.8069	1.6109	1.6123	1.7903	1.7915
		-2	0.4940	0.3584	0.3638	0.4893	0.4895
		2	0.1615	0.3587	0.3982	0.1625	0.1625
		10	1.4651	1.6144	1.7870	1.4571	1.4570

and

$$\hat{Var}(\hat{\delta}_i^*) = \frac{2n_i^2(\bar{x}_i^* - x_{(1)i}^*)^2 - n_i(\bar{x}_i^* - x_{(1)i}^*)^2 + 2n_i^2(\bar{x}_i^* - x_{(1)i}^*)x_{(1)i}^* + n_i^2 {x_{(1)i}^*}^2}{(n_i-1)^3(\bar{x}_i^* - x_{(1)i}^*)^2}. \tag{20}$$

The unbiased estimator is

$$\tilde{\delta}_i^* = \frac{(n_i-1)^2}{n_i^2}\hat{\delta}_i^*. \tag{21}$$

Table 3. Coverage probabilities of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions: 5 sample cases.

$(n_1, n_2, n_3, n_4, n_5)$	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	θ	GCI	LS	AM	SB	PB
(10,10,10,10,10)	(1,1,1,1,1)	-10	0.8070	0.7158	0.8720	0.8312	0.8814
		-2	0.7834	0.6326	NA	0.7846	0.8580
		2	0.9384	0.9988	1.0000	0.9318	0.9546
		10	0.8350	0.8348	0.9314	0.8522	0.9004
(15,15,15,15,15)	(1,1,1,1,1)	. -10	0.8346	0.7572	0.8502	0.8572	0.8940
		-2	0.8220	0.6880	0.9670	0.8268	0.8720
		2	0.9426	1.0000	0.9994	0.9392	0.9528
		10	0.8652	0.8784	0.9268	0.8770	0.9110
(30,30,30,30,30)	(1,1,1,1,1)	-10	0.8828	0.8396	0.8778	0.8976	0.9156
		-2	0.8654	0.7514	0.9044	0.8786	0.9044
		2	0.9452	0.9996	0.9994	0.9446	0.9516
		10	0.9030	0.9230	0.9390	0.9116	0.9272
(10,10,15,30,30)	(1,1,1,1,1)	-10	0.8640	0.7964	0.8728	0.8822	0.9060
		-2	0.8360	0.7030	NA	0.8392	0.8790
		2	0.9500	0.9996	0.9994	0.9434	0.9542
		10	0.8866	0.8974	0.9068	0.8960	0.9232
(30,30,30,50,50)	(1,1,1,1,1)	-10	0.8954	0.8580	0.8842	0.9096	0.9230
		-2	0.8896	0.7780	0.8914	0.9014	0.9204
		2	0.9468	1.0000	0.9986	0.9464	0.9486
		10	0.9194	0.9416	0.9004	0.9264	0.9408
(50,50,50,50,50)	(1,1,1,1,1)	-10	0.9108	0.8794	0.9006	0.9214	0.9318
		-2	0.8948	0.7938	0.8834	0.9102	0.9228
		2	0.9534	1.0000	0.9996	0.9518	0.9540
		10	0.9198	0.9422	0.9486	0.9260	0.9362
(15,15,30,50,50)	(1,1,1,1,1)	-10	0.8826	0.8376	0.8792	0.9004	0.9150
		-2	0.8814	0.7720	0.9016	0.8896	0.9134
		2	0.9490	1.0000	0.9978	0.9438	0.9496
		10	0.9122	0.9272	0.8732	0.9164	0.9324
(50,50,50,100,100)	(1,1,1,1,1)	-10	0.9158	0.8852	0.8994	0.9272	0.9334
		-2	0.9026	0.8030	0.8664	0.9126	0.9246
		2	0.9496	1.0000	1.0000	0.9508	0.9550
		10	0.9296	0.9520	0.9416	0.9302	0.9376
(100,100,100,100,100)	(1,1,1,1,1)	-10	0.9254	0.9040	0.9108	0.9300	0.9372
		-2	0.9218	0.8308	0.8728	0.9288	0.9368
		2	0.9536	1.0000	0.9998	0.9520	0.9538
		10	0.9350	0.9570	0.9516	0.9362	0.9406
(30,30,50,100,100)	(1,1,1,1,1)	-10	0.9150	0.8848	0.9016	0.9216	0.9324
		-2	0.8980	0.7958	0.8710	0.9104	0.9250
		2	0.9530	0.9998	0.9982	0.9526	0.9532
		10	0.9254	0.9468	0.9026	0.9308	0.9376
(100,100,100,200,200)	(1,1,1,1,1)	-10	0.9270	0.9012	0.9104	0.9328	0.9394
		-2	0.9230	0.8198	0.8524	0.9318	0.9364
		2	0.9466	1.0000	1.0000	0.9478	0.9468
		10	0.9390	0.9608	0.9586	0.9420	0.9480
(200,200,200,200,200)	(1,1,1,1,1)	-10	0.9308	0.9040	0.9088	0.9356	0.9380
		-2	0.9326	0.8360	0.8570	0.9360	0.9406
		2	0.9524	1.0000	1.0000	0.9536	0.9536
		10	0.9376	0.9612	0.9628	0.9410	0.9430

The variance of $\tilde{\delta}_i^*$ is

$$Var(\tilde{\delta}_i^*) = \frac{(n_i - 1)^4}{n_i^4} Var(\hat{\delta}_i^*). \tag{22}$$

According to Graybill and Deal [27], the common SNR is a pooled estimated unbiased estimator of the SNR based on k individual samples. The common SNR is defined by

$$\tilde{\delta}^* = \sum_{i=1}^{k} \frac{\tilde{\delta}_i^*}{V\hat{a}r(\tilde{\delta}_i^*)} \Bigg/ \sum_{i=1}^{k} \frac{1}{V\hat{a}r(\tilde{\delta}_i^*)}, \tag{23}$$

Table 4. Average lengths of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions: 5 sample cases.

$(n_1, n_2, n_3, n_4, n_5)$	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$	θ	GCI	LS	AM	SB	PB
(10,10,10,10,10)	(1,1,1,1,1)	-10	7.8535	5.7157	6.0647	6.9312	6.9149
		-2	1.7669	1.2341	NaN	1.7335	1.7091
		2	0.6634	1.2990	2.4511	0.6858	0.6810
		10	5.9721	5.8182	8.6631	5.5577	5.5304
(15,15,15,15,15)	(1,1,1,1,1)	-10	6.2285	4.6084	4.7248	5.5681	5.5756
		-2	1.5017	1.0079	1.3023	1.4300	1.4202
		2	0.5066	1.0437	5.1733	0.5268	0.5250
		10	4.7334	4.6898	23.8459	4.4328	4.4259
(30,30,30,30,30)	(1,1,1,1,1)	-10	4.1074	3.2354	3.2618	3.8296	3.8380
		-2	1.0711	0.7125	0.7971	1.0169	1.0161
		2	0.3369	0.7259	1.8926	0.3467	0.3465
		10	3.1631	3.2635	8.6781	3.0382	3.0388
(10,10,15,30,30)	(1,1,1,1,1)	-10	5.6826	4.0840	4.1638	4.9174	4.9285
		-2	1.3110	0.8929	NaN	1.2714	1.2626
		2	0.4396	0.9210	2.1792	0.4529	0.4520
		10	4.1810	4.1380	9.5228	3.8884	3.8847
(30,30,30,50,50)	(1,1,1,1,1)	-10	3.5794	2.8715	2.8883	3.3679	3.3739
		-2	0.9481	0.6364	0.6921	0.9035	0.9031
		2	0.2962	0.6432	1.2132	0.3034	0.3031
		10	2.7717	2.8934	5.4594	2.6795	2.6808
(50,50,50,50,50)	(1,1,1,1,1)	-10	3.0356	2.5030	2.5134	2.9057	2.9098
		-2	0.8136	0.5528	0.5892	0.7795	0.7799
		2	0.2556	0.5592	0.9017	0.2608	0.2607
		10	2.3723	2.5121	4.0515	2.3052	2.3058
(15,15,30,50,50)	(1,1,1,1,1)	-10	4.0860	3.1297	3.1540	3.7287	3.7380
		-2	1.0206	0.6904	0.7597	0.9785	0.9764
		2	0.3258	0.7024	1.3118	0.3336	0.3333
		10	3.0644	3.1510	5.7860	2.9238	2.9247
(50,50,50,100,100)	(1,1,1,1,1)	-10	2.5096	2.1116	2.1172	2.4212	2.4251
		-2	0.6760	0.4667	0.4873	0.6538	0.6542
		2	0.2141	0.4714	0.6374	0.2173	0.2173
		10	1.9748	2.1180	2.8343	1.9333	1.9338
(100,100,100,100,100)	(1,1,1,1,1)	-10	2.0456	1.7641	1.7673	1.9978	1.9990
		-2	0.5583	0.3916	0.4037	0.5438	0.5443
		2	0.1779	0.3937	0.4892	0.1798	0.1798
		10	1.6342	1.7711	2.1923	1.6104	1.6108
(30,30,50,100,100)	(1,1,1,1,1)	-10	2.7167	2.2465	2.2531	2.5957	2.5994
		-2	0.7226	0.4971	0.5210	0.6971	0.6974
		2	0.2283	0.5014	0.6870	0.2317	0.2317
		10	2.1171	2.2545	3.0269	2.0630	2.0633
(100,100,100,200,200)	(1,1,1,1,1)	-10	1.7022	1.4883	1.4901	1.6714	1.6731
		-2	0.4641	0.3306	0.3376	0.4559	0.4562
		2	0.1498	0.3324	0.3840	0.1509	0.1509
		10	1.3661	1.4932	1.7126	1.3526	1.3530
(200,200,200,200,200)	(1,1,1,1,1)	-10	1.4066	1.2467	1.2478	1.3899	1.3900
		-2	0.3842	0.2768	0.2810	0.3791	0.3793
		2	0.1249	0.2777	0.3083	0.1255	0.1255
		10	1.1345	1.2480	1.3815	1.1269	1.1272

where

$$\tilde{\delta}_i^* = \frac{(n_i-1)^2}{n_i^2}\left(\frac{\bar{x}_i^*}{\bar{x}_i^* - x_{(1)i}^*}\right),$$

$$\hat{Var}(\tilde{\delta}_i^*) = \frac{(n_i-1)^4}{n_i^4}\hat{Var}(\hat{\delta}_i^*),$$

and

$$\hat{Var}(\hat{\delta}_i^*) = \frac{2n_i^2(\bar{x}_i^* - x_{(1)i}^*)^2 - n_i(\bar{x}_i^* - x_{(1)i}^*)^2 + 2n_i^2(\bar{x}_i^* - x_{(1)i}^*)x_{(1)i}^* + n_i^2(x_{(1)i}^*)^2}{(n_i-1)^3(\bar{x}_i^* - x_{(1)i}^*)^2}.$$

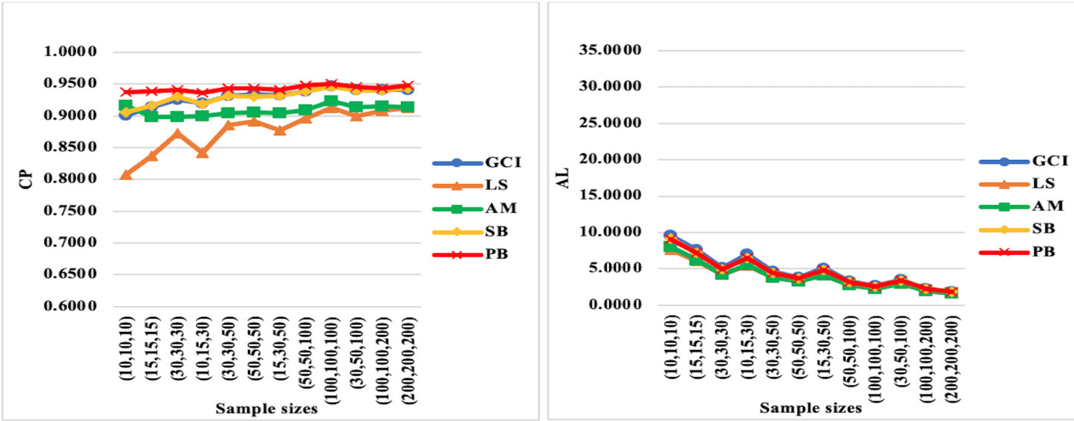

Figure 1. CP and AL of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions for $k = 3$ and $\theta =$ -10.

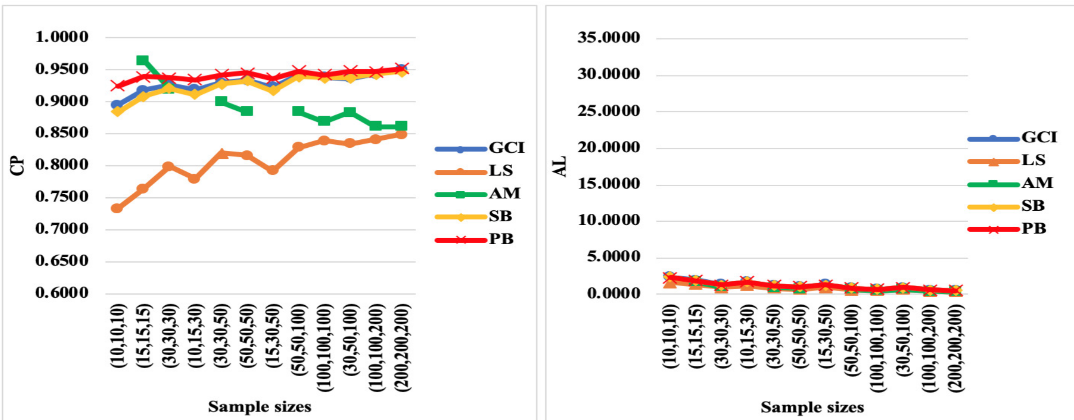

Figure 2. CP and AL of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions for $k = 3$ and $\theta =$ -2.

The B bootstrap statistics are used to construct the sampling distribution for estimating the confidence interval for the common SNR. Therefore, the $100(1-\alpha)\%$ SB confidence interval for common SNR is obtained by

$$\begin{aligned} CI_{\delta.SB} &= [L_{\delta.SB}, U_{\delta.SB}] \\ &= [\bar{\tilde{\delta}}^* - z_{1-\alpha/2}S^*, \bar{\tilde{\delta}}^* + z_{1-\alpha/2}S^*], \end{aligned} \tag{24}$$

where $\bar{\tilde{\delta}}^*$ and S^* denote the mean and the standard deviation of $\tilde{\delta}^*$ defined in Equation (23) and $z_{1-\alpha/2}$ denotes the $100(1-\alpha/2)$-th percentile of the standard normal distribution.

Furthermore, the $100(1-\alpha)\%$ PB confidence interval for common SNR is obtained by

$$\begin{aligned} CI_{\delta.PB} &= [L_{\delta.PB}, U_{\delta.PB}] \\ &= [\tilde{\delta}^*(\alpha/2), \tilde{\delta}^*(1-\alpha/2)], \end{aligned} \tag{25}$$

where $\tilde{\delta}^*(\alpha/2)$ and $\tilde{\delta}^*(1-\alpha/2)$ denote the $100(\alpha/2)$-th and the $100(1-\alpha/2)$-th percentiles of $\tilde{\delta}^*$, respectively.

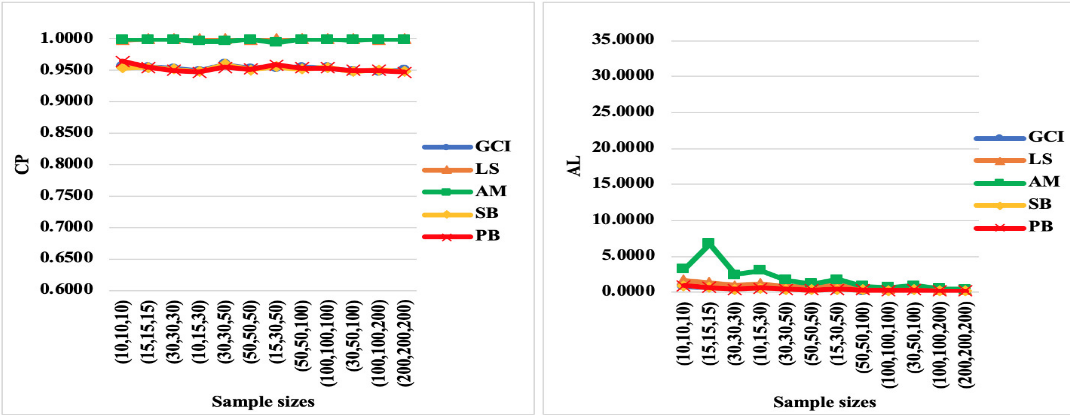

Figure 3. CP and AL of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions for $k = 3$ and $\theta = 2$.

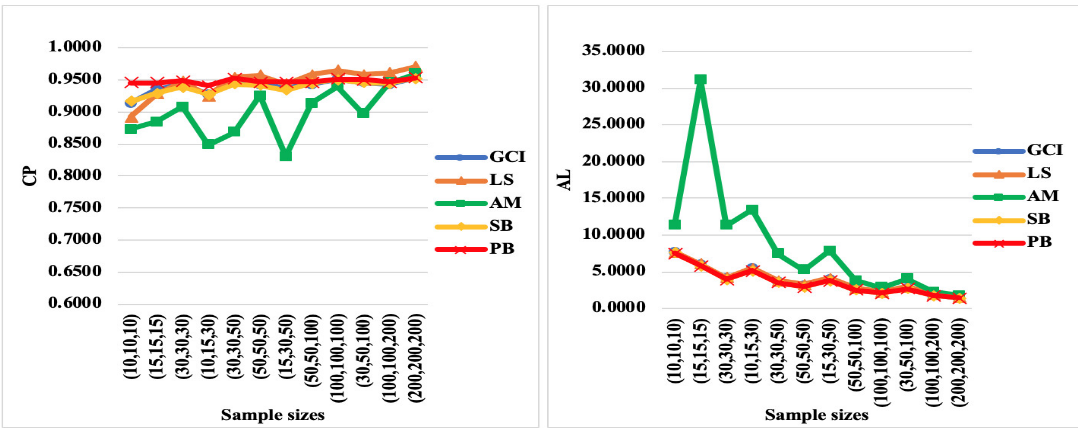

Figure 4. CP and AL of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions for $k = 3$ and $\theta = 10$.

3. Simulation studies

Simulation study is performed to compare the proposed approaches for estimating confidence intervals of common SNR. The simulation is performed to evaluate the proposed confidence intervals for common SNR by coverage probability and average length. The data are generated from two-parameter exponential distributions with scale parameters λ_i and location parameters β_i, where $i = 1, 2, ..., k$. The sample cases $k = 3$ and $k = 5$ are used. The scale parameters $\lambda_1, \lambda_2, ..., \lambda_k$ are fixed equal to be 1. The location parameters $\beta_1, \beta_2, ..., \beta_k$ are applied to get the common SNR δ = -10, -2, 2, and 10. For each parameter¡¯s combination, 5000 simulation data sets are generated. For each simulation data set, 2500 random variables are generated to compute the generalized pivotal quantities and bootstrap pivotal quantity. Hence, the 95% confidence intervals based on the GCI approach, the LS approach, the adjusted MOVER approach, the SB approach, and the PB approach can be obtained. The coverage probabilities and average lengths for 3 and 5 sample cases are presented in Tables 1 - 4. From Tables 1 - 2, the coverage probabilities and average lengths for $k = 3$ sample cases clearly indicate that the coverage probabilities of all approaches are below the nominal confidence level of 0.95 when the common SNR is negative value. For $\delta = 2$, the coverage

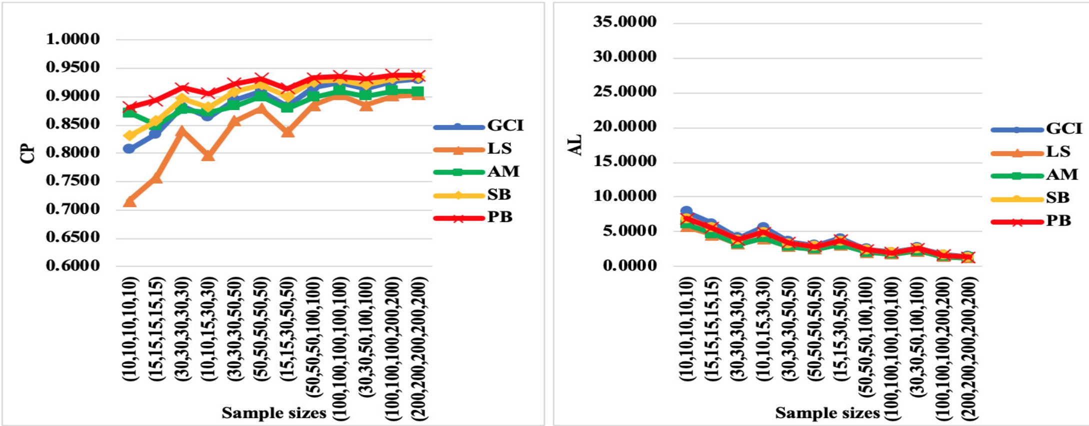

Figure 5. CP and AL of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions for $k = 5$ and θ = -10.

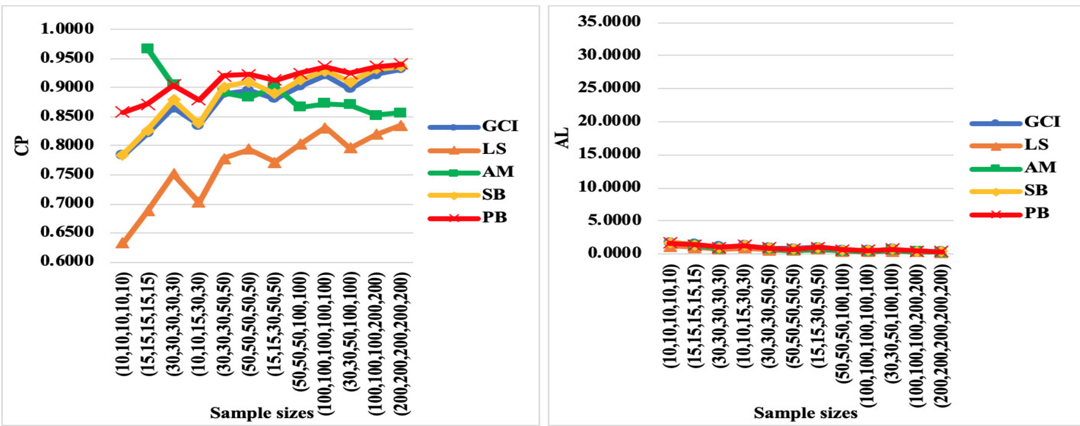

Figure 6. CP and AL of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions for $k = 5$ and θ = -2.

probabilities of all approaches are greater than the nominal confidence level of 0.95, but the LS and AM approaches provide the conservative confidence intervals. For $\delta = 10$, the PB appraoch is satisfactorily in terms of the coverage probabilities and average lengths. From Tables 3 - 4, the coverage probabilities of LS, AM, and SB approaches are below the nominal confidence level of 0.95 for all sample sizes, except the coverage probabilities of these approach are greater than the nominal confidence level of 0.95 when the common SNR is equal to two. The LS and AM approaches are conservative confidence intervals when the common SNR is equal to two. The GCI and PB approaches have the coverage probabilities less than the nominal confidence level of 0.95 when the common SNR is negative value for all sample sizes, whereas two these approaches have the coverage probabilities close to the nominal confidence level of 0.95 when the common SNR is positive value for large sample sizes. The GCI and PB approaches are recommended to construct the confidence interval for the common SNR of two-parameter exponential distributions when the sample sizes are large and the common SNR is positive value.

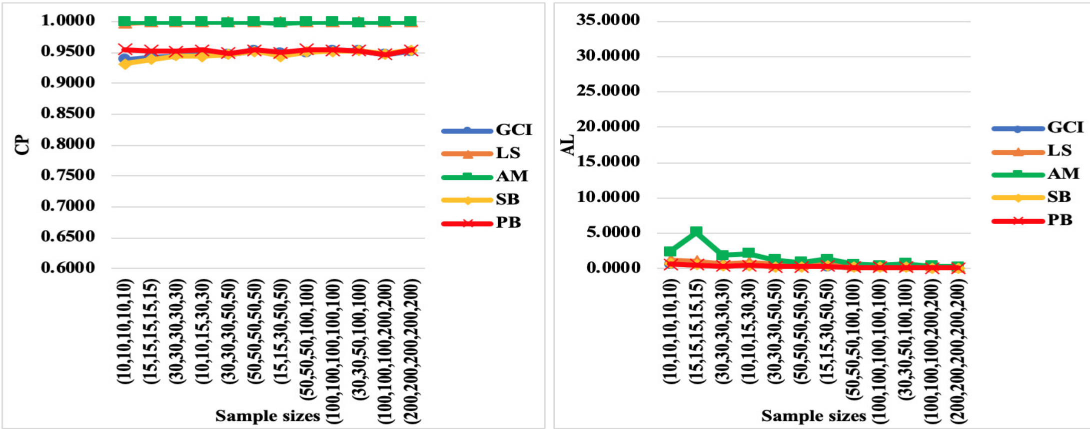

Figure 7. CP and AL of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions for $k = 5$ and $\theta = 2$.

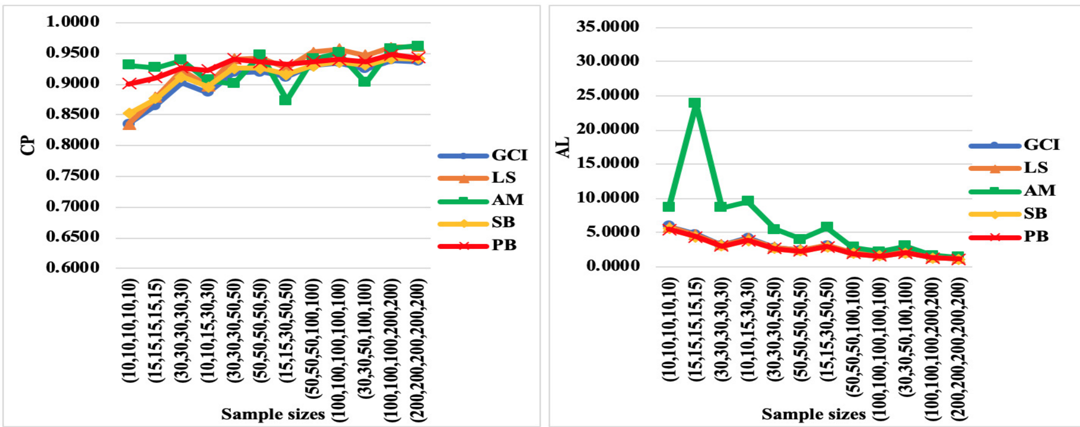

Figure 8. CP and AL of 95% two-sided confidence intervals for common SNR of two-parameter exponential distributions for $k = 5$ and $\theta = 10$.

4. Empirical application

Maurya et al. [28] presented the data of the survival days of patients for different types of lung cancer. The summary statistics of squamous type are as follows $n_1 = 9$, $\bar{x}_1 = 51.0000$, and $x_{(1)1} = 8.0000$. The summary statistics of small type are as follows $n_2 = 9$, $\bar{x}_2 = 22.1111$, and $x_{(1)2} = 13.0000$. The summary statistics of adeno type are as follows $n_3 = 9$, $\bar{x}_3 = 72.8889$, and $x_{(1)3} = 3.0000$. And the summary statistics of large type are as follows $n_4 = 9$, $\bar{x}_4 = 197.8889$, and $x_{(1)4} = 103.0000$. The common SNR of two-parameter exponential distributions is 1.1114. The 95% confidence intervals are constructed using five proposed approaches. Using the GCI approach, the 95% generalized confidence interval is [1.0224,1.3770] with interval length 0.3546. Using the LS approach, the 95% LS confidence interval is [0.5670,1.6559] with interval length 1.0889. Using the adjusted MOVER approach, the 95% adjusted MOVER confidence interval is [0.0316,2.1761] with interval length 2.1445. Using the SB approach, the 95% SB confidence interval is [1.0295,1.3203] with interval length 0.2908. And using the PB approach, the 95% PB confidence interval is [1.0777,1.3641] with interval length 0.2864. For all five approaches, the confidence interval contains the true common SNR of two-parameter exponential distributions. The PB approach is better than the others. This is because the interval length of the PB confidence interval is shorter than the interval lengths of the other approaches.

5. Discussion and conclusions

In this study, the research paper of Saothayanun and Thangjai [6] is extended to develop new approaches for estimating the confidence interval for common SNR of two-parameter exponential distributions. The confidence intervals are constructed using the GCI approach, the large sample approach, the adjusted MOVER approach, and the bootstrap approaches (SB approach and PB approach). A Monte Carlo simulation study is conducted to evaluate the performance of the proposed approaches. The coverage probabilities and the average lengths of each confidence interval is estimated using simulation. The coverage probabilities of all approaches are below the nominal confidence level of 0.95 when the common SNR is negative value. The coverage probabilities of the GCI and PB approaches are close to the nominal confidence level of 0.95 when the sample sizes are large and the common SNR is positive value. Therefore, the GCI approach and the PB approach are recommended to estimate the confidence interval for the common SNR of two-parameter exponential distributions.

Acknowledgement

This research was funded by Faculty of Applied Science, King Mongkuts University of Technology North Bangkok. Contract No. 6345102.

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