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The Topp-Leone odd log-logistic Gumbel Distribution: Properties and Applications

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Abstract In this article, the Topp-Leone odd log-logistic Gumbel (TLOLL-Gumbel) family of distribution have been studied. This family, contains the very flexible skewed density function. We study many aspects of the new model like hazard rate function, asymptotics, useful expansions, moments, generating Function, Rényi entropy and order statistics. We discuss maximum likelihood estimation of the model parameters. Further, we study flexibility of the proposed family are illustrated of two real data sets.

Keywords Gumbel Distributin, Maximum likelihood, Moment, Odd log-logistic-G family, Rényi Entropy, ToppCLeone distribution

AMS 2010 subject classifications 60E05, 62F10, 62G05

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1. Introduction

In many real-world examples, known classical distributions do not provide a good fit to real data. Several continuous univariate distributions have been extensively used in literature for modeling data in many areas such as economics, engineering, biological studies and environmental sciences. However, applied areas such as finance, lifetime analysis and insurance clearly require extended forms of these distributions. So, several classes of distributions have been constructed by extending common families of continuous distributions. These generalized distributions provide more exibility by adding one (or more) parameters to a baseline model.
The main aim of the paper is to propose a new family of distributions from the Topp and Leone's distribution that can have the bathtub shaped hazard rate to be used for lifetime modeling. Recently, several properties of the Topp and Leone's distribution have been studied by several authors. We mention some of them: moments Nadarajah and Kotz [20], reliability measures and stochastic orderings Ghitany et al. [9], distributions of sums, products and ratios Zhou et al. [31], behavior of kurtosis Kotz and Seier [17]; record values Zghoul [30], moments of order statistics Genc [7], stress-strength modeling Genc [8] and Bayesian estimation under trimmed samples Sindhu [27].
The distribution Topp and Leone it is a one-parameter distribution with the cumulative distribution function (cdf) specified by

$$F_{TL}(x) = \left[1 - (1-x)^2\right]^b,$$

where $0 < x < 1$ and $b > 0$. Having only one parameter and its domain restricted to $(0, 1)$, Topp and Leone's distribution is not very flexible.
Consider starting from a parent continuous distribution function $G(x)$. A natural way of generating families of distributions on some other support from a simple starting parent distribution with pdf $g(x) = dG(x)/dx$ is to

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apply the quantile function to a family of distributions on the interval $(0,1)$.
Brito et. al., [5] combined the work of Topp and Leone [29] and cdf of the odd log-logistic distribution to construct a new class of TLOLL generalized distributions. The cdf of the OLL-G family is given by

$$F_{OLL-G}(x) = \frac{G(x;\boldsymbol{\xi})^a}{G(x;\boldsymbol{\xi})^a + \bar{G}(x;\boldsymbol{\xi})^a},$$

for $a > 0$, $G(x)$ is a baseline cdf and $\bar{G}(x) = 1 - G(x)$. From an arbitrary parent cdf $G(x)$, the cdf F(x) of the TLOLL-Gumble distribution is defined by

$$F(x;a,b,\boldsymbol{\xi}) = \left\{1 - \left[1 - \frac{G(x;\boldsymbol{\xi})^a}{G(x;\boldsymbol{\xi})^a + \bar{G}(x;\boldsymbol{\xi})^a}\right]^2\right\}^b, \tag{1}$$

where a and $b > 0$ are shape parameters and $\boldsymbol{\xi}$ is the vector of parameters of the baseline G distribution.
The probability density function (pdf) corresponding to (1) is given by

$$\begin{aligned} f(x;a,b,\boldsymbol{\xi}) &= \frac{2\,a\,b\,g(x;\boldsymbol{\xi})G(x;\boldsymbol{\xi})^{a-1}\bar{G}(x;\boldsymbol{\xi})^{2a-1}}{\left[G(x;\boldsymbol{\xi})^a + \bar{G}(x;\boldsymbol{\xi})^a\right]^3} \\ &\times \left\{1 - \left[1 - \frac{G(x;\boldsymbol{\xi}))^a}{G(x;\boldsymbol{\xi}))^a + \bar{G}(x;\boldsymbol{\xi}))^a}\right]^2\right\}^{b-1}. \end{aligned} \tag{2}$$

We denote by $X \sim TLOLL - G(a,b,\boldsymbol{\xi})$ a random variable having density function (2). The TLOLL-Gumbel model is obtained by inserting the Gumbel distribution as a parent model in the Topp-Leone odd log-logistic family.

Furthermore, the basic motivations for using the TLOLL-Gumble family in practice are the following:

- to produce a skewness and kurtosis distribution;
- to construct heavy-tailed distributions for modeling real data;
- to generate distributions with symmetric, left-skewed, right-skewed or reversed-J shape;
- to provide consistently better fits than other generated models under the same underlying distribution.

Simulating the TLOLL-Gumble random variable is straightforward. If U has a uniform distribution on the unit interval $(0,1)$, the solution of the equation,

$$x_u = G^{-1}\left\{\frac{[(1-u^{1/b})^{-1/2} - 1]^{1/a}}{1 + [(1-u^{1/b})^{-1/2} - 1]^{1/a}}\right\}. \tag{3}$$

This paper is organized as follows. In Section 2, we defined The Topp-Leone Odd Log-Logistic Gumbel family of distributions. In Section 3, some of its mathematical properties are obtained including asymptotics, useful expansions, moments, generating Function, Rényi entropy and order statistics. Inference for the model parameters is performed in Section 4. A simulation study is presented in Section 5. Applications to two real data sets illustrate the performance of the proposed models in Section 6. The paper is concluded in Section 7.

2. The Topp-Leone odd log-logistic Gumbel Distribution

Gumbel (1958) gave detailed results on extreme value theory in his book Statistics of Extremes. Furthermore, Gumbel [10] has been referred to by Johnson et al. [13] as the first to bring attention to the possibility of using the Gumbel distribution to model extreme values of random data. For more information on extreme value distributions, see Johnson et al. [13], Gumbel [10], Kotz and Nadarajah [16] and Beirlant et al. [2].
The Gumbel distribution is perhaps the most widely applied statistical distribution for problems in engineering. It

is also known as the extreme value distribution of type I. Some of its recent application areas in engineering include flood frequency analysis, network engineering, nuclear engineering, offshore engineering, risk-based engineering, space engineering, software reliability engineering, structural engineering, and wind engineering.
In this section we introduce the new distribution. The Gumble distribution pdf has,

$$G(x;\mu,\sigma) = \exp\left(-e^{-z}\right) \qquad \& \qquad z = \frac{x-\mu}{\sigma},$$

and it's cdf is,

$$g(x;\mu,\sigma) = \frac{1}{\sigma}\exp\left[-\left(z+e^{-z}\right)\right],$$

and $\bar{G}(x;\mu,\sigma) = 1 - G(x;\mu,\sigma)$. where $x \in R$, $\mu \in R$ and $\sigma > 0$.

The cdf of TLOLL-Gumbel distribution is,

$$F(x;a,b,\mu,\sigma) = \left[1 - \left(\frac{(1-\exp(-e^{-z}))^a}{[\exp(-e^{-z})]^a + [1-\exp(-e^{-z})]^a}\right)^2\right]^b,$$

where $a > 0$ and $b > 0$.
The pdf of TLOLL-Gumbel distribution is,

$$\begin{aligned} f(x;a,b,\mu,\sigma) &= \frac{2ab\exp(-(z+e^{-z}))\exp(-(a-1)e^{-z})(1-\exp(-e^{-z}))^{2a-1}}{\sigma\{[\exp(-e^{-z})]^a + [1-\exp(-e^{-z})]^a\}^3} \\ &\times \left[1 - \left(\frac{(1-\exp(-e^{-z}))^a}{[\exp(-e^{-z})]^a + [1-\exp(-e^{-z})]^a}\right)^2\right]^{b-1}. \end{aligned} \tag{4}$$

The special cases of the proposed family is:
Taking $a = 1$ in TLOLL-Gumbel family, we obtain TL-Gumbel distributions with the following pdf.

$$f(x;b,\mu,\sigma) = 2\frac{b}{\sigma}\exp\left[-\left(z+e^{-z}\right)\right]\left(1-\exp\left(-e^{-z}\right)\right)\left\{1-\left[1-\exp\left(-e^{-z}\right)\right]^2\right\}^{b-1}.$$

Some shapes of the TLOLL-Gumble pdf are displayed in Figures 1. This Figures shows the plots of pdf TLOLL-Gumble for selected parameter values. Different skewed density functions including mild and high skewed ones (positive and negative) for selected parameter values. The TLOLL-Gumble family contains the very flexible skewed density function (unimodal and bimodal) that is useful in fitting real data sets (see Section 6).

3. Some mathematical properties

In this section, we obtain some statistical properties such as : asymptotics, useful expansions, moments, generating function, Rényi entropy and order statistic Brito et al., [5] and Andrade et al., [1]. In addition, plots of the skewness, kurtosis and entropy are presented.

3.1. Asymptotics

Corollary 1
The asymptotics of $F(x)$, $f(x)$ and $\tau(x)$ as $x \to -\infty$ are given by

$$F(x) \sim (2a)^b \exp\left(-be^{-z}\right) \qquad as \qquad x \to -\infty,$$

$$f(x) \sim b(2a)^b z\exp\left[-\left(z+be^{-z}\right)\right] \qquad as \qquad x \to -\infty,$$

and

$$\tau(x) \sim bze^{-z} \qquad as \qquad x \to -\infty.$$

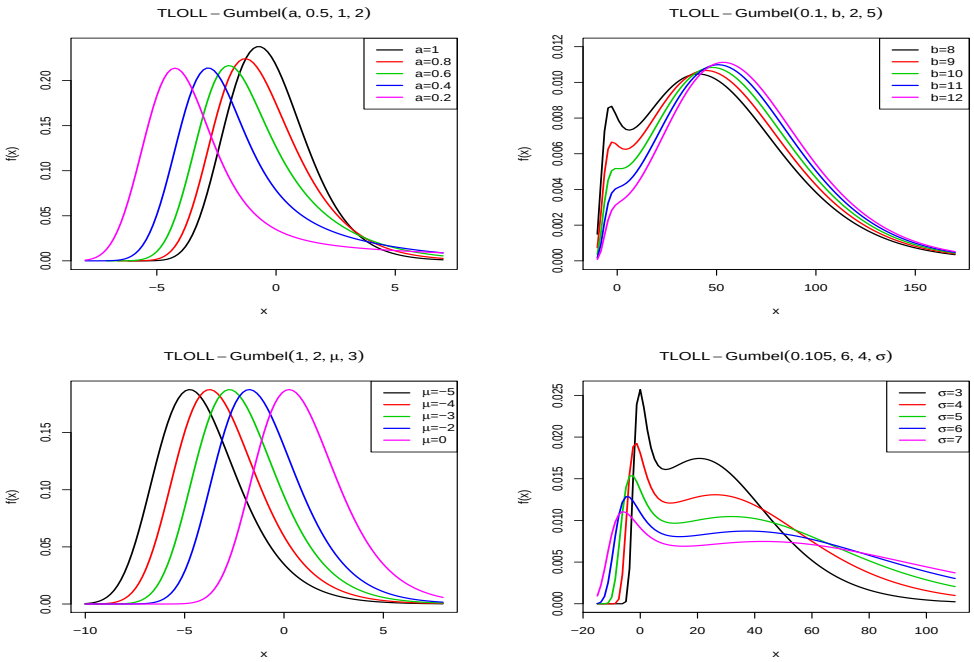

Figure 1. The sample curves of density function of TLOLL-Gumbel.

Corollary 2
The asymptotics of $F(x)$, $f(x)$ and $\tau(x)$ as $x \to \infty$ are given by

$$1 - F(x) \sim be^{-2az} \qquad as \qquad x \to \infty,$$

$$f(x) \sim 2abze^{-2az} \qquad as \qquad x \to \infty,$$

and

$$\tau(x) \sim 2az \qquad as \qquad x \to \infty.$$

3.2. Useful expansions

First the binomial epansions,

$$(1-u)^a = \sum_{j=0}^{\infty} (-1)^j \binom{a}{j} u^j, \tag{5}$$

holds for $-1 < u < 1$ and any $a > 0$ any real non-integer.

Second, using the generalized binomial expansion (5) we can rewrite (4),

$$\begin{aligned} F(x) &= \left\{1 - \left[\frac{(1-\exp(-e^{-z}))^a}{[\exp(-e^{-z})]^a + [1-\exp(-e^{-z})]^a}\right]^2\right\}^b \\ &= \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} \left[\frac{(1-\exp(-e^{-z}))^a}{[\exp(-e^{-z})]^a + [1-\exp(-e^{-z})]^a}\right]^{2i} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{2i} (-1)^{i+j} \binom{b}{i} \binom{2i}{j} \frac{\exp(-jae^{-z})}{\{[\exp(-e^{-z})]^a + [1-\exp(-e^{-z})]^a\}^j}. \end{aligned} \tag{6}$$

Third, we can expand $G(x)^{ja} = \exp\left(-jae^{-z}\right)$ as,

$$\exp\left(-jae^{-z}\right) = \sum_{k=0}^{\infty} a_{k,j} \exp\left(-ke^{-z}\right),$$

where

$$a_{k,j} = \sum_{l=k}^{\infty} (-1)^{l+k} \binom{aj}{l} \binom{l}{k},$$

and

$$\left\{\exp\left(-ae^{-z}\right) + \left[1 - \exp\left(-e^{-z}\right)\right]^a\right\}^j = \sum_{k=0}^{\infty} b_{k,j} \exp\left(-ke^{-z}\right),$$

where

$$b_{k,j} = h_k(a, j),$$

is defined in Appendix **A**.

Fourth, we can write,

$$\frac{\exp\left(-jae^{-z}\right)}{\left\{\left[\exp\left(-e^{-z}\right)\right]^a + \left[1 - \exp\left(-e^{-z}\right)\right]^a\right\}^j} = \frac{\sum_{k=0}^{\infty} a_{k,j} \exp\left(-ke^{-z}\right)}{\sum_{k=0}^{\infty} b_{k,j} \exp\left(-ke^{-z}\right)} = \sum_{k=0}^{\infty} c_{k,j} \exp\left(-ke^{-z}\right),$$

where $c_{0,j} = a_{0,j}/b_{0,j}$ and, for $k \geq 1$, we have,

$$c_{k,j} = \frac{1}{b_{0,j}} \left(a_{k,j} - \frac{1}{b_{0,j}} \sum_{r=1}^{k} b_{k,j} c_{k-r,j}\right).$$

Then, the cdf of the TLOLL-Gumbel distribution can be expressed as,

$$F(x) = \sum_{i,k=0}^{\infty} \sum_{j=0}^{2i} (-1)^{i+j} \binom{b}{i} \binom{2i}{j} c_{k,j} \exp\left(-ke^{-z}\right) = \sum_{k=0}^{\infty} d_k \Pi_k(x), \tag{7}$$

where $\Pi_k(x) = \exp\left(-ke^{-z}\right)$ and, $d_k = \sum_{i=0}^{\infty} \sum_{j=0}^{2i} (-1)^{i+j} \binom{b}{i} \binom{2i}{j} c_{k,j}$. The corresponding TLOLL-Gumbel density function follows by differentiating equation (7)

$$f(x) = \sum_{k=0}^{\infty} w_{k+1} \pi_{k+1}(x). \tag{8}$$

where $\pi_{k+1}(x)$ is given by

$$\pi_{k+1}(x) = \frac{(k+1)}{\sigma} \exp\left\{-\left[z + (k+1)\exp\left(-z\right)\right]\right\}.$$

In Table 1, for matching the power series expansion in equation (8) with the density $f(x)$, we produced a table for multiple values of parameters and different values of x to compare the sum to the density. As shown in the Table 1, with increasing the value of the b parameter, the accuracy of the density function estimation is increased. This is due to the fact that in the combination $\binom{b}{i}$ of $F(x)$ in equation (6), more sentences are considered together and, as a result, accuracy increases.

Table 1. Real and Estimated Values of $f(x;a,b,\mu,\sigma)$ for some values of $(x;a,b,\mu,\sigma)$

$(x;a,b,0,1)$	$\hat{f}$	f	$(x;a,b,2,3)$	$\hat{f}$	f
(1;1,5)	0.6561102	0.6263780	(5;1,5)	0.2187034	0.1954593
(1;1,10)	0.6499662	0.6400196	(5;1,10)	0.2166554	0.2133399
(1;2,5)	0.5101664	0.4870044	(5;2,5)	0.1700555	0.1623348
(1;2,10)	0.8484403	0.8483221	(5;2,10)	0.2828134	0.282774
(1;3,5)	0.2111149	0.2096979	(5;3,5)	0.07037165	0.06989931
(1;3,10)	0.4058773	0.4058771	(5;3,10)	0.1352924	0.1352924

3.3. Moments

It is hardly necessary to emphasize the importance of calculating the moments of a random variable in statistical analysis, particularly in applied work. Some key features of a distribution such as skewness and kurtosis can be studied through its moments. The nth moment of X can be determined from (7) as,

$$\begin{aligned}\mu'_n = E(X^n) &= \int_{-\infty}^{\infty} x^n \sum_{k=0}^{\infty} w_{k+1}\pi_{k+1}(x)dx \\ &= \frac{1}{\sigma}\sum_{k=0}^{\infty}(k+1)w_{k+1}\int_{-\infty}^{\infty} x^n \exp\left\{-\left[\frac{x-\mu}{\sigma}+(k+1)\exp\left(-\frac{x-\mu}{\sigma}\right)\right]\right\}dx,\end{aligned}$$

which, on setting $u = \exp\left[-(x-\mu)/\sigma\right]$, it reduces to,

$$E(X^n) = \sum_{k=0}^{\infty}(k+1)w_{k+1}\int_0^{\infty}(\mu-\sigma\log(u))^n \exp\left\{-u(k+1)\right\}du,$$

Using the binomial expansion for $(\mu-\sigma\log(z))^n$, $E(X^n)$ can be expressed as,

$$E(X^n) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(k+1)\binom{n}{i}(-\sigma)^i\mu^{n-i}w_{k+1}\int_0^{\infty}\left[\log(u)\right]^i \exp\left[-u(k+1)\right]du. \tag{9}$$

Using a result by Nadarajah (2006), $I(i,k) = \int_0^{\infty}\left[\log(u)\right]^i \exp\left[-u(k+1)\right]du$ reduces to,

$$I(i,k) = \left(\frac{\partial}{\partial a}\right)^i\left[(k+1)^{-a}\Gamma(a)\right]|_{a=1}. \tag{10}$$

By combining (9) and (10), the nth moment of X becomes,

$$E(X^n) = \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}(k+1)\binom{n}{i}(-\sigma)^i\mu^{n-i}w_{k+1}\left(\frac{\partial}{\partial a}\right)^i\left[(k+1)^{-a}\Gamma(a)\right]|_{a=1}.$$

The measures of skewness and kurtosis of the TLOLL-Gumbel distribution can be obtained as,

$$Skewness(X) = \frac{\mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3}{(\mu'_2-(\mu'_1)^2)^{\frac{3}{2}}},$$

and

$$Kurtosis(X) = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6(\mu'_1)^2\mu'_3 - 3(\mu'_1)^4}{(\mu'_2-(\mu'_1)^2)},$$

respectively. Plots of skewness and kurtosis of the TLOLL-Gumbel distribution are displayed in Figure 2. As shown in this figure, when σ increases, skewness and kurtosis decreases. Also, When μ increases, $skewnessandkurtosis$ does not change and for a and b values, there is no fixed pattern either.

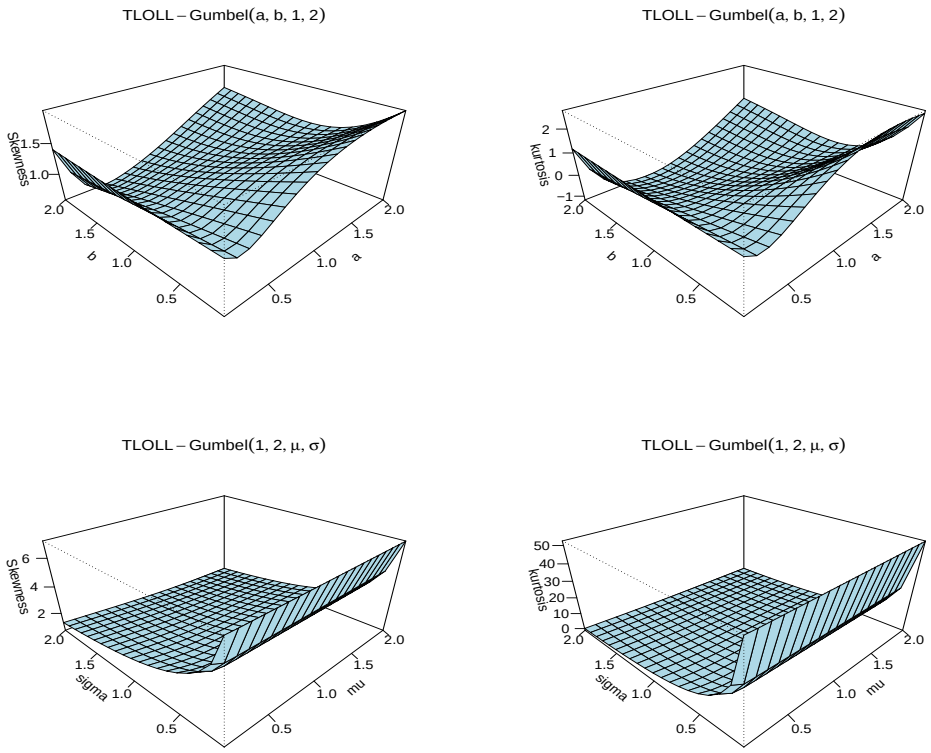

Figure 2. Skewness and Kurtosis for TLOLL-Gumbel.

3.4. Generating Function

The moment generating function (mgf) of X is computed as follows,

$$\begin{aligned} M(t) &= \sum_{k=0}^{\infty} w_{k+1} \int_{\infty}^{\infty} e^{tx} \pi_{k+1}(x) dx, \\ &= \frac{1}{\sigma} \sum_{k=0}^{\infty} (k+1) w_{k+1} \int_{\infty}^{\infty} e^{tx} \exp\left\{ - \left[\frac{x-\mu}{\sigma} + (k+1) \exp\left(- \frac{x-\mu}{\sigma} \right) \right] \right\} dx, \end{aligned}$$

Setting $u = \exp\left[-(x-\mu)/\sigma\right]$, $M(t)$ reduces to,

$$M(t) = \frac{1}{\sigma} \sum_{k=0}^{\infty} (k+1) w_{k+1} \int_{0}^{\infty} u^{-t\sigma} \exp[-(k+1)u] du.$$

Using a result by Cordeiro et al. [6]), we have,

$$I(k) = \int_{0}^{\infty} u^{-t\sigma} \exp[-(k+1)u] du = \Gamma(1-2\sigma)(k+1)^{t\sigma}.$$

and then

$$M(t) = e^{t\mu} \Gamma(1-t\sigma) \sum_{k=0}^{\infty} (k+1)^{t\sigma} w_{k+1}.$$

3.5. *Rényi entropy*

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by,

$$I_\delta(X) = (1-\delta)^{-1}\log\int_\infty^\infty f(x)^\delta hx, \quad \delta > 0 \quad and \quad \delta \neq 1.$$

Then we have,

$$\begin{aligned} f(x)^r &= (\frac{2ab}{\sigma})^\delta \sum_{i,k=0}^{\infty}\sum_{j=0}^{2i}(-1)^{i+j+k}\binom{\delta(b-1)}{i}\binom{2i}{j}\binom{\delta(2a-1)}{k}[\exp(-(z+e^{-z}))]^\delta \\ &\times \frac{\exp(-(\delta(a-1)+aj+k)e^{-z})}{\{[\exp(-e^{-z})]^a + [1-\exp(-e^{-z})]^a\}^{j+2}}, \end{aligned}$$

but,

$$\frac{\exp(-(\delta(a-1)+aj+k)e^{-z})}{\{\exp[-e^{-z})]^a + [1-\exp(-e^{-z}right)]^a\}^{j+2}} = \frac{\sum_{l=0}^\infty a^*_{l,j,k}(\exp(-le^{-z})}{\sum_{l=0}^\infty b^*_{l,j,k}(\exp(-le^{-z})} = \sum_{l=0}^\infty \delta^*_{l,j,k}(\exp(-le^{-z}),$$

where,

$$a^*_{l,j,k} = \sum_{l=0}^\infty (-1)^{j+1}\binom{\delta(a-1)+aj+k}{i}\binom{i}{l}, \quad \delta^*_{0,j,k} = \frac{a^*_{0,j,k}}{b^*_{0,j,k}},$$

and $b^*_{l,j,k} = h_l(a, j+2)$ is deffined in appendix, and for $l \geq 1$,

$$\delta^*_{j,k,l} = \frac{1}{b^*_{j,k,0}}\left[a^*_{j,k,l} - \frac{1}{b^*_{l,j,0}}\sum_{r=0}^{l} b^*_{j,k,r}\delta^*_{j,k,1-r}\right].$$

Then

$$I_\delta(X) = \frac{\delta}{1-\delta}\log(2\alpha\beta) + (1-\delta)^{-1}\log\left[\sum_{i,k,l=0}^\infty\sum_{j=0}^{2i} w^*_{l,i,j,k}I(\delta,l)\right],$$

where,

$$w^*_{l,i,j,k} = (-1)^{i+j}\binom{\delta(b-1)}{i}\binom{2i}{j}\binom{\delta(2\alpha-1)}{l}\delta^*_{l,j,k}(a,\delta,i,j,k),$$

and

$$I(\delta,l) = \int_{-\infty}^\infty \frac{1}{\sigma}\exp\left\{-\left[\delta z + e^{-z}(\delta+l)\right]\right\}dx,$$

which, on setting $u = \exp(-z)$, it reduces to,

$$I(\delta,l) = \int_0^\infty \frac{u^{\delta-1}}{\sigma}\exp\{-u(\delta+l)\}\,dx = \frac{\Gamma(\delta)}{\sigma(\delta+l)^\delta}.$$

In Figure 3 one can see some curves of the entropy function of the TLOLL-Gumbel distribution for some parameters.

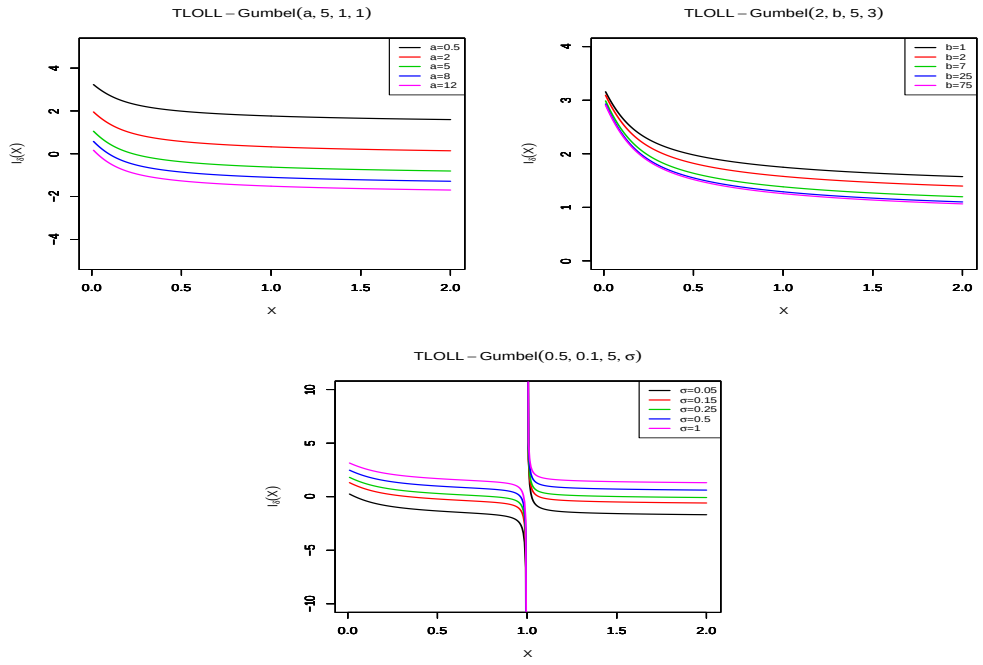

Figure 3. Plots of Rényi entropy of TLOLL-Gumbel distribution for parameters.

3.6. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice.
Suppose $X_1, \ldots, X_n$ is a random sample from any TLOLL-Gumbel distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as,

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i, n-i+1)} f(x)\, F^{i-1}(x)\, [1-F(x)]^{n-i} \\ &= \frac{1}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1}, \end{aligned}$$

We use the result 0.314 of Gradshteyn and Ryzhik [11] for a power series raised to a positive integer n (for $n \geq 1$),

$$\left(\sum_{i=0}^{\infty} a_i\, u^i \right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i,$$

where the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation (with $c_{n,0} = a_0^n$),

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}.$$

We can demonstrate that the density function of the ith order statistic of any TLOLL-Gumbel distribution can be expressed as,

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k}\, \pi_{r+k+1}(x) = \sum_{r,k=0}^{\infty} m_{r,k} \frac{(r+k+1)}{\sigma} \exp\left\{ -\left[z + (r+k+1) \exp\left(-z\right) \right] \right\},$$

where,

$$m_{r,k} = \frac{n!\,(r+1)\,(i-1)!\,c_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\, f_{j+i-1,k}}{(n-i-j)!\,j!},$$

where c_r is given by (7) and the quantities $f_{j+i-1,k}$ can be determined given that $f_{j+i-1,0} = b_0^{j+i-1}$ and recursively we have:

$$f_{j+i-1,k} = (k\,d_0)^{-1} \sum_{m=1}^{k} [m\,(j+i) - k]\,d_m\,f_{j+i-1,k-m}, k \geq 1.$$

4. Inference

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used when constructing confidence intervals and also in test statistics.
The normal approximation for these estimators in large sample theory is easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters from complete samples only by maximum likelihood.
Let Θ be the $(p \times 1)$ parameter vector. Under standard regularity conditions when $n \to \infty$, the distribution of $\hat{\Theta}$ can be approximated by a multivariate normal $N_p(0, J(\hat{\Theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\hat{\Theta})$ is the total observed information matrix evaluated at $\hat{\Theta}$.

4.1. Likelihood of the TLOLL-Gumbel family

Let $x_1, \ldots, x_n$ be the observed values from the TLOLL-Gumbel distribution with parameters a, b, μ and σ. Let $\Theta = (\alpha, \beta, \lambda, \sigma)^\top$ be the $r \times 1$ parameter vector. Then, the log-likelihood function for the vector of parameters $\Theta = (a, b, \mu, \sigma)$, say $L(\Theta)$, is given by To simplify, assume that the distribution and density of Gumbel function is as follows:

$$t_i = G(x_i; \mu, \sigma) = \exp\left(-e^{-z_i}\right) \qquad \& \qquad z_i = \frac{x_i - \mu}{\sigma},$$

and

$$g(x; \mu, \sigma) = \frac{t_i}{\sigma} e^{-z_i},$$

Then, the log-likelihood function can be written as follows:

$$\begin{aligned} \ell(\Theta) = log\,[L(\Theta)] \quad &= \quad n\log 2 + n\log a + n\log b - n\log\sigma - \sum_{i=0}^{n} z_i - a\sum_{i=0}^{n} e^{-z_i} \\ &\quad + (2a-1)\sum_{i=0}^{n}\log(1-t_i) - 3\sum_{i=0}^{n}\log w(z_i) + (b-1)\sum_{i=0}^{n}\log p_i, \end{aligned} \tag{11}$$

where $w(z_i) = [\exp(-e^{-z_i})]^a + [1 - \exp(-e^{-z_i})]^a$ and $p_i = 1 - (\frac{1-t_i}{w(z_i)})^2$.
The log-likelihood function (11) can be maximized either directly by using the SAS (PROC NLMIXED) or the Ox program (sub-routine MaxBFGS) or by solving the nonlinear likelihood equations obtained by differentiating (11). The score vector components, say $U_n(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left(\frac{\partial \ell_n}{\partial a}, \frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \mu}, \frac{\partial \ell_n}{\partial \sigma}\right)^\top = (U_a, U_b, U\mu, U_\sigma)^\top$ are given by,

$$\begin{cases} U_a = \frac{n}{a} + \sum_{i=1}^{n} e^{-z_i} + 2\sum_{i=0}^{n} \log(1-t_i) - 3\sum_{i=1}^{n} \frac{w_i^{(a)}}{w(z_i)} + (b-1)\sum_{i=1}^{n} \frac{p_i^{(a)}}{p_i} = 0 \\ U_b = \frac{n}{b} + \sum_{i=0}^{n} \log p_i = 0 \\ U_\mu = \frac{n}{\sigma} - \frac{a}{\sigma}\sum_{i=0}^{n} e^{-z_i} - (2a-1)\sum_{i=0}^{n} \frac{t_i^{(\mu)}}{t_i} - 3\sum_{i=0}^{n} \frac{w_i^{(\mu)}}{w(z_i)} + (b-1)\sum_{i=1}^{n} \frac{p_i^{(\mu)}}{p_i} = 0 \\ U_\sigma = -\frac{n}{\sigma} + \frac{1}{\sigma}\sum_{i=0}^{n} z_i + \frac{a}{\sigma}\sum_{i=0}^{n} z_i e^{-z_i} - (2a-1)\sum_{i=0}^{n} \frac{t_i^{(\sigma)}}{t_i} - 3\sum_{i=0}^{n} \frac{w_i^{(\sigma)}}{w(z_i)} + (b-1)\sum_{i=1}^{n} \frac{p_i^{(\sigma)}}{p_i} = 0 \end{cases}$$

where,

$$\begin{cases} t_i^{(\mu)} = \frac{\partial t_i}{\partial \mu} = -\frac{1}{\sigma}\exp(-z_i - e^{-z_i}) \\ t_i^{(\sigma)} = \frac{\partial t_i}{\partial \sigma} = -\frac{z_i}{\sigma}\exp(-z_i - e^{-z_i}) \\ w_i^{(a)} = \frac{\partial w_i}{\partial a} = (1-t_i)^a \ln(1-t_i) - exp(-z_i - ae^{-z_i}) \\ w_i^{(\mu)} = \frac{\partial w_i}{\partial \mu} = \frac{-a}{\sigma}\left[exp(-z_i - ae^{-z_i})(1-t_i)^{a-1} - exp(-z_i - ae^{-z_i})\right] \\ w_i^{(\sigma)} = \frac{\partial w_i}{\partial \sigma} = \frac{-az_i}{\sigma^2}\left[exp(-z_i - ae^{-z_i}) - \exp(-z_i - e^{-z_i})(1-t_i)^{a-1}\right] \\ p_i^{(a)} = \frac{\partial p_i}{\partial a} = -2\exp(-2e^{-z_i})\frac{(1-t_i)^a \ln(1-t_i) - exp\left(-z_i - ae^{-z_i}\right)}{\left(exp\left(ae^{-z_i}\right) + (1-t_i)^a\right)^3} \\ p_i^{(\mu)} = \frac{\partial p_i}{\partial \mu} = \frac{\left(\left(\exp(e^{-z_i}) + a - 1\right)(1-t_i)^a \exp(ae^{-z_i}) + (1-a)\exp(e^{-z_i}) + a - 1\right)}{\sigma\left(\exp(e^{-z_i}) - 1\right)\left(\left(1 - \exp(e^{-z_i})\right)^a \exp(ae^{-z_i}) + 1\right)^3} \\ \qquad \times 2\exp(-z_i + 2(a-1)e^{-z_i}) \\ p_i^{(\sigma)} = \frac{\partial p_i}{\partial \sigma} = \frac{\left(e^{e^{-z_i}} - 1\right)^a \left(\exp(e^{-z_i}) + a - 1\right) - (a-1)\exp(e^{-z_i}) + a - 1}{\sigma^2\left(\left(\exp(e^{-z_i}) - 1\right)^a + 1\right)^3 \left(\exp(e^{-z_i}) - 1\right)} \\ \qquad \times 2(\sigma z_i)\exp\left(-\frac{e^{-z_i}((\sigma z_i)e^{z_i} - 2(a-1)\sigma)}{\sigma}\right) \end{cases}$$

Setting the nonlinear system of equations $U_a = U_b = U_\mu = U_\sigma$ and solving them simultaneously yields the MLE of $\Theta = (a, b, \mu, \sigma)^\top$, i.e. $\hat{\Theta} = (\hat{a}, \hat{b}, \hat{\mu}, \hat{\sigma})^\top$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods such as the Newton-Raphson type algorithms.

For interval estimation of (a, b, μ, σ) and hypothesis tests on these parameters, we obtain the observed information matrix since the expected information matrix is very complicated and requires numerical integration. The 4×4 observed information matrix $J(\Theta)$, becomes as follows:

$$J(\Theta) = -\begin{pmatrix} \mathbf{J}_{aa} & \mathbf{J}_{ab} & \mathbf{J}_{a\mu} & \mathbf{J}_{a\sigma} \\ . & \mathbf{J}_{bb} & \mathbf{J}_{b\mu} & \mathbf{J}_{b\sigma} \\ . & . & \mathbf{J}_{\mu\mu} & \mathbf{J}_{\mu\sigma} \\ . & . & . & \mathbf{J}_{\sigma\sigma} \end{pmatrix},$$

whose elements are given in Appendix B.

Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of $(\widehat{\Theta} - \Theta)$ is $N_4(\mathbf{0}, I(\Theta)^{-1})$, where $I(\Theta)$ is the expected information matrix. The multivariate normal $N_4(\mathbf{0}, J(\widehat{\Theta})^{-1})$ distribution, where $I(\Theta)$ is replaced by $J(\widehat{\Theta})$, i.e., the observed information matrix evaluated at $\widehat{\Theta}$, can be used to construct approximate confidence intervals for the individual parameters.

5. Simulation study

In this section, the Maximum likelihood estimators of parameters of purpose density function has been assessed by simulating: $(a, b, \mu, \sigma) = (2, 0.5, 2, 1)$. To examine the performance of the MLEs for the TLOLL-Gumbel distribution, similar to karamikabir et al. [15], we perform a simulation study:

1. Generate r samples of size n from (4) using (3).
2. Compute the MLEs for the r samples, say $(\hat{a}, \hat{b}, \hat{\mu}, \hat{\sigma})$ for $i = 1, 2, \ldots, r$.
3. Compute the standard errors of the MLEs for r samples, say $(s_{\hat{a}}, s_{\hat{b}}, s_{\hat{\mu}}, s_{\hat{\sigma}})$ for $i = 1, 2, \ldots, r$. The standard errors are computed by inverting the observed information matrices.
4. Compute the biases and mean squared errors given by,

$$Bias_{\widehat{\boldsymbol{\theta}}}(n) = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i),$$

$$MSE_{\widehat{\boldsymbol{\theta}}}(n) = \frac{1}{r}\sum_{i=1}^{r}(\widehat{\boldsymbol{\theta}}_i - \boldsymbol{\theta}_i)^2,$$

for $\theta = (a, b, \mu, \sigma)$.

We repeat these steps for $r = 1000$ and $n = 15, 16, \ldots 90$ with different values of (a, b, μ, σ), so computing $Bias_{\widehat{\boldsymbol{\theta}}}(n)$ and $MSE_{\widehat{\boldsymbol{\theta}}}(n)$.
Figure 4, reveals how the four biases and mean squared errors vary with respect to n. The biases and mean squared errors for the estimates decrease to zero when $n \to \infty$ $(n > 50)$, as expected. The reported observations are for only one choice for (a, b, μ, σ) , namely that $(a, b, \mu, \sigma) = (2, 0.5, 2, 1)$.
Figure 4, shows how the four biases and mean squared errors vary with respect to for $(a, b, \mu, \sigma) = (2, 0.5, 2, 1)$. The biases and MSEs for each parameter similar to the previous state decrease to zero when $n \to \infty$.
Finally figure 4 same as before shows how the four biases and MSEs with $(a, b, \mu, \sigma) = (2, 0.5, 2, 1)$ decrease to zero.
The reported observations are for only above cases for (a, b, μ, σ), but the results are similar for a wide range of other choices for (a, b, μ, σ).

6. Empirical illustration

In this section we illustrate the flexibility of the TLOLL-Gumbel model using three data sets. Similar investigations could be performed for other distributions. The computations are performed using the **R** software version 3.3.2 (package AdequacyModel). The **R** codes are in Appendix C. The maximization is performed by the BFGS method. The algorithm used to estimate the model parameters has achieved convergence for all current models.
In addition, we present three applications by fitting the TLOLL-Gumbel model and some famous models. The Akaike information criterion (AIC), Bayesian information criterion (BIC), Cramér–von Mises (W^*), Anderson-Darling (A^*) and KolmogorovCSmirnov (K.S), have been chosen for comparison of models for the first three examples.
The Lomax-Gumbel Distribution (LGu), Gupta et al. [12], The McDonald Gumbel model (McGu), de Brito [4], Marshall-Olkin Gumbel distribution (MOGu), Jose [14], The beta Gumbel distribution (BGu), Nadarajah and Kotz [21], The Beta Exponentiated Gumbel Distribution (BEGu), Ownuk [25], The Kumaraswamy Gumbel distribution (KwGu), Cordeiro et al., [6], The Exponentiated Gumbel Type Distributions (EGu) Nadarajah and Kotz [23], The Zografos-Balakrishnan Gumbel Family of Distributions (ZBGu), Nadarajah et al. [24], The Topp-Leone Gumbel (TLGu) and The odd log-logistic Gumbel (OLLGu) have been selected for comparison in the three examples. The parameters of models have been estimated by the MLE method.

6.1. The Survival times data

This sub-section is related to study of the 72 survival times of guinea pigs injected with difierent doses of tubercle bacilli by Kundu et al. [18] and Leiva et al. [19]. The data represent the survival times of guinea pigs injected with different doses of tubercle bacilli. It is known that guinea pigs have high susceptibility to human tuberculosis and that is why they were used in this study. Here, we are primarily concerned with the animals in the same cage that were under the same regimen. The regimen number is the common logarithm of the number of

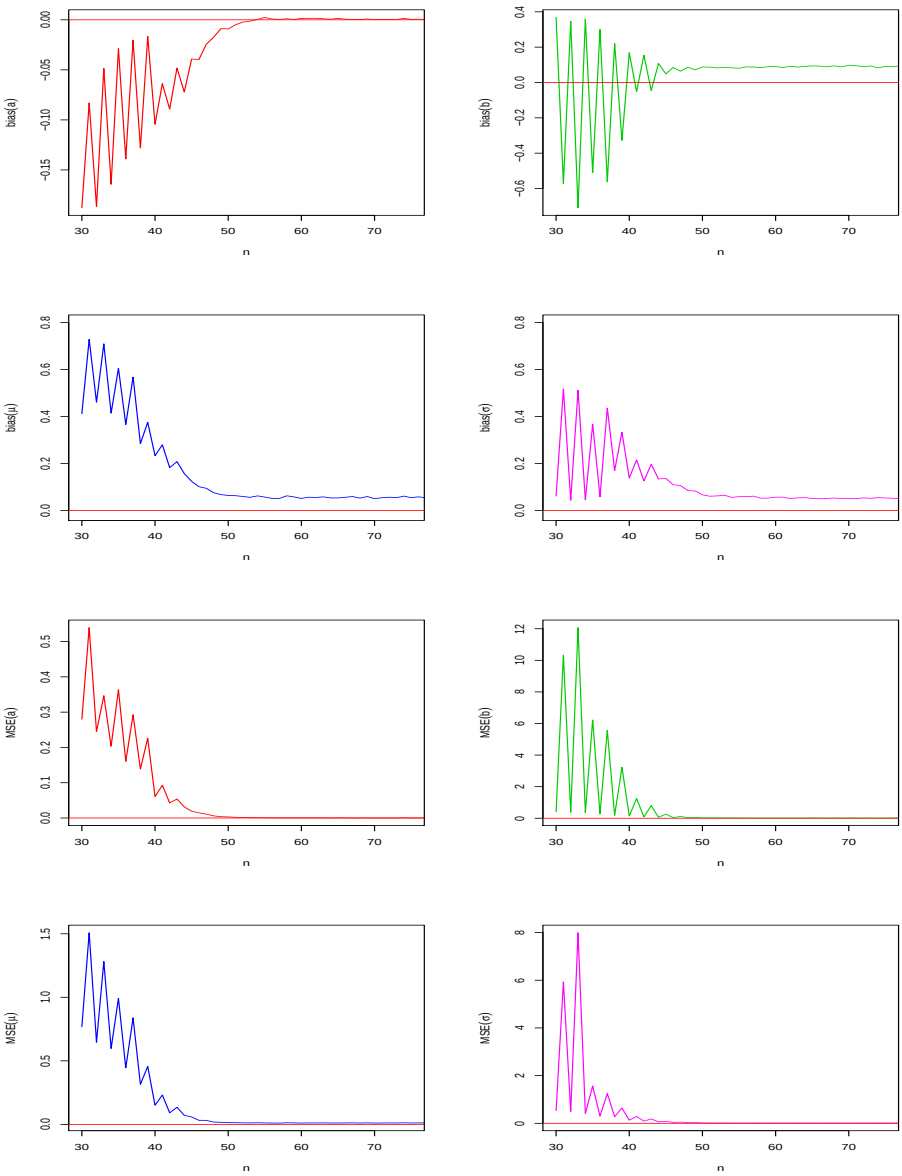

Figure 4. Biases and MSE's of $\hat{a}, \hat{b}, \hat{\mu}, \hat{\sigma}$ versus n when $(a, b, \mu, \lambda) = (2, 0.5, 2, 1)$.

bacillary units in 0.5 ml of challenge solution; i.e., regimen 6.6 corresponds to 4.0×10^6 bacillary units per 0.5 ml ($\log(4.0 \times 10^6) = 6.6$). Corresponding to regimen 6.6, there were 72 observations listed below:
12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.
In the Tables 2 and 3, a summary of the fitted information criteria and estimated MLE's for this data with different models have come, respectively. Models have been sorted from the lowest to the highest value of AIC. As you see,

the TLOLL-Gumbel is selected as the best model with all criteria. The histogram of the Survival times data and the plots of fitted pdf are displayed in Figure 5.

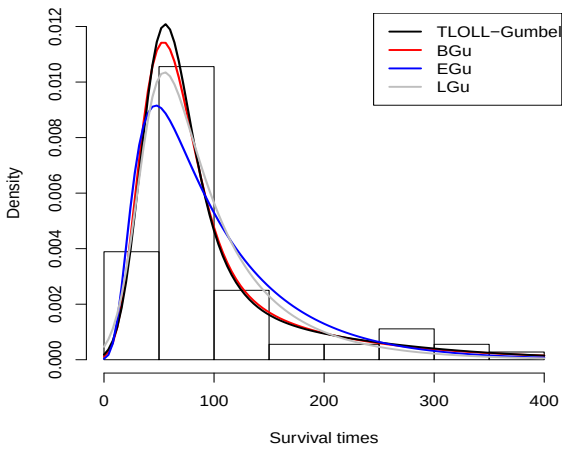

Figure 5. Histogram for Survival times data.

Table 2. Information criteria for the Survival times data.

Model	**AIC**	**BIC**	**W***	**A***	**K-S**
TLOLL-Gumbel	781.56	790.67	0.05	0.31	0.08
BGu	783.03	792.14	0.06	0.36	0.09
EGu	787.51	794.34	0.19	1.01	0.11
LGu	788.85	797.96	0.15	0.83	0.08
KwGu	789.52	798.62	0.19	1.01	0.11
BEGu	789.92	799.03	4.20	21.04	0.85
OLLGu	790.95	797.78	0.23	1.23	0.15
MOGu	791.22	798.05	0.21	1.17	0.09
McGu	791.42	802.80	0.15	0.84	0.29
Gumbel	800.22	804.77	0.42	2.36	0.15
ZBGu	802.98	809.81	0.14	0.88	0.25
TLGu	811.25	818.07	0.58	3.28	0.18

6.2. The strengths of 1.5 cm glass fibers data

This sub-section is related to study of 63 strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. It is obtained from Smith and Naylor [28] and also analysed by Barreto-Souza et al. [3]. The sample are experimental data of the strength of glass fibres of two lengths, 1.5 cm, from the National Physical Laboratory in England. The data set are the following:
0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 ,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.
In the Tables 4 and 5, a summary of the fitted information criteria and estimated MLE's for this data with different models have come, respectively. Models have been sorted from the lowest to the highest value of AIC. As you see,

Table 3. Estimated MLE's and Standard errors for the Survival times data.

Model	**MLE**	**Standard errors**
TLOLL-Gumbel (a, b, μ, σ)	$(0.24, 0.64, 135.07, 34.46)$	$(0.06, 0.23, 17.24, 4.36)$
BGu(a, b, μ, σ)	$(0.057, 0.24, 129.71, 26.49)$	$(0.03, 0.07, 22.94, 3.80)$
EGu(a, μ, σ)	$(0.21, 30.87, 14.69)$	$(0.05, 6.31, 2.38)$
LGu(a, b, μ, σ)	$(0.22, 0.01, 154.24, 69.27)$	$(0.21, 0.01, 30.73, 11.62)$
KwGu(a, b, μ, σ)	$(6.55, 0.20, 3.61, 14.30)$	$(1.07, 0.03, 0.48, 0.87)$
BEGu(a, b, μ, σ)	$(39.75, 0.01, 26.78, 18.16)$	$(28.83, 0.003, 11.19, 9.60)$
OLLGu(a, μ, σ)	$(0.41, 76.15, 25.33)$	$(0.09, 6.36, 3.88)$
MOGu(a, μ, σ)	$(0.09, 141.91, 72.99)$	$(0.08, 33.00, 14.70)$
OLLGu(a, μ, σ)	$(0.09, 141.91, 72.99)$	$(0.08, 33.00, 14.70)$
McGu(a, b, c, μ, σ)	$(10.25, 0.19, 0.21, 15.33, 13.65)$	$(11.28, 0.02, 0.23, 0.12, 0.24)$
MOGu(a, μ, σ)	$(0.09, 141.91, 72.99)$	$(0.08, 33.00, 14.70)$
Gumbel(μ, σ)	$(67.87, 46.95)$	$(5.74, 4.76)$
ZBGu(a, μ, σ)	$(0.74, 87.44, 50.48)$	$(0.51, 4.75, 7.68)$
TLGu(a, μ, σ)	$(2.60, 54.60, 85.06)$	$(1.10, 43.40, 9.69)$

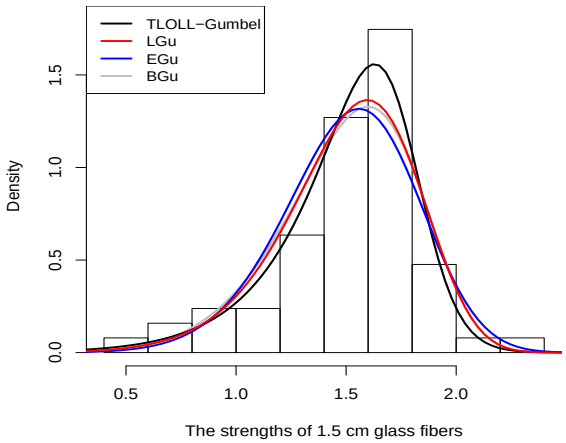

Figure 6. Histogram for the strengths of 1.5 cm glass fibers data.

the TLOLL-Gumbel is selected as the best model with all criteria. The histogram of the strengths of 1.5 cm glass fibers data. and the plots of fitted pdf are displayed in Figure 8.

6.3. The logarithm of the ratio of received light data

This sub-section is related to study of the logarithm of the ratio of received light (logratio) in LIDAR (light detection and ranging) data set presented by Ruppert et al. [26] that include 221 observations. The technique known as LIDAR (light detection and ranging) uses the reflection of laser-emitted light to detect chemical compounds in the atmosphere. The LIDAR technique has proven to be an efficient tool for monitoring the distribution of several atmospheric pollutants of importance

Similar to previous sub-section, in the Tables 6 and 7, a summary of the fitted information criteria and estimated MLE's for this data with different models have come, respectively. As you see, the TLOLL-Gumbel is selected as

Table 4. Information criteria for the strengths of 1.5 cm glass fibers data.

Model	AIC	BIC	W*	A*	K-S
TLOLL-Gumbel	32.76	41.33	0.10	0.57	0.11
LGu	36.47	45.04	0.18	1.01	0.14
EGu	36.73	43.16	0.25	1.36	0.16
BGu	36.83	45.40	0.19	1.07	0.15
KwGu	37.47	46.04	0.21	1.18	0.15
BEGu	38.17	46.75	0.24	1.30	0.16
MOGu	38.18	44.61	0.28	1.53	0.13
OLLGu	38.26	44.69	0.26	1.45	0.13
McGu	38.77	49.49	0.19	1.04	0.17
TLGu	56.03	62.46	0.59	3.25	0.22
ZBGu	61.57	68.00	1.38	7.34	0.76
Gumbel	65.04	69.32	0.75	4.12	0.22

Table 5. Estimated MLE's and Standard errors for the Sstrengths of 1.5 cm glass fibers data.

Model	MLE	Standard errors
TLOLL-Gumbel (a, b, μ, σ)	$(10.02, 0.47, 1.12, 1.91)$	$(24.24, 0.27, 1.81, 4.80)$
LGu(a, b, μ, σ)	$(55.66, 0.02, 6.05, 2.13)$	$(147.04, 0.03, 2.25, 0.75)$
EGu(a, μ, σ)	$(74.24, 3.44, 1.24)$	$(60.45, 0.55, 0.23)$
BGu(a, b, μ, σ)	$(0.53, 175.02, 3.65, 1.13)$	$(0.35, 263.87, 0.96, 0.47)$
KwGu(a, b, μ, σ)	$(1.72, 230.54, 3.39, 1.52)$	$(32.21, 291.87, 28.61, 0.35)$
BEGu(a, b, μ, σ)	$(0.61, 83.04, 3.09, 0.97)$	$(0.43, 69.14, 0.61, 0.38)$
MOGu(a, μ, σ)	$(217.94, 0.60, 0.17)$	$(272.65, 0.20, 0.02)$
OLLGu(a, μ, σ)	$(80.54, -5.48, 19.14)$	$(171.97, 14.96, 40.83)$
McGu(a, b, c, μ, σ)	$(0.54, 200.51, 1.07, 3.66, 1.16)$	$(0.36, 320.97, 14.95, 16.31, 0.51)$
TLGu(a, μ, σ)	$(0.49, 1.79, 0.45)$	$(0.39, 0.23, 0.06)$
ZBGu(a, μ, σ)	$(0.23, 1.96, 0.40)$	$(0.03, 0.01, 0.02)$
Gumbel(μ, σ)	$(1.33, 0.38)$	$(0.05, 0.03)$

the best model with all criteria. The histogram of the logarithm of the ratio of received light data and the plots of fitted pdf are displayed in Figure 7.

Table 6. Information criteria for the logarithm of the ratio of received light data.

Model	AIC	BIC	W*	A*	K-S
TLOLL-Gumbel	-49.05	-35.46	1.94	10.57	0.22
BGu	4.01	17.60	2.30	12.65	0.28
McGu	5.29	22.28	2.22	12.16	0.46
BEGu	31.63	45.22	2.49	13.79	0.26
LGu	46.92	60.51	2.65	14.73	0.24
EGu	52.66	62.86	2.68	14.89	0.24
KwGu	52.70	66.29	2.67	14.85	0.24
MOGu	88.48	98.68	2.82	15.69	0.22
OLLGu	92.79	102.98	2.89	16.15	0.22
ZBGu	99.74	109.94	3.71	20.36	1
TLGu	100.83	111.03	2.79	15.50	0.25
Gumbel	125.12	131.91	2.88	16.03	0.24

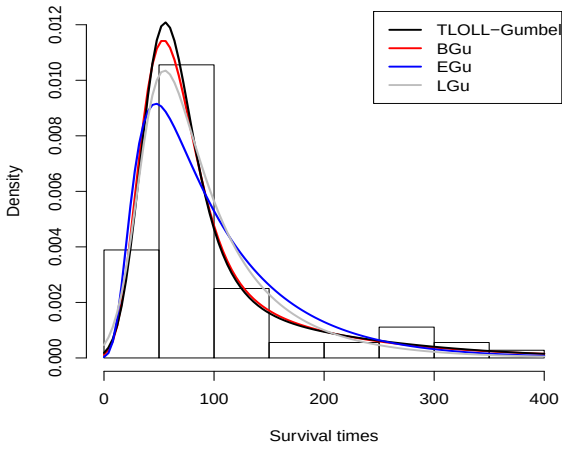

Figure 7. Histogram for the logarithm of the ratio of received light data.

Table 7. Estimated MLE's and Standard errors for the logarithm of the ratio of received light data.

Model	**MLE**	**Standard errors**
TLOLL-Gumbel (a, b, μ, σ)	$(18.40, 0.10, -0.30, 0.88)$	$(0.01, 0.01, 0.003, 0.003)$
BGu(a, b, μ, σ)	$(0.11, 700.05, 1.02, 0.50)$	$(0.02, 338.38, 0.11, 0.05)$
McGu(a, b, c, μ, σ)	$(0.11, 773.54, 4.19, 0.30, 0.51)$	$(0.02, 377.25, 47.12, 5.70, 0.05)$
BEGu(a, b, μ, σ)	$(0.19, 256.97, 0.77, 0.51)$	$(0.02, 77.04, 0.03, 0.02)$
LGu(a, b, μ, σ)	$(59.32, 0.01, 3.92, 1.92)$	$(37.26, 0.01, 0.54, 0.18)$
EGu(a, μ, σ)	$(202.85, 1.98, 1.30)$	$(116.09, 0.35, 0.13)$
KwGu(a, b, μ, σ)	$(1.89, 320.03, 1.37, 1.39)$	$(10.49, 197.82, 7.70, 0.14)$
MOGu(a, μ, σ)	$(24.47, -0.84, 0.18)$	$(16.73, 0.09, 0.01)$
OLLGu(a, μ, σ)	$(62.31, -5.68, 14.77)$	$(76.70, 6.65, 18.15)$
ZBGu(a, μ, σ)	$(12.57, -1.43, 0.09)$	$(3.96, 0.16, 0.02)$
TLGu(a, μ, σ)	$(0.23, 0.11, 0.33)$	$(0.10, 0.10, 0.02)$
Gumbel(μ, σ)	$(-0.44, 0.29)$	$(0.02, 0.01)$

7. Conclusions

In this article, we introduce the Topp-Leone odd log-logistic Gumbel distributions. Some of its various properties including explicit expansions, moment, entropies, order statistics and maximum likelihood estimator, are provided. The TLOLL-Gumbel is applied to fit three real data sets. Three applications to real data demonstrate the importance of the TLOLL-Gumbel family. These applications show that the TLOLL-Gumbel has the ability to fit skewed (left or right) and heavy-tailed data due to its flexibility. We can conclude that the TLOLL-Gumbel distribution is a very suitable model to the current data and it was always one of the best models. The results of tables and figures illustrate the importance of the new distribution to analyze of real data. Therefore, using of TLOLL-Gumbel distribution in the real examples is suggested.

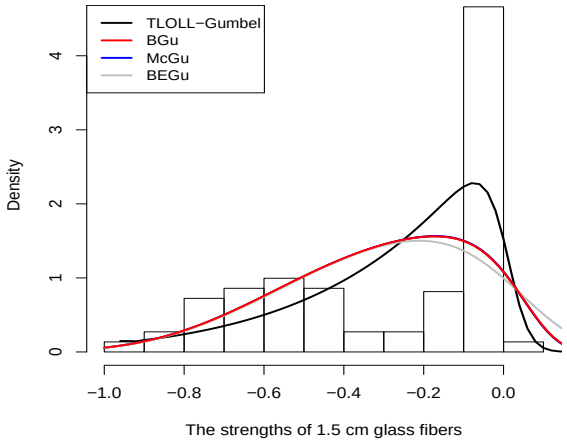

Figure 8. Histogram for the strengths of 1.5 cm glass fibers data.

Acknowledgement

The authors wish to thank the Editor for the thorough reading.

Appendix A: Two useful power series

We present two power series required for the algebraic developments in Section 2.
First, expanding z^λ in Taylor series, we can write,

$$z^\lambda = \sum_{k=0}^{\infty} (\lambda)_k \, (z-1)^k / k! = \sum_{i=0}^{\infty} f_i \, z^i, \tag{12}$$

where

$$f_i = f_i(\lambda) = \sum_{k=i}^{\infty} \frac{(-1)^{k-i}}{k!} \binom{k}{i} (\lambda)_k,$$

and $(\lambda)_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$ denotes the descending factorial.
Second, we obtain an expansion for $[G(x)^a + \bar{G}(x)^a]^c$ as,

$$[G(x)^a + \bar{G}(x)^a] = \sum_{j=0}^{\infty} t_j \, G(x)^j,$$

where $t_j = t_j(a) = a_j(a) + (-1)^j \binom{a}{j}$. Then, using (12) , we have,

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{i=0}^{\infty} f_i \left(\sum_{j=0}^{\infty} t_j \, G(x)^j \right)^i,$$

where $f_i = f_i(c)$.
Finally, we obtain,

$$[G(x)^a + \bar{G}(x)^a]^c = \sum_{j=0}^{\infty} h_j(a,c)\, G(x)^j,$$

where $h_j(a,c) = \sum_{i=0}^{\infty} f_i\, m_{i,j}$ and (for $i \geq 0$) $m_{i,j} = (j\, t_0)^{-1} \sum_{m=1}^{j} [m(j+1) - j]\, t_m\, m_{i,j-m}$ (for $j \geq 1$) and $m_{i,0} = t_0^i$.

Appendix B

The elements of the observed information matrix $J(\theta)$ for the parameters (a, b, μ, σ) are given by the following:

$$\begin{aligned}
J_{aa} &= \frac{\partial^2 \ell(\Theta)}{\partial a^2} = -\frac{n}{a^2} - 3\sum_{i=1}^{n} \frac{w_i^{(aa)}}{w(z_i)} + (b-1)\sum_{i=1}^{n} \frac{p_i^{(aa)}}{p_i} \\
J_{ab} &= \frac{\partial^2 \ell(\Theta)}{\partial a \partial b} = \sum_{i=1}^{n} \frac{p_i^{(a)}}{p_i} \\
J_{a\mu} &= \frac{\partial^2 \ell(\Theta)}{\partial a \partial \mu} = \sigma^{-1}\sum_{i=1}^{n} e^{-z_i} + 2\sum_{i=0}^{n} \frac{t_i^{(\mu)}}{1-t_i} - 3\sum_{i=1}^{n} \frac{w_i^{(a\mu)} w_i - w_i^{(a)} w_i^{(\mu)}}{w_i^2} \\
&\quad +(b-1)\sum_{i=1}^{n} \frac{p_i^{(a\mu)} p_i - p_i^{(a)} p_i^{(\mu)}}{p_i^2} \\
J_{a\sigma} &= \frac{\partial^2 \ell(\Theta)}{\partial a \partial \sigma} = \sigma^{-1}\sum_{i=1}^{n} z_i e^{-z_i} + 2\sum_{i=0}^{n} \frac{t_i^{(\sigma)}}{1-t_i} - 3\sum_{i=1}^{n} \frac{w_i^{(a\sigma)} w_i - w_i^{2(a)}}{w_i^2} \\
&\quad +(b-1)\sum_{i=1}^{n} \frac{p_i^{(a\sigma)} p_i - p_i^{2(a)}}{p_i^2} \\
J_{bb} &= \frac{\partial^2 \ell(\Theta)}{\partial b^2} = -\frac{n}{b^2} + \sum_{i=0}^{n} \frac{p_i^{(b)}}{p_i} \\
J_{b\mu} &= \frac{\partial^2 \ell(\Theta)}{\partial b \partial \mu} = \sum_{i=0}^{n} \frac{p_i^{(\mu)}}{p_i} \\
J_{b\sigma} &= \frac{\partial^2 \ell(\Theta)}{\partial b \partial \sigma} = \sum_{i=0}^{n} \frac{p_i^{(\sigma)}}{p_i} \\
J_{\mu\mu} &= \frac{\partial^2 \ell(\Theta)}{\partial \mu^2} = -a\sigma^{-2}\sum_{i=0}^{n} e^{-z_i} - (2a-1)\sum_{i=0}^{n} \frac{t_i^{(\mu\mu)} t_i - t_i^{2(\mu)}}{t_i^2} \\
&\quad -3\sum_{i=0}^{n} \frac{w_i^{(\mu\mu)} w_i - w_i^{2(\mu)}}{w_i^2} + (b-1)\sum_{i=1}^{n} \frac{p_i^{(\mu\mu)} p_i - p_i^{2(\mu)}}{p_i^2} \\
J_{\mu\sigma} &= \frac{\partial^2 \ell(\Theta)}{\partial \mu \partial \sigma} = -\frac{n}{\sigma^2} - a\sigma^{-2}\sum_{i=0}^{n} z_i e^{-z_i} - (2a-1)\sum_{i=0}^{n} \frac{t_i^{\mu\sigma} t_i - t_i^{\mu} t_i^{\sigma}}{t_i^2} \\
&\quad -3\sum_{i=0}^{n} \frac{w_i^{\mu\sigma} w_i - w_i^{\mu} w_i^{\sigma}}{w_i^2} + (b-1)\sum_{i=1}^{n} \frac{p_i^{\mu\sigma} p_i - p_i^{\mu} p_i^{\sigma}}{t_p^2} \\
J_{\sigma\sigma} &= \frac{\partial^2 \ell(\Theta)}{\partial \sigma^2} = \frac{n}{\sigma^2} + 2\sigma^{-2}\sum_{i=0}^{n} z_i + a\sum_{i=0}^{n} -\frac{\left(\sigma^{-1} z_i\right)\left(2\sigma - \sigma^{-1} z_i\right) \mathrm{e}^{-\frac{x-\mu}{\sigma}}}{\sigma^4}
\end{aligned}$$

$$-(2a-1)\sum_{i=0}^{n}\frac{t_i^{(\sigma\sigma)}t_i-t_i^{2(\sigma)}}{t_i^2}-3\frac{w_i^{(\sigma\sigma)}w_i-w_i^{2(\sigma)}}{w_i^2}$$
$$+(b-1)\frac{p_i^{(\sigma\sigma)}p_i-p_i^{2(\sigma)}}{p_i^2}$$

where,

$$t_i^{(\mu\mu)} = \frac{\partial^2 t_i}{\partial\mu^2}=\sigma^{-2}\left(\mathrm{e}^{z_i}-1\right)\mathrm{e}^{z_i-\mathrm{e}^{z_i}}$$
$$t_i^{(\sigma\sigma)} = \frac{\partial^2 t_i}{\partial\sigma^2}=\sigma^{-4}\left(\sigma z_i\right)\left(\left(\sigma z_i\right)\mathrm{e}^{z_i}-2\sigma-z_i\right)\mathrm{e}^{z_i-\mathrm{e}^{z_i}}$$
$$t_i^{(\mu\sigma)} = \frac{\partial^2 t_i}{\partial\mu\partial\sigma}=\sigma^{-3}\left(\left(\sigma z_i\right)\mathrm{e}^{z_i}-\sigma-z_i\right)\mathrm{e}^{z_i-\mathrm{e}^{z_i}}$$
$$w_i^{(aa)} = \frac{\partial^2 w_i}{\partial a^2}=\mathrm{e}^{\frac{2(\sigma z_i)}{\sigma}-\mathrm{e}^{z_i}a}+\left(1-t_i\right)^a\ln^2\left(1-t_i\right)$$
$$w_i^{(a\mu)} = \frac{\partial^2 w_i}{\partial a\partial\mu}=-\sigma^{-1}\left[a\mathrm{e}^{z_i-\mathrm{e}^{z_i}}\left(1-t_i\right)^{a-1}\ln\left(1-t_i\right)-\left(\frac{a\mathrm{e}^{z_i}}{\sigma}-\frac{1}{\sigma}\right)\mathrm{e}^{z_i-a\mathrm{e}^{z_i}}-\frac{\mathrm{e}^{z_i-\mathrm{e}^{z_i}}\left(1-t_i\right)^{a-1}}{\sigma}\right]$$
$$w_i^{(a\sigma)} = \frac{\partial^2 w_i}{\partial a\partial\sigma}=-\frac{\left(\sigma z_i\right)\mathrm{e}^{z_i-a\mathrm{e}^{z_i}}\left(a\left(1-t_i\right)^a\mathrm{e}^{a\mathrm{e}^{z_i}}\ln\left(1-t_i\right)+\left(1-t_i\right)^a\mathrm{e}^{a\mathrm{e}^{z_i}}+\left(a\mathrm{e}^{z_i}-1\right)\mathrm{e}^{\mathrm{e}^{z_i}}-a\mathrm{e}^{z_i}+1\right)}{\sigma^2\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)}$$
$$w_i^{(\mu\sigma)} = \frac{\partial^2 w_i}{\partial\mu\partial\sigma}=\sigma^{-3}\Big[\sigma\left(a\mathrm{e}^{z_i-\mathrm{e}^{z_i}}\left(1-t_i\right)^{a-1}-a\mathrm{e}^{z_i-a\mathrm{e}^{z_i}}\right)+a\left(a\left(\sigma z_i\right)\mathrm{e}^{z_i}-z_i\right)\mathrm{e}^{z_i-a\mathrm{e}^{z_i}}$$
$$-a\left(\left(\sigma z_i\right)\mathrm{e}^{z_i}-z_i\right)\mathrm{e}^{z_i-\mathrm{e}^{z_i}}\left(1-t_i\right)^{a-1}+(a-1)\,a\left(\sigma z_i\right)\mathrm{e}^{\frac{2(\sigma z_i)}{\sigma}-2\mathrm{e}^{z_i}}\left(1-t_i\right)^{a-2}\Big]$$
$$p_i^{(aa)} = \frac{\partial^2 p_i}{\partial a^2}=\left(\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+1\right)^4\right)^{-1}2\Big(\left(\ln\left(\mathrm{e}^{t_i}\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\right)\left(\ln\left(\mathrm{e}^{t_i}\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\right)\left(2\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a-1\right)\right.\right.$$
$$\left.\left.-6\mathrm{e}^{z_i}\right)-\mathrm{e}^{\frac{2(\sigma z_i)}{\sigma}}\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+2\mathrm{e}^{\frac{2(\sigma z_i)}{\sigma}}\Big)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^2\mathrm{e}^{2\mathrm{e}^{z_i}(a-1)}$$
$$p_i^{(a\mu)} = \frac{\partial^2 p_i}{\partial a\partial\mu}=-\left(\sigma\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+1\right)^4\right)^{-1}2\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\mathrm{e}^{\frac{2(a-1)\sigma\mathrm{e}^{z_i}-\mu+x}{\sigma}}\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a\right.$$
$$\times\Big(\left(-a\left(3\mathrm{e}^{\mathrm{e}^{z_i}}-2\right)+2\left(a-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a-2\right)\ln\left(\mathrm{e}^{-\mathrm{e}^{z_i}}\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\right)-\mathrm{e}^{z_i}\left(a\left(\mathrm{e}^{\mathrm{e}^{z_i}}+2\right)-2\right)$$
$$\left.+\mathrm{e}^{\mathrm{e}^{z_i}}-\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a\right)+\mathrm{e}^{\mathrm{e}^{z_i}}+2\mathrm{e}^{z_i}\left(a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+1\right)-2\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a-1\Big)$$
$$p_i^{(a\sigma)} = \frac{\partial^2 p_i}{\partial a\partial\sigma}=-\left(\sigma^2\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+1\right)^4\right)^{-1}2\left(\sigma z_i\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\mathrm{e}^{\frac{2(a-1)\sigma\mathrm{e}^{z_i}+\sigma z_i}{\sigma}}\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a\right.$$
$$\times\Big(\left(-a\left(3\mathrm{e}^{\mathrm{e}^{z_i}}-2\right)+2\left(a-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a-2\right)\ln\left(\mathrm{e}^{t_i}\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\right)t_i\left(a\left(\mathrm{e}^{\mathrm{e}^{z_i}}+2\right)-2\right)+\mathrm{e}^{\mathrm{e}^{z_i}}$$
$$\left.-\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a\right)+\mathrm{e}^{\mathrm{e}^{z_i}}+2\mathrm{e}^{z_i}\left(a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+1\right)-2\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a-1\Big)$$
$$p_i^{(\mu\mu)} = \frac{\partial^2 p_i}{\partial\mu^2}=\Big(\sigma^2\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+1\right)^4\Big)^{-1}2\Big(a\left(\mathrm{e}^{-z_i}\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+1\right)-a\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a-2\right)\right)\mathrm{e}^{2\mathrm{e}^{z_i}}$$
$$-\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+1\right)\left((a-1)\left(\mathrm{e}^{-z_i}-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+(2a-1)\,\mathrm{e}^{-z_i}+(2a-1)^2\right)\mathrm{e}^{\mathrm{e}^{z_i}}$$
$$+(a-1)\left(\mathrm{e}^{-z_i}+2a-2\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{2a}+2\left(a-1\right)\left(\mathrm{e}^{-z_i}+2a-2\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+(a-1)\,\mathrm{e}^{-z_i}$$
$$+2a^2-4a+2\Big)\exp\left(-2\mathrm{e}^{z_i}\left(\left(-z_i\right)\mathrm{e}^{-z_i}-(a-1)\right)\right)$$
$$p_i^{(\mu\sigma)} = \frac{\partial^2 p_i}{\partial\mu\partial\sigma}=\Big(\sigma^3\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+1\right)^4\Big)^{-1}2\left(-\sigma z_i\right)\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a+1\right)$$
$$\left(-a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+(a-1)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a-1\right)\exp\left(\sigma^{-1}\left(2\left(a-1\right)\sigma\mathrm{e}^{z_i}+\sigma z_i\right)+\mathrm{e}^{z_i}z_i\right)$$
$$+6a\sigma z_i\left(-a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+(a-1)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^a\exp\left(2\left(a-1\right)\mathrm{e}^{z_i}+\sigma z_i+\mathrm{e}^{z_i}z_i\right)$$

$$
\begin{aligned}
&+\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}+1\right)\left(2\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\left((a-1)\,a\,(-\sigma z_i)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a-1}\mathrm{e}^{\mathrm{e}^{z_i}z_i}-a\,(-\sigma z_i)\,\mathrm{e}^{\mathrm{e}^{z_i}z_i}\right)\right.\\
&\left.\times\exp\left(2\,(a-1)\,\sigma\mathrm{e}^{z_i}+\sigma z_i\right)\right)+2\left(2\,(a-1)\,(-\sigma z_i)\,\mathrm{e}^{z_i}-z_i\right)\\
&\times\left(-a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+(a-1)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\exp\left(2\,(a-1)\,\mathrm{e}^{z_i}+z_i\right)\\
&-2\sigma\left(-a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+(a-1)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\exp\left(2\,(a-1)\,\mathrm{e}^{z_i}+z_i\right)\\
p_i^{(\sigma\sigma)} &= \frac{\partial^2 p_i}{\partial\sigma^2}=-\left(\sigma^4\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}+1\right)^{4}\right)^{-1}2\,(\sigma z_i)^2\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}+1\right)\\
&\left(-a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+(a-1)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}-1\right)\exp\left(\frac{2\,(a-1)\,\sigma\mathrm{e}^{z_i}+\sigma z_i}{\sigma}+\mathrm{e}^{z_i}+z_i\right)+6a\,(\sigma z_i)^2\\
&\times\left(-a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+(a-1)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}\exp\left(\frac{2\,(a-1)\,\sigma\mathrm{e}^{z_i}+\sigma z_i}{\sigma}+\mathrm{e}^{z_i}+z_i\right)\\
&+\left(\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}+1\right)\left(2\,(\sigma z_i)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\left(a\,(\sigma z_i)\,\mathrm{e}^{\mathrm{e}^{z_i}+z_i}-(a-1)\,a\,(\sigma z_i)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a-1}\mathrm{e}^{\mathrm{e}^{z_i}+z_i}\right)\right.\\
&\left.\times\exp\left(2\sigma^{-1}\,(a-1)\,\sigma\mathrm{e}^{z_i}+z_i\right)+2\,(\sigma z_i)\left(-2\,(a-1)\,(\sigma z_i)\,\mathrm{e}^{z_i}-z_i\right)\right.\\
&\times\left(-a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+(a-1)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\exp\left(2\,(a-1)\,\mathrm{e}^{z_i}+z_i\right)\\
&\left.-4\,(\sigma z_i)\,\sigma\left(-a\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)+(a-1)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)^{a}-1\right)\left(\mathrm{e}^{\mathrm{e}^{z_i}}-1\right)\exp\left(2\,(a-1)\,\mathrm{e}^{z_i}+z_i\right)\right)
\end{aligned}
$$

Appendix C: R code

The program is developed in R to obtain the value of CDF function:

```
F = function(x,par){
 a <-par[1];b<-par[2];mu<-par[3];sigma<-par[4]
 z = (x-mu)/sigma
 G = exp(-exp(-z))
 Gbar = 1-G
 D = Gbar^a/(G^a+Gbar^a)
 A = (1-D^2)^b
 return(A)}
```

PDF function:

```
f = function(x,par){
 a <-par[1];b<-par[2];mu<-par[3];sigma<-par[4]
 z = (x-mu)/sigma
 g = (1/sigma) * exp(-(z+exp(-z)))
 G = exp(-exp(-z))
 Gbar = 1-G
 N = 2*a*b*g*(G^(a-1))*(Gbar^(2*a-1))
 M = (G^a+Gbar^a)^3
 D = (Gbar^a/(G^a+Gbar^a))
 O = (1-D^2)^(b-1)
 A = (N/M)*O
 return(A)}
```

The program is developed in R to obtain the value of skewness:

```
library(moments)
a=seq(0.1,2,.1)
```

```
b=seq(0.1,2,.1)
G_skew <- function(a, b) {
x <- seq(0, 10, le=2*10^4)
U = f(x,c(a,b,1,2))
Skew=skewness(U)
return(Skew)
}
G_skew <- Vectorize(G_skew, c('a', 'b'))
Skew <- outer(a,b,G_skew)
```

The program is developed in R to generate of TLOLL-G distribution (for example TLOLL-G(2,0.5,2,1):

```
r=5000
n=seq(30,90,1)
p_a=2; p_b=0.5; p_mu=2; p_sigma=1
sim = c(p_a,p_b,p_mu,p_sigma)

F_x = function(x) F(x,sim)
inverse = function(u, lower, upper) {
uniroot((function (x) F_x(x)- u), lower = lower, upper = upper)$root}
X_generator <- function(n, lower, upper){
U <- runif(n,0,1)
X <- c()
for (i in 1:n){
X[i]=inverse(U[i], lower, upper)}
return(X)}
```

The program is developed in R to obtain the value of likelihood function:

```
LikFun_f <- function(x,par){
     a <- par[1];b <- par[2];mu <- par[3];sigma <- par[4]
     z <- (x-mu)/sigma
     g = (1/sigma) * exp(-(z+exp(-z)))
     G = exp(-exp(-z))
     Gbar = 1-G
     N = 2*a*b*g*G^(a-1)*Gbar^(2*a-1)
     M = (G^a+Gbar^a)^3
     D = (Gbar^a/(G^a+Gbar^a))
     O = (1-D^2)^(b-1)
     A = prod((N/M)*O)
return(A)}
```

The program is developed in R to compute the value of Bias and MSE with r iterate and sample size n (for example for a parameter):

```
for (k in 1:length(n)){
 for (j in 1:r){
  X = X_generator(n[k],-500,500)
  MLE = optim(par = sim,fn=LikFun_f,x=X)
  a_mle[j]= MLE$par[1]
  bias1_a[j] = a_mle[j]-p_a
  MSE1_a[j]=(bias1_a[j])^2}
 bias_a[k]= mean(bias1_a)
 MSE_a[k]= mean(MSE1_a)}
```

REFERENCES

1. T. Andrade, H. Rodrigues, M. Bourguignon, and G. M. Cordeiro, *The Exponentiated Generalized Gumbel Distribution*, Revista Colombiana de Estadłstica, vol. 38(1), pp. 123–143, 2015.
2. J. Beirlant, Y. Goegebeur, J. Segers, and J. Teugels, D. De Waal, and C. Ferro, *Statistics of Extremes: Theory and Applications*, West Sussex, England: John Wiley and Sons Ltd, 2006.
3. W. Barreto-Souza, A. H. S. Santos, and G. M. Cordeiro, *The beta generalized exponential distribution*, Journal of Statistical Computation and Simulation, vol. 80(2), pp. 159–172, 2010.
4. E. Brito, G. O. Silva, G. M. Cordeiro, and C. G. B. Demtrio, *The McDonald Gumbel model,* Communications in Statistics - Theory and Methods, vol. 45(11), pp. 3367–3382, 2016.
5. E. Brito, G. M. Cordeiro, H. M. Yousof, M. Alizadeh, and G. O. Silva, *The ToppCLeone odd log-logistic family of distributions*, Journal of Statistical Computation and Simulation, vol. 87(15), pp. 3040–3058, 2017.
6. G. M. Cordeiro, S. Nadarajah, and E. M. M. Ortega, *The Kumaraswamy Gumbel distribution*, Statistical Methods and Applications, vol. 21, pp. 139–168, 2012.
7. A. I. Genec, *Moments of order statistics of Topp-Leone distribution*, Statistical Papers, vol. 53(1), pp. 117–131, 2012.
8. A. I. Genec, *Estimation of P($X > Y$) with Topp-Leone distribution.*, Journal of Statistical Computation and Simulation, vol. 83(2), pp. 326–339, 2013.
9. M. E. Ghitany, S. Kotz, and M. Xie, *On some reliability measures and their stochastic orderings for the Topp-Leone distribution*, Journal of Applied Statistics, vol. 32(7), pp. 715–722, 2005.
10. E. J. Gumbel, *Statistics of Extremes*, New York: Columbia University Press, 1958.
11. I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, series, and products*, Seventh ed. San Diego: Academic Press, 2007.
12. J. Gupta, M. Garg, and M. Gupta, *The Lomax-Gumbel Distribution*, Palestine Journal of Mathematics, vol. 5(1), pp. 35–42, 2016.
13. N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*, Vol. 2 (2nd ed.). New York: John Wiley and Sons, Inc, 1995.
14. K. K. Jose, *Marshall-Olkin Family of Distributions and their applications in reliability theory, time series modeling and stress-strength analysis*, Int. Statistical Inst.: Proc. 58th World Statistical Congress, 2011, Dublin (Session CPS005), pp. 3918–3923, 2011.
15. H. Karamikabir, M. Afshari, M. Alizadeh, and G. G. Hamedani, *A new extended generalized Gompertz distribution with statistical properties and simulations*, Communications in Statistics - Theory and Methods, Published online, 2019.
16. S. Kotz, and S. Nadarajah, *Extreme Value Distributions*, London, Imperial College Press, 2000.
17. S. Kotz, and E. Seier, *Kurtosis of the Topp-Leone distributions*, InterStat, pp. 1–15, 2007.
18. D. Kundu, N. Kannan, and N. Balakrishnan, *On the hazard function of Birnbaum-Saunders distribution and associated inference*, Computational Statistics and Data Analysis, vol. 52, pp. 2692–2702, 2008.
19. V. Leiva, M. Barros, and G. A. Paula, *Generalized Birnbaum-Saunders Models Using R. XI Escola de Modelos de Regressao*, Recife, Brazil, 2009.
20. S. Nadarajah, and S. Kotz, *Moments of some J-shaped distributions*, Journal of Applied Statistics, vol. 30(3), pp. 311-317, 2003.
21. S. Nadarajah, and S. Kotz, *The beta Gumbel distribution*, Mathematical Problems in Engineering, vol. 2004(4), pp. 323–332, 2004.
22. S. Nadarajah, *The exponentiated Gumbel distribution with climate application*, Environmetrics, vol. 17, 13–23, 2006.
23. S. Nadarajah, and S. Kotz, *The Exponentiated Type Distributions*, Acta Applicandae Mathematicae, vol. 92, pp. 97–111, 2006.
24. S. Nadarajah, G. M. Cordeiro, and E. M. M. Ortega, *The Zografos Balakrishnan-G Family of Distributions: Mathematical Properties and Applications*, Communications in Statistics - Theory and Methods, vol. 44, 186–215, 2014.
25. J. Ownuk, *The Beta Exponentiated Gumbel Distribution*, Journal of The Iranian Statistical Society, vol. 14(2), pp. 1–14, 2015.
26. D. Ruppert, M. P. Wand, and R. J. Carroll, *Semiparametric regression*, Cambridge university press, Number 12, 2003.
27. T. N. Sindhu, M. Saleem, and M. Aslam, *Bayesian estimation for Topp-Leone distribution under trimmed samples*, Journal of Basic and Applied Scientific Research, vol. 3, pp. 347–360, 2013.
28. R. L. Smith, and J. C. Naylor, *A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution*, Journal of Applied Statistics, vol. 36, pp. 358–369, 1987.
29. C. W. Topp, and F. C. Leone, *A family of J-shaped frequency functions*, Journal of the American Statistical Association, vol. 50(269), pp. 209–219, 1955.
30. A. A. Zghoul *Record values from a family of J-shaped distributions*, Statistica, vol. 71(3):355, 2011.
31. M. Zhou, D. W. Yang, Y. Wang, and S. Nadarajah, *Some J-shaped distributions: sums, products and ratios. In: Reliability and Maintainability Symposium. RAMS'06.*, Annual; IEEE, pp. 175–181, 2006.