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Semi-Infinite Mathematical Programming Problems Involving Generalized Convexity

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Abstract In this paper, we consider semi-infinite mathematical programming problems with equilibrium constraints (SIMPEC). By using the notion of convexificators, we establish sufficient optimality conditions for the SIMPEC. We formulate Wolfe and Mond-Weir type dual models for the SIMPEC under generalized convexity assumptions. Moreover, weak and strong duality theorems are established to relate the SIMPEC and two dual programs in the framework of convexificators.

Keywords Duality, Convexificators, Generalized convexity, Constraint qualification

AMS 2010 subject classifications 90C46, 49J52, 90C30

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1. Introduction

A semi-infinite programming (SIP) is an optimization problem in finitely many variables on a feasible set described by infinitely many constraints. There are many applications of SIP in various fields such as robotics, mathematical physics, Chebyshev approximation, optimal control, robust optimization, transportation problems, fuzzy sets, cooperative games, engineering design etc. (see [41, 15]). For basic theory, survey articles on SIP we refer to [30, 35, 18] and for monograph [11].

The notion of convexificators can be seen as a generalization of subdifferentials. Jeyakumar and Luc [17] has shown that the Clarke subdifferentials[6], Michel-Penot subdifferentials[32], Ioffe-Mordukhovich subdifferentials[16] and Treiman subdifferentials[47] of a locally Lipschitz real-valued function are convexificators and these known subdifferentials may contain the convex hull of a convexificator. Unlike some of the subdifferentials which are compact and convex objects, convexificators are not necessarily compact or convex. We refer to the recent results [26, 29, 27, 4].

Generalized convex functions have been introduced in order to weaken the convexity requirements as much as possible to obtain results related to optimization theory [19, 24, 25]. One of the significant generalization of convex function is invex function [12, 8]. The class of invex functions preserves many properties of the class of convex functions and has shown to be very useful in a variety of applications [33]. It is well known that optimality and duality [43, 28, 44] theory provide the foundation of algorithms for a solution of an optimization problem and hence constitute an important portion in the study of mathematical programming. Mond-Weir [37] and Wolfe [48] dual models have been studied for semi-infinite programming problems [36, 39], bi-level problems [46], mathematical programs with vanishing constraints [34], and mathematical programming problem with equilibrium constraints [40, 20, 21, 22, 23].

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Mathematical programs with equilibrium constraints is an extension of the class of bilevel programming problems[45, 31, 7, 9], which is also known as mathematical programs with complementary constraints. MPEC plays a vital role in many fields such as capacity enhancement in traffic, dynamic pricing in telecommunication networks, engineering design, economic equilibrium [42, 5, 13, 14]. Many practical problems in these fields have been modeled using the MPEC formulation. In practice, it is natural that an MPEC problem may arise where infinitely many restrictions are present rather than finite many restrictions in finitely many variables. This gives us a motivation to formulate semi-infinite mathematical program with equilibrium constraints (SIMPEC).

Motivated by the earlier work of T. Antczak over (p, r)- invex functions [1], B-(p,r)-invex and generalized invex functions [2] and nonsmooth version of r-invex functions [3], we will introduce the definitions of p-invex function and generalized p-invex functions under the framework of convexificators. We will establish sufficient optimality condition for the SIMPEC involving p-invex function and generalized p-invex functions and we will also provide supportive example corresponding to the sufficient optimality condition. We will also derive weak and strong duality theorems relating to the SIMPEC and two dual models (Wolfe and Mond-Weir) under the framework of convexificators using generalized invexity assumptions.

The organization of this paper is as follows: in Section 2, we give some preliminaries, definitions, and results. In Section 3, we establish sufficient optimality condition for the SIMPEC, using convexificators. In Section 4, we derive weak and strong duality theorems relating to the SIMPEC and two dual models using generalized invex functions. In Section 5, we conclude the results of this paper.

2. Preliminaries

Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. Let C be a nonempty subset of $\mathbb{R}^n$. The convex hull of C and the convex cone generated by C are denoted by $co\ C$ and $cone\ C$, respectively.

The negative polar cone is defined by $C^- = \{u \in \mathbb{R}^n | \, \forall\, x \in C, \ \langle x, u\rangle \leqslant 0\}$.
Let C be a nonempty subset of $\mathbb{R}^n$ and $x \in cl\ C$ (closure of C) then the contingent cone $T(x, C)$ to C at x is defined by

$$T(x, C) = \{u \in \mathbb{R}^n \mid \exists\, t_n \downarrow 0, \ \exists\, u_n \to u \text{ such that } x + t_n u_n \in C\}.$$

We consider SIMPEC in the following form:

$$\begin{aligned} \text{SIMPEC} \quad \min \quad & F(x) \\ \text{subject to}: \quad & g(x, t) \leqslant 0 \ \forall\, t \in T, \ \ h(x) = 0, \\ & \Phi(x) \geqslant 0, \ \Psi(x) \geqslant 0, \ \langle \Phi(x), \Psi(x)\rangle = 0, \end{aligned}$$

where the index set T is an infinite compact subset of $\mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \times T \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $\Phi : \mathbb{R}^n \to \mathbb{R}^l$ and $\Psi : \mathbb{R}^n \to \mathbb{R}^l$ are given functions.

Along the lines of [36] for a given feasible vector $\tilde{x}$ for the SIMPEC, we define the following index sets:

$$\begin{aligned} & T_g := T_g(\tilde{x}) := \{t \in T : g(\tilde{x}, t) = 0\}, \\ & \delta := \delta(\tilde{x}) := \{i \in \{1, 2, \dots, l\} : \Phi_i(\tilde{x}) = 0, \Psi_i(\tilde{x}) > 0\}, \\ & \alpha := \alpha(\tilde{x}) := \{i \in \{1, 2, \dots, l\} : \Phi_i(\tilde{x}) = 0, \Psi_i(\tilde{x}) = 0\}, \\ & \kappa := \kappa(\tilde{x}) := \{i \in \{1, 2, \dots, l\} : \Phi_i(\tilde{x}) > 0, \Psi_i(\tilde{x}) = 0\}, \end{aligned}$$

where the set α is known as degenerate set.

Definition 1
Let $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, $x \in \mathbb{R}^n$ and let $F(x)$ be finite. Then, the *lower and upper Dini directional derivatives* of F at x in the direction v are defined, respectively, by

$$F_d^-(x, v) := \liminf_{t \to 0^+} \frac{F(x + tv) - F(x)}{t},$$

and

$$F_d^+(x,v) := \limsup_{t\to 0^+} \frac{F(x+tv)-F(x)}{t}.$$

Definition 2
(see [17]) A function $F:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is said to admit an upper convexificator, $\partial^* F(x)$ at $x\in\mathbb{R}^n$ if $\partial^* F(x)\subseteq \mathbb{R}^n$ is a closed set and, for every $v\in\mathbb{R}^n$,

$$F_d^-(x,v) \leqslant \sup_{\xi\in\partial^* F(x)} \langle \xi, v\rangle .$$

Definition 3
(see [17]) A function $F:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is said to admit a lower convexificator, $\partial_* F(x)$ at $x\in\mathbb{R}^n$ if $\partial_* F(x)\subseteq \mathbb{R}^n$ is a closed set and, for every $v\in\mathbb{R}^n$,

$$F_d^+(x,v) \geqslant \inf_{\xi\in\partial_* F(x)} \langle \xi, v\rangle .$$

The function F is said to have a convexificator $\partial^*_* F(x)\subseteq\mathbb{R}^n$ at $x\in\mathbb{R}^n$, if $\partial^* F(x)$ is both upper and lower convexificator of F at x.

Definition 4
(see [10]) A function $F:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ is said to admit an upper semi-regular convexificator, $\partial^* F(x)$ at $x\in\mathbb{R}^n$ if $\partial^* F(x)\subseteq \mathbb{R}^n$ is a closed set and, for every $v\in\mathbb{R}^n$

$$F_d^+(x,v) \leqslant \sup_{\xi\in\partial^* F(x)} \langle \xi, v\rangle . \tag{1}$$

If equality holds in (1), then $\partial^* F(x)$ is called an upper regular convexificator of F at x.

Motivated by the definition of (p,r)- invex functions [1] and nonsmooth version of r-invex functions [3], we are introducing the definitions of p-invex function and generalized p-invex functions in terms of convexificators.

Definition 5
Let $\eta:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be a kernel function and let $F:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be an extended real valued function, which admit convexificator at $\tilde{x}\in\mathbb{R}^n$. Then F is said to be
(i) ∂^*_*-p-invex at $\tilde{x}$ with respect to η if for every $x\in\mathbb{R}^n$,

$$F(x)\geqslant F(\tilde{x}) + \frac{1}{p}\langle \xi, e^{p\eta(x,\tilde{x})}-\mathbf{1}\rangle, \forall\, \xi\in\partial^*_* F(\tilde{x}), p\neq 0.$$

(ii) ∂^*_*-p-pseudoinvex at $\tilde{x}$ with respect to η if for every $x\in\mathbb{R}^n$,

$$\exists\, \xi\in\partial^*_* F(\tilde{x}),\ \frac{1}{p}\langle \xi, e^{p\eta(x,\tilde{x})}-\mathbf{1}\rangle \geqslant 0 \Rightarrow F(x)\geqslant F(\tilde{x}), p\neq 0.$$

(iii) ∂^*_*-p-quasiinvex at $\tilde{x}$ with respect to η if for every $x\in\mathbb{R}^n$,

$$F(x)\leqslant F(\tilde{x}) \Rightarrow \frac{1}{p}\langle \xi, e^{p\eta(x,\tilde{x})}-\mathbf{1}\rangle \leqslant 0, \forall\, \xi\in\partial^*_* F(\tilde{x}), p\neq 0.$$

Now we provide following examples in support of the definitions given above.

Example 2.1 Consider the function $F:\mathbb{R}\to\mathbb{R}$ is given by

$$F(x)=\begin{cases} 1+x; & x\geqslant 0,\\ 1+|\sin x|; & x<0,\end{cases}$$

then the function becomes ∂_*^*-p-invex at $\tilde{x} = 0$ with respect to the kernel function, $\eta(x, \tilde{x}) = \cos x \sin \tilde{x}$ and $\partial_*^* F(0) = \{-1, 1\}$.
Example 2.2 Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by

$$F(x) = \begin{cases} 2 + x; & x \geqslant 0, \\ 3 - |\cos x|; & x < 0, \end{cases}$$

if we take point $\tilde{x} = 0$, then the function becomes ∂_*^*-p-pseudoinvex function at $\tilde{x} = 0$ with respect to the kernel function, $\eta(x, \tilde{x}) = 1 + x \sin x$ and $\partial_*^* F(0) = \{0, 1\}$.

Example 2.3 Consider the function $F : \mathbb{R} \to \mathbb{R}$ is given by

$$F(x) = \begin{cases} 1 - \frac{x}{2}; & x \geqslant 0, \\ |\cos x|; & x < 0, \end{cases}$$

if we take point $\tilde{x} = 0$ then the function becomes ∂_*^*-p-quasiinvex function at $\tilde{x} = 0$ with respect to the kernel function, $\eta(x, \tilde{x}) = \cos x \sin \tilde{x}$, and $\partial_*^* F(0) = \{-\frac{1}{2}, 0\}$.

Remark 1
Based on the Definition 5, the definition of ∂_*^*-p-invex function and generalized ∂_*^*-p-invex functions can also be given in terms of upper semi-regular convexificators.

Pandey and Mishra[38] presented the following notations for SIMPEC:

$$\begin{aligned}
&g = \bigcup_{i=1}^{m} co\partial^* g(\tilde{z}, t_i), \quad h = \bigcup_{i=1}^{p} co\partial^* h_i(\tilde{z}) \cup co\partial^*(-h_i)(\tilde{z}), \\
&\Phi_\delta = \bigcup_{i\in\delta} co\partial^* \Phi_i(\tilde{z}) \cup co\partial^*(-\Phi_i)(\tilde{z}), \quad \Phi_\alpha = \bigcup_{i\in\alpha} co\partial^* \Phi_i(\tilde{z}), \\
&\Psi_\kappa = \bigcup_{i\in\kappa} co\partial^* \Psi_i(\tilde{z}) \cup co\partial^*(-\Psi_i)(\tilde{z}), \quad \Psi_\alpha = \bigcup_{i\in\alpha} co\partial^* \Psi_i(\tilde{z}), \\
&(\Phi\Psi)_\alpha = \bigcup_{i\in\alpha} co\partial^*(-\Phi_i)(\tilde{z}) \cup co\partial^*(-\Psi_i)(\tilde{z}), \\
&\Gamma(\tilde{z}) := g^- \cap h^- \cap \Phi_\delta^- \cap \Psi_\kappa^- \cap (\Phi\Psi)_\alpha^-,
\end{aligned}$$

where, $t_1, t_2, \ldots, t_m \in T_g(\tilde{z}),\ m \leqslant n+1$, and $\tilde{z}$ is a feasible point of the problem SIMPEC.

The following definitions are taken from Pandey and Mishra [38] for SIMPEC.

Definition 6
Let $\tilde{z}$ be a feasible point of SIMPEC, and assume that all functions have upper convexificators considered above at $\tilde{z}$. We say that the generalized standard Abadie constraint qualification (GS Abadie CQ) holds at $\tilde{z}$ if at least one of the dual sets used in the definition of $\Gamma(\tilde{z})$ is nonzero and

$$\Gamma(\tilde{z}) \subset T(C, \tilde{z}).$$

Definition 7
A feasible point $\tilde{z}$ of SIMPEC is called the generalized alternatively stationary point (GA-stationary) point if there exist $\beta = (\beta^g, \beta^h, \beta^\Phi, \beta^\Psi) \in \mathbb{R}^{k+p+2l}, \gamma \in (\gamma^h, \gamma^\Phi, \gamma^\Psi) \in \mathbb{R}^{p+2l}$ and $t_1, t_2, \ldots, t_m \in T_g(\tilde{z}),\ m \leqslant n+1$, such that

the following conditions hold:

$$0 \in co\partial^* F(\tilde{z}) + \sum_{i=1}^{m} \beta_i^g co\partial^* g(\tilde{z}, t_i) + \sum_{r=1}^{p} \left[\beta_r^h co\partial^* h_r(\tilde{z}) + \gamma_r^h co\partial^*(-h_r)(\tilde{z})\right]$$
$$+ \sum_{i=1}^{l} \left[\beta_i^\Phi co\partial^*(-\Phi_i)(\tilde{z}) + \beta_i^\Psi co\partial^*(-\Psi_i)(\tilde{z})\right]$$
$$+ \sum_{i=1}^{l} \left[\gamma_i^\Phi co\partial^*(\Phi_i)(\tilde{z}) + \gamma_i^\Psi co\partial^*(\Psi_i)(\tilde{z})\right],$$
$$\beta_i^g \geqslant 0,\ (i = 1, 2, \dots, m),\ \beta_r^h,\ \gamma_r^h \geqslant 0,\ (r = 1, 2, \dots, p), \tag{2}$$
$$\beta_i^\Phi, \beta_i^\Psi, \gamma_i^\Phi, \gamma_i^\Psi \geqslant 0,\ (i = 1, 2, \dots, l), \tag{3}$$
$$\beta_\kappa^\Phi = \beta_\delta^\Psi = \gamma_\kappa^\Phi = \gamma_\delta^\Psi = 0,\ \forall\, i \in \alpha,\ \gamma_i^\Phi = 0 \text{ or } \gamma_i^\Psi = 0. \tag{4}$$

Definition 8
A feasible point $\tilde{z}$ of SIMPEC is called the generalized strong stationary (GS-stationary) point if there exist $\beta = (\beta^g, \beta^h, \beta^\Phi, \beta^\Psi) \in \mathbb{R}^{k+p+2l}, \gamma \in (\gamma^h, \gamma^\Phi, \gamma^\Psi) \in \mathbb{R}^{p+2l}$ and $t_1, t_2, \dots, t_m \in T_g(\tilde{z}),\ \ m \leqslant n+1,$ satisfying conditions (2) and (3) together with the following condition:

$$\beta_\kappa^\Phi = \beta_\delta^\Psi = \gamma_\kappa^\Phi = \gamma_\delta^\Psi = 0,\ \forall\, i \in \alpha,\ \gamma_i^\Phi = 0,\ \gamma_i^\Psi = 0.$$

The following result shows that GS-stationarity is a necessary optimality condition for SIMPEC.

Theorem 1
[38] Let $\tilde{z}$ be a local optimal solution of SIMPEC. Suppose that F is locally Lipschitz function at $\tilde{z}$, which admits a bounded upper semi-regular convexificator $\partial^* F(\tilde{z})$. Assume also that GS-ACQ holds at $\tilde{z}$ and that the cone

$$\delta = cone\ co\ g + cone\ co\ h + cone\ co\ \Phi_\delta + cone\ co\ \Psi_\kappa + cone\ co\ (\Phi\Psi)_\alpha,$$

is closed, then $\tilde{z}$ is a GS-stationary point.

Corollary 1
[38] Let $\tilde{z}$ be a local optimal solution of SIMPEC. Suppose that F is locally Lipschitz near $\tilde{z}$. Assume also that F and effective constraint functions admit bounded upper semi-regular convexificators at $\tilde{z}$. If GS-ACQ holds at $\tilde{z}$, then $\tilde{z}$ is a GS-stationary point.

Note: The following index sets will be used in Section 3 and 4, respectively:

$$\alpha_\gamma^\Phi := \{i \in \alpha : \gamma_i^\Psi = 0, \gamma_i^\Phi > 0\},$$
$$\alpha_\gamma^\Psi := \{i \in \alpha : \gamma_i^\Psi > 0, \gamma_i^\Phi = 0\},$$
$$\delta_\gamma^+ := \{i \in \delta : \gamma_i^\Phi > 0\},$$
$$\kappa_\gamma^+ := \{i \in \kappa : \gamma_i^\Psi > 0\}.$$

In the next section, we will obtain sufficient optimality condition under generalized invexity assumptions using the notion of convexificators.

3. Sufficient Optimality condition

The following theorem shows that under generalized ∂^*-p-invexity assumptions, GA-stationarity turns into a global sufficient optimality condition.

Theorem 2
Let $\tilde{z}$ be a feasible GA-stationary point of SIMPEC and assume there is a real number $p \neq 0$ such that F is ∂^*-p-pseudoinvex at $\tilde{z}$ with respect to the kernel η and $g(.,t)\ (t \in T_g), \pm h_r\ (r = 1, 2, \dots, p),\ -\Phi_i\ (i \in \delta \cup \alpha), -\Psi_i\ (i \in \alpha \cup \kappa)$ are ∂^*-p- quasiinvex at $\tilde{z}$ with respect to the common kernel η. If $\alpha^{\Phi}_{\gamma} \cup \alpha^{\Psi}_{\gamma} \cup \delta^{+}_{\gamma} \cup \kappa^{+}_{\gamma} = \varnothing$ *then* $\tilde{z}$ is a global optimal solution of SIMPEC.

Proof
Let us consider that x be any arbitrary feasible point of SIMPEC. Then for any $t_i \in T_g(\tilde{z})$.

$$g(x, t_i) \leqslant g(\tilde{z}, t_i),$$

by ∂^*-p-quasiinvexity of $g(x, t_i)$ at $\tilde{z}$, it follows that

$$\frac{1}{p}\langle \xi^g_i, e^{p\eta(x,\tilde{z})} - \mathbf{1}\rangle \leqslant 0,\ \forall\, \xi^g_i \in \partial^* g(\tilde{z}, t_i),\ \forall\, t_i \in T_g(\tilde{z}). \tag{5}$$

Similarly, we have

$$\frac{1}{p}\left\langle \zeta_r, e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0,\ \forall\, \zeta_r \in \partial^* h_r(\tilde{z}),\ \forall\, r = \{1, 2, \dots, p\}, \tag{6}$$

$$\frac{1}{p}\left\langle \nu_r, e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0,\ \forall\, \nu_r \in \partial^*(-h_r)(\tilde{z}),\ \forall\, r = \{1, 2, \dots, p\}, \tag{7}$$

$$\frac{1}{p}\left\langle \xi^{\Phi}_i, e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0,\ \forall\, \xi^{\Phi}_i \in \partial^*(-\Phi_i)(\tilde{z}),\ \forall\, i \in \delta \cup \alpha, \tag{8}$$

$$\frac{1}{p}\left\langle \xi^{\Psi}_i, e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0,\ \forall\, \xi^{\Psi}_i \in \partial^*(-\Psi_i)(\tilde{z}),\ \forall\, i \in \alpha \cup \kappa. \tag{9}$$

If $\alpha^{\Phi}_{\gamma} \cup \alpha^{\Psi}_{\gamma} \cup \delta^{+}_{\gamma} \cup \kappa^{+}_{\gamma} = \varnothing$, multiplying (5)-(9) by $\beta^g_i \geqslant 0\ (i = 1, 2, \dots, m),\ \beta^h_r > 0\ (r = 1, 2, \dots, p),\ \gamma^h_r > 0\ (r = 1, 2, \dots, p),\ \beta^{\Phi}_i > 0\ (i \in \delta \cup \alpha),\ \beta^{\Psi}_i > 0\ (i \in \alpha \cup \kappa)$, respectively and adding, we obtain

$$\frac{1}{p}\left\langle \sum_{i=1}^{m} \beta^g_i \xi^g_i, e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0,\ \ \frac{1}{p}\left\langle \sum_{r=1}^{p} \left[\beta^h_r \zeta_r + \gamma^h_r \nu_r\right], e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0,$$
$$\frac{1}{p}\left\langle \sum_{\delta\cup\alpha} \beta^{\Phi}_i \xi^{\Phi}_i, e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0,\ \ \frac{1}{p}\left\langle \sum_{\alpha\cup\kappa} \beta^{\Psi}_i \xi^{\Psi}_i, e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0.$$

Therefore,

$$\frac{1}{p}\left\langle \left(\sum_{i=1}^{m} \beta^g_i \xi^g_i + \sum_{r=1}^{p} \left[\beta^h_r \zeta_r + \gamma^h_r \nu_r\right] + \sum_{\delta\cup\alpha} \beta^{\Phi}_i \xi^{\Phi}_i + \sum_{\alpha\cup\kappa} \beta^{\Psi}_i \xi^{\Psi}_i\right), e^{p\eta(x,\tilde{z})} - \mathbf{1}\right\rangle \leqslant 0,$$

for all $\xi^g_i \in co\partial^* g(\tilde{z}, t_i),\ \zeta_r \in co\partial^* h_r(\tilde{z}),\ \nu_r \in co\partial^*(-h_r)(\tilde{z}),\ \xi^{\Phi}_i \in co\partial^*(-\Phi_i)(\tilde{z})$ and $\xi^{\Psi}_i \in co\partial^*(-\Psi_i)(\tilde{z})$. Thus by GA-stationarity of $\tilde{z}$, we can choose $\xi \in co\partial^* F(\tilde{z})$, such that,

$$\frac{1}{p}\langle \xi, e^{p\eta(x,\tilde{z})} - \mathbf{1}\rangle \geqslant 0.$$

By ∂^*-p-pseudoinvexity of F at $\tilde{z}$ with respect to the common kernel η and for the same real number $p \neq 0$, we have $F(x) \geqslant F(\tilde{z})$ for all feasible points x. Hence $\tilde{z}$ is a global optimal solution of SIMPEC. This completes the proof. □

The following example illustrates Theorem 2.

Example 1
Consider the following SIMPEC problem

SIMPEC

$$\min F(x) = \begin{cases} x^2|cos\frac{\pi}{x}|, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

,

subject to :

$$g(x,t) = \begin{cases} -2x^2 - t \leqslant 0, \ \forall\, t \in [0,1], x \in [-2,2] \\ 0, \quad x \notin [-2,2] \end{cases}$$

$$\theta(x) = \begin{cases} 0, \ x = \{-2,2\} \\ |x|, \quad otherwise \end{cases}$$

$$\psi(x) = \begin{cases} 0, \ x = \{-2,2\} \\ x^2, \ otherwise \end{cases}$$

$\langle \theta(x), \Psi(x) \rangle = 0.$

Now, one can see that the feasible points of SIMPEC are {-2,0,2}. Here F is ∂^*-p-pseudoinvex at $\tilde{z} = 0$ with respect to the kernel, $\eta(x, \tilde{z}) = \cos x \sin \tilde{z}$ and for a nonzero real no $p = 1$. Further, $g, -\theta$ and $-\ \Psi$ are ∂^*-p-quasiinvex at $\tilde{z} = 0$ with respect to the common kernel, $\eta(x, \tilde{z}) = \cos x \sin \tilde{z}$ and for the same nonzero real no $p = 1$. We have $co\partial^* F(0) = [-\pi, \pi],\ co\partial^* g(0, t_1) = \{0\},\ t_1 = 0,\ co\partial^*(-\theta)(0) = \{-1, 1\}$ and $co\partial^*(-\Psi)(0) = \{0\}$. One can easily verify that, there exist $\beta^g = 1, \beta^\theta = 1$ and $\beta^\Psi = 1$ such that the feasible point $\tilde{z} = 0$ is a GA- stationary point and $\tilde{z} = 0$ is a global optimal solution for the given primal problem SIMPEC. Hence the assumptions of the Theorem 2, are satisfied.

4. Duality

In this section, we formulate and study a Wolfe type dual problem for SIMPEC using ∂^*-p-invexity. We also formulate Mond-Weir type dual problem and study SIMPEC using ∂^*-p-invexity and generalized ∂^*-p-invexity assumptions.

The Wolfe type dual for the problem SIMPEC is formulated as follows:

$$\text{WD}(\tilde{x}) \qquad \max_{z,\beta,\gamma} \left\{ F(z) + \sum_{i=1}^{m} \beta_i^g g(z,t_i) + \sum_{r=1}^{p} \rho_r^h h_r(z) - \sum_{i=1}^{l} \left[\beta_i^\Phi \Phi_i(z) + \beta_i^\Psi \Psi_i(z) \right] \right\}$$

subject to :

$$\begin{aligned} 0 \in co\partial^* F(z) &+ \sum_{i=1}^{m} \beta_i^g co\partial^* g(z,t_i) + \sum_{r=1}^{p} \left[\beta_r^h co\partial^* h_r(z) + \gamma_r^h co\partial^*(-h_r)(z) \right] \\ &+ \sum_{i=1}^{l} \left[\beta_i^\Phi co\partial^*(-\Phi_i)(z) + \beta_i^\Psi co\partial^*(-\Psi_i)(z) \right], \\ &\beta_i^g \geqslant 0,\ (i = 1,2,\ldots,m),\ \beta_r^h,\ \gamma_r^h \geqslant 0,\ (r = 1,2,\ldots,p), \\ &\beta_i^\Phi, \beta_i^\Psi, \gamma_i^\Phi, \gamma_i^\Psi \geqslant 0,\ (i = 1,2,\ldots,l), \\ &\beta_\kappa^\Phi = \beta_\delta^\Psi = \gamma_\kappa^\Phi = \gamma_\delta^\Psi = 0,\ \forall\, i \in \alpha,\ \gamma_i^\Phi = 0,\ \gamma_i^\Psi = 0, \end{aligned} \tag{10}$$

where, $\rho_r^h = \beta_r^h - \gamma_r^h, \beta = (\beta^g, \beta^h, \beta^\Phi, \beta^\Psi) \in \mathbb{R}^{k+p+2l}$, $\gamma = (\gamma^h, \gamma^\Phi, \gamma^\Psi) \in \mathbb{R}^{p+2l}$ and $t_1, t_2, \ldots, t_m \in T_g,\ m \leqslant n+1$.

Theorem 3
(Weak Duality) Let $\tilde{x}$ be feasible for SIMPEC, (z, β) be feasible for the dual WD and the index sets $T_g, \delta, \alpha, \kappa$ are defined accordingly. Suppose there is a real number $p \neq 0$ such that $F,\ g(.,t)\ (t \in T),\ \pm h_r\ (r = 1,2,\ldots,p),\ -\Phi_i\ (i \in \delta \cup \alpha),\ -\Psi_i\ (i \in \alpha \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-invex functions at z, with respect to the common kernel η. If $\alpha_\gamma^\Phi \cup \alpha_\gamma^\Psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \varnothing$, then for any x feasible for the SIMPEC, we have

$$F(x) \geqslant F(z) + \sum_{i=1}^{m} \beta_i^g g(z,t_i) + \sum_{r=1}^{p} \rho_r^h h_r(z) - \sum_{i=1}^{l} \left[\beta_i^\Phi \Phi_i(z) + \beta_i^\Psi \Psi_i(z)\right].$$

Proof
Suppose that x be any arbitrary feasible point for the SIMPEC. It follows that

$$g(x,t) \leqslant 0,\ \forall\, t \in T \text{ and } h_r(x) = 0,\ \ r = 1,2,\ldots,p.$$

Since F is ∂^*-p-invex at z, with respect to the kernel η, we have

$$F(x) - F(z) \geqslant \frac{1}{p}\langle \xi, e^{p\eta(x,z)} - \mathbf{1}\rangle,\ \forall\, \xi \in \partial^* F(z). \tag{11}$$

Similarly,

$$g(x,t_i) - g(z,t_i) \geqslant \frac{1}{p}\left\langle \xi_i^g, e^{p\eta(x,z)} - \mathbf{1}\right\rangle, \quad \forall\, \xi_i^g \in \partial^* g(z,t_i),\ \forall\, t_i \in T_g(\tilde{x}), \tag{12}$$

$$h_r(x) - h_r(z) \geqslant \frac{1}{p}\left\langle \zeta_r, e^{p\eta(x,z)} - \mathbf{1}\right\rangle, \quad \forall\, \zeta_r \in \partial^* h_r(z),\ \forall\, r = \{1,2,\ldots,p\}, \tag{13}$$

$$-h_r(x) + h_r(z) \geqslant \frac{1}{p}\left\langle \nu_r, e^{p\eta(x,z)} - \mathbf{1}\right\rangle, \quad \forall\, \nu_r \in \partial^*(-h_r)(z),\ \forall\, r = \{1,2,\ldots,p\}, \tag{14}$$

$$-\Phi_i(x) + \Phi_i(z) \geqslant \frac{1}{p}\left\langle \xi_i^\Phi, e^{p\eta(x,z)} - \mathbf{1}\right\rangle, \quad \forall\, \xi_i^\Phi \in \partial^*(-\Phi_i)(z),\ \forall\, i \in \delta \cup \alpha, \tag{15}$$

$$-\Psi_i(x) + \Psi_i(z) \geqslant \frac{1}{p}\left\langle \xi_i^\Psi, e^{p\eta(x,z)} - \mathbf{1}\right\rangle, \quad \forall\, \xi_i^\Psi \in \partial^*(-\Psi_i)(z),\ \forall\, i \in \alpha \cup \kappa. \tag{16}$$

If $\alpha_\gamma^\Phi \cup \alpha_\gamma^\Psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \varnothing$, then multiplying (12)-(16) by $\beta_i^g \geqslant 0\ \ (i = 1,2,\ldots,m)$, $\beta_r^h > 0\ \ (r = 1,2,\ldots,p)$, $\gamma_r^h > 0\ (r = 1,2,\ldots,p)$, $\beta_i^\Phi > 0\ (i \in \delta \cup \alpha)$, $\beta_i^\Psi > 0\ (i \in \alpha \cup \kappa)$, respectively and adding (11)- (16), it follows that

$$\begin{aligned} F(x) - F(z) &+ \sum_{i=1}^{m} \beta_i^g g(x,t_i) - \sum_{i=1}^{m} \beta_i^g g(z,t_i) + \sum_{r=1}^{p} \beta_r^h h_r(x) - \sum_{r=1}^{p} \beta_r^h h_r(z) - \sum_{r=1}^{p} \gamma_r^h h_r(x) \\ &+ \sum_{r=1}^{p} \gamma_r^h h_r(z) - \sum_{i=1}^{l} \beta_i^\Phi \Phi_i(x) + \sum_{i=1}^{l} \beta_i^\Phi \Phi_i(z) - \sum_{i=1}^{l} \beta_i^\Psi \Psi_i(x) + \sum_{i=1}^{l} \beta_i^\Psi \Psi_i(z) \\ &\geqslant \frac{1}{p}\left\langle \xi + \sum_{i=1}^{m} \beta_i^g \xi_i^g + \sum_{r=1}^{p} \left[\beta_r^h \zeta_r + \gamma_r^h \nu_r\right] + \sum_{i=1}^{l} \left[\beta_i^\Phi \xi_i^\Phi + \beta_i^\Psi \xi_i^\Psi\right], e^{p\eta(x,z)} - \mathbf{1} \right\rangle. \end{aligned}$$

From (10), $\exists\ \tilde{\xi} \in co\partial^* F(z),\ \ \tilde{\xi}_i^g \in co\partial^* g(z, t_i)\ \ (t_i \in T_g),\ \ \tilde{\zeta}_r \in co\partial^* h_r(z),\ \ \tilde{\nu}_r \in co\partial^*(-h_r)(z),\ \ \tilde{\xi}_i^\Phi \in co\partial^*(-\Phi_i)(z)$ and $\tilde{\xi}_i^\Psi \in co\partial^*(-\Psi_i)(z)$, such that

$$\tilde{\xi} + \sum_{i=1}^{m} \beta_i^g \tilde{\xi}_i^g + \sum_{r=1}^{p} \left[\beta_r^h \tilde{\zeta}_r + \gamma_r^h \tilde{\nu}_r\right] + \sum_{i=1}^{l} \left[\beta_i^\Phi \tilde{\xi}_i^\Phi + \beta_i^\Psi \tilde{\xi}_i^\Psi\right] = 0.$$

Therefore,

$$F(x) - F(z) + \sum_{i=1}^{m} \beta_i^g g(x, t_i) - \sum_{i=1}^{m} \beta_i^g g(z, t_i) + \sum_{r=1}^{p} \beta_r^h h_r(x) - \sum_{r=1}^{p} \beta_r^h h_r(z) - \sum_{r=1}^{p} \gamma_r^h h_r(x)$$

$$+ \sum_{r=1}^{p} \gamma_r^h h_r(z) - \sum_{i=1}^{l} \beta_i^\Phi \Phi_i(x) + \sum_{i=1}^{l} \beta_i^\Phi \Phi_i(z) - \sum_{i=1}^{l} \beta_i^\Psi \Psi_i(x) + \sum_{i=1}^{l} \beta_i^\Psi \Psi_i(z) \geqslant 0.$$

Using the feasibility of x for SIMPEC, i.e., $g(x, t_i) \leqslant 0,\ h_r(x) = 0,\ \Phi_i(x) \geqslant 0,\ \Psi_i(x) \geqslant 0$, it follows that

$$F(x) - F(z) - \sum_{i=1}^{m} \beta_i^g g(z, t_i) - \sum_{r=1}^{p} \beta_r^h h_r(z) + \sum_{r=1}^{p} \gamma_r^h h_r(z) + \sum_{i=1}^{l} \beta_i^\Phi \Phi_i(z) + \sum_{i=1}^{l} \beta_i^\Psi \Psi_i(z) \geqslant 0.$$

Hence,

$$F(x) \geqslant F(z) + \sum_{i=1}^{m} \beta_i^g g(z, t_i) + \sum_{r=1}^{p} \rho_r^h h_r(z) - \sum_{i=1}^{l} \left[\beta_i^\Phi \Phi_i(z) + \beta_i^\Psi \Psi_i(z)\right],$$

where, $\rho_r^h = \beta_r^h - \gamma_r^h$. This completes the proof. □

Theorem 4

(Strong Duality) Let $\tilde{x}$ be a local optimal solution of SIMPEC and let F be locally Lipschitz near $\tilde{x}$. Suppose that $F,\ g(., t)\ (t \in T),\ \pm h_r(r = 1, 2, \ldots, p), -\Phi_i(i \in \delta \cup \alpha), -\Psi_i(i \in \alpha \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-invex functions at $\tilde{x}$ with respect to the common kernel η and for the same real number $p \neq 0$. If GS-ACQ holds at $\tilde{x}$, then there exists $\tilde{\beta} = (\tilde{\beta}^g, \tilde{\beta}^h, \tilde{\beta}^\Phi, \tilde{\beta}^\Psi) \in \mathbb{R}^{k+p+2l}$, such that $(\tilde{x}, \tilde{\beta})$ is an optimal solution of the dual WD and the respective objective values are equal.

Proof

Since, $\tilde{x}$ is a local optimal solution of SIMPEC and GS-ACQ is satisfied at $\tilde{x}$, now, using Corollary 1, $\exists\ \tilde{\beta} = (\tilde{\beta}^g, \tilde{\beta}^h, \tilde{\beta}^\Phi, \tilde{\beta}^\Psi) \in \mathbb{R}^{k+p+2l}, \tilde{\gamma} \in (\tilde{\gamma}^h, \tilde{\gamma}^\Phi, \tilde{\gamma}^\Psi) \in \mathbb{R}^{p+2l}$, and indices $t_1, t_2, \ldots, t_m \in T_g(\tilde{x}),\ \ m \leqslant n+1$, such that GS-stationarity conditions for SIMPEC are satisfied, that is, $\exists\ \tilde{\xi} \in co\partial^* F(\tilde{x}),\ \tilde{\xi}_i^g \in co\partial^* g(\tilde{x}, t_i),\ \tilde{\zeta}_r \in co\partial^* h_r(\tilde{x}),\ \tilde{\nu}_r \in co\partial^*(-h_r)(\tilde{x}),\ \tilde{\xi}_i^\Phi \in co\partial^*(-\Phi_i)(\tilde{x})$ and $\tilde{\xi}_i^\Psi \in co\partial^*(-\Psi_i)(\tilde{x})$, such that

$$\begin{aligned}
&\tilde{\xi} + \sum_{i=1}^{m} \tilde{\beta}_i^g \tilde{\xi}_i^g + \sum_{r=1}^{p} \left[\tilde{\beta}_r^h \tilde{\zeta}_r + \tilde{\gamma}_r^h \tilde{\nu}_r\right] + \sum_{i=1}^{l} \left[\tilde{\beta}_i^\Phi \tilde{\xi}_i^\Phi + \tilde{\beta}_i^\Psi \tilde{\xi}_i^\Psi\right] = 0,\\
&\tilde{\beta}_i^g \geqslant 0,\ (i = 1, 2, \ldots, m),\ \tilde{\beta}_r^h,\ \tilde{\gamma}_r^h \geqslant 0,\ \ (r = 1, 2, \ldots, p),\\
&\tilde{\beta}_i^\Phi, \tilde{\beta}_i^\Psi, \tilde{\gamma}_i^\Phi, \tilde{\gamma}_i^\Psi \geqslant 0,\ (i = 1, 2, \ldots, l),\\
&\tilde{\beta}_\kappa^\Phi = \tilde{\beta}_\delta^\Psi = \tilde{\gamma}_\kappa^\Phi = \tilde{\gamma}_\delta^\Psi = 0,\ \forall\ i \in \alpha,\ \tilde{\gamma}_i^\Phi = 0,\ \tilde{\gamma}_i^\Psi = 0.
\end{aligned}$$

Therefore $(\tilde{x}, \tilde{\beta})$ is feasible for the dual WD. Now, using Theorem 3, we obtain

$$F(\tilde{x}) \geqslant F(z) + \sum_{i=1}^{m} \beta_i^g g(z, t_i) + \sum_{r=1}^{p} \rho_r^h h_r(z) - \sum_{i=1}^{l} \left[\beta_i^\Phi \Phi_i(z) + \beta_i^\Psi \Psi_i(z)\right], \tag{17}$$

where, $\rho_r^h = \beta_r^h - \gamma_r^h$, for any feasible solution (z, β) for the dual WD. Using the feasibility condition of SIMPEC and dual WD, that is, for $t_i \in T_g(\tilde{x}),\ g(\tilde{x}, t_i) = 0,\ h_r(\tilde{x}) = 0, (r = 1, 2, \dots, p),\ \Phi_i(\tilde{x}) = 0, \forall i \in \delta \cup \alpha$, and $\Psi_i(\tilde{x}) = 0, \forall i \in \alpha \cup \kappa$, then, we have

$$F(\tilde{x}) = F(\tilde{x}) + \sum_{i=1}^{m} \tilde{\beta}_i^g g(\tilde{x}, t_i) + \sum_{r=1}^{p} \tilde{\rho}_r^h h_r(\tilde{x}) - \sum_{i=1}^{l} \left[\tilde{\beta}_i^\Phi \Phi_i(\tilde{x}) + \tilde{\beta}_i^\Psi \Psi_i(\tilde{x})\right], \tag{18}$$

where, $\tilde{\rho}_r^h = \tilde{\beta}_r^h - \tilde{\gamma}_r^h$. Using (17) and (18), it follows that

$$\begin{aligned} F(\tilde{x}) &+ \sum_{i=1}^{m} \tilde{\beta}_i^g g(\tilde{x}, t_i) + \sum_{r=1}^{p} \tilde{\rho}_r^h h_r(\tilde{x}) - \sum_{i=1}^{l} \left[\tilde{\beta}_i^\Phi \Phi_i(\tilde{x}) + \tilde{\beta}_i^\Psi \Psi_i(\tilde{x})\right] \\ &\geqslant F(z) + \sum_{i=1}^{m} \beta_i^g g(z, t_i) + \sum_{r=1}^{p} \rho_r^h h_r(z) - \sum_{i=1}^{l} \left[\beta_i^\Phi \Phi_i(z) + \beta_i^\Psi \Psi_i(z)\right]. \end{aligned}$$

Hence, $(\tilde{x}, \tilde{\beta})$ is an optimal solution for the dual WD and the respective objective values are equal. This completes the proof. □

Now, we formulate the Mond-Weir type dual problem (MWD) for SIMPEC and establish duality theorems using convexificators.

$$\text{MWD}(\tilde{x}) \qquad \max_{z,\beta,\gamma} F(z)$$

subject to:

$$\begin{aligned} 0 \in co\partial^* F(z) &+ \sum_{i=1}^{m} \beta_i^g co\partial^* g(z, t_i) + \sum_{r=1}^{p} \left[\beta_r^h co\partial^* h_r(z) + \gamma_r^h co\partial^* (-h_r)(z)\right] \\ &+ \sum_{i=1}^{l} \left[\beta_i^\Phi co\partial^* (-\Phi_i)(z) + \beta_i^\Psi co\partial^* (-\Psi_i)(z)\right], \\ &g(z, t_i) \geqslant 0\ (t_i \in T_g(\tilde{x})),\quad h_r(z) = 0\ (r = 1, 2, \dots, p), \\ &\Phi_i(z) \leqslant 0\ (i \in \delta \cup \alpha),\quad \Psi_i(z) \leqslant 0\ (i \in \alpha \cup \kappa), \\ &\beta_i^g \geqslant 0\ (i = 1, 2, \dots, m),\ \beta_r^h, \gamma_r^h \geqslant 0\ (r = 1, 2, \dots, p), \\ &\beta_i^\Phi, \beta_i^\Psi, \gamma_i^\Phi, \gamma_i^\Psi \geqslant 0\ (i = 1, 2, \dots, l), \\ &\beta_\kappa^\Phi = \beta_\delta^\Psi = \gamma_\kappa^\Phi = \gamma_\delta^\Psi = 0\ \forall\, i \in \alpha, \gamma_i^\Phi = 0, \gamma_i^\Psi = 0, \end{aligned} \tag{19}$$

where, $\beta = (\beta^g, \beta^h, \beta^\Phi, \beta^\Psi) \in \mathbb{R}^{k+p+2l}$, $\gamma = (\gamma^h, \gamma^\Phi, \gamma^\Psi) \in \mathbb{R}^{p+2l}$ and $t_1, t_2, \dots, t_m \in T_g(\tilde{x}),\ m \leqslant n + 1$.

Theorem 5
(Weak Duality) Let $\tilde{x}$ be feasible for SIMPEC, (z, β) be feasible for the dual MWD and the index sets $T_g, \delta, \alpha, \kappa$ be defined accordingly. Suppose that $F, g(., t)\ (t \in T), \pm h_r (r = 1, 2, \dots, p), -\Phi_i (i \in \delta \cup \alpha), -\Psi_i (i \in \alpha \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-invex functions at z, with respect to the common kernel η and for the same real number $p \neq 0$. If $\alpha_\gamma^\Phi \cup \alpha_\gamma^\Psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \varnothing$, then for any x feasible for SIMPEC, we have

$$F(x) \geqslant F(z).$$

Proof
Since F is ∂^*-p-invex at z, with respect to the kernel η, then, we have

$$F(x) - F(z) \geqslant \frac{1}{p}\left\langle \xi, e^{p\eta(x,z)} - \mathbf{1}\right\rangle,\ \forall\, \xi \in \partial^* F(z). \tag{20}$$

Similarly, we have

$$g(x,t_i)-g(z,t_i)\geqslant \frac{1}{p}\Big\langle \xi_i^g, e^{p\eta(x,z)}-\mathbf{1}\Big\rangle,\quad \forall\, \xi_i^g\in\partial^* g(z,t_i),\ \forall\, t_i\in T_g, \tag{21}$$

$$h_r(x)-h_r(z)\geqslant \frac{1}{p}\Big\langle \zeta_r, e^{p\eta(x,z)}-\mathbf{1}\Big\rangle,\quad \forall\, \zeta_r\in\partial^* h_r(z),\ \forall\, r=\{1,2,\ldots,p\}, \tag{22}$$

$$-h_r(x)+h_r(z)\geqslant \frac{1}{p}\Big\langle \nu_r, e^{p\eta(x,z)}-\mathbf{1}\Big\rangle,\quad \forall\, \nu_r\in\partial^* (-h_r)(z),\ \forall\, r=\{1,2,\ldots,p\}, \tag{23}$$

$$-\Phi_i(x)+\Phi_i(z)\geqslant \frac{1}{p}\Big\langle \xi_i^\Phi, e^{p\eta(x,z)}-\mathbf{1}\Big\rangle,\quad \forall\, \xi_i^\Phi\in\partial^* (-\Phi_i)(z),\ \forall\, i\in\delta\cup\alpha, \tag{24}$$

$$-\Psi_i(x)+\Psi_i(z)\geqslant \frac{1}{p}\Big\langle \xi_i^\Psi, e^{p\eta(x,z)}-\mathbf{1}\Big\rangle,\quad \forall\, \xi_i^\Psi\in\partial^* (-\Psi_i)(z),\ \forall\, i\in\alpha\cup\kappa. \tag{25}$$

If $\alpha_\gamma^\Phi\cup\alpha_\gamma^\Psi\cup\delta_\gamma^+\cup\kappa_\gamma^+=\varnothing$, multiplying (21)-(25) by $\beta_i^g\geqslant 0\ (i=1,2,\ldots,m), \beta_r^h>0\ (r=1,2,\ldots,p),\ \gamma_r^h> 0\ (r=1,2,\ldots,p),\ \beta_i^\Phi>0\ (i\in\delta\cup\alpha),\ \beta_i^\Psi>0\ (i\in\alpha\cup\kappa)$, respectively and adding (20)-(25), we obtain

$$\begin{aligned}F(x)-F(z)&+\sum_{i=1}^m\beta_i^g g(x,t_i)-\sum_{i=1}^m\beta_i^g g(z,t_i)+\sum_{r=1}^p\beta_r^h h_r(x)-\sum_{r=1}^p\beta_r^h h_r(z)-\sum_{r=1}^p\gamma_r^h h_r(x)\\&+\sum_{r=1}^p\gamma_r^h h_r(z)-\sum_{i=1}^l\beta_i^\Phi\Phi_i(x)+\sum_{i=1}^l\beta_i^\Phi\Phi_i(z)-\sum_{i=1}^l\beta_i^\Psi\Psi_i(x)+\sum_{i=1}^l\beta_i^\Psi\Psi_i(z)\\&\geqslant\frac{1}{p}\Big\langle \xi+\sum_{i=1}^m\beta_i^g\xi_i^g+\sum_{r=1}^p\left[\beta_r^h\zeta_r+\gamma_r^h\nu_r\right]+\sum_{i=1}^l\left[\beta_i^\Phi\xi_i^\Phi+\beta_i^\Psi\xi_i^\Psi\right], e^{p\eta(x,z)}-\mathbf{1}\Big\rangle.\end{aligned}$$

From (19), $\exists$ $\tilde\xi\in co\partial^*F(z)$, $\tilde\xi_i^g\in co\partial^*g(z,t_i), \tilde\zeta_r\in co\partial^*h_r(z)$, $\tilde\nu_r\in co\partial^*(-h_r)(z)$, $\tilde\xi_i^\Phi\in co\partial^*(-\Phi_i)(z)$ and $\tilde\xi_i^\Psi\in co\partial^*(-\Psi_i)(z)$, such that

$$\tilde\xi+\sum_{i=1}^m\beta_i^g\tilde\xi_i^g+\sum_{r=1}^p\left[\beta_r^h\tilde\zeta_r+\gamma_r^h\tilde\nu_r\right]+\sum_{i=1}^l\left[\beta_i^\Phi\tilde\xi_i^\Phi+\beta_i^\Psi\tilde\xi_i^\Psi\right]=0.$$

Therefore,

$$\begin{aligned}F(x)-F(z)&+\sum_{i=1}^m\beta_i^g g(x,t_i)-\sum_{i=1}^m\beta_i^g g(z,t_i)+\sum_{r=1}^p\beta_r^h h_r(x)-\sum_{r=1}^p\beta_r^h h_r(z)-\sum_{r=1}^p\gamma_r^h h_r(x)\\&+\sum_{r=1}^p\gamma_r^h h_r(z)-\sum_{i=1}^l\beta_i^\Phi\Phi_i(x)+\sum_{i=1}^l\beta_i^\Phi\Phi_i(z)-\sum_{i=1}^l\beta_i^\Psi\Psi_i(x)+\sum_{i=1}^l\beta_i^\Psi\Psi_i(z)\geqslant 0.\end{aligned}$$

Using the feasibility of x and z for SIMPEC and MWD, respectively, we obtain

$$F(x)\geqslant F(z).$$

This completes the proof. □

Theorem 6
(Strong Duality): Let $\tilde x$ be a local optimal solution of SIMPEC and let F be locally Lipschitz near $\tilde x$. Suppose there is a real number $p\neq 0$ such that $F,\ g(.,t)\ (t\in T),\ \pm h_r\ (r=1,2,\ldots,p),\ -\Phi_i\ (i\in\delta\cup\alpha),\ -\Psi_i\ (i\in\alpha\cup\kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-invex functions at $\tilde x$ with respect to the common kernel η. If GS-ACQ holds at $\tilde x$, then there exists $\tilde\beta$, such that $(\tilde x,\tilde\beta)$ is an optimal solution of the dual MWD and the respective objective values are equal.

Proof
Since, $\tilde{x}$ is a local optimal solution of SIMPEC and the GS-ACQ is satisfied at $\tilde{x}$, now using Corollary 1, $\exists\ \tilde{\beta}=(\tilde{\beta}^g,\tilde{\beta}^h,\tilde{\beta}^\Phi,\tilde{\beta}^\Psi)\in\mathbb{R}^{k+p+2l},\tilde{\gamma}\in(\tilde{\gamma}^h,\tilde{\gamma}^\Phi,\tilde{\gamma}^\Psi)\in\mathbb{R}^{p+2l}$, and indices $t_1,t_2,\ldots,t_m\in T_g(\tilde{x}),\ m\leqslant n+1$, such that the GS-stationarity conditions for SIMPEC are satisfied, that is, $\exists\,\tilde{\xi}\in co\partial^*F(\tilde{x}),\ \tilde{\xi}_i^g\in co\partial^*g(\tilde{x},t_i),\ \tilde{\zeta}_r\in co\partial^*h_r(\tilde{x}),\ \tilde{\nu}_r\in co\partial^*(-h_r)(\tilde{x}),\ \tilde{\xi}_i^\Phi\in co\partial^*(-\Phi_i)(\tilde{x})$ and $\tilde{\xi}_i^\Psi\in co\partial^*(-\Psi_i)(\tilde{x})$, such that

$$\begin{aligned}
&\tilde{\xi}+\sum_{i=1}^m\tilde{\beta}_i^g\tilde{\xi}_i^g+\sum_{r=1}^p\left[\tilde{\beta}_r^h\tilde{\zeta}_r+\tilde{\gamma}_r^h\tilde{\nu}_r\right]+\sum_{i=1}^l\left[\tilde{\beta}_i^\Phi\tilde{\xi}_i^\Phi+\tilde{\beta}_i^\Psi\tilde{\xi}_i^\Psi\right]=0,\\
&\tilde{\beta}_i^g\geqslant 0,(i=1,2,\ldots,m),\ \tilde{\beta}_r^h,\ \tilde{\gamma}_r^h\geqslant 0,\ \ (r=1,2,\ldots,p),\\
&\tilde{\beta}_i^\Phi,\tilde{\beta}_i^\Psi,\tilde{\gamma}_i^\Phi,\tilde{\gamma}_i^\Psi\geqslant 0,\ (i=1,2,\ldots,l),\\
&\tilde{\beta}_\kappa^\Phi=\tilde{\beta}_\delta^\Psi=\tilde{\gamma}_\kappa^\Phi=\tilde{\gamma}_\delta^\Psi=0,\forall\, i\in\alpha,\ \tilde{\gamma}_i^\Phi=0,\ \tilde{\gamma}_i^\Psi=0.
\end{aligned}$$

Since $\tilde{x}$ is an optimal solution for SIMPEC, we have

$$\sum_{i=1}^m\tilde{\beta}_i^g g(\tilde{x},t_i)=0,\sum_{i=1}^p\tilde{\beta}_i^h h_i(\tilde{x})=0,\sum_{i=1}^l\tilde{\beta}_i^\Phi\Phi_i(\tilde{x})=0,\sum_{i=1}^l\tilde{\beta}_i^\Psi\Psi_i(\tilde{x})=0.$$

Therefore $(\tilde{x},\tilde{\beta})$ is feasible for MWD. By Theorem 4.3, for any feasible (z,β), we have

$$F(\tilde{x})\geqslant F(z).$$

It follows that $(\tilde{x},\tilde{\beta})$ is an optimal solution for MWD and the respective objective values are equal. This completes the proof. □

Now, we establish weak and strong duality theorems for SIMPEC and its Mond-Weir type dual problem (MWD) under generalized ∂^*-p-invexity assumptions.

Theorem 7
(Weak Duality) Let $\tilde{x}$ be feasible for SIMPEC, (z,β) be feasible for the dual MWD and the index sets T_g,δ,α,κ are defined accordingly. Suppose that F is ∂^*-p-pseudoinvex at z, with respect to the kernel η and $g(.,t)$ $(t\in T)$, $\pm h_r$ $(r=1,2,\ldots,p)$, $-\Phi_i$ $(i\in\delta\cup\alpha)$, $-\Psi_i$ $(i\in\alpha\cup\kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-quasiinvex functions at z, with respect to the common kernel η and for the same real number $p\neq 0$. If $\alpha_\gamma^\Phi\cup\alpha_\gamma^\Psi\cup\delta_\gamma^+\cup\kappa_\gamma^+=\varnothing$, then for any x feasible for the problem $SIMPEC$, we have

$$F(x)\geqslant F(z).$$

Proof
Suppose that for some feasible point x, such that $F(x)<F(z)$, then by ∂^*-p-pseudoinvexity of F at z, with respect to the kernel η, we have

$$\frac{1}{p}\langle\xi,e^{p\eta(x,z)}-\mathbf{1}\rangle<0,\forall\,\xi\in\partial^*F(z).\tag{26}$$

From (19), $\exists$ $\tilde{\xi}\in co\partial^*F(z)$, $\tilde{\xi}_i^g\in co\partial^*g(z,t_i)$, $\tilde{\zeta}_r\in co\partial^*h_r(z)$, $\tilde{\nu}_r\in co\partial^*(-h_r)(z)$, $\tilde{\xi}_i^\Phi\in co\partial^*(-\Phi_i)(z)$ and $\tilde{\xi}_i^\Psi\in co\partial^*(-\Psi_i)(z)$, such that

$$-\sum_{i=1}^m\beta_i^g\tilde{\xi}_i^g-\sum_{r=1}^p\left[\beta_r^h\tilde{\zeta}_r+\gamma_r^h\tilde{\nu}_r\right]-\sum_{\delta\cup\alpha}\beta_i^\Phi\tilde{\xi}_i^\Phi-\sum_{\alpha\cup\kappa}\beta_i^\Psi\tilde{\xi}_i^\Psi\in\partial^*F(z).\tag{27}$$

By (26), we have

$$\frac{1}{p}\left\langle\left(\sum_{i=1}^m\beta_i^g\tilde{\xi}_i^g+\sum_{r=1}^p\left[\beta_r^h\tilde{\zeta}_r+\gamma_r^h\tilde{\nu}_r\right]+\sum_{\delta\cup\alpha}\beta_i^\Phi\tilde{\xi}_i^\Phi+\sum_{\alpha\cup\kappa}\beta_i^\Psi\tilde{\xi}_i^\Psi\right),e^{p\eta(x,z)}-\mathbf{1}\right\rangle>0.\tag{28}$$

For each $t_i \in T_g,\ g(x,t_i) \leqslant 0 \leqslant g(z,t_i)$. Hence, by ∂^*-p-quasiinvexity, it follows that

$$\frac{1}{p}\left\langle \xi_i^g, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0, \forall\, \xi_i^g \in \partial^* g(z,t_i),\ \forall\, t_i \in T_g. \tag{29}$$

Similarly, we have

$$\frac{1}{p}\left\langle \zeta_r, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0,\ \forall\, \zeta_r \in \partial^* h_r(z),\ \forall\, r = \{1,2,\ldots,p\}. \tag{30}$$

For any feasible point z of the dual MWD, and for every $r, -h_r(z) = h_r(x) = 0$. On the other hand, $-\Phi_i(x) \leqslant -\Phi_i(z), \forall i \in \delta \cup \alpha$, and $-\Psi_i(x) \leqslant -\Psi_i(z), \forall i \in \alpha \cup \kappa$. By ∂^*-p- quasiinvexity, we have

$$\frac{1}{p}\left\langle \nu_r, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0,\ \forall\, \nu_r \in \partial^*(-h_r)(z),\ \forall\, r = \{1,2,\ldots,p\}, \tag{31}$$

$$\frac{1}{p}\left\langle \xi_i^\Phi, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0, \forall\, \xi_i^\Phi \in \partial^*(-\Phi_i)(z), \forall\, i \in \delta \cup \alpha, \tag{32}$$

$$\frac{1}{p}\left\langle \xi_i^\Psi, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0, \forall\, \xi_i^\Psi \in \partial^*(-\Psi_i)(z), \forall\, i \in \alpha \cup \kappa. \tag{33}$$

From Eqs, (29)-(33), we have

$$\frac{1}{p}\left\langle \tilde{\xi}_i^g, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0,\ (i = 1,2,\ldots,m),$$

$$\frac{1}{p}\left\langle \tilde{\zeta}_r, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0,\ \frac{1}{p}\left\langle \tilde{\nu}_r, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0,\ (r = 1,2,\ldots,p),$$

$$\frac{1}{p}\left\langle \tilde{\xi}_i^\Phi, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0, \forall\, i \in \delta \cup \alpha,\ \ \frac{1}{p}\left\langle \tilde{\xi}_i^\Psi, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0, \forall\, i \in \alpha \cup \kappa.$$

Since $\alpha_\gamma^\Phi \cup \alpha_\gamma^\Psi \cup \delta_\gamma^+ \cup \kappa_\gamma^+ = \varnothing$, we have

$$\frac{1}{p}\left\langle \sum_{i=1}^{m} \beta_i^g \tilde{\xi}_i^g, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0,\ \ \frac{1}{p}\left\langle \sum_{r=1}^{p} \left[\beta_r^h \tilde{\zeta}_r + \gamma_r^h \tilde{\nu}_r\right], e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0,$$

$$\frac{1}{p}\left\langle \sum_{\delta\cup\alpha} \beta_i^\Phi \tilde{\xi}_i^\Phi, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0,\ \ \frac{1}{p}\left\langle \sum_{\alpha\cup\kappa} \beta_i^\Psi \tilde{\xi}_i^\Psi, e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0.$$

Therefore,

$$\frac{1}{p}\left\langle \left(\sum_{i=1}^{m} \beta_i^g \tilde{\xi}_i^g + \sum_{r=1}^{p} \left[\beta_r^h \tilde{\zeta}_r + \gamma_r^h \tilde{\nu}_r\right] + \sum_{\delta\cup\alpha} \beta_i^\Phi \tilde{\xi}_i^\Phi + \sum_{\alpha\cup\kappa} \beta_i^\Psi \tilde{\xi}_i^\Psi \right), e^{p\eta(x,z)} - \mathbf{1} \right\rangle \leqslant 0.$$

which contradicts (28). Hence $F(x) \geqslant F(z)$. This completes the proof. □

Theorem 8
(Strong Duality) Let $\tilde{x}$ be a local optimal solution of SIMPEC and let F be locally Lipschitz near $\tilde{x}$. Suppose that F is ∂^*-p-pseudoinvex at $\tilde{x}$, with respect to the kernel η, further $g(.,t)\ (t \in T),\ \pm h_r\ (r = 1,2,\ldots,p),\ -\Phi_i\ (i \in \delta \cup \alpha),\ -\Psi_i\ (i \in \alpha \cup \kappa)$ admit bounded upper semi-regular convexificators and are ∂^*-p-quasiinvex functions at $\tilde{x}$ with respect to the common kernel η. If GS-ACQ holds at $\tilde{x}$, then there exists $\tilde{\beta}$, such that $(\tilde{x}, \tilde{\beta})$ is an optimal solution of the dual MWD and the respective objective values are equal.

Proof
The proof follows on the lines of the proof of Theorem 4.4 by using Theorem 4.5. □

5. Conclusion

We have studied SIMPEC and established sufficient optimality condition under generalized invexity assumptions. We have introduced Wolfe and Mond –Weir type dual models for the SIMPEC in the framework of convexificators. We have established weak and strong duality theorems relating to the SIMPEC and two dual models using ∂^*-p-invexity, ∂^*-p-pseudoinvexity and ∂^*-p-quasiinvexity assumptions.

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